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November 2006
LSE-CDAM-2006-18

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Abstract. Rufus Isaacs introduced Princess and Monster games in the final chapter of his classic book. The value of the Princess and Monster game on an interval is as of yet unknown. We present some numerical results to estimate this value.

In the final chapter of his classic book Differential Games, Rufus Isaacs introduced the 'Princess and Monster' games. A Monster and a Princess may move about in a restricted space, more specifically in a network, and the Monster tries to catch the Princess. They are not able to see each other and that is why this type of game is known as a Search Game [3, 7, 9]. It is different from the more familiar Game of Pursuit [5], in which both players have visual contact. In most of the Search Games that have been solved so far, the Princess is immobile; e.g., [6, 12]. The only Search Game with a mobile Princess that has been solved is the Princess and Monster game on a circle, and this was done a long time ago [1, 13]. In a complementary paper [4] we have shown that the Princess and Monster game on an interval $[-1, 1]$ is not trivial (not trivial in the sense that for the Monster it is not optimal to start at one random end and then go as fast as possible to the other) and that the value $\mathcal{V}$ of the game is bounded by $15/11 < \mathcal{V} < 13/9$. These bounds were obtained by analytical considerations and computations that can be checked by hand. In this paper we consider a restricted game that has a value $\mathcal{V}_r \leq \mathcal{V}$. By numerical simulations we show that $\mathcal{V}_r \approx 1.373$.

1. Rules of the game

The rules of the game are as follows. The Monster $\mathcal{M}$ and the Princess $\mathcal{P}$ may choose an arbitrary initial point on the closed interval $[-1, 1]$. The Monster moves at speed bounded by 1, so the trajectory of $\mathcal{M}$, $\mathcal{M}(t)$ is a continuous function with Lifschitz constant 1. The Princess may move at arbitrary speed. In [4] we have shown that $\mathcal{M}$ always moves at maximum speed and that $\mathcal{P}$ need not move at speed greater than 1. If a player moves at speed 1 then we say that the player *runs*

The minimax theorem in [2] implies that the value of the game $\mathcal{V}$ exists. The precise optimal strategies for the Monster and the Princess are not known, but we have derived some properties of optimal strategies in [4]. We have

This research was carried out while the first author visited Delft University on NWO visitor grant B61-590.

shown that it is optimal for the Princess to never cross the mid point and if she reaches an end, then she should stay there. If the Monster reaches an end, however, then he should turn and run to the opposite end.

2. A restricted search game on the interval

We have indicated in [4] that $\mathcal{V}$ can in principle be computed in an iterative manner, but the convergence is very slow and not numerically feasible. In this paper we take a different approach and restrict the number of pure strategies of the players. For the Monster we allow pure strategies of two types (the strategies of $\mathcal{M}$ and $\mathcal{P}$ are indicated by M and P respectively):

- M_1 Choose an arbitrary initial point on $[-1,+1]$ and choose a direction, right or left. $\mathcal{M}$ runs in that direction until the end and and then he runs back. In particular, if $\mathcal{M}$ starts at an end then he runs to the opposite end. This is an important pure strategy and if the Monster moves in this way, then we say that he is a *sweeper*, as in [10].
- M_2 Choose an arbitrary initial point on $[-1,+1]$ and choose a direction. $\mathcal{M}$ runs in that direction until he meets the sweeper coming from the opposite end. Then $\mathcal{M}$ turns and joins the sweeper. Subsequently he keeps running in this direction till the end (at either $+1$ or -1) has been reached and then returns to the opposite end.

For the Princess we allow the following pure strategies:

- P The Princess chooses an infinitesimal small $\varepsilon > 0$. She either hides at an end point and remains immobile, or she choose an arbitrarily initial point in the intervals $[-1+\varepsilon, -\varepsilon] \cup [\varepsilon, 1-\varepsilon]$ and remains immobile there until the sweeper coming from the nearest end is ε close. Then she runs to the middle and turns when she gets ε close, where she turns and runs back to the nearest end.

We conjecture that the optimal strategies of both players are mixed strategies which are based upon the pure strategies given above, but we were unable to prove this. We give some evidence for this conjecture at the end of the paper. However, we are able to show that against P the optimal response is to use a mixed strategy on $\{M_1, M_2\}$:

Theorem 1. *If the Princess uses a mixed strategy that consists of the pure strategies of the type listed above, then the optimal response of the Monster is to use a mixed strategy that consists of pure strategies of the two types that are listed above.*

It is possible to prove this theorem following the approach of Section 5 in [4]. We only give a sketch of the proof here. If the Princess uses these pure strategies, then she remains immobile and runs only if the sweeper is ε-close, or she hides at an end and remains immobile all the time. Therefore at time $t > 0$, either the Princess is immobile and $\mathcal{P}(t) = \pm 1$ or $\mathcal{P}(t) \in (-1+t+\varepsilon, 1-t-\varepsilon)$, or she is running and $\mathcal{P}(t) = \pm(-1+t+\varepsilon)$. The

Monster knows this. If he starts at an end, then his only sensible strategy is to run to the other end. If he starts in the middle, then he should run in one direction to increase the chance of catching the Princess while she is still immobile. Then if he is ε-close to a sweeper, the Monster may turn and then it is sensible only to run to the other side, or he may continue his run to the end, turn there and run to the other side. If we ignore ε, which we may in the limit, then we get the pure strategies given above for the Monster.

The value of this version of the search game, with the given restricted classes of strategies, is indicated by $\mathcal{V}_r$. Because of Thm 1 it satisfies $\mathcal{V}_r \leq \mathcal{V}$.

3. Approximation of $\mathcal{V}_r$ by Discretization

As a first approximation of $\mathcal{V}$ discretize the interval $[-1, 1]$ and take two grid points only -1 and $+1$: the mesh of this simple grid is $\Delta x = 2$. Discretize time accordingly into time steps of $\Delta t = 2$. It is not hard to see that it is optimal for both players to choose a grid point at random. The Monster moves to the other grid point, the Princess remains where she is. The value of this simple discretized game is 1.

As a second approximation discretize by three grid points $-1, 0, 1$. Discretize time accordinly into time-steps of $\Delta t = 1$ allowing the Princess an ε-advantage: the Princess may move on time $n - \varepsilon$ while the Monster moves on time n for $n \in \mathbb{N}$. There is an obvious symmetry in the game: each player choses -1 and $+1$ equally likely. Let's call this the end point strategy E as opposed to the mid point strategy C (of "Center") in which the initial point is 0. If the Monster chooses an end point, then he runs the opposite end as quickly as possible. If the Princess chooses an end point, then she stays there. If the Monster chooses the mid point, then he runs to a random end and runs back. If the Princess chooses the mid point, she runs to a random side at time $1 - \epsilon$. So we get a 2×2 matrix game (ignoring ε):

	E	C
E	1	1.5
C	2	0

in which the Monster chooses a row and the Princess a column. The value of this game is slightly larger: 6/5.

We discretize the game. The Princess and the Monster may only choose an initial position on a fixed equidistant grid with $2n$ points, both endpoints included in this counting ($n = 1, 2, \ldots$). Hence the mesh size Δ equals $2/(2n - 1)$. The case with $n = 1$ coincides with the first approximation given above. The Princess moves at the time steps $\Delta - \varepsilon, 2\Delta - \varepsilon, \ldots$ and the Monster moves at time steps $0, \Delta, 2\Delta, \ldots$. The game is over as soon as they occupy the same grid point. In particular, if the Princess and the Monster choose the same initial grid point, then the game is immediately over and the capture time is 0.

Remark 2. *It is obvious that the probability for both players choosing the same initial point is positive for the finite grid case. Because of the assumption that in such a case capture is immediate, this is to the disadvantage for the Princess and thus the value of this game is a lower bound for $\mathcal{V}_r$; with increasing n it will converge from below to $\mathcal{V}_r$.*
With the original game on the continuous interval $[-1,+1]$ in mind, the probability of both players starting at the same interior point is probably zero. In this vein the capture time, if $\mathcal{M}$ and $\mathcal{P}$ would start at the same point of the finite grid, could be defined differently; for instance as the average of the two possibilities of the position of $\mathcal{M}$ being an infinitesimal distance to the left, or right, of $\mathcal{P}$'s position.

We discretize the interval by putting a symmetric grid with respect to the mid point. There are n equidistant grid points smaller than 0 and there are n equidistant grid points greater than 0 at:

$$\{-1,-1+\Delta,\ldots,-1+(n-1)\Delta\} \;\cup\; \{1,1-\Delta,\ldots,1-(n-1)\Delta\}.$$

We denote the value of the discretized game by $\mathcal{V}_n$. Obviously $\mathcal{V}_n \to \mathcal{V}_r$ as $n \to \infty$.

n	$\mathcal{V}_n$
1	1
2	1.266
4	1.330
8	1.354
16	1.365
32	1.370
64	1.373

Table 1 Numerical approximation of $\mathcal{V}_r$

The matrix in the discretized game has size $8n \times 2n$ and we are unable to compute $\mathcal{V}_n$ for larger n. Our results seem to suggest that the limit value is $\mathcal{V}_r = 11/8$.

The results of our simulations show that the Princess uses three types of strategies: either she hides at either end point, with a positive probability ≈ 0.127 at each end point, or she hides at $\pm\varepsilon$ until time $1-\varepsilon$ and runs to a random end with total probability ≈ 0.236 evenly divided over the two points, or she takes an initial position in $(-1,1)$ according to a continuous probability distribution as depicted in Figure 1.
The Monster also uses two discrete strategies and one continuous strategy. He uses the sweeper strategy with probability ≈ 0.80, evenly divided between the two sweep options. He starts at $\pm(-1+\varepsilon)$ and runs to the opposite end and back with probability ≈ 0.075 for each option, or he picks an initial position in $(-1,1)$ according to a continuous probability distribution as depicted in Figure 2. If the initial position < 0 then he runs to the right end and subsequently back; if the initial position is > 0 then he runs first to the left end and then back. Note that the Monster only uses the pure

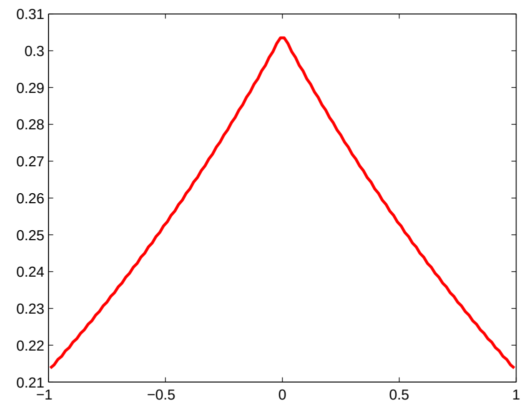

FIGURE 1. Probability density of the continuous Princess strategy

strategies of M_1 and he only uses half of these: if $\mathcal{M}(0) < 0$ then he runs to $+1$ and back to -1; if $\mathcal{M}(0) > 0$ then he runs to -1 and back to $+1$

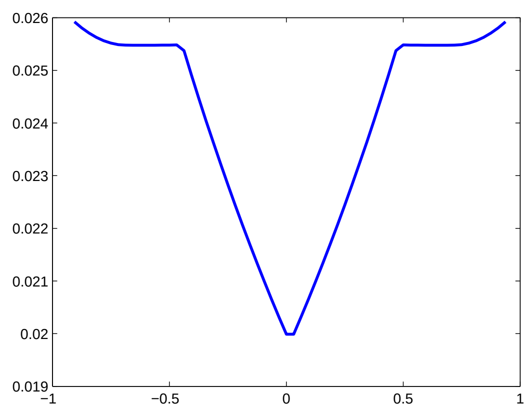

FIGURE 2. Probability density of the continuous Monster strategy

4. Approximation by means of Taylor expansions

The starting assumptions here are that both players use a continuous initial distribution, with delta functions (concentrated masses) at both ends. For the Monster we assume a continuous initial density distribution M_l on $(-1,+1)$ for immediate left turns after the start and a continuous initial density distribution M_r on $(-1,+1)$ for immediate right turns after the start. Because of symmetry, $M_l(x) = M_r(-x)$. The remaining mass,

$1-\int_1^{+1}(M_l(x)+M_r(-x))dx$, is split into two equal parts which will refer to the mass concentrations at both ends. The functions P_l and P_r are likewise defined for the Princess, the only difference being that the Princess does not immediately after the start turn left or right, but first does not move and then, at moment $\min\{1+x, 1-x\}-\epsilon$, with x being the initial position of $\mathcal{P}$, turns left or right. This ϵ-value is arbitrarily small (and positive) and indicates the fact that the Princess is followed on her heels by the Monster if both run in the same direction. Obvious constraints are that all these densities are nonnegative.
Assume the functions P_l and M_l are linear, i.e.

$$P_l(x) = c + dx,\ M_l(x) = a + bx,$$

on the interval $[-1, +1]$ and where the constants a, b must be chosen by the Monster,and $(c, d$ must be chosen by the Princess, subject to the constraints already mentioned. Thus we face the game

$$\min_{a,b} \max_{c,d} T,$$

where T denotes the time of capture. Expressing T in terms of the variables a, b, c, d and subsequently solving the minmax problem (using Maple) we find

$$a = \frac{654}{4327},\ b = \frac{624}{4327},\ c = \frac{2240}{4327},\ d = \frac{140}{4327},$$

and the value of the game is 1.345.
Please note that the admissible class of strategies for the Monster does not include the optimal strategies as obtained in the discretized game (where, for instance, $\mathcal{M}$, starting from a point in $(-1, 0)$, would only move to the right and not to the left). Note also, that the class of admissible strategies for the Princess are different from the one before (after an initial rest, she simply runs to one of the two ends and stays there). In spite of the fact that the optimal strategies (as dealt with in the previous section) are outside the current admissible classes used, it is surprising to see how close the numerically obtained values are.
This method can easily be adapted to include different classes of strategies for the Monster and Princess, such as for instance to include the possibilities of the previous section. Moreover, the method can be extended to include second and even higher-order terms in the expansions of M_l and P_l.

5. Conclusions

Our numerical simulations show that in the restricted game the Monster plays one of the two sweeper strategies S with a total probability ≈ 0.8. The Monster starts in an internal point of the interval with the complementary probability ≈ 0.2, let us call this strategy $I_{\mathcal{M}}$. The Princess chooses to hide at an end E with probability around ≈ 0.25 and starts at an internal point with probability ≈ 0.75. Let us call this latter strategy $I_{\mathcal{P}}$. The payoff of S against E and $I_{\mathcal{P}}$ is easy to compute. Our simulations indicate that in the strategy $I_{\mathcal{M}}$ the Monster starts near one end and that he runs to the other end and back. The payoff of this strategy against E is ε close to 3. The payoff of $I_{\mathcal{M}}$ against $I_{\mathcal{P}}$ is unclear. If the probabilities are right, however, then the matrix of the game should be as follows:

	E	$I_{\mathcal{P}}$
S	1	1.5
$I_{\mathcal{M}}$	3	4/5(?)

The value of this game is 11/8.

Some preliminary steps taken with respect to Taylor expansions of the optimal mixed strategies indicated that this approach seems worthwhile to be pursued further.

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