# On the Complexity of Ordered Colorings

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**Abstract**

We introduce two variants of proper colorings with imposed partial ordering on the set of colors. One variant shows very close connections to some fundamental problems in graph theory, e.g., directed graph homomorphism and list colorings. We study the border between tractability and intractability for both variants.

## 1 Introduction

We introduce two variants of proper colorings with imposed partial ordering on the set of colors. Vertices of all considered graphs $G$ are labeled with integers from 1 to $|V(G)|$, and we will use the normal order of the integers. In particular, the vertices form a completely ordered set. The set of colors forms a partially ordered set. The following ordered colorings are proper colorings satisfying additional requirements.

In the first coloring problem, we require for every two colors $A, B$ for which $A$ is smaller than $B$ in the partial order, that every vertex colored $A$ is smaller than every vertex colored $B$. We will show that this problem is in P if the set of colors contains at most two independent colors by reduction to 2-SAT, and that otherwise it is intractable.

In the second coloring problem, we require for every two colors $A, B$ for which $A$ is smaller than $B$ in the partial order, that for every edge whose end vertices are colored by $A$ and $B$, that the vertex with color $A$ is smaller than the vertex with color $B$. Note that a vertex colored $A$ can

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be greater than a vertex colored $B$ provided they are not adjacent. We show that this problem is NP-complete if the partial ordering on colors contains at least two incomparable pairs. Otherwise, the problem is in P.

## 2 Basic definitions

Most of our terminology and notation will be standard and can be found in any textbook on graph theory such as [2], and on computational complexity such as [4].

Throughout this paper a graph $G = (V, E)$ is finite, simple and loopless. We will always use $n$ for the number of vertices and assume that the vertices of the input graph $G$ are integers from 1 to $n$ ( in other words we set $V = [1, n]$ ). With this assumption we can consider the vertex set as a linearly ordered set $(V, \leq)$.

The set of available colors is denoted by *Col*. A *coloring* of a graph $G$ is a function $\varphi : V \to Col$ so that for every edge $uv \in E$ we have $\varphi(u) \neq \varphi(v)$. We additionally assume that $\preceq$ is an antisymmetric, reflexive and transitive relation on *Col*, i.e., $\mathcal{C} = (Col, \preceq)$ is a partially ordered set. We sometimes call $(Col, \preceq)$ the *color-poset*. We will study the complexity of two coloring problems in which we will color vertices of the input graph $G$ with colors in *Col* satisfying further requirements. We will refer to a coloring that satisfies these requirements as a feasible coloring.

Let $C \in Col$, the set $V_C$ of vertices in $G$ colored by $C$ will be called a *color class*.

If $(P, \preccurlyeq)$ is a poset, then an *antichain* is a set $S \subseteq P$ so that no two elements from $S$ are comparable ( i.e., for all $x, y \in S$ we have $x \not\preccurlyeq y$ and $y \not\preccurlyeq x$ ).

Given $X, Y \subseteq V$, an edge with one end-vertex in $X$ and the other in $Y$ is called an $X, Y$*-edge*. Let $E(X, Y)$ be the set of all $X, Y$-edges.

## 3 Coloring with ordered color classes

In this section we study the complexity of the following decision problem :

**Problem 3.1.**
*Fix a color-poset $(Col, \preceq)$. Given a graph $G = (V, E)$ with $V = [1, n]$, determine whether $G$ can be colored with Col such that for any two colors $A, B$ with $A \preceq B$ and for any two vertices $u \in V_A$ and $v \in V_B$ we have $u \leq v$.*

If a coloring as in Problem 3.1 exists, we say that the graph $G$ can be *feasibly colored with* $(Col, \preceq)$.

Note that if $\preceq$ is the empty relation on *Col*, then Problem 3.1 is the well-known graph coloring problem.

**Theorem 3.2.**
*If the poset $(Col, \preceq)$ does not contain an antichain of size 3, then Problem 3.1 can be solved in polynomial time. Otherwise, the problem is NP-complete.*

The proof that Problem 3.1 is in P if the longest antichain in $(Col, \preceq)$ has length at most 2 is quite technical. For this reason we first give the proof for a specific small poset with all the details. The proof of the general case will be done with less detail after that. The fairly straightforward proof that the problem is NP-complete if $(Col, \preceq)$ contains an antichain of length 3 will be the final proof in this section.

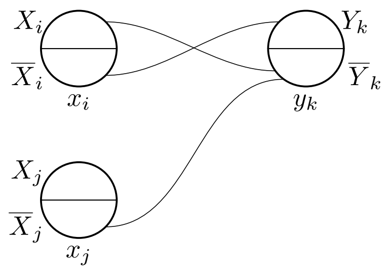
**Lemma 3.3.**
*Let $Col = \{A, B, C\}$ and suppose the only relation between the colors is $A \preceq B$. Problem 3.1 can be solved in polynomial time for this poset $(\{A, B, C\}, \preceq)$.*

*Proof.* Suppose a graph $G$ can be feasibly colored with $(\{A, B, C\}, \preceq)$. It follows that there exists a vertex $v \in V$ such that vertices in the set $V_{AC} = [1, v-1]$ are colored $A$ or $C$, and vertices in the set $V_{BC} = [v, n]$ are colored $B$ or $C$. Therefore, the subgraphs $G_{AC}$ and $G_{BC}$ of $G$, induced by $V_{AC}$ and $V_{BC}$, respectively, are bipartite. Suppose $G_{AC}$ has $k$ components and $G_{BC}$ has $\ell$ components. Let $\{(X_i, \overline{X_i})\}_{i=1}^{k}$ be the set of bipartitions of connected components $\{\mathcal{X}_i\}_{i=1}^{k}$ of $G_{AC}$, and let $\{(Y_j, \overline{Y_j})\}_{j=1}^{\ell}$ be the set of bipartitions of connected components $\{\mathcal{Y}_j\}_{j=1}^{\ell}$ of $G_{BC}$.

Every bipartite component $\mathcal{X}_i$ (respectively $\mathcal{Y}_j$) has exactly two 2-colorings. Thus, we can associate each $\mathcal{X}_i$ with a boolean variable $x_i$ as follows: if the bipartition $X_i$ of $\mathcal{X}_i$ is colored $C$, then $x_i = 0$; otherwise, $x_i = 1$. Similarly, we associate each $\mathcal{Y}_j$ with a boolean variable $y_j$: if the bipartition $Y_j$ of $\mathcal{Y}_j$ is colored $C$, then $y_j = 0$; otherwise, $y_j = 1$.

Every $(V_{AC}, V_{BC})$-edge of $G$ imposes a constraint on a feasible coloring of $G$. This constraint can be equivalently expressed in terms of the above boolean variables:
- for every $(X_i, Y_j)$-edge of $G$, the clause $x_i \vee y_j$ should be satisfied;
- for every $(X_i, \overline{Y_j})$-edge of $G$, the clause $x_i \vee \overline{y_j}$ should be satisfied;
- for every $(\overline{X_i}, Y_j)$-edge of $G$, the clause $\overline{x_i} \vee y_j$ should be satisfied;
- for every $(\overline{X_i}, \overline{Y_j})$-edge of $G$, the clause $\overline{x_i} \vee \overline{y_j}$ should be satisfied.

Let $f_v$ be the conjunction of the above 2-literal disjunctions for all $(V_{AC}, V_{BC})$-edges of $G$, see Figure 1 for an example.




$$\iff (x_i \vee \bar{y}_k) \wedge (\bar{x}_i \vee y_k) \wedge (\bar{x}_j \vee \bar{y}_k)$$

Figure 1: An example of the transformation of a graph $G$ into a 2-CNF formula.

Obviously, there exists a feasible coloring of $G$ if and only if the formula $f_v$ is satisfiable. An assignment of boolean values to variables in $f_v$ can be translated into a coloring of $G$ in linear time. Hence, to find a coloring for $G$ feasible with $(\{A, B, C\}, \preceq)$, we can use the following algorithm

- For each $v \in V$, check if the subgraphs induced on $[1, v-1]$ and on $[v, n]$ are bipartite. If so, construct the formula $f_v$ and use a 2-SAT solver to check the satisfiability of the formula. The graph $G$ has a feasible coloring if and only if for at least one $v \in V$ the subgraphs induced on $[1, v-1]$ and on $[v, n]$ are bipartite and the formula $f_v$ is satisfiable.

In [3], a 2-SAT solver working in linear time in the size of the formula is described. Note that each $f_v$ has $O(n)$ variables and $O(n^2)$ clauses. Since there are $n$ choices for $v$, Problem 3.1 can be solved in $O(n^3)$ steps. $\square$

We next give the general proofs of the two parts of Theorem 3.2.

**Theorem 3.2 A**
*If the poset $(Col, \preceq)$ does not contain an antichain of size 3, then Problem 3.1 can be solved in polynomial time.*

*Proof.* If $\mathcal{P} = (P, \preccurlyeq)$ is a poset and $u \in P$, then by $(P - u, \preccurlyeq)$ we denote the poset with the same ordering $\preccurlyeq$ on the set $P \setminus \{u\}$. For a poset $\mathcal{P}$ with $|P| = p$ elements that does not contain an antichain of size 3, we will construct of collection $\Lambda_{\mathcal{P}}$ of $p$-tuples $\overline{\lambda} = (\lambda_1, \ldots \lambda_p)$, where each $\lambda_i$ is a set of one or two elements from $P$. The *concatenation* of two tuples $\overline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\overline{\mu} = (\mu_1, \ldots, \mu_\ell)$ is $\overline{\lambda} * \overline{\mu} = (\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell)$.

The construction of $\Lambda_{\mathcal{P}}$ is recursively as follows:

- If $P$ has one element, say $P = \{a\}$, then $\Lambda_{\mathcal{P}} = \{\, \{a\}\, \}$.
- If $|P| \geq 2$ and $P$ has a maximum element $a$, then $\Lambda_{\mathcal{P}} = \{\, \overline{\lambda} * (\{a\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-a} \,\}$.
- If $|P| \geq 2$ and $P$ has no maximum element, then, since the longest antichain has length 2, there are two maximal elements $a, b$ ( so $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$, while $p \preccurlyeq a$ or $p \preccurlyeq b$ for all $p \in P$ ). In that case we set $\Lambda_{\mathcal{P}} = \{\, \overline{\lambda} * (\{a, b\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-a} \,\} \cup \{\, \overline{\lambda} * (\{a, b\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-b} \,\}$.

Note that the number of tuples in $\Lambda_{\mathcal{P}}$ is finite.

Now we go back to a color poset $\mathcal{C} = (Col, \preceq)$ with no antichain longer than 3. Set $c = |Col|$. Consider a graph $G = (V, E)$ with $V = [1, n]$. Suppose $G$ can be feasibly colored with $\mathcal{C}$. With any such coloring we can associate a $c$-tuple $\overline{\lambda} = (\lambda_1, \ldots \lambda_c) \in \Lambda_{\mathcal{C}}$ and a $(c+1)$-tuple of integers $(v_1, v_2 \ldots, v_{c+1})$ so that:

1. We have $1 = v_1 \leq v_2 \leq \cdots \leq v_c \leq v_{c+1} = n + 1$.
2. For all $i = 1, \ldots, c$, the subgraph of $G$ induced on $[v_i, v_{i+1} - 1]$ is an independent set or a bipartite graph.
3. For all $i = 1, \ldots, c$, the colors appearing on the vertices $[v_i, v_{i+1} - 1]$ are exactly the colors in $\lambda_i$.

The existence of these tuples follows directly from the fact that each antichain in $\mathcal{C}$ has length 1 or 2, and hence we cannot "mix" three or more colors from *Col* when coloring the graph.

Given a $(c+1)$-tuple of integers $(v_1, v_2 \ldots, v_{c+1})$, it is easy to check if this tuple satisfies conditions 1 and 2 above. If it does, then we can choose a $c$-tuple $\overline{\lambda} = (\lambda_1, \ldots \lambda_c) \in \Lambda_{\mathcal{C}}$ and use the same technique as applied in the proof of Lemma 3.3 to transform the problem of the existence of a feasible coloring of $G$ in accordance with condition 3 to the existence of a solution of a certain 2-SAT problem. Hence the question if there exists a coloring of $G$ in accordance with two chosen tuples satisfying conditions 1 – 3 can be done in time polynomial in $n$.

The number of $(c+1)$-tuples $(v_1, v_2 \ldots, v_{c+1})$ with $1 = v_1 \leq v_2 \leq \cdots \leq v_c \leq v_{c+1} = n + 1$ is $O(n^c)$. And the number of tuples in $\Lambda_{\mathcal{C}}$ is $O(2^c)$. And so, to check if a graph $G$ on $n$ ordered vertices $[1, n]$ can be feasibly colored with $\mathcal{C}$, we need to consider at most $O(n^c)$ 2-SAT problems. Using estimates for the numbers of variables and clauses in each of the 2-SAT problems similar to those in the proof of Lemma 3.3, we can conclude that Problem 3.1 can be solved in polynomial time. $\square$

**Theorem 3.2 B**
*If the poset $(Col, \preceq)$ contains an antichain of size 3 or more, then Problem 3.1 is NP-complete.*

*Proof.* Let $\mathcal{C} = (Col, \preceq)$ be a poset containing an antichain of size 3 or more. It is obvious that Problem 3.1 is in NP. To prove the problem is NP-complete we give a reduction from the proper $K$-coloring problem, which is well known to be NP-complete for any fixed $K \geq 3$.

Let $S \subseteq Col$ be an antichain in $\mathcal{C}$ with $|S| \geq 3$, chosen such that $S$ is *maximal.* I.e., for all $C \in Col \setminus S$ we have that $C \preceq A$ for some $A \in S$, or $A \preceq C$ for some $A \in S$, but not both (since $S$ is an antichain). Let $Col_D \subseteq Col$ be the set of colors $C \in Col \setminus S$ such that $C \preceq A$ for some $A \in S$, and define $Col_U \subseteq Col$ similarly for colors in $Col \setminus S$ which are larger than some color in $S$. Set $K = |S|$, $n_D = |Col_D|$ and $n_U = |Col_U|$.

Given a graph $G' = (V', E')$ on $n'$ vertices, we construct a graph $G$ with ordered vertex set $[1, n]$ where $n = n_D + n' + n_U$, so that $G'$ has a $K$-coloring if and only if $G$ can be feasibly colored with $\mathcal{C}$. Let the vertices of $G$ be $[1, n_D] \cup [n_D + 1, n_D + n'] \cup [n_D + n' + 1, n_D + n' + n_U]$. Add edges so that the graph on $[n_D + 1, n_D + n']$ is isomorphic to $G'$, and vertices in $[1, n_D] \cup [n_D + n' + 1, n_D + n' + n_U]$ are joined to all other vertices.

Suppose $G'$ has a $K$-coloring. Then we can color $G$ in accordance with $\mathcal{C}$ as follows:

- Give each vertex in $[1, n_D]$ its own color from $Col_D$, using some linear extension of the order imposed by $\mathcal{C}$ on $Col_D$.
- Color the vertices in $[n_D + 1, n_D + n']$ with colors from $S$, according to the $K$-coloring possible on $G'$.
- Give each vertex in $[n_D + n' + 1, n_D + n' + n_U]$ its own color from $Col_U$, using some linear extension of the order imposed by $\mathcal{C}$ on $Col_U$.

It's easy to check that this coloring of $G$ is feasible with $\mathcal{C}$, where the crucial observation is that $S$ is an antichain in $\mathcal{C}$ and hence every proper coloring with colors from $S$ is always in accordance with the poset order.

Next suppose that $G$ has a feasible coloring with $\mathcal{C}$. Such a coloring must have the following properties:

- Each vertex in $[1, n_D]$ has a unique color from $Col_D$, and this color is smaller than any color appearing on $[n_D + 1, n_D + n'] \cup [n_D + n' + 1, n_D + n' + n_U]$.
- Each vertex in $[n_D + n' + 1, n_D + n' + n_U]$ has a unique color from $Col_U$, and this color is larger than any color appearing on $[1, n_D] \cup [n_D + 1, n_D + n']$.

From this it follows that the vertices in $[n_D + 1, n_D + n']$ are colored with colors from $S$. Since $S$ is an antichain, the only requirement to color those vertices with $S$ is that is must be a proper coloring. Such a proper coloring immediately gives a $K$-coloring of $G'$. $\square$

## 4 Coloring with ordered edges

In this section we study the complexity of the following decision problem:

**Problem 4.1.**
*Fix a color-poset $(Col, \preceq)$. Given a graph $G = (V, E)$ with $V = [1, n]$, determine whether $G$ can be colored with $Col$ such that for any two colors $A, B$ with $A \preceq B$ and for any edge $(u, v) \in E$ with $u \in V_A$ and $v \in V_B$ we have $u \leq v$.*

We will transform this problem into a directed graph homomorphism problem. In this section, by $(u, v)$ we mean a directed edge (arc) from $u$ to $v$. We will use $E$ to denoted the set of directed edges of a digraph as well.

The three semi-complete digraphs depicted in Figure 2 play a crucial role in our characterization.

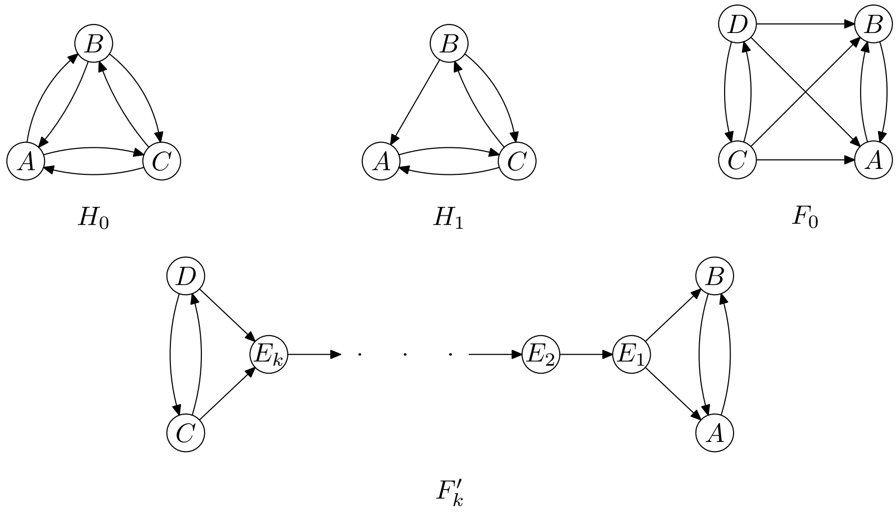



Figure 2: The crucial semi-complete digraphs in the dichotomy characterization of Problem 4.1. $F_k$ is obtained by taking the transitive closure of $F'_k$.

**Definition 4.2.**
*Let $H$ be a fixed directed graph. An $H$-*coloring *of a digraph $G$ is a homomorphism from $G$ to $H$ (a mapping $\varphi$ from $V(G)$ to $V(H)$ such that if $(x,y) \in E(G)$ then $(\varphi(x), \varphi(y)) \in E(H)$). The* acyclic $H$-coloring problem *asks whether for a given acyclic digraph $G$ there is an $H$-coloring of $G$.*

**Definition 4.3.**
*A* semi-complete digraph *is a digraph $H$ which contains a spanning tournament, i.e., between any two vertices there is either one arc or both opposite arcs. We say that a semi-complete digraph is* quasi-acyclic *if after removal of all pairs of oppositely directed arcs we obtain a transitive acyclic subgraph of $H$.*

An alternative definition of a quasi-acyclic semi-complete digraph is as a semi-complete digraph in which every directed cycle contains at least two arcs that come from pairs of opposite arcs. It is easy to see that there is a one-to-one correspondence between posets and quasi-acyclic semi-complete digraphs. Given a color-poset $\mathcal{C} = (Col, \preceq)$, to obtain the corresponding digraph take $V = Col$. And for every two colors $A$ and $B$, $A \neq B$, if $A \prec B$ add an arc $(A, B)$, if $B \prec A$ add an arc $(B, A)$, and otherwise add a pair of opposite arcs $(A, B)$ and $(B, A)$. The resulting semi-complete digraph is quasi-acyclic and will be denoted $H_{\mathcal{C}}$.

Similarly, a graph $G$ with ordered vertex set $[1, n]$ can be considered as a digraph $\overrightarrow{G}$ if we replace all edges $uv \in E(G)$ with $u < v$ by arcs $(u, v)$.

Without a proof we note that, with this notation, we have that $G$ has a coloring by $\mathcal{C}$ feasible for Problem 4.1 if and only if there is an $H_{\mathcal{C}}$-coloring of $\overrightarrow{G}$.

**Theorem 4.4.**
*Let $H$ be a quasi-acyclic semi-complete digraph. The acyclic $H$-coloring problem can be solved in polynomial time if $H$ contains at most one directed 2-cycle (a pair of oppositely directed arcs). Otherwise, the problem is NP-complete.*

This is exactly the same dichotomy as for the general $H$-coloring problem where $H$ is any semi-complete digraph, cf. [1]. In particular, the first part of the theorem follows from the polynomiality of this problem. But for completeness we give a short proof below as well.

The second part of the theorem follows from the Lemmas 4.6 – 4.10 that make up most of the remainder of this section. In those lemma we first prove the NP-completeness of the acyclic $H$-coloring problem for three crucial quasi-acyclic semi-complete digraphs and then for any quasi-acyclic semi-complete digraph containing any of those three as an induced subdigraph.

**Lemma 4.5.**
*Let $H$ be a quasi-acyclic semi-complete digraph. The acyclic $H$-coloring problem can be solved in polynomial time if $H$ contains at most one directed 2-cycle.*

*Proof.* The problem is trivial if $|V(H| = 1$. If $|V(H)| = 2$ and there is a 2-cycle in $H$, then an acyclic digraph $G$ has a homomorphism from $G$ to $H$ if and only if the underlying graph of $G$ (obtained by ignoring the directions on the arcs) is bipartite. Similarly, if $|V(H)| = 2$ and there is no 2-cycle in $H$, then an acyclic digraph $G$ has a homomorphism from $G$ to $H$ if and only if we can partition $V(H) = H_D \cup H_U$ and every arc in $H$ has its tail in $H_D$ and its head in $H_U$.

So assume $|V(H)| \geq 3$. Since $H$ has an acyclic spanning tournament, there are both a universal sink and a universal source in $H$, i.e., there is a vertex $u \in V(H)$ so that for all $v \in V(H)$, $v \neq u$, we have $(v, u) \in E(H)$; and there is a vertex $u' \in V(H)$ so that for all $v \in V(H)$, $v \neq u'$, we have $(u', v) \in E(H)$. Since $H$ has at most one 2-cycle $\{(p, q), (q, p)\}$ and after removing that 2-cycle we obtain a transitive acyclic subgraph of $H$, it is easy to check that we can choose at least one of $u, u'$ different from both $p$ and $q$.

Suppose that the universal sink $u$ in $H$ is different from $p$ and $q$. Let $G$ be an acyclic digraph and let $G_i$ be the sets of sinks in $G$. Then there is a homomorphism from $G$ to $H$ if and only if there is a homomorphism from $G - G_i$ to $H - u$. Indeed, homomorphisms from $G - G_i$ to $H - u$ are easily extendable to homomorphisms from $G$ to $H$. The converse follows since in any homomorphism from $G$ to $H$, the neighbors of vertices in $G_i$ are not mapped to $u$, and hence we can remap the vertices in $G_i$ to $u$ if necessary.

A similar observation holds if the universal source in $H$ is different from $p$ and $q$. We use induction on $|V(H)|$ to get a straightforward polynomial algorithm to check if there is a homomorphism from $H$ to $G$. $\square$

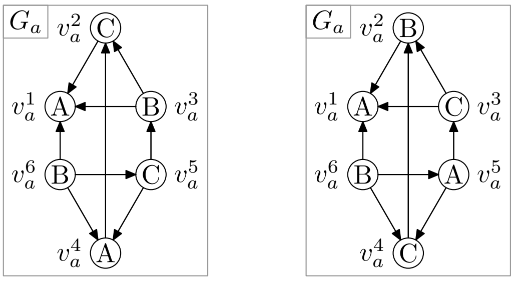
The following follows immediately from the NP-completeness of the ordinary 3-coloring problem.

**Lemma 4.6.**
*Let $H_0$ be the quasi-acyclic semi-complete digraph depicted in Figure 2. The acyclic $H_0$-coloring problem is NP-complete.*

**Lemma 4.7.**
*Let $H_1$ be the quasi-acyclic semi-complete digraph depicted in Figure 2. The acyclic $H_1$-coloring problem is NP-complete.*

*Proof.* The problem is obviously in NP. We will show that given a boolean formula $f(x_1, x_2, \ldots, x_m)$ in conjunctive normal form (CNF), we can construct an acyclic directed graph $\widetilde{G}(f)$ which has an $H_1$-coloring $\varphi$ if and only if $f$ is satisfiable. For every boolean variable $a$ of $f$ (respectively, every auxiliary variable encoding the boolean value of a subformula of $f$), we will construct the gadget

$G_a = (V_a, E_a)$ defined as follows: $V_a = \{v_a^1, \ldots, v_a^6\}$ and $E_a = \{\, (v_a^2, v_a^1),\ (v_a^3, v_a^1),\ (v_a^3, v_a^2),\ (v_a^4, v_a^2),\ (v_a^5, v_a^3),\ (v_a^5, v_a^4),\ (v_a^6, v_a^1),\ (v_a^6, v_a^4),\ (v_a^6, v_a^5)\,\}$, see Figure 3.




$Pat(G_a) = P^0$ $\quad$ $Pat(G_a) = P^1$

Figure 3: The two possible homomorphisms (coloring patterns) from $G_a$ to $H_1$: on the left hand side $\mathrm{Pat}(G_a) = P^0$; on the right hand side $\mathrm{Pat}(G_a) = P^1$.

Each gadget $G_a$ is acyclic and has only two $H_1$-colorings, called *color patterns*, see Figure 3. Let the color pattern $P^0$ of $G_a$ with $\varphi(v_a^1) = \varphi(v_a^4) = A$, $\varphi(v_a^3) = \varphi(v_a^6) = B$ and $\varphi(v_a^2) = \varphi(v_a^5) = C$, represent the fact that the value of the boolean variable $a$ is 0. Similarly, let the color pattern $P^1$ of $G_a$ with $\varphi(v_a^1) = \varphi(v_a^5) = A$, $\varphi(v_a^2) = \varphi(v_a^6) = B$ and $\varphi(v_a^3) = \varphi(v_a^4) = C$, represent the fact that $a = 1$.

Now let us give an overview of the main ideas in the construction of $\widetilde{G}(f)$. Firstly, we construct a gadget $G_{x_i}$ for every variable $x_i$ occurring in the formula $f$. Secondly, for every disjunctive subformula $d_i$ of $f = d_1 \wedge d_2 \wedge \cdots \wedge d_k$, we construct an acyclic digraph $G(d_i)$ containing a gadget $G_{d_i}$ and using the gadgets $G_{x_1}, \ldots, G_{x_m}$ such that for every $H_1$-coloring of $G(d_i)$, $\mathrm{Pat}(G_{d_i}) = P^1$ if and only if color patterns of $G_{x_1}, \ldots, G_{x_m}$ correspond to a true assignment for $d_i$. Finally, we extend each $G(d_i)$ to a graph $\widetilde{G}(d_i)$ such that the gadget $G_{d_i}$ will have the color pattern $P^1$ in every $H_1$-coloring of $\widetilde{G}(d_i)$. Since the gadgets $G_{x_1}, \ldots, G_{x_m}$ are common for every $\widetilde{G}(d_i)$, it will follow that there is an $H_1$-coloring of $\widetilde{G}(f) = \bigcup_{i=1}^{k} \widetilde{G}(d_i)$ if and only if there is a true assignment for $f$, i.e., if $f$ is satisfiable.

In the following we describe a recursive construction of the graph $G(d)$, where $d = d_i$. If $d$ has no $\vee$'s, then either $d = x_j$ or $d = \overline{x_j}$, for some $j = 1, \ldots, m$. As described above, add a gadget $G_d$ to $G(d)$. To complete the construction add the arc $(v_d^3, v_{x_j}^2)$ if $d = x_j$, and the arc $(v_d^2, v_{x_j}^2)$ if $d = \overline{x_j}$, see Figure 4. First suppose $d = x_j$. If the color pattern of $G_{x_j}$ is $P^0$, then because of the arc $(v_d^3, v_{x_j}^2)$, the only possible color pattern for $G_d$ is $P^0$, and similarly if the color pattern of $G_{x_j}$ is $P^1$, the only possible color pattern for $G_d$ is $P^1$. Since every true assignment of $d$ has $x_j = 1$ and the remaining variables can have arbitrary values, for every $H_1$-coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if color patterns of $G_{x_1}, \ldots, G_{x_m}$ correspond to such a true assignment. If $d = \overline{x_j}$, the argument is similar.

Assume now that $d$ contains an $\vee$. Then there are disjunctive subformulas $d'$ and $d''$ in $f$ such that $d = d' \vee d''$. We recursively construct graphs $G(d')$ and $G(d'')$ (in polytime). Construct $G(d)$ as follows. Take a union of $G(d')$ and $G(d'')$ and add a new variable gadget $G_d$ and disjunction gadget consisting of three new vertices $w_d^1, w_d^2, w_d^3$ and the arcs $(w_d^2, w_d^1)$, $(w_d^3, w_d^2)$. Furthermore, add the arcs $(w_d^1, v_{d'}^2)$, $(w_d^1, v_{d'}^5)$, $(w_d^3, v_{d''}^2)$, $(w_d^3, v_{d''}^5)$ and $(v_d^4, w_d^2)$ connecting disjunction gadget with the variable gadgets, see Figure 5.

It is easy to check that in every $H_1$-coloring of $G(d')$ (respectively $G(d'')$), the gadget $G_{d'}$

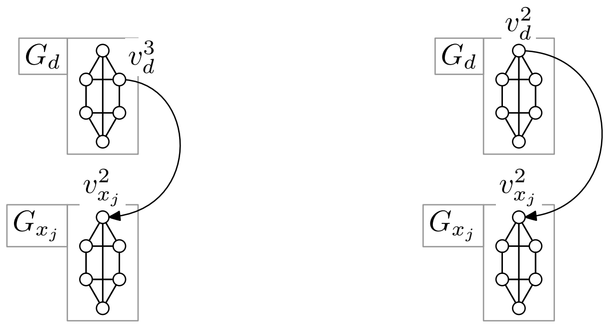



Figure 4: On the left: $G(d)$ for the boolean formula $d = x_j$; on the right: $G(d)$ for the boolean formula $d = \overline{x_j}$.

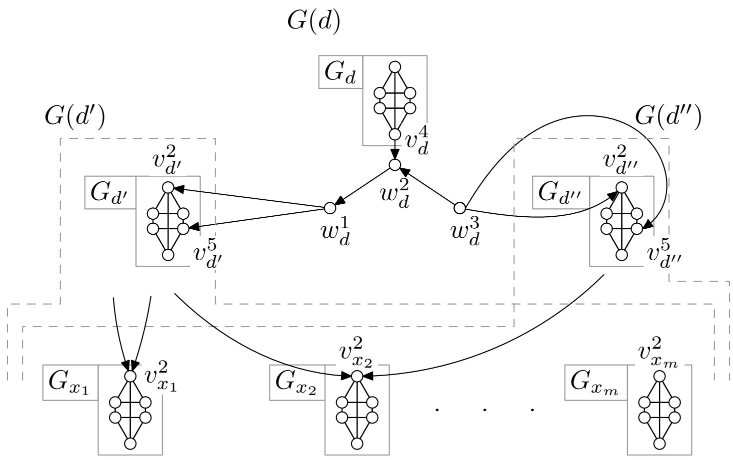



Figure 5: The construction of $G(d)$ for the boolean formula $d = d' \vee d''$ using the union of the graphs $G(d')$ and $G(d'')$. In the example above we assume that $d'$ contains both $x_1$ and $\overline{x_1}$ which is indicated by two arcs leaving the gadget $G_{x_1}$. Similarly, the arcs leaving the gadget $G_{x_2}$ indicate that $x_2$ or its complement is in both $d'$ and $d''$. Finally, no arc leaving the gadget $G_{x_m}$ indicates that neither $d'$ nor $d''$ contain $x_m$ or $\overline{x_m}$.

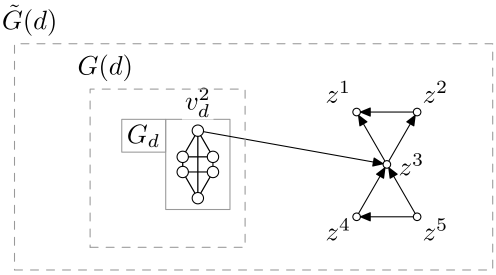



Figure 6: The construction of $\widetilde{G}(d)$.

in $G(d')$ (respectively $G_{d''}$ in $G(d'')$) has color pattern $P^1$ if and only if the color patterns of gadgets $G_{x_1}, \ldots, G_{x_m}$ represent a true assignment for $d'$ (respectively $d''$). Otherwise it has color pattern $P^0$. Since the color patterns of gadgets $G_{x_1}, \ldots, G_{x_m}$ represent a true assignment for $d$ if and only if they represent a true assignment for $d'$ or $d''$, it is enough to show that in any $H_1$-coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if $\mathrm{Pat}(G_{d'}) = P^1$ or $\mathrm{Pat}(G_{d''}) = P^1$.

Consider an $H_1$-coloring of $G(d)$. If $\mathrm{Pat}(G_{d'}) = \mathrm{Pat}(G_{d''}) = P^1$, then $\varphi(w_d^1) = \varphi(w_d^3) = C$ and $\varphi(w_d^2) = A$ or $B$, and hence $\mathrm{Pat}(G_d) = P^1$. (Note that this is true even if $\varphi(w_d^2) = B$; the vertex $v_d^4$ cannot be colored $A$ because of the arc $(v_d^4, w_d^2)$.) If $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^0$, then $\varphi(w_d^1) = C$, and since $(w_d^3, w_d^2)$ is an arc, $\varphi(w_d^2) = A$ and $\varphi(w_d^3) = B$. Hence, $\mathrm{Pat}(G_d) = P^1$. The case $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^1$ is analogous to the previous one. If $\mathrm{Pat}(G_{d'}) = \mathrm{Pat}(G_{d''}) = P^0$, then $\varphi(w_d^1)$ and $\varphi(w_d^3)$ is either $A$ or $B$. Since $G(d)$ contains the arcs $(w_d^2, w_d^1)$ and $(w_d^3, w_d^2)$, we must have $\varphi(w_d^2) = C$, and hence $\mathrm{Pat}(G_d) = P^0$. This verifies the construction of the graph $G(d)$.

Finally, we extend each $G(d)$ to a graph $\widetilde{G}(d)$ such that the gadget $G_d$ will have the color pattern $P^1$ in every $H_1$-coloring of $\widetilde{G}(d)$. Then, for every $H_1$-coloring of $\widetilde{G}(d)$, the color patterns of $G_{x_1}, \ldots, G_{x_m}$ must represent a true assignment for $d$. It follows that $\widetilde{G}(f)$ has an $H_1$-coloring if and only if $f$ is satisfiable.

Let $Z$ be the graph with $V(Z) = \{z^1, \ldots, z^5\}$, and $E(Z) = \{(z^2, z^1), (z^3, z^2), (z^3, z^1), (z^4, z^3), (z^5, z^3), (z^5, z^4)\}$. Since $z^3$ has two incoming (outgoing) arcs from vertices connected by an arc, it cannot be colored $B$ (respectively $A$). Hence, $z^3$ must be colored $C$. Consequently, $\varphi(z^1) = \varphi(z^4) = A$ and $\varphi(z^2) = \varphi(z^5) = B$. Now, we are ready to construct $\widetilde{G}(d)$. Take the union of $G(d)$ and $Z$ and add an arc $(v_d^2, z^3)$, see Figure 6. Obviously, in any $H_1$-coloring of $\widetilde{G}(d)$, $\mathrm{Pat}(G_d) = P^1$.

Finally, let us deduce that the directed graph $\widetilde{G}(f)$ is acyclic. Firstly, by induction, we show that for every subformula $d$ used in the construction of $G(d)$ is acyclic. This is certainly true for every $d = x_j$ or $d = \overline{x_j}$. Now, consider the formula $d = d' \vee d''$, see Figure 5. By induction $G(d')$ and $G(d'')$ are acyclic and since all arcs incident with the common part of $G(d')$ and $G(d'')$ (the input variable gadgets) end in the common part, the union $G(d') \cup G(d'')$ is acyclic as well. The graph $G(d)$ contains this union and two new gadgets and all arcs connecting these two parts start in the new gadgets and end in the union. Hence, $G(d)$ is also acyclic. Thus, for every $d_i$ in $f$, $G(d_i)$ and obviously also $\widetilde{G}(d_i)$ is acyclic. By the same argument as above their union $\widetilde{G}(f)$ is also acyclic. $\square$

**Lemma 4.8.**
*Let $F_0$ be the quasi-acyclic semi-complete digraph depicted in Figure 2. The acyclic $F_0$-coloring problem is NP-complete.*

*Proof.* The problem is obviously in NP. The proof of NP-hardness follows the lines of the proof of Lemma 4.7. We will again show that given a boolean formula $f(x_1, x_2, \ldots, x_m)$ in CNF, we can construct an acyclic directed graph $\widetilde{G}(f)$ which has a $F_0$-coloring if and only if $f$ is satisfiable.

The main differences with the proof of Lemma 4.7 is that the new variable gadgets will share a four-vertex subdigraph and will have eight color patterns. This is unavoidable since in every $F_0$-coloring, the colors $A$ and $B$ (respectively, $C$ and $D$) are interchangeable. The four-vertex digraph $Z$, common to all gadgets, has $V(Z) = \{z^1, \ldots, z^4\}$ and $E(Z) = \{ (u,v) \mid u, v \in V(Z) \wedge u > v \}$, see Figure 7. Note that the digraph $Z$ has a unique $F_0$-coloring up to swapping $A$ and $B$, or $C$

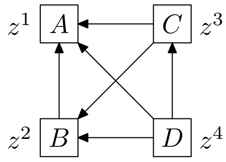



Figure 7: The graph $Z$.

and $D$. Without loss of generality, it is enough to consider only those $F_0$-colorings $\varphi$ of $\widetilde{G}(f)$ which have $\varphi(z^1) = A$, $\varphi(z^2) = B$, $\varphi(z^3) = C$ and $\varphi(z^4) = D$. This assumption will reduce the number of color patterns of variable gadgets to two.

For every boolean variable $a$ of $f$ (input or auxiliary), we will construct the following acyclic gadget $G_a = (V_a, E_a)$ with $V_a = \{v_a^1, \ldots, v_a^6\}$ and $E_a = \{ (v_a^2, v_a^1), (v_a^3, v_a^2), (v_a^4, v_a^1), (v_a^5, v_a^4), (v_a^6, v_a^3), (v_a^6, v_a^5) \}$. As proposed above, we complete the construction of the gadget by connecting it to the common digraph $Z$ by arcs $(v_a^3, z^1)$, $(v_a^4, z^1)$, $(v_a^5, z^1)$, $(v_a^6, z^1)$, $(v_a^2, z^2)$, $(z^3, v_a^3)$ and $(z^3, v_a^4)$, see Figure 8. The figure shows two coloring patterns of $G_a$: $P^0$ with $\varphi(v_a^1) = \varphi(v_a^3) = B$, $\varphi(v_a^2) =$

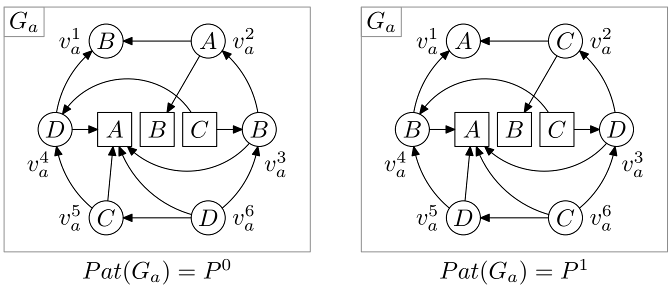



Figure 8: Two coloring patterns of the gadget $G_a$: $\mathrm{Pat}(G_a) = P^0$ (on the left); and $\mathrm{Pat}(G_a) = P^1$ (on the right). The squares labeled with $A$, $B$ and $C$ represent vertices $z^1$, $z^2$ and $z^3$ of the digraph $Z$ common to all gadgets, respectively. For simplicity, we omitted the fourth vertex of $Z$ and the arcs inside $Z$.

$A, \varphi(v_a^4) = \varphi(v_a^6) = D$ and $\varphi(v_a^5) = C$, which will represent the fact that $a = 0$; and $P^1$ with $\varphi(v_a^1) = A$, $\varphi(v_a^2) = \varphi(v_a^6) = C$, $\varphi(v_a^3) = \varphi(v_a^5) = D$ and $\varphi(v_a^4) = B$ corresponding with $a = 1$.

For each disjunctive subformula $d = d_i$ of $f = d_1 \wedge \cdots \wedge d_k$, the construction of $\widetilde{G}(d)$ is analogous to the construction in the proof of Lemma 4.7. Therefore, we only describe the main ingredients of the construction: the base step $d = x_j$ or $d = \overline{x_j}$, the inductive step $d = d' \vee d''$, and forcing the color pattern of $G_d$ to $P^1$. In the base step, add the arc $(v_{x_j}^6, v_d^5)$, if $d = x_j$; or the arc $(v_{x_j}^5, v_d^5)$, if $d = \overline{x_j}$, joining the input variable gadget $G_{x_j}$ to variable gadget $G_d$, see Figure 9. One can easily

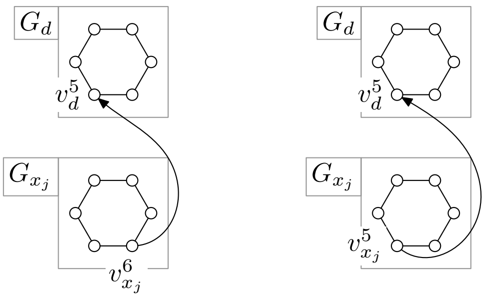



Figure 9: On the left: $G(d)$ for the boolean formula $d = x_j$; on the right: $G(d)$ for the boolean formula $d = \overline{x_j}$.

check that $G(d)$ is acyclic and has the required property.

In the inductive step, construct $G(d)$ as follows. Take the union of $G_{d'}$ and $G_{d''}$, add one new gadget $G_d$ and three new vertices $w_d^1, w_d^2, w_d^3$. Furthermore, add the arcs $(v_{d'}^2, w_d^1)$, $(v_{d'}^5, w_d^2)$, $(v_{d'}^5, v_d^4)$, $(w_d^3, v_{d''}^5)$, $(v_{d''}^5, v_d^4)$, $(w_d^2, w_d^1)$, $(w_d^2, v_d^1)$, and $(w_d^3, w_d^2)$, see Figure 10.

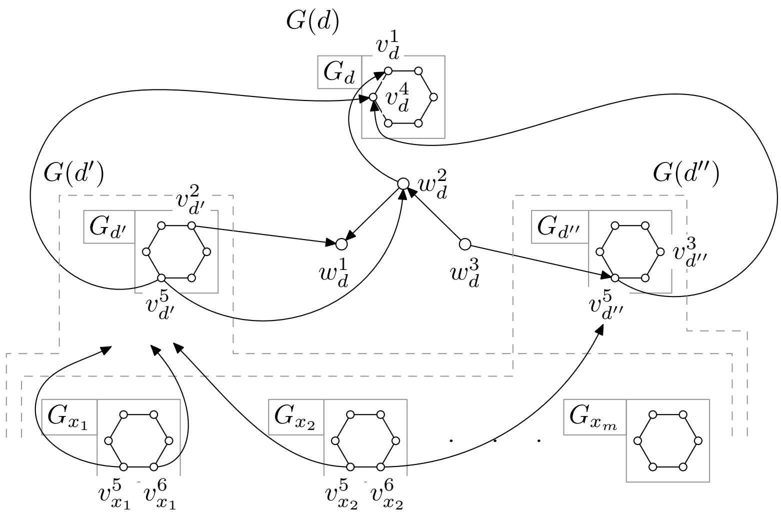



Figure 10: The construction of $G(d)$ for the boolean formula $d = d' \vee d''$ using the union of graphs $G(d')$ and $G(d'')$. In the example we assume that $d'$ contains both $x_1$ and $\overline{x_1}$ which is indicated by two arcs leaving the gadget $G_{x_1}$. Similarly, the arcs leaving the gadget $G_{x_2}$ indicate that $x_2$ or $\overline{x_2}$ are in both $d'$ and $d''$. Finally, no arc leaving the gadget $G_{x_m}$ indicates that neither $d'$ nor $d''$ contains $x_m$ or $\overline{x_m}$.

We show that in any feasible coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if $\mathrm{Pat}(G_{d'}) = P^1$ or $\mathrm{Pat}(G_{d''}) = P^1$. Notice that the arcs $(v_{d'}^5, v_d^4)$ and $(v_{d''}^5, v_d^4)$ guarantee that if one of the gadgets $G_{d'}$ or $G_{d''}$ has color pattern $P^1$, gadget $G_d$ must have color pattern $P^1$. Now, it is enough to show that there is a feasible coloring for the three $w-$vertices in each one of these three cases. Indeed we can color the $w-$vertices as follows: if $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^0$, $\varphi(w^1) = A$, $\varphi(w^2) = B$, $\varphi(w^3) = D$; if $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^1$, $\varphi(w^1) = B$, $\varphi(w^2) = D$, $\varphi(w^3) = C$; and if $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^1$, $\varphi(w^1) = A$, $\varphi(w^2) = B$, $\varphi(w^3) = C$. It remains to consider

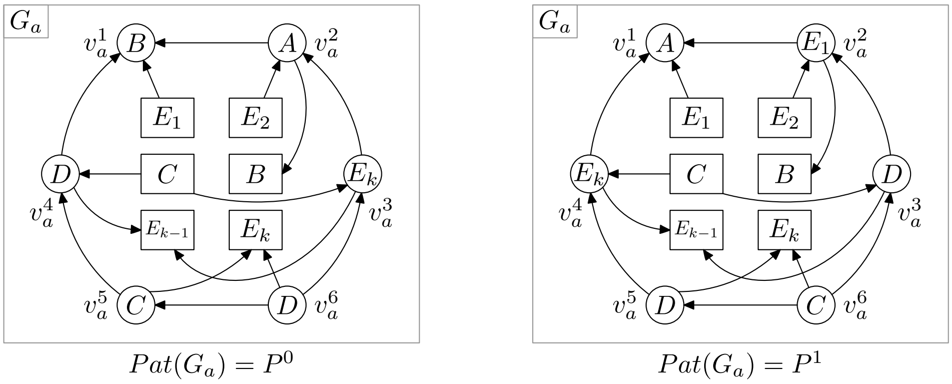



Figure 11: Two coloring patterns of the gadget $G_a$: $\mathrm{Pat}(G_a) = P^0$ (on the left); and $\mathrm{Pat}(G_a) = P^1$ (on the right). The squares labeled with $B$, $C$, $E_1$, $E_2$, $E_{k-1}$ and $E_k$ represent vertices $z^2$, $z^3$, $q^1$, $q^2$, $q^{k-1}$ and $q^k$ of the common digraph $Z$, respectively. For the case $k = 1$, the vertices $E_1, E_2, E_{k-1}, E_k$ are the same vertex. For the cases $k = 2, 3$, some of these vertices are naturally identified with each other.

the case when $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^0$. We have $\varphi(v_{d'}^2) = A$ and $\varphi(v_{d'}^5) = \varphi(v_{d''}^5) = C$. The arc $(v_{d'}^2, w_d^1)$ forces $\varphi(w_d^1) = B$ and the arc $(w_d^3, v_{d''}^5)$ forces $\varphi(w_d^3) = D$. Furthermore, the arcs $(v_{d'}^2, w_d^1)$, $(w_d^3, v_{d''}^5)$ and $(v_{d''}^5, w_d^2)$ force $\varphi(w_d^2) = A$. Because of the arc $(w_d^2, v_d^1)$, $\varphi(v_d^1) = B$. Hence, $\mathrm{Pat}(G_d) = P^0$. This verifies the construction of the graph $G(d)$.

Finally, to extend $G(d)$ to $\widetilde{G}(d)$ forcing $G_d$ to color pattern $P^1$, add the arc $(v_d^1, z^2)$. Let us observe that the directed graph $\widetilde{G}(f)$ is acyclic. Note that if a directed graph contains a sink (only incoming arcs) or a source (only outgoing arcs) then it can be removed from the graph without affecting acyclicity of the graph. Therefore, it is enough to show that the graph $\widetilde{G}(f)'$ obtained from $\widetilde{G}(f)$ by removing the vertices of $Z$ and the vertices $w_d^1$ and $w_d^3$, for every subformula $d$ used in the construction, is acyclic. That can be easily seen using a similar argument as in the proof of Lemma 4.7. □

**Lemma 4.9.**
*Let $k \geq 1$ and let $F_k$ be the quasi-acyclic semi-complete digraph obtained by forming the transitive closure of the digraph $F_k'$ depicted in Figure 2. The acyclic $F_k$-coloring is NP-complete.*

*Proof.* The reduction from SAT is very similar to the reduction described in the proof of Lemma 4.8. In fact, all the gadgets and connections among them require just slight modifications. The graph $Z$ in the proof of Lemma 4.8 can be viewed as the transitive closure of the directed path $(z^4, z^3, z^2, z^1)$. In this proof the graph $Z$ will be the transitive closure of the path $(z^4, z^3, q^k, \ldots, q^1, z^2, z^1)$. Again, $Z$ has a unique $F^k$-coloring up to swapping $A$ and $B$, or $C$ and $D$. In any coloring, $\varphi(q^i) = E_i$ and without loss of generality, let $\varphi(z^1) = A$, $\varphi(z^2) = B$, $\varphi(z^3) = C$ and $\varphi(z^4) = D$. There are $k + 4$ colors available to color the variable gadget $G_a$. In order to restrict the number of color patterns of $G_a$ to two, we replace the following four arcs $(v_a^3, z^1)$, $(v_a^4, z^1)$, $(v_a^5, z^1)$ and $(v_a^6, z^1)$ with arcs $(v_a^3, q^{k-1})$, $(v_a^4, q^{k-1})$, $(v_a^5, q^k)$ and $(v_a^6, q^k)$, and add the following two arcs $(q^1, v_a^1)$, $(q^2, v_a^2)$. The color patterns $P^0$ and $P^1$ of the resulting gadget are depicted in Figure 11.

In the inductive construction of $\widetilde{G}_d$, there are no differences in the base step. In the inductive step, we add a directed path $Q$ of length $k$ from the vertex $v_{d''}^3$ to the vertex $w_d^2$. The purpose of this path is to forbid colors $E_1, \ldots, E_k$ at vertex $w_d^2$ in the case when $\mathrm{Pat}(G_{d''}) = P^0$. In this case,

$\varphi(v^2_{d''}) = E_k$. If at the same time $\varphi(w^2_d) \in \{E_1, \ldots, E_k\}$, then all vertices of $Q$ must be in this set of colors which is not possible. On the other hand, if $\mathrm{Pat}(G_{d''}) = P^1$, and hence $\varphi(v^3_{d''}) = D$, then the path $Q$ does not restrict any color at $w^2_d$. The following table shows all possible colorings of vertices crucial for determining the color pattern of $G_d$ depending on $\mathrm{Pat}(G_{d'})$ and $\mathrm{Pat}(G_{d''})$.

| $\mathrm{Pat}(G_{d'})$ | $\mathrm{Pat}(G_{d''})$ | $v^2_{d'}$ | $v^5_{d'}$ | $v^3_{d''}$ | $v^5_{d''}$ | $w^1_d$ | $w^2_d$ | $w^3_d$ | $v^1_d$ | $v^4_d$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $P^0$ | $P^0$ | $A$ | $C$ | $E_k$ | $C$ | $B$ | $A$ | $D$ | $B$ | $D$ |
| $P^0$ | $P^1$ | $A$ | $C$ | $D$ | $D$ | $B$ | $E_1, \ldots, E_k, D$ | $C$ | $A$ | $E_k$ |
| $P^1$ | $P^0$ | $E_1$ | $D$ | $E_k$ | $C$ | $A$ | $B$ | $D$ | $A$ | $E_k$ |
| $P^1$ | $P^1$ | $E_1$ | $D$ | $D$ | $D$ | $A/B$ | $B, E_1, \ldots, E_k / E_1, \ldots, E_k$ | $C$ | $A$ | $E_k$ |

Thus, one can easily see that again $\mathrm{Pat}(G_d) = P^1$ if and only if $\mathrm{Pat}(G_{d'}) = P^1$ or $\mathrm{Pat}(G_{d''}) = P^1$. The rest of the proof is analogous to the proof of Lemma 4.8. ☐

It can be easily seen that a quasi-acyclic semi-complete digraph $H$ contains at least two directed 2-cycles, if and only if it contains either $H_0$, $H_1$ or $F_0$ as an induced subdigraph. Sufficiency is trivial. For the necessity suppose that $H$ does not contain $H_0$. Now, if there are two 2-cycles in $H$ which share a vertex then the three vertices on these two 2-cycles induce $H_1$. Otherwise, take any two 2-cycles. Since they are independent, the four vertices on them induce $F_0$. Therefore, to prove the NP-completeness part of Theorem 3.2, it is enough to show the following lemma.

**Lemma 4.10.**
*Suppose $H$ is a quasi-acyclic semi-complete digraph. If $H$ contains either $H_0$, $H_1$ or $F_0$ (see Figure 2), as an induced subdigraph, then the acyclic $H$-coloring problem is NP-complete.*

*Proof.* We will distinguish three cases:
**Case 1.** We first prove that when $H$ contains $H_0$ as an induced subdigraph, the acyclic $H$-coloring problem is NP-complete by reduction from the proper 3-coloring problem. In particular, given a graph $G$, we will construct an acyclic digraph $\overrightarrow{K}$ which has an $H$-coloring if and only if $G$ is 3-colorable.

Let $A$, $B$ and $C$ be the vertices of $H_0$. Find a directed Hamilton path $P$ in $H$ on which vertices $A$, $B$ and $C$ are consecutive (by using the topological sort on the poset corresponding to $H$). Let $P_\ell$ (respectively $P_u$) denote the subpath of $P$ containing all vertices preceding (respectively following) the three vertices $A$, $B$ and $C$ on $P$. Note that $P$ has the property that its transitive closure is $H$-colorable. This will be used later in the proof.

We construct an acyclic digraph $\overrightarrow{K}$ as follows. Start with subpaths $P_\ell$ and $P_u$ and vertices in $V(G)$. Add an arc from the last vertex of $P_\ell$ to every vertex in $V(G)$ and from every vertex in $V(G)$ to the first vertex of $P_u$, see Figure 12.

Next take the transitive closure of this graph. Observe that the resulting graph has many $H$-colorings in which vertices on $P_\ell$ and $P_u$ are mapped to corresponding vertices in $H$, and vertices in $V(G)$ are arbitrarily mapped to vertices $A$, $B$ and $C$. Consider any acyclic orientation of $G$ and add every arc of this orientation to the constructed digraph joining corresponding vertices in $V(G)$. The resulting acyclic digraph is $\overrightarrow{K}$.

Firstly, suppose that $G$ is 3-colorable with colors $A$, $B$ and $C$. For an $H$-coloring of $\overrightarrow{K}$ choose the $H$-coloring from the previous paragraph that agrees on vertices in $V(G)$ with the 3-coloring.

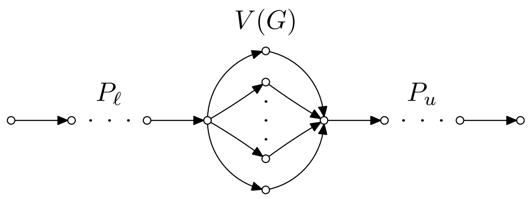



Figure 12: First step in the construction of the acyclic digraph $\vec{K}$.

Secondly, suppose that $\overrightarrow{K}$ has an $H$-coloring. Since $V_P = V(P_\ell) \cup V(P_u)$ induces a transitive tournament in $\overrightarrow{K}$, any such $H$-coloring must use all but three vertices of $H$. Since every vertex of $V(G)$ is incident ( is a tail or a head of an arc ) with every vertex in $V_P$, in any $H$-coloring of $\overrightarrow{K}$ the vertices in $V(G)$ are mapped to the remaining three vertices of $H$, i.e., $G$ is 3-colorable.

**Case 2.** Next, we prove that when $H$ contains $H_1$ but not $H_0$ as an induced subdigraph, the $H$-coloring problem is NP-complete by reduction from the acyclic $H_1$-coloring problem ( see Lemma 4.7 ). Let $\overrightarrow{G}$ be an acyclic digraph for which we want to decide whether it is $H_1$-colorable. We will construct an acyclic digraph $\overrightarrow{K}$ which is $H$-colorable if and only if $\overrightarrow{G}$ is $H_1$-colorable. The construction of $\overrightarrow{K}$ is exactly as in the previous case with the only difference that $\overrightarrow{G}$ already fixes an acyclic orientation used in the construction. However, it is not obvious that there exists a directed Hamilton path in $H$ with three consecutive vertices $B$, $C$ and $A$ inducing $H_1$.

Consider a poset $(V(H), \preceq)$ corresponding to $H$. We say that a triple $[B, A, C]$ of different vertices in $H$ is *nice* if $B \preceq A$, and the pairs $A, C$ and $B, C$ are both incomparable. Obviously, there is a nice triple since $H_1$ is an induced subdigraph of $H$. Take a directed Hamilton path $P$ in $H$ on which there is a nice triple $[B, A, C]$ such that the distance of $B$ and $A$ along $P$ is the smallest possible. We will show that the distance of $B$ and $A$ on $P$ is one. Suppose by contradiction that $E \neq B$ is the immediate predecessor of $A$ on $P$. Note that for any two $X, Y$ such that $X$ precedes $Y$ on $P$, $H$ contains the arc $(X, Y)$. ( This follows since either $X \prec Y$ or $X$ and $Y$ are incomparable. ) Hence, if $H$ contains both arcs $(A, E)$ and $(E, A)$ ( a double arc $(A, E)$ ), then by exchanging $A$ and $E$ we obtain another directed Hamilton path in $H$ with smaller distance of $A$ and $B$, a contradiction. Thus, we may assume that $H$ contains the arc $(E, A)$ but not $(A, E)$, i.e., $E \prec A$. Moreover, either $E \prec C$ or they are incomparable, since otherwise $C \prec A$.

Suppose that $B \prec E$. It follows that $C$ and $E$ are incomparable, and hence the triple $[B, E, C]$ is nice and has a smaller distance between $B$ and $E$ along $P$. Hence, $B$ and $E$ must be incomparable. Let $F$ be an immediate successor of $B$ on $P$. By a similar argument as above, it follows that $B \prec F$, and $A$ and $F$ are incomparable. Therefore, $E \neq F$ and they are incomparable. Now, $[B, F, E]$ is a nice triple with distance one between $B$ and $F$ on $P$, a contradiction.

We say that a nice triple $[B, A, C]$ on a directed Hamilton path is *very nice* if the distance between $B$ and $A$ along $P$ is one. By the above argument, there is a directed Hamilton path with a very nice triple. Take such a path $P$ and a very nice triple $[B, A, C]$ such that the sum of the distances from $B$ to $C$ and from $A$ to $C$ on $P$ is the smallest possible. We will show that this sum of distances is three. Without loss of generality suppose that $C$ follows $A$ on $P$. Suppose by contradiction that $E \neq C$ is the immediate successor of $A$ on $P$. If $E$ is incomparable with both $A$ and $B$ then $[B, A, E]$ is a very nice triple with a smaller sum of distances. It follows that $C$ and $E$ must be incomparable ( otherwise, by transitivity, at least one of $A, C$ or $B, C$ would be comparable ). If $A$ and $E$ are incomparable as well then $A, C, E$ induce $H_0$ in $H$, a contradiction.

Hence, $[A, E, C]$ is a very nice triple with a smaller sum of distances on $P$, a contradiction.

Obviously, if $\overrightarrow{G}$ is $H_1$-colorable, we can use this coloring to construct an $H$-coloring of $\overrightarrow{K}$ as above. Conversely, suppose now that $\overrightarrow{K}$ has an $H$-coloring. As before, it follows that $V(\overrightarrow{G})$ is mapped to some three vertices of $H$. Since $H_0$ is not an induced subdigraph of $H$, the three vertices induce a subdigraph of $H_1$. Therefore, $\overrightarrow{G}$ is $H_1$-colorable.

**Case 3.** Finally, we prove that when $H$ contains $F_0$ but neither $H_0$ nor $H_1$ as an induced subdigraph, the $H$-coloring problem is NP-complete by reduction from one of the following problems: the acyclic $F_0$-coloring problem (Lemma 4.8) or the acyclic $F_k$-coloring problem for some $k \geq 1$ (Lemma 4.9).

Take a directed Hamilton path $P$ in $H$. Since $H_0$ and $H_1$ are not induced subdigraphs of $H$, it is easy to see that we can make sure that any two incomparable vertices in the corresponding poset $(V(H), \preceq)$ are consecutive on $P$. Since $F_0$ is an induced subdigraph of $H$ there are at least two incomparable pairs. Take two such pairs $D, C$ and $B, A$ which are in this order on $P$ and are closest to each other. Similarly, it is easy to see that for any vertex $E$ in between $C$ and $B$ on $P$, $D \prec E$, $C \prec E$, $E \prec B$ and $E \prec A$. Therefore, vertices of the subpath $P_m$ of $P$ from $D$ to $A$ induce $F_k$ where $k$ is the number of vertices between $C$ and $D$.

We will show that the $H$-coloring problem is NP-complete by reduction from the acyclic $F_k$-coloring problem. Let $P_\ell$ (respectively $P_u$) denote the subpath of $P$ containing all vertices preceding $D$ (respectively following $A$) on $P$. Given $\overrightarrow{G}$, construct the acyclic digraph $\overrightarrow{K}$ similarly as in the previous cases. Obviously, if $\overrightarrow{G}$ is $F_k$-colorable, we can use this coloring to construct an $H$-coloring of $\overrightarrow{K}$ as in the previous cases. Conversely, suppose now that $\overrightarrow{K}$ has an $H$-coloring. We will show that vertices of $V(\overrightarrow{G})$ in $\overrightarrow{K}$ are colored with colors on $P_m$ in $H$. By contradiction, suppose $x \in V(\overrightarrow{G})$ is colored by $y$ not on $P_m$. Without loss of generality let $y \in P_\ell$. Since for every vertex $z$ on $P_\ell$, there is an arc $(z, x)$ in $\overrightarrow{K}$, the color of $z$ must be either incomparable with $y$ or a predecessor of $y$ on $P$. Since there is most one incomparable vertex with $y$ and it must lie on $P_\ell$ of $H$, we conclude that the color of $z$ is on $P_\ell$ in $H$ and different from $y$. Since each vertex on $P_\ell$ of $\overrightarrow{K}$ must have a different color, there is not enough colors for them, a contradiction. Therefore, $\overrightarrow{G}$ is $F_k$-colorable. $\square$

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