ESTIMATES IN CORONA THEOREMS FOR SOME SUBALGEBRAS OF H^∞

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Abstract. If n is a nonnegative integer, then denote by $\partial^{-n}H^\infty$ the space of all complex valued functions f defined on $\mathbb{D}$ such that $f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ belong to H^∞, with the norm

$$\|f\| = \sum_{j=0}^{n} \frac{1}{j!}\|f^{(j)}\|_\infty.$$

We prove bounds on the solution in the corona problem for $\partial^{-n}H^\infty$. As corollaries, we obtain estimates in the corona theorem also for some other subalgebras of the Hardy space H^∞.

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Notation

$:=$	equal by definition;
$\mathbb{C}$	the complex plane;
$\mathbb{D}$	the unit disk, $\mathbb{D} := \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$;
$\overline{\mathbb{D}}$	the closed unit disk, $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$;
$\mathbb{T}$	the unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$;
dm	normalized Lebesgue measure on $\mathbb{T}$, $m(\mathbb{T}) = 1$;
$\partial, \overline{\partial}$	derivatives with respect to z and $\overline{z}$ respectively: $\partial := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, $\overline{\partial} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$;
Δ	Laplacian, $\Delta := 4\partial\overline{\partial}$;
$\lvert\!\lvert \cdot \rvert\!\rvert$, $\lVert \cdot \rVert$	When dealing with vector valued functions with values in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we use $\lvert \cdot \rvert$ for the norm in H induced by the inner product $\langle\cdot,\cdot\rangle$. We will use the symbol $\lVert \cdot \rVert$ (usually with a subscript) for the norm in the function space; thus for a vector valued function f, the symbol $\lVert f\rVert_\infty$ denotes its L^∞ norm, which is the essential supremum of $\lvert f(z)\rvert$ over z in the domain of definition of f. On the other hand, the symbol $\lvert f\rvert$ stands for the scalar valued function whose value at a point z is the norm of the vector $f(z)$;
$\cdot^\top$, $\overline{\cdot}$, $\cdot^*$	If M is a matrix (possibly infinite), then $M^\top$ denotes the transpose of M. The complex conjugate of M is denoted by $\overline{M}$, and $M^* := (\overline{M})^\top$;
H^∞	space of bounded holomorphic functions on $\mathbb{D}$ with the supremum norm;
H^1	space of analytic functions f on $\mathbb{D}$ such that $\lVert f\rVert_1 := \sup_{0\le r<1} \int_{\mathbb{T}} \lvert f(r\zeta)\rvert dm(\zeta) < \infty$;
A	space of bounded holomorphic functions on $\mathbb{D}$ with a continuous extension to $\mathbb{T}$ with the supremum norm.

1. Introduction

The paper is devoted to the estimates in the corona problem in some smooth subalgebras of the algebra H^∞ of bounded analytic functions in the unit disc $\mathbb{D}$.

There main motivation for studying this problem comes from the idea of "visibility" or "δ-visibility" of the spectrum, introduced by N. Nikolski [5].

Let us recall the main definitions. Let $\mathcal{A}$ be a commutative unital Banach algebra continuously embedded into the space $C(X)$ of all continuous functions on a Hausdorff topological space X, $\mathcal{A} \subset C(X)$. The point evaluations δ_x $(x \in X)$ given by

$$\delta_x(f) = f(x), \quad f \in \mathcal{A},$$

are multiplicative linear functionals on $\mathcal{A}$. Hence if $\mathcal{A}$ distinguishes points of X, then we can identify X with a subset of the maximal ideal space of $\mathcal{A}$ (the spectrum $\mathfrak{M}(\mathcal{A})$ of $\mathcal{A}$), that is, $X \subset \mathfrak{M}(\mathcal{A})$.

Definition 1.1. Let $0 < \delta \le 1$. The spectrum of $\mathcal{A}$ is said to be (δ, m)-*visible* (from X) if there exists a constant $C(m)$ such that for any vector $f = (f_1, \dots, f_m) \in \mathcal{A}^m$ satisfying

$$\inf_{x \in X} \sum_{k=1}^{m} |f_k(x)|^2 \ge \delta^2 > 0 \tag{1.1}$$

and the normalizing condition

$$\|f\|^2 := \sum_{k=1}^{m} \|f_k\|_{\mathcal{A}}^2 \le 1,$$

the Bezout equation

$$g \cdot f := \sum_{k=1}^{m} g_k f_k = e \tag{1.2}$$

has a solution $g = (g_1, \dots, g_m) \in \mathcal{A}^m$ with

$$\|g\| = \left(\sum_{k=1}^{m} \|g_k\|_{\mathcal{A}}^2 \right)^{1/2} \le C(m).$$

The spectrum is called *completely* δ-*visible* if it is (δ, m)-visible for all $m \ge 1$ and the constants $C(m)$ can be chosen in such a way that $\sup_{m \ge 1} C(m) < \infty$.

This is a norm refinement of the usual corona problem for Banach algebras, and the motivations for the consideration of this problem can be found in Nikolski [5].

The classical corona theorem for the algebra H^∞, see [1], says that if the functions $f_k \in H^\infty = H^\infty(\mathbb{D})$ satisfy

$$1 \ge \sum_{k=1}^{m} |f_k(z)|^2 \ge \delta^2 > 0, \qquad \forall z \in \mathbb{D}, \tag{1.3}$$

then the Bezout equation

$$\sum_{k=1}^{m} g_k f_k = 1 \tag{1.4}$$

has a solution $g_1, g_2, \dots, g_m$, and moreover the solution satisfies the estimates

$$\sum_{k=1}^{m} |g_k(z)|^2 \le C(\delta, m)^2, \qquad \forall z \in \mathbb{D}.$$

Later refinements obtained independently by M. Rosenblum [7] and V. Tolokonnikov [11], got the estimate independent on m and allowed the case $m = \infty$, see Appendix 3 of [6] for modern treatment.

Note that having estimates that are independent of m in the corona theorem in fact gives us something slightly more than the complete δ-visibility of the spectrum of H^∞, since the normalizing condition in (1.3) is weaker than the corresponding normalizing condition in Definition 1.1.

On the other hand there many algebras with invisible spectrum. For example, for the Wiener algebra W of analytic functions,

$$f = \sum_{k=0}^{\infty} \widehat{f}(k) z^k, \quad \|f\|_W := \sum |\widehat{f}(k)| < \infty,$$

the Corona Theorem holds trivially, that is, the unit disc $\mathbb{D}$ is dense in the maximal ideal space $\mathfrak{M}(W)$, but it is in general impossible to control the norms of solution of the Bezout equation: the algebra W is not even $(\delta, 1)$-visible for small δ.

It is general understanding among experts that the estimates hold for local norms, and may (generally) fail for non-local norms, for example for norms given in terms of Fourier coefficients.

In this article, we study the following subalgebras of H^∞. Let us recall that A denotes the *disc algebra* of all bounded analytic functions continuous up to the boundary, $A = H^\infty \cap C(\mathbb{T})$.

Definition 1.2. For a positive integer n define the following algebras:

(1) $\partial^{-n} H^\infty$ is the set of all analytic functions f defined on $\mathbb{D}$ such that f, f', ..., $f^{(n)}$ belong to H^∞.
(2) $\partial^{-n} A$ is the set of all analytic functions f defined on $\mathbb{D}$ such that $f, f', \dots, f^{(n)}$ belong to the disk algebra A.
(3) More generally, if S be an open subset of $\mathbb{T}$, then $\partial^{-n} A_S$ is the set of all analytic functions f defined on $\mathbb{D}$ such that $f, f', \dots, f^{(n)}$ belong to A_S, where A_S denotes the class of functions defined on the disk that are holomorphic and bounded in $\mathbb{D}$ and extend continuously to S.

The above spaces are Banach algebras with the norm given by

$$\|f\| = \sum_{j=0}^{n} \frac{1}{j!} \|f^{(j)}\|_\infty.$$

The factor $1/j!$ is chosen so the norm satisfies the estimate $\|fg\| \le \|f\| \cdot \|g\|$.[1]

For a Hilbert space H, one can consider the H-valued spaces $\mathcal{A}(H)$, where $\mathcal{A}$ is one of the spaces $\partial^{-n} H^\infty$, $\partial^{-n} A$, $\partial^{-n} A_S$ defined above. Namely, for an analytic H-valued function f we define its norm as

$$\|f\| = \sum_{j=0}^{n} \frac{1}{j!} \|f^{(j)}\|_\infty, \tag{1.5}$$

where the norm is understood as the L^∞ norm of the vector-valued function with values in H. For example, if $H = \ell^2$ (or $H = \mathbb{C}^m$), then for $f = \{f_k\}_{k=1}^\infty = (f_1, f_2, \dots, f_k, \dots)$,

$$\|f^{(j)}\|_\infty = \operatorname{essup}_{z \in \mathbb{T}} \left\| f^{(j)}(z) \right\| = \operatorname{essup}_{z \in \mathbb{T}} \left(\sum_k |f_k^{(j)}(z)|^2 \right)^{\frac{1}{2}}.$$

[1]In the definition of the Banach algebra it is usually required that the norm satisfies the estimate $\|fg\| \le \|f\| \cdot \|g\|$. However, in a unital Banach algebra, if one is given the norm that only satisfies a weaker inequality $\|fg\| \le C\|f\| \cdot \|g\|$ (so the multiplication is continuous), there is a standard way to replace the norm by an equivalent one, satisfying the inequality with $C = 1$. Namely, the new norm of an element f is defined as the operator norm of multiplication by f. It is an easy exercise to show that the new norm is equivalent to the original one; one needs the fact that the algebra is unital to get one of the estimates.

We prove in the paper that the corona theorem with estimates holds for all these algebras, and that the estimate does not depend on the number of functions f_k. This fact implies complete δ-visibility of the spectrum for all $\delta > 0$.

One of the motivations for studying these algebras comes from control theory. Namely, for a system (plant) G with coprime factorization $G = f_1/f_2$, the construction of a stabilizing feedback is equivalent to solving the Bezout equation

$$g_1 f_1 + g_2 f_2 \equiv 1,$$

with the stabilizing controller given by $-g_1/g_2$. And assuming that the original plant G (more precisely, its coprime factorization) has some smoothness, we want to be able to construct the stabilizing controller with the same smoothness and to be sure that the smoothness of this stabilizer is controlled by the smoothness of G.

Before proving the corona theorem with bounds for the subalgebras of H^∞ introduced above in Definition 1.2 , we remark that the corona theorem itself is trivial for them. Indeed we show below that the maximal ideal space $\mathfrak{M}(\partial^{-n} A_S)$ $(n \in \mathbb{N})$ is the closed unit disk, which gives the equivalence of (1.1) (with $X = \mathbb{D}$) and the solvability of the Bezout equation (1.2).

Theorem 1.3. *Let $\mathcal{A}$ be one of the algebras $\partial^{-n} H^\infty$, $\partial^{-n} A$, $\partial^{-n} A_S$ defined above $(n \geq 1)$. The maximal ideal space of $\mathcal{A}$ is the closed unit disk.*

Proof. Note that $\partial^{-n} H^\infty \subset A$, and so point evaluation at a fixed $\lambda \in \overline{\mathbb{D}}$ gives a multiplicative linear functional on $\partial^{-n} A_S$. We will show that every multiplicative linear functional arises in this manner.

Let L be a multiplicative linear functional and let $\lambda := L(z)$ (the value of L on the function $f(z) \equiv z$). Then clearly $L(f) = f(\lambda)$ for polynomials f. We show that for any polynomial f

$$|L(f)| \leq \|f\|_\infty. \tag{1.6}$$

This estimate immediately implies that $|\lambda| \leq 1$ (apply (1.6) to the function $f(z) \equiv z$). Since $\mathcal{A} \subset A$, any function f in $\mathcal{A}$ can be approximated by polynomials in the L^∞-norm. But (1.6) implies that L is continuous in L^∞ norm, so formula (1.6) holds for all $f \in \mathcal{A}$. Note that in this reasoning we do not need the density of polynomials in the norm of $\mathcal{A}$ (which happens only if $\mathcal{A} = \partial^{-n} A$).

To prove (1.6) let us notice that if $f \in \mathcal{A}$ and $\inf_{z \in \mathbb{D}} |f(z)| > 0$, then f is invertible in $\mathcal{A}$. Indeed, since $\mathcal{A} \subset A$, the condition $\inf_{z \in \mathbb{D}} |f(z)| > 0$ implies that f is invertible in A.

Differentiating $1/f$ n times we get that all its derivatives up to the order n are in the algebra H^∞ or A or A_S, depending on the algebra $\mathcal{A}$ we are considering.

Therefore, if $0 \notin \operatorname{clos} \operatorname{range}(f) = \operatorname{range}(f)$, then f is invertible in $\mathcal{A}$, and so f does not belong to any proper ideal of $\mathcal{A}$. Thus $L(f) \neq 0$ for any maximal ideal (multiplicative linear functional) L. Replacing f by $f - a$, $a \in \mathbb{C}$, we get that if $a \notin \operatorname{range}(f)$, then for any multiplicative linear functional L, $L(f) \neq a$, that is, $L(f) \subset \operatorname{range}(f)$. Thus $|L(f)| \leq \|f\|_\infty$, and (1.6) is proved. □

Plan of the paper. In section 2 we prove the corona theorem with estimates on the norm of the solution for the algebra $\partial^{-n} H^\infty$, see Theorem 2.1 below.

This result is stronger than the complete δ-visibility of the spectrum of $\partial^{-n} H^\infty$.

We will use this result to show that the corona theorem with the same estimates holds for the algebras $\partial^{-n} A$ and $\partial^{-n} A_S$ as well. That of course would imply that the spectrum of these algebras in completely δ-visible for all $\delta > 0$.

The estimates for the algebra $\partial^{-n}A$ will be obtained from the estimates for $\partial^{-n}H^\infty$ by a simple approximation argument. The same argument will be used to get the estimates for $\partial^{-n}A_S$, with the essential difference that the construction of the approximating functions is quite involved in this case: the reasoning "modulo the approximation" is very similar to the one for the $\partial^{-n}A$.

Note that the results for $n = 0$ are quite known. While we cannot give the exact reference, the fact that the estimates in the corona theorem for the disc algebra as the same as the estimates for H^∞ is known to the specialists. The estimates in the corona theorem for the algebra A_S were considered by the first author, [8], although the equality of these estimates to ones for H^∞ was not mentioned there.

2. Estimates in the corona theorem for $\partial^{-n}H^\infty$

Theorem 2.1. *Let n be a nonnegative integer, and let $\mathcal{A} = \partial^{-n}H^\infty$. There exists a constant $C(\delta, n)$ such that for all $f = (f_1, f_2, \dots, f_k, \dots) \in \mathcal{A}(\ell^2)$ satisfying*

$$0 < \delta \leq \| f(z) \|_{\ell^2} \quad \text{for all } z \in \mathbb{D}, \tag{2.1}$$

and

$$\|f\|_{\mathcal{A}(\ell^2)} \leq 1, \tag{2.2}$$

there exist $g = (g_1, g_2, \dots, g_k, \dots) \in \mathcal{A}(\ell^2)$ such that

$$\sum_k g_k(z) f_k(z) = 1 \quad \text{for all } z \in \mathbb{D}, \tag{2.3}$$

and

$$\|g\|_{\mathcal{A}(\ell^2)} \leq C(\delta, n). \tag{2.4}$$

Note that by considering sequences $f = (f_1, f_2, \dots, f_n, \dots)$ with finitely many non-zero entries, one can get the result about m-tuples as an elementary corollary.

2.1. **Preliminaries for the proof.** We want to introduce a different equivalent norm on the space $\partial^{-n}H^\infty$. Namely, for smooth functions on the circle $\mathbb{T}$ let us consider the differential operator D

$$(Df)(e^{it}) := -i\frac{d}{dt} f(e^{it}).$$

Define the space $D^{-n}L^\infty := \{f \in L^\infty \mid D^k f \in L^\infty, k = 1, 2, \dots, n\}$. A natural norm on this class is given by

$$\sum_{k=0}^{n} \|f^{(k)}\|_\infty. \tag{2.5}$$

Of course, one can also define this space for the functions with values in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$. For our purposes it is more convenient to consider a different equivalent norm on $D^{-n}L^\infty$

$$\|f\| := \| \widehat{f}(0) \| + \|D^n f\|_\infty, \quad f \in D^{-n}L^\infty, \tag{2.6}$$

where $\widehat{f}(k)$ $(k \in \mathbb{Z})$ denotes the kth Fourier coefficient of f,

$$\widehat{f}(k) = (2\pi)^{-1} \int_\pi^\pi f(e^{it}) e^{-ikt} dt.$$

To show the equivalence of two norms, let us notice that for $\zeta \in [0, 2\pi)$,

$$f(e^{i\zeta}) = \frac{1}{2\pi}\int_{\zeta-\pi}^{\zeta+\pi} [f(e^{i\zeta}) - f(e^{i\theta})]d\theta + \widehat{f}(0).$$

Since

$$\left| f(e^{i\zeta}) - f(e^{i\theta}) \right| \le \|Df\|_\infty |\theta - \zeta|,$$

we get by integrating this estimate

$$\|f\|_\infty \le \frac{1}{4}\|Df\|_\infty + \left|\widehat{f}(0)\right|. \tag{2.7}$$

As $\widehat{Df}(0) = 0$, $\|Df\|_\infty \le \frac{1}{4}\|D^2 f\|_\infty$. Proceeding in a similar manner we get

$$\|D^k f\|_\infty \le 4^{k-n}\|D^n f\|_\infty, \quad k \in \{1, \dots, n\}, \quad f \in D^{-n}L^\infty,$$

so the norms of all derivatives can be estimated by $\|D^n f\|_\infty$ and $\left|\widehat{f}(0)\right|$. Therefore the norms (2.5) and (2.6) are equivalent.

Now we want to find the predual to $D^{-n}L^\infty$. It is easy to see that if one writes an appropriate duality, then $D^{-n}L^\infty$ is dual to L^1. Namely, it follows from the standard L^1-L^∞ duality that any bounded linear functional on L^1 can be represented as

$$L(f) = \langle \widehat{f}(0), \widehat{g}(0)\rangle + \int_{\mathbb{T}} \langle f, D^n g\rangle dm, \qquad f \in L^1, \tag{2.8}$$

where g is a function in $D^{-n}L^\infty$. Moreover, the norm of L is comparable with the norm $\|g\|_{D^{-n}L^\infty}$. Indeed, the functional L can be represented as

$$L(f) = \int_{\mathbb{T}} \langle f, F\rangle dm, \quad f \in L^1,$$

where $F \in L^\infty$, $\|F\|_\infty = \|L\|$. Let D^{-1} denote the integration operator, $D^{-1}e^{int} = \frac{1}{n}e^{int}$, $n \ne 0$. Then $D^{-n}(F - \widehat{F}(0)) + \widehat{F}(0) =: g \in D^{-n}L^\infty$ with the norm $\|g\|_{D^{-n}L^\infty}$ comparable to $\|F\|_\infty$, which immediately implies the representation (2.8).

And finally, it is easy to see that $\partial^{-n}H^\infty = H^\infty \cap D^{-n}L^\infty$ and the norm $\|\cdot\|_{D^{-n}L^\infty}$ is equivalent to the norm in $\partial^{-n}H^\infty$. Indeed, since $D(e^{ikt}) = ke^{ikt}$ we conclude that $Df(z) = zf'(z)$ for analytic polynomials $f = \sum_{k=0}^N a_k z^k$. Iterating the formula $Df(z) = zf'(z)$ and using the fact that multiplication by z does not change the norm in $L^\infty(\mathbb{T})$ we get the estimate

$$\|D^k f\|_\infty \le C\sum_{j=1}^k \|f^{(j)}\|_\infty, \qquad k = 1, 2, \dots, n,$$

which implies that $\|f\|_{D^{-n}L^\infty} \le C\|f\|_{\partial^{-n}H^\infty}$.

To get the opposite inequality, we iterate the identity $f'(z) = z^{-1}DF(z)$, and since the multiplication by z^{-1} does not change the $L^\infty(\mathbb{T})$ norm we get the estimate

$$\|f^{(k)}\|_\infty \le C\sum_{j=1}^k \|D^j f\|_\infty, \qquad k = 1, 2, \dots, n.$$

Using standard approximation reasoning we get that the norms are equivalent for functions $f \in \mathrm{Hol}(\overline{\mathbb{D}})$, where $\mathrm{Hol}(\overline{\mathbb{D}})$ is the set of all functions analytic in a neighborhood of the closed disc $\overline{\mathbb{D}}$. It is also easy to see that $\partial^{-n}H^\infty \cap \mathrm{Hol}(\overline{\mathbb{D}}) = D^{-n}L^\infty \cap H^\infty \cap \mathrm{Hol}(\overline{\mathbb{D}})$.

Finally, for both $X = \partial^{-n} H^\infty$ and $X = D^{-n} L^\infty \cap H^\infty$ we have that $f \in X$ iff $f_r \in X$ for all r, $0 \le r < 1$, where $f_r(z) := f(rz)$. Moreover $\|f\|_X = \lim_{r\to 1-} \|f_r\|_X$.

Note that the operator D is symmetric, namely, for smooth f, g, integration by parts or use of the Fourier series representations yields

$$\int_{\mathbb{T}} \langle Df, g\rangle dm = \int_{\mathbb{T}} \langle f, Dg\rangle dm. \tag{2.9}$$

Therefore, for smooth functions f the duality (2.8) can be rewritten as

$$L(f) = \langle \widehat{f}(0), \widehat{g}(0)\rangle + \int_{\mathbb{T}} \langle D^n f, g\rangle dm, \qquad f \in L^1, \tag{2.10}$$

Remark 2.2. Given a $\Phi \in C^\infty(\overline{\mathbb{D}})$, there always exists a $\Psi \in C^\infty(\overline{\mathbb{D}})$ such that $\overline{\partial}\Psi = \Phi$ on some neighbourhood of $\overline{\mathbb{D}}$. Indeed, let O be open and let $\overline{\mathbb{D}} \subset O$. Let $\alpha \in C_0^\infty(O)$ be such that $\alpha = 1$ on a neighbourhood of $\overline{\mathbb{D}}$. Defining Ψ by

$$\Psi(z) = -\frac{1}{\pi} \iint_{\mathbb{R}^2} \frac{\alpha(\zeta)\Phi(\zeta)}{\zeta - z} dxdy, \quad z \in \mathbb{C},$$

it can be seen that $\Psi \in C^\infty(\mathbb{C})$ and $\overline{\partial}\Psi = \Phi$.

2.2. **Setting up the $\overline{\partial}$-equation.** We will follow the standard way of setting up the $\overline{\partial}$-equations to solve the corona problem, as presented for example in [6]. We assume that we are given a column vector $f = (f_1, f_2, \ldots, f_m, \ldots)^\top$ and we want to find a row vector $g = (g_1, g_2, \ldots, g_m, \ldots)$ satisfying

$$g \cdot f = \sum_k g_k f_k \equiv 1.$$

We will use the standard linear algebra conventions, for example for a matrix A, $A^* = \overline{A}^\top$. In particular, f^* is a row vector $f^* = (\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m, \ldots)$. Also, for two vectors $f, g \in \ell^2$ we will use the notation $g \cdot f$ for the "dot product", $g \cdot f := g^\top f = \sum_k g_k f_k$.

As usual, it is sufficient to prove the theorem under the additional assumption that f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. Let $0 < r < 1$, and set $f_r(z) = f(rz)$, $z \in \mathbb{D}$. Then $f_r \in \operatorname{Hol}(\overline{\mathbb{D}})$, and we have $\|f_r\| \le 1$, and $|f_r(z)| \ge \delta$ for all $z \in \mathbb{D}$. If the statement of the theorem is true for f's in $\operatorname{Hol}(\overline{\mathbb{D}})$, then there exists a $g_r \in \operatorname{Hol}(\mathbb{D})$ such that $g_r(z) f_r(z) = 1$ for all $z \in \mathbb{D}$, and $\|g_r\| \le C(\delta)$. If we choose $r_k \to 1$ such that $g_r \to g$ uniformly on compact subsets of $\mathbb{D}$ (possible by Montel's theorem), then the g satisfies (2.3) and (2.4) of the theorem.

We suppose therefore that $f \in \operatorname{Hol}(\overline{\mathbb{D}})$ and (2.1) holds.

Define the row vector φ,

$$\varphi = \frac{f^*}{|f|^2}.$$

Then $\varphi \in C^\infty(\overline{\mathbb{D}})$, and $\varphi f \equiv 1$ on a neighbourhood of $\overline{\mathbb{D}}$. So φ solves the Bezout equation $\varphi f \equiv 1$, but it is not analytic in $\mathbb{D}$. Note that

$$\overline{\partial}\varphi = \frac{(f')^*}{|f|^2} - \frac{(f')^* f}{|f|^4} f^*.$$

If we find a matrix Ψ solving the $\overline{\partial}$-equation

$$\overline{\partial}\Psi = \varphi^\top \overline{\partial}\varphi =: \Phi,$$

then

$$g := \varphi + f^\top(\Psi^\top - \Psi)$$

will be analytic in $\mathbb{D}$, since

$$\begin{aligned}
\overline{\partial} g &= \overline{\partial}\varphi + f^\top(\overline{\partial}\Psi^\top - \overline{\partial}\Psi) \quad (\text{since } \overline{\partial} f = 0)\\
&= \overline{\partial}\varphi + f^\top((\overline{\partial}\varphi)^\top\varphi - \varphi^\top\overline{\partial}\varphi) \quad (\text{using } \overline{\partial}\Psi = \varphi^\top\overline{\partial}\varphi)\\
&= \overline{\partial}\varphi + ((\overline{\partial}\varphi)f)^\top\varphi - \overline{\partial}\varphi \quad (\text{using } \varphi f \equiv 1)\\
&= ((\overline{\partial}\varphi)f)^\top\varphi = (\overline{\partial}(\varphi f))^\top\varphi \quad (\text{since } \overline{\partial} f = 0)\\
&= 0
\end{aligned}$$

where the last equality follows from the fact that $\varphi f \equiv 1$. Moreover, since the matrix $\Xi = \Psi - \Psi^\top$ is antisymmetric ($\Xi^\top = -\Xi$), we have $f^\top(\Psi - \Psi^\top)f = 0$, so $gf = \varphi f \equiv 1$.

2.3. **Estimates of the solution of the $\overline{\partial}$-equation from the boundedness of L.** Let us see what we need to get the estimate of the norm of the solution. Since $D^n(\Xi f) = \sum_{k=0}^n \binom{n}{k}(D^k\Xi)D^{n-k}f$, the estimates

$$\operatorname*{esssup}_{\zeta\in\mathbb{T}} \|\Psi^{(k)}(\zeta)\| \le C < \infty, \qquad k = 1, 2, \dots, n$$

where $\|\cdot\|$ denotes the operator norm of a matrix, imply the solution g is in the space $D^{-n}L^\infty(\ell^2)$. Since the solution g we get is analytic, that is exactly what we need.

Since the operator norm of a matrix is dominated by the Hilbert–Schmidt norm $\|\cdot\|_{\mathfrak{S}_2}$, it is sufficient to estimate the Hilbert–Schmidt norms of the derivatives, that is, to estimate the norm of the solution Ψ in the space $D^{-n}L^\infty(\mathfrak{S}_2)$. Note that the space $\mathfrak{S}_2$ of Hilbert–Schmidt operators (matrices) is a Hilbert space with the inner product $\langle A, B\rangle_{\mathfrak{S}_2} := \operatorname{tr} AB^* \equiv \operatorname{tr} B^*A$, so all the previous discussions about norms and duality for the space $D^{-n}L^\infty$ do apply here.

We estimate the norm of the solution of the $\overline{\partial}$-equation by duality. Let Ψ_0 be any smooth solution of the $\overline{\partial}$-equation

$$\overline{\partial}\Psi = \Phi := \varphi^\top\overline{\partial}\varphi = \frac{\overline{f}}{\|f\|^2}\left(\frac{(f')^*}{\|f\|^2} - \frac{(f')^* f}{\|f\|^4} f^*\right), \tag{2.11}$$

Define the linear functional L on $H^1_0(\mathfrak{S}_2) := zH^1(\mathfrak{S}_2)$,

$$L(h) = \int_\mathbb{T} \operatorname{tr}\{(D^n h)\Psi_0\} dm = \int_\mathbb{T} \langle D^n h, \Psi_0^*\rangle_{\mathfrak{S}_2} dm.$$

Note that the above expression is well defined on a dense subspace of smooth functions in $H^1_0(\mathfrak{S}_2)$, for example on the subspace $X_0 = H^1_0(\mathfrak{S}_2) \cap \operatorname{Hol}(\overline{D}, \mathfrak{S}_2)$.

If we prove that L is a bounded functional on $H^1_0(\mathfrak{S}_2)$, it can be extended by Hahn–Banach Theorem to a bounded functional on the whole space $L^1(\mathfrak{S}_2)$. That means, according to our discussions of duality, see (2.8), (2.10), that there exists a function $\Psi \in D^{-n}L^\infty$, $\|\Psi\|_{D^{-n}L^\infty} \asymp \|L\|$, such that

$$L(h) = \int_\mathbb{T} \operatorname{tr}\{(D^n h)\Psi_0\} dm = \int_\mathbb{T} \operatorname{tr}\{(D^n h)\Psi\} dm \qquad \forall h \in X_0.$$

Note that $\widehat{h}(0) = 0$ for $h \in X_0$, so the term corresponding $\langle \widehat{f}(0), \widehat{g}(0)\rangle$ from (2.8), (2.10) disappears.

Since $\int_\mathbb{T} \operatorname{tr}\{(D^n h)(\Psi - \Psi_0)\} = 0$ on a dense set X in H^1_0, the function $\Psi - \Psi_0$ is analytic in $\mathbb{D}$, so Ψ solves the $\overline{\partial}$-equation $\overline{\partial}\Psi = \Phi$.

2.4. **Estimates of the functional L.** To estimate $L(h)$, we use Green's formula,

$$\int_{\mathbb{T}} u\, dm - u(0) = \frac{2}{\pi} \int_{\mathbb{D}} (\partial\overline{\partial} u(z)) \ln \frac{1}{|z|} dxdy$$

which holds for C^2-smooth functions u in the closed disc $\overline{\mathbb{D}}$ (recall that $\partial\overline{\partial} = \frac{1}{4}\Delta$). Applying this formula to $u = \operatorname{tr}\{(D^n h)\Psi\}$, where $D^n h$ in the disc is defined as the harmonic (analytic) extension from the boundary, we get

$$\begin{aligned} L(h) &= \int_{\mathbb{T}} \operatorname{tr}\{(D^n h)\Psi\}\, dm \\ &= \frac{2}{\pi} \iint_{\mathbb{D}} (\partial\overline{\partial} \operatorname{tr}\{(D^n h)\Psi\}) \log \frac{1}{|z|} dxdy \quad \text{(because } D^n h(0) = 0\text{)} \\ &= \frac{2}{\pi} \iint_{\mathbb{D}} (\partial \operatorname{tr}\{(D^n h)\Phi\}) \log \frac{1}{|z|} dxdy \quad \text{(because } \overline{\partial}(D^n h) = 0 \text{ and } \overline{\partial}\Psi = \Phi\text{)} \\ &= \frac{2}{\pi}(I_1 + I_2), \end{aligned}$$

where

$$I_1 := \iint_{\mathbb{D}} \operatorname{tr}\{(D^n h)\partial\Phi\} \log \frac{1}{|z|} dxdy \quad \text{ and } I_2 := \iint_{\mathbb{D}} \operatorname{tr}\{(\partial D^n h)\Phi\} \log \frac{1}{|z|} dxdy.$$

To estimate the integrals I_1, I_2 we would like to move the derivatives to Φ. To do this, let us extend the operator D to the whole disc as follows:

$$Dw(re^{i\theta}) = -i \frac{d}{d\theta} w(re^{i\theta}).$$

Then $Dz^n = nz^n$ and $D\overline{z}^n = -n\overline{z}^n$ for $n \geq 0$, and so for holomorphic w, $Dw(z) = zw'(z)$ and $D\overline{w} = -\overline{zw'(z)}$.

Note that if we treat $D^n h$ as the "extended" operator D^n applied to the function in the disc, we get the same result as before, when we defined $D^n h$ in the disc as the harmonic (analytic) extension from the boundary.

2.4.1. *Estimates of I_1.* Using the symmetry of D, see (2.9), we get for I_1

$$\begin{aligned} I_1 = \iint_{\mathbb{D}} \operatorname{tr}\{(D^n h)\partial\Phi\} \log \frac{1}{|z|} dxdy &= \iint_{\mathbb{D}} \langle D^n h, \overline{\partial}\Phi^* \rangle_{\mathfrak{S}_2} \log \frac{1}{|z|}\, dxdy \\ &= \iint_{\mathbb{D}} \langle h, D^n \overline{\partial}\Phi^* \rangle_{\mathfrak{S}_2} \log \frac{1}{|z|}\, dxdy \end{aligned}$$

where the last equality can be seen as follows: we write the integral in polar coordinates, then, in the integral with respect to $d\theta$ we apply the formula (2.9) and finally we go back to $dxdy$. Note that we used the inner product notation, because the symmetry of the operator D is more transparent and is easier to write this way.

Applying n times the operator D we get that $D^n\overline{\partial}\Phi^*$ can be represented as a sum of terms of form

$$\frac{\text{product of analytic and antianalytic factors}}{\| f \|^{2r}} \tag{2.12}$$

where (up to the transpose) the antianalytic factors can be only of the form $(f^{(j)})^*$, the analytic ones can be only of the form $f^{(l)}$, $j, l = 0, 1, \ldots, n+1$. Moreover, if one looks at the derivatives of the maximal possible order $k = n + 1$, each term of form (2.12) can have at most one factor $f^{(k)}$ and at most one factor $(f^{(k)})^*$ (it can have both $f^{(k)}$ and $(f^{(k)})^*$). Indeed, the direct computations show that the function $\overline{\partial}\Phi^*$ clearly is represented as such a sum, with the maximal order of each derivative being 1. Each differentiation D preserves the form, and increases the maximal order of the derivative at most[2] by 1.

The terms in the decomposition (2.12) of $D^n\overline{\partial}\Phi^*$ containing both factors $f^{(k)}$ and $(f^{(k)})^*$ of maximal possible order $k = n+1$ can be estimated by $C\|f^{(n+1)}\|^2_{\ell^2}$. Note that $f^{(n)} \in H^\infty(\ell^2)$. Since for a bounded analytic function F with values in a Hilbert space the measure $\|F'(z)\|^2 \log\frac{1}{|z|}\,dxdy$ is Carleson (with the Carleson norm estimated by $C\|F\|^2_\infty$), we can conclude that the measure $\|f^{(n+1)}\|_{\ell^2} \log\frac{1}{|z|}\,dxdy$ is Carleson. Therefore

$$\iint_{\mathbb{D}} \|h(z)\|_{\mathfrak{S}_2} \cdot \|f^{(n+1)}\|^2_{\ell^2} \log\frac{1}{|z|}\,dxdy \le C\|h\|_{H^1(\mathfrak{S}_2)}$$

so the terms of I_1 containing both $f^{(n+1)}$ and $(f^{(n+1)})^*$ are estimated.

The terms in the decomposition (2.12) of $D^n\overline{\partial}\Phi^*$ containing only the derivatives of order $k < n+1$ are bounded, so the corresponding terms in I_1 are easily estimated, because the measure $\log\frac{1}{|z|}\,dxdy$ is trivially Carleson.

Finally, the terms in (2.12) containing only one of the factors $f^{(n+1)}$ or $(f^{(n+1)})^*$ can be estimated by $C\|f^{(n+1)}\|_{\ell^2}$, and since by the Cauchy–Schwartz inequality

$$\begin{aligned} &\iint_{\mathbb{D}} \|h(z)\|_{\mathfrak{S}_2} \|f^{(n+1)}(z)\|_{\ell^2} \log\frac{1}{|z|}\,dxdy \\ &\quad\le \left(\iint_{\mathbb{D}} \|h(z)\|_{\mathfrak{S}_2} \|f^{(n+1)}(z)\|^2_{\ell^2} \log\frac{1}{|z|}\,dxdy\right)^{1/2} \left(\iint_{\mathbb{D}} \|h(z)\|_{\mathfrak{S}_2} \log\frac{1}{|z|}\,dxdy\right)^{1/2} \\ &\quad\le C\|h\|_{H^1(\mathfrak{S}_2)} \end{aligned}$$

(as we discussed above, the measures in both integrals in the second line are Carleson), so the corresponding terms in I_1 are also easily estimated.[3]

2.4.2. *Estimates of I_2.* Let us now estimate I_2. By trivial estimates for we have $|z| < 1/2$,

$$|\operatorname{tr}\{(\partial D^n h)\Phi\}| \le C\|h\|_{H^1(\mathfrak{S}_2)},$$

so we need only to estimate the integral I_2', where one integrates over $1/2 \le |z| < 1$.

Indeed, the derivatives of h can be estimated by the standard estimates for power series, it one recalls that $\|\widehat{h}(k)\|_{\mathfrak{S}_2} \le \|h\|_{H^1(\mathfrak{S}_2)}$. We also have $\|\Phi(z)\| \le C\|f'(z)\|$, and using the similar reasoning with power series one can show that $\|f'(z)\| \le C$ for $|z| < 1/2$.

Note that for analytic f we have $\partial f = z^{-1}Df$, and so we can replace $\partial D^n h$ by $z^{-1}D^{n+1}h$ in I_2'. Thus

$$I_2' = \iint\limits_{1/2\le|z|<1} \operatorname{tr}\{(\partial D^n h)\Phi\} \log\frac{1}{|z|}\,dxdy = \iint\limits_{1/2<|z|<1} \langle z^{-1}D^{n+1}h, \Phi^*\rangle_{\mathfrak{S}_2} \log\frac{1}{|z|}\,dxdy.$$

---

[2]It can be shown by more careful analysis, that no cancellation happens, and the maximal order of the derivative increases *exactly* by 1, but we do not need this for the proof: we only need that it cannot increase by more than 1.

[3]A careful analysis of $D^n\overline{\partial}\Phi^*$ can show that the terms containing only one derivative of the maximal order are impossible here, but the above reasoning is significantly simpler than the careful analysis of derivatives.

Using the symmetry of D we get as in the case of I_1

$$
\begin{aligned}
I_2' &= \iint\limits_{1/2<|z|<1} \langle Dh, D^n((\overline{z})^{-1}\Phi^*)\rangle_{\mathfrak{S}_2} \log\frac{1}{|z|}\,dxdy \\
&= \iint\limits_{1/2<|z|<1} \langle z^{-1}h'(z), D^n((\overline{z})^{-1}\Phi^*)\rangle_{\mathfrak{S}_2} \log\frac{1}{|z|}\,dxdy.
\end{aligned}
$$

Applying the operator D repeatedly to $(\overline{z})^{-1}\Phi^*$, we get the representation of $D^n((\overline{z})^{-1}\Phi^*)$ as the sum of terms of form (2.12), with slight differences. Namely, the analytic factors, as in the case of I_1 can be of the form $f^{(l)}$, $l = 1, 2, \ldots, n+1$, and the antianalytic factors (and that is the difference with the case of I_1) can only be of the form $(f^{(j)})^*$, $j = 1, 2, \ldots, n$ or $(\overline{z})^{-\kappa}$, $\kappa \geq 1$. And again, any term containing the derivative $f^{(n+1)}$ of the highest possible order can contain it only once.

We notice that $(\overline{z})^{-1}\Phi^*$ has such representation with $n = 0$, and each differentiation preserves the form of the decomposition and increases the maximal possible order of derivatives $f^{(l)}$ and $(f^{(j)})^*$ by at most 1.

To estimate I_2', let h_1 be a scalar-valued outer function in H^2 such that $|h_1(\zeta)|^2 = \| h(\zeta) \|$ a.e. on $\mathbb{T}$. Then $h \in H^1(\mathfrak{S}_2)$ can be represented as $h = h_1 h_2$, where $h_1 \in H^2$ (scalar), $h_2 \in H^2(\mathfrak{S}_2)$, and $\|h_1\|_{H^2}^2 = \|h_2\|_{H^2(\mathfrak{S}_2)}^2 = \|h\|_{H^1(\mathfrak{S}_2)}$.

Since $h' = h_1 h_2' + h_1' h_2$, we can estimates the terms of I_2' containing the derivative $f^{(n+1)}$ of the highest possible order by

$$
\iint_{\mathbb{D}} \| h(z) \|_{\mathfrak{S}_2} \| f^{(n+1)} \|_{\ell^2} \log\frac{1}{|z|}\,dxdy \leq \iint_{\mathbb{D}} (|h_1| \cdot \| h_2' \| + |h_1'| \cdot \| h_2 \|) \| f^{(n+1)} \|_{\ell^2} \log\frac{1}{|z|}\,dxdy.
$$

Since, as we discussed before, when treating I_1, the measure $\| f^{(n+1)}(z) \|_{\ell^2} \log\frac{1}{|z|}\,dxdy$ is Carleson, with its Carleson norm bounded by $C\|f\|_{H^\infty(\ell^2)}$, we get

$$
\begin{aligned}
&\iint_{\mathbb{D}} |h_1| \cdot \| h_2' \| \cdot \| f^{(n+1)} \|_{\ell^2} \log\frac{1}{|z|}\,dxdy \\
&\qquad \leq \left(\iint_{\mathbb{D}} |h_1|^2 \| f^{(n+1)} \|_{\ell^2}^2 \log\frac{1}{|z|}\,dxdy \right)^{1/2} \left(\iint_{\mathbb{D}} \| h_2' \|^2 \log\frac{1}{|z|}\,dxdy \right)^{1/2} \\
&\qquad \leq C\|h_1\|_{H^2}\|h_2\|_{H^2(\mathfrak{S}_2)} = C\|h\|_{H^1(\mathfrak{S}_2)};
\end{aligned}
$$

here the first integral in the second line is estimated because the measure is Carleson, and the second integral is simply the Littlewood–Paley representation of the norm $\|h_2\|_{H^2(\mathfrak{S}_2)}$. The integral $\iint_{\mathbb{D}} |h_1'| \cdot \| h_2 \| \cdot \| f^{(n+1)} \|_{\ell^2} \log\frac{1}{|z|}\,dxdy$ is estimated similarly.

The terms in the decomposition (2.12) of $D^n((\overline{z})^{-1}\Phi^*)$ which contain only derivatives of order at most n are bounded. Therefore to estimate the rest of I_2' it is sufficient to estimate $\iint_{\mathbb{D}} \| h' \| \log\frac{1}{|z|}\,dxdy$. Decomposing as above $h = h_1 h_2$ and using the fact that the measure

$\log \frac{1}{|z|}\, dxdy$ is trivially Carleson, we get the estimate

$$\begin{aligned}\iint_{\mathbb{D}} |h_1|\cdot \| h_2' \| \log \frac{1}{|z|}\, dxdy \\ &\le \left(\iint_{\mathbb{D}} |h_1|^2 \log \frac{1}{|z|}\, dxdy \right)^{1/2} \left(\iint_{\mathbb{D}} \| h_2' \|^2 \log \frac{1}{|z|}\, dxdy \right)^{1/2} \\ &\le C \|h_1\|_{H^2} \|h_2\|_{H^2(\mathfrak{S}_2)} = C\|h\|_{H^1(\mathfrak{S}_2)};\end{aligned}$$

The integral $\iint_{\mathbb{D}} |h_1'| \cdot \| h_2 \| \log \frac{1}{|z|}\, dxdy$, and thus the rest of I_2' is estimated similarly. □

3. Estimates in the corona theorem for other algebras: preliminaries and the case of $\partial^{-n} A$

3.1. **Continuity of the best estimate.** For a function algebra $\mathcal{A}$ (one should think about one of the algebras from Definition 1.2) let $C(\mathcal{A}, \delta)$, $\delta > 0$ denote the best possible estimate on the norm of the solution of the Bezout equation,

$$C(\mathcal{A}, \delta) := \sup_f \inf\{ \|g\|_{\mathcal{A}(\ell^2)} \,|\, g\cdot f := \sum_k g_k f_k \equiv 1\},$$

where the supremum is taken over all $f = (f_1, f_2, \dots, f_m, \dots) \in \mathcal{A}(\ell^2)$, $\|f\|_{\mathcal{A}(\ell^2)} \le 1$ and such that

$$\| f(z) \|_{\ell^2} := \left(\sum_k |f_k(z)|^2 \right)^{1/2} \ge \delta.$$

We will show in the rest of the paper that for the function algebras from the Definition 1.2 the constants $C(\mathcal{A}, \delta)$ coincide,

$$C(\partial^{-n} H^\infty, \delta) = C(\partial^{-n} A, \delta) = C(\partial^{-n} A_S, \delta).$$

Note that the inequalities

$$C(\partial^{-n} H^\infty, \delta) \le C(\partial^{-n} A, \delta), \qquad C(\partial^{-n} H^\infty, \delta) \le C(\partial^{-n} A_S, \delta)$$

are trivial. Indeed, if $f \in \partial^{-n} H^\infty(\ell^2)$ and satisfies the estimates $\|f\| \le 1$, $\| f(z) \| \ge \delta$, the functions f_r, $f_r(z) = f(rz)$ are in $\partial^{-n} A$ (and in $\partial^{-n} A_S$) and satisfy the same estimates. Therefore, for any $\varepsilon > 0$ one can find solutions $g^r \in \partial^{-n} A(\ell^2)$, $g^r \cdot f_r \equiv 1$, $\|g^r\| \le C(\partial^{-n} A, \delta) + \varepsilon$. Picking a uniformly convergent on compact sets subsequence $g^{r_k} \to g$, $r_k \to 1-$ (which is possible by Montel's theorem), we get the $\partial^{-n} H^\infty(\ell^2)$ solution g, $g\cdot f \equiv 1$, $\|g\| \le C(\partial^{-n} A, \delta) + \varepsilon$. Since ε is arbitrary, we get the estimate $C(\partial^{-n} H^\infty, \delta) \le C(\partial^{-n} A, \delta)$. The estimate for the algebra $\partial^{-n} A_S$ is obtained in absolutely the same way.

Clearly, if $\mathcal{A}$ is one of the algebras we are considering in the paper, the functions $\delta \mapsto C(\mathcal{A}, \delta)$ are non-increasing. We can say even more:

Lemma 3.1. *Let $\mathcal{A}$ be one of the algebras $\partial^{-n} H^\infty$, $\partial^{-n} A$, $\partial^{-n} A_S$ ($n \ge 0$). Then the function $\delta \mapsto C(\mathcal{A}, \delta)$ is continuous on $(0, 1)$.*

Note that this lemma holds for $n = 0$, which corresponds to the case of agebras H^∞, A and A_S.

Proof of Lemma 3.1. To prove the continuity it is sufficient to only prove uniform right semi-continuity, that is, that $C(\mathcal{A}, \delta) = \lim_{\alpha \to \delta+} C(\mathcal{A}, \alpha)$ uniformly in $\delta \in [\delta_0, 1)$, for all $\delta_0 > 0$.

Because $C(\mathcal{A},\delta)$ is a non-increasing function of δ, it will be sufficient only to prove only "$\le$" estimate (but still uniformly in $\delta \ge \delta_0$).

Let $f = (f_1, f_2, \dots, f_m, \dots) \in \mathcal{A}(\ell^2)$, $\|f\| \le 1$, $|f(z)|_{\ell^2} \ge \delta\ \forall z \in \mathbb{D}$.

Consider a new vector $\widetilde{f}^\gamma$, which is obtained from f by adding an extra entry $f_0 \equiv \gamma$, $\widetilde{f}^\gamma = (\gamma, f_1, f_2, \dots, f_m, \dots)$, where $\gamma > 0$ is small. Clearly

$$|\widetilde{f}^\gamma(z)| \ge \sqrt{\delta^2 + \gamma^2} \qquad \forall z \in \mathbb{D}.$$

Also,

$$\|\widetilde{f}^\gamma\|_{\mathcal{A}(\ell^2)} = \sqrt{a^2+\gamma^2} + \|f\|_{\mathcal{A}(\ell^2)} - a \le \sqrt{a^2+\gamma^2} + 1 - a,$$

where $a = \|f\|_{H^\infty(\ell^2)}$. Note that trivially $a \ge \delta$.

The expression $\sqrt{a^2+\gamma^2} + 1 - a$ is a decreasing function of a, so taking into account that $a \ge \delta$ we can estimate

$$\|\widetilde{f}^\gamma\|_{\mathcal{A}(\ell^2)} \le \sqrt{\delta^2+\gamma^2} + 1 - \delta.$$

Therefore the $\mathcal{A}(\ell^2)$ norm of the vector $(\sqrt{\delta^2+\gamma^2}+1-\delta)^{-1}\widetilde{f}^\gamma$ is at most 1, and we have

$$(\sqrt{\delta^2+\gamma^2}+1-\delta)^{-1}|\widetilde{f}^\gamma(z)|_{\ell^2} \ge \widetilde{\delta} = \widetilde{\delta}(\gamma) := \frac{\sqrt{\delta^2+\gamma^2}}{\sqrt{\delta^2+\gamma^2}+1-\delta}.$$

Note that $\widetilde{\delta}(\gamma) > \delta$ for $\gamma > 0$. That can be checked by noticing that $\widetilde{\delta}(0) = \delta$ and that $\frac{d\widetilde{\delta}(\gamma)}{d\gamma} > 0$ if $\gamma > 0$. Also, trivially, $\widetilde{\delta}(\gamma) \to \delta$ as $\gamma \to 0+$ uniformly in $\delta \in [\delta_0, 1)$ for all $\delta_0 > 0$.

Applying the definition of $C(\mathcal{A},\delta)$ to the rescaled function $(\sqrt{\delta^2+\gamma^2}+1-\delta)^{-1}\widetilde{f}^\gamma$ and then scaling everything back, we can find a vector $\widetilde{g}^\gamma = (g_0^\gamma, g_1^\gamma, g_2^\gamma, \dots, g_m^\gamma, \dots) \in \mathcal{A}(\ell^2)$ such that $\widetilde{g}^\gamma \cdot \widetilde{f}^\gamma \equiv 1$ and

$$\|\widetilde{g}^\gamma\|_{\mathcal{A}(\ell^2)} \le (\sqrt{\delta^2+\gamma^2}+1-\delta)^{-1} C(\mathcal{A},\widetilde{\delta}(\gamma)) + \gamma \le C(\mathcal{A},\widetilde{\delta}(\gamma)) + \gamma.$$

Since $C(\mathcal{A},\delta)$ is non-increasing, $C(\mathcal{A},\widetilde{\delta}(\gamma)) + \gamma \le C(\mathcal{A},\delta_0) + 1 =: M$ for $\delta \ge \delta_0$, so we have uniform (in γ and $\delta \ge \delta_0$) bound on the norm of $\widetilde{g}^\gamma$.

Define $g^\gamma := (g_1^\gamma, g_2^\gamma, \dots, g_m^\gamma, \dots) \in \mathcal{A}(\ell^2)$. Since $1 = \widetilde{g}^\gamma \cdot \widetilde{f}^\gamma = g_0^\gamma\gamma + g^\gamma \cdot f$ and also $\|\gamma g_0^\gamma\|_{\mathcal{A}} \le \gamma\|\widetilde{g}^\gamma\|_{\mathcal{A}(\ell^2)} \le \gamma \cdot (C(\mathcal{A},\delta_0)+1) =: M\gamma$, we conclude that $\|1 - g^\gamma \cdot f\|_{\mathcal{A}} \le M\gamma \to 0$ as $\gamma \to 0+$. Therefore for small γ the scalar function $g^\gamma \cdot f$ is invertible in $\mathcal{A}$ and moreover $\|(g^\gamma \cdot f)^{-1}\|_{\mathcal{A}} \le 1/(1 - M\gamma) \to 1$ as $\gamma \to 0+$. Then the function $(g^\gamma \cdot f)^{-1} g^\gamma$ solves the Bezout equation $(g^\gamma \cdot f)^{-1} g^\gamma \cdot f \equiv 1$, and

$$\|(g^\gamma \cdot f)^{-1} g^\gamma\|_{\mathcal{A}(\ell^2)} \le (C(\mathcal{A},\widetilde{\delta}(\gamma)) + \gamma)/(1 - M\gamma).$$

This inequality implies right semi-continuity of $C(\delta)$. Indeed, since the right side of the equation

$$\widetilde{\delta} := \frac{\sqrt{\delta^2+\gamma^2}}{\sqrt{\delta^2+\gamma^2}+1-\delta}$$

is an increasing function of γ, then for $\delta_0 \le \delta \le \widetilde{\delta} \le 1$ this equation has a unique solution $\gamma = \gamma(\delta,\widetilde{\delta})$. Moreover, the function $\gamma(\delta,\widetilde{\delta})$ is clearly continuous (and thus uniformly continuous) on $\delta_0 \le \delta \le \widetilde{\delta} \le 1$.

Therefore, given $\delta_0 > 0$ and $\varepsilon > 0$ one can find $\kappa > 0$ such that for all δ, $\widetilde{\delta}$ satisfying $\delta_0 \le \delta \le \widetilde{\delta} \le \delta + \kappa$ the inequality $C(\mathcal{A},\delta) \le C(\mathcal{A},\widetilde{\delta}) + \varepsilon$. The inequality $C(\mathcal{A},\widetilde{\delta}) \le C(\mathcal{A},\delta)$ is trivial because of monotonicity of $C(\mathcal{A},\delta)$. □

3.2. **Estimate in the algebra $\partial^{-n}A$.** We are going to prove that $C(\partial^{-n}A,\delta) = C(\partial^{-n}H^\infty,\delta)$ for $n \geq 0$. We only need to prove that $C(\partial^{-n}A,\delta) \leq C(\partial^{-n}H^\infty,\delta)$, since, as it was discussed above, the opposite inequality is trivial. Note that here we do not need the continuity of $C(\mathcal{A},\delta)$ proved above in Section 3.1.

Let $f \in (\partial^{-n}A)(\ell^2)$ satisfy

$$\| f(z) \|_{\ell^2} \geq \delta, \qquad \forall z \in \mathbb{D},$$

and $\|f\| \leq 1$. By the definition of $C(\partial^{-n}H^\infty,\delta)$, for any $\varepsilon > 0$ there exists $g \in \partial^{-n}H^\infty(\ell^2)$ solving the Bezout equation $g \cdot f \equiv 1$ and such that $\|g\| \leq C(\partial^{-n}H^\infty,\delta) + \varepsilon$.

If $0 < r < 1$, then $g_r \cdot f_r \equiv 1$, where $f_r(z) := f(rz)$ and $g_r(z) = g(rz)$, $z \in \mathbb{D}$. So we can write

$$g_r f = g_r \cdot f_r + g_r \cdot (f - f_r) = 1 + \alpha_r,$$

where $\alpha_r := g_r \cdot (f - f_r) \in \partial^{-n}A$. Since $\|f - f_r\| \to 0$ as $r \nearrow 1$ and $\|g_r\| \leq \|g\|$, we can conclude that $\|\alpha_r\| \to 0$ as $r \nearrow 1$. Thus for r close to 1, we have that $1 + \alpha_r$ is invertible in $\partial^{-n}A$ and $\|(1 + \alpha_r)^{-1}\| \to 1$ as $r \nearrow 1$.

Then $(1+\alpha_r)^{-1}g_r f \equiv 1$, and so $(1+\alpha_r)^{-1}g_r \in \partial^{-n}A$ is a left inverse of f. Moreover, since $\|g_r\| \leq \|g\| \leq C(\partial^{-n}H^\infty,\delta)+\varepsilon$ and $\|(1+\alpha_r)^{-1}\| \to 1$ as $r \nearrow 1$, it follows that for r sufficiently close to 1, $\|(1 + \alpha_r)^{-1}g_r\| \leq C(\partial^{-n}H^\infty,\delta) + 2\varepsilon$. Therefore $C(\partial^{-n}A,\delta) \leq C(\partial^{-n}H^\infty,\delta) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get the desired estimate.

3.3. **Preliminary estimates in the algebra $\partial^{-n}A_S$.** In this section we will show that $C(\partial^{-n}A_S,\delta) \leq 3C(\partial^{-n}H^\infty,\delta)^2$. To get the sharp estimate $C(\partial^{-n}A_S,\delta) \leq C(\partial^{-n}H^\infty,\delta)$ one needs to use more delicate reasoning, presented in Section 4 below.

We should emphasize that the reasoning below works only for $n \geq 1$, that is, that it does not work for the algebra A_S.

Let $f \in \partial^{-n}A_S(\ell^2)$, $\|f\| \leq 1$ satisfy

$$\| f(z) \|_{\ell^2} \geq \delta, \qquad \forall z \in \mathbb{D}.$$

Let $\varepsilon > 0$. By the definition of $C(\partial^{-n}H^\infty,\delta)$, there exists $g \in \partial^{-n}H^\infty(\ell^2)$ solving the Bezout equation $g \cdot f \equiv 1$ and such that $\|g\| \leq C(\partial^{-n}H^\infty,\delta) + \varepsilon$. Then, as before, $g_r \cdot f_r \equiv 1$ for $0 < r < 1$, where $f_r(z) := f(rz)$, $g_r(z) = g(rz)$, $z \in \mathbb{D}$. We cannot claim that $f_r \to f$ as $r \nearrow 1$ in the norm of $\partial^{-n}H^\infty$, but, since $\partial^{-1}H^\infty \subset A$, one can easily see that the convergence in the weaker norm of $\partial^{-n+1}H^\infty$ takes place (or, equivalently, in the norm of $\partial^{-n+1}A$, which is the same):

$$\|f_r - f\|_{\partial^{-n+1}H^\infty(\ell^2)} \to 0 \qquad \text{as } r \to 0+ .$$

Therefore,

$$g_r f = g_r \cdot f_r + g_r \cdot (f - f_r) = 1 + \alpha_r,$$

where $\alpha_r := g_r \cdot (f - f_r) \in \partial^{-n}A_S$, and $\|\alpha_r\|_{\partial^{-n+1}H^\infty(\ell^2)} \to 0$ as $r \nearrow 1$. We can see that $1+\alpha_r \in \partial^{-n+1}A$, so $1+\alpha_r$ is invertible in this algebra $\partial^{-n+1}A$ and $\|(1+\alpha_r)^{-1}-1\|_{\partial^{-n+1}A} \to 0$ as $r \nearrow \infty$.

We can show even more, namely that $1+\alpha_r$ is invertible in $\partial^{-n}A_S$ and estimate its norm in this algebra. Namely, let $\varphi_r = (1-\alpha_r)^{-2}$. Then clearly $\varphi_r \in \partial^{-n+1}A$ and $\|\varphi_r - 1\|_{\partial^{-n+1}A} \to 0$ as $r \nearrow 1$. Differentiating we get

$$((1 + \alpha_r)^{-1})' = -(1 + \alpha_r)^{-2}\alpha_r' = -\varphi_r\alpha_r',$$

so for the nth derivative

$$((1+\alpha_r)^{-1})^{(n)} = \sum_{k=0}^{n-1} \binom{n-1}{k} \varphi_r^{(k)} \alpha_r^{(n-k)}.$$

Note that this derivative is continuous on S (because $\alpha_r \in \partial^{-n} A_S$, $\varphi_r \in \partial^{-n+1} A$), so that $(1+\alpha_r)^{-1} \in \partial^{-n} A_S$. Since $\|\alpha_r\|_{\partial^{-n+1}A} \to 0$ as $r \nearrow 1$,

$$\left\| \sum_{k=1}^{n-1} \binom{n-1}{k} \varphi_r^{(k)} \alpha_r^{(n-k)} \right\|_\infty \to 0, \qquad \text{as } r \nearrow 1,$$

and so

$$\limsup_{r\to 1-} \|((1+\alpha_r)^{-1})^{(n)}\|_\infty \le \limsup_{r\to 1-} \|\varphi_r\|_\infty \|\alpha_r^{(n)}\|_\infty \le \limsup_{r\to 1-} n! \|\alpha_r\|_{\partial^{-n}H^\infty}$$

But it follows from the definition of α_r that

$$\|\alpha_r\|_{\partial^{-n}H^\infty} \le 2\|g_r\|_{\partial^{-n}H^\infty} \le 2\|g\|_{\partial^{-n}H^\infty} \le 2(C(\partial^{-n}H^\infty, \delta) + \varepsilon).$$

Using the fact that $\|(1+\alpha_r)^{-1} - 1\|_{\partial^{-n+1}A} \to 0$ as $r \nearrow 1$, we can estimate

$$\begin{aligned} \limsup_{r\to 1-} \|(1+\alpha_r)^{-1}\|_{\partial^{-n}H^\infty} &\le \limsup_{r\to 1-} \left(1 + \frac{1}{n!} \left\| ((1+\alpha_r)^{-1})^{(n)} \right\|_\infty \right) \\ &\le 1 + 2(C(\partial^{-n}H^\infty, \delta) + \varepsilon) \le 3(C(\partial^{-n}H^\infty, \delta) + \varepsilon) \end{aligned}$$

Note that the function $(1+\alpha_r)^{-1} g_r$ solves the Bezout equation, $(1+\alpha_r)^{-1} g_r \cdot f \equiv 1$ and belongs to $\partial^{-n} A_S$. We can estimate the norm

$$\begin{aligned} \limsup_{r\to 1-} \|(1+\alpha_r)^{-1} g_r\|_{\partial^{-n}H^\infty} &\le \limsup_{r\to 1-} \|(1+\alpha_r)^{-1}\|_{\partial^{-n}H^\infty} \|g\|_{\partial^{-n}H^\infty} \\ &\le 3(C(\partial^{-n}H^\infty, \delta) + \varepsilon)(C(\partial^{-n}H^\infty, \delta) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we get $C(\partial^{-n}A, \delta) \le 3C(\partial^{-n}H^\infty, \delta)^2$. □

3.4. **Remark on the stable rank of the algebras $\partial^{-n}H^\infty$, $\partial^{-n}A$, $\partial^{-n}A_S$.** Recall that if R is any ring, then its *Bass stable rank*, denoted by $\mathrm{bsr}(R)$, is by definition the least m such that whenever $r_1, \dots, r_{m+1} \in R$ and $\{r_j\}$ generate R as a left ideal, there are $b_1 \dots, b_m \in R$ such that $r_1 + b_1 r_{m+1}, \dots, r_m + b_m r_{m+1}$ generate R as a left ideal.

The Bass stable rank of each algebra for the function algebras from the Definition 1.2 is equal to 1. For $n \in \mathbb{N}$, this can be deduced easily from the fact that the Bass stable rank of the disk algebra A is 1, as follows. (That $\mathrm{bsr}(A) = 1$ was shown in [4].) Suppose that $f_1, f_2 \in \partial^{-n}A_S$ generate $\partial^{-n}A_S$. Then $f_1, f_2 \in A$ and for all $z \in \mathbb{D}$, $|f_1(z)| + |f_2(z)| > \delta > 0$. Using $\mathrm{bsr}(A) = 1$, it follows that there exists a $g_2 \in A$ such that $f_1 + f_2 g_2$ is invertible in A. If $r \in (0,1)$, define $g_{2,r} \in \partial^{-n}A_S$ by $g_{2,r}(z) := g_2(rz)$, $z \in \mathbb{D}$. Choosing r close enough to 1, we can ensure that $f_1 + f_2 g_{2,r}$ is invertible in A, and hence also in $\partial^{-n}A_S$.

4. Equality of the best estimate in the corona theorem for $\partial^{-n}A_S$ with that for $\partial^{-n}H^\infty$

In this section we will show that $C(\partial^{-n}A_S, \delta) = C(\partial^{-n}H^\infty, \delta)$ for $n \ge 0$. The method is similar to the one used in the previous section for $\partial^{-n}A$, except that we will need a more elaborate approximation scheme (given in Subsection 4.1) below.

The main idea is that we are going approximate the corona data f by the function $\widetilde{f}$ that extends analytically across S to a bigger (simply connected) domain $\Omega \supset \mathbb{D}$. The solution $\widetilde{g}$ of the Bezout equation $\widetilde{g} \cdot \widetilde{f} \equiv 1$ restricted to $\mathbb{D}$ automatically belongs to the class $\partial^{-n} A_S$ and "almost solves" the equation $g \cdot f \equiv 1$. Then, applying the reasoning similar to the one in Section 3.2 we get the estimate on the norm of the solution.

To carry out this plan we first of all need to construct such an approximation, which is done below in Section 4.1. We will also need to show that we can keep under control changes of the estimates when we conformally map Ω to the disc $\mathbb{D}$.

4.1. **Approximation result.** In this subsection, we prove a result about uniform approximation of a function from $\partial^{-n} A_S(\ell^2)$ by a function holomorphic across S, in Theorem 4.3. This result is a consequence of the following Lemma 4.2.

Definition 4.1. For an open set $\Omega \subset \mathbb{C}$ let $H^\infty(\Omega)$ denote the set of all bounded analytic functions on Ω. If n is a nonnegative integer, let $\partial^{-n} H^\infty(\Omega)$ be the set of all analytic functions f on Ω such that $f, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ belong to $H^\infty(\Omega)$, with the norm given by

$$\|f\|_{\partial^{-n} H^\infty(\Omega)} = \sum_{k=0}^{n} \frac{1}{k!} \|f^{(k)}\|_{H^\infty(\Omega)}.$$

Note that the space $\partial^{-n} H^\infty(\Omega; \ell^2)$ of ℓ^2-valued functions is defined similarly. Sometimes, when it is clear from the context that we are dealing with vector-valued functions, we will use $\partial^{-n} H^\infty(\Omega)$ instead of $\partial^{-n} H^\infty(\Omega; \ell^2)$

Lemma 4.2. *Let Ω be an open bounded subset of $\mathbb{C}$ containing 0 and with boundary $\partial\Omega$ that has a C^N-smooth polar parameterization $r = \rho(\theta)$. Suppose that C is a closed subarc in $\partial\Omega$, and K is an open (in $\partial\Omega$) set containing C. Let R be the open sector corresponding to K, $R = \{r\zeta : r \geq 0,\ \zeta = \rho(\theta) \in K\}$.*

Suppose that $f \in \partial^{-n} H^\infty(\Omega) = \partial^{-n} H^\infty(\Omega; \ell^2)$, where $n \leq N$, is such that f and all its derivatives $f^{(k)}$ for $k = 1, 2, \dots, n$ extend continuously to $K = R \cap \partial\Omega$.

Then given any $\varepsilon > 0$, there exists a domain $\widetilde{\Omega} = \Omega \cup O$, where O is an open neighborhood of C in $\mathbb{C}$ and a holomorphic function $F : \widetilde{\Omega} \to \ell^2$ with the following properties:

(S1) $\|F|_\Omega - f\| < \varepsilon$.
(S2) *The derivatives $F^{(k)}$, $k = 0, 1, 2, \dots, n$ extend continuously to $\widetilde{K} := \partial\widetilde{\Omega} \cap R$.*
(S3) $\left| \|F\|_{\partial^{-n} H^\infty(\widetilde{\Omega})} - \|f\|_{\partial^{-n} H^\infty(\Omega)} \right| < \varepsilon$.
(S4) *The boundary $\partial\widetilde{\Omega}$ of $\widetilde{\Omega}$ has a C^N-smooth polar parameterization $r = \widetilde{\rho}(\theta)$, and moreover $\|\rho - \widetilde{\rho}\|_{C^N} < \varepsilon$.*

Proof. Define a (trivial radial) C^n extension of f (denoted by the same letter) to $\Omega \cup R$ by

$$f(rz) = f(z), \qquad z \in \partial\Omega,\ r > 1.$$

Let φ be a compactly supported C^∞-function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on a neighbourhood U of C (in $\mathbb{C}$), and $\varphi = 0$ outside a slightly larger neighbourhood W; see Figure 1.

Define a function h (with values in ℓ^2) by

$$h(\zeta) = \frac{1}{\pi} \iint (\overline{\partial}\varphi(z)) \frac{f(z)}{z - \zeta} dxdy + \varphi(\zeta) f(\zeta) =: u + \varphi f \tag{4.1}$$

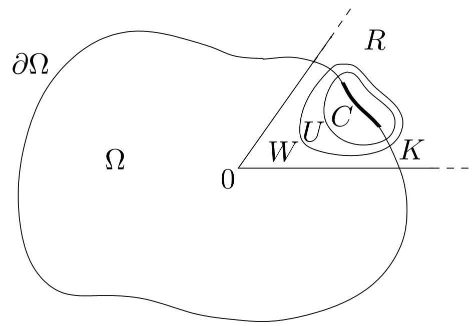

FIGURE 1. Support of the cut-off function φ is contained in W.

Note that the function h is well-defined for all $z \in \mathbb{C}$, if we put $\varphi f = (\overline{\partial}\varphi)f = 0$ outside of $\Omega \cup R$, where f is not defined.

Moreover $h \in C^n(\mathbb{C})$. Indeed, the integral u belongs to $C^n(\mathbb{C})$ since the convolution of the locally integrable function $z \mapsto \frac{1}{z}$ with the compactly supported C^n-function $(\overline{\partial}\varphi)f$ is $C^n(\mathbb{C})$, and trivially $\varphi f \in C^n(\mathbb{C})$.

Using Green's Theorem, one can see that the formula

$$u(\zeta) = \frac{1}{2\pi i} \iint \frac{\psi(z)}{\zeta - z} dz \wedge d\overline{z} = \frac{1}{\pi} \iint \frac{\psi(z)}{z - \zeta} dx dy$$

gives, for a continuous compactly supported ψ, a solution u of a $\overline{\partial}$-equation $\overline{\partial}u = \psi$; see for instance §1 in Chapter VIII of Garnett [3]. Hence, u satisfies the $\overline{\partial}$-equation

$$\overline{\partial}u = (\overline{\partial}\varphi)f. \tag{4.2}$$

We claim that h is holomorphic in Ω. Indeed, since f is holomorphic in Ω, the $\overline{\partial}$-equation (4.2) implies

$$\overline{\partial}h = \overline{\partial}(u - \varphi f) = (\overline{\partial}\varphi)f - (\overline{\partial}\varphi)f = 0.$$

Furthermore, we show that $f - h$ is holomorphic in U. Using again (4.2) and recalling that $\varphi \equiv 1$ in U, we get $\overline{\partial}u \equiv 0$, $\overline{\partial}\varphi \equiv 0$ on U, so $\overline{\partial}h = \overline{\partial}(u - \varphi f) = \varphi\overline{\partial}f = \overline{\partial}f$ in U. But that exactly means $f - h$ is analytic in U.

We observe that if we take the function F to be $f - h$, then it is holomorphic in $\Omega \cup U$, but it does not necessarily satisfy condition (S1). We rectify this situation by adding a shifted version of h (which is close to h).

For $0 < r < 1$ define $h_r(z) := h(rz)$. Since $h \in C^n(\mathbb{C})$,

$$h^{(k)}(rz) \to f^{(k)}(z) \qquad \text{as } r \to 1, \qquad k = 0, 1, 2, \dots, n$$

uniformly on compact subsets of $\mathbb{C}$. Therefore, we can find $r < 1$ sufficiently close to 1 so that

$$\|(h_r - h)|_\Omega\| \le \varepsilon/2 < \varepsilon. \tag{4.3}$$

Define $F = f - h + h_{r_0}$ on $\Omega \cup R$. The condition (S1) is satisfied since $\|F - f\| = \|h_r - h\| < \varepsilon$ on Ω. Moreover, F is holomorphic in $(\Omega \cup U) \cap \frac{1}{r}\Omega = \Omega \cup (U \cap \frac{1}{r}\Omega) = \Omega \cup O_1$ because f, h, h_{r_0} are all holomorphic in Ω, $f - h$ is holomorphic in U, and h_r is holomorphic in $\frac{1}{r}\Omega$.

Clearly, if $O \Subset O_1$ is an arbitrary open neighborhood of C, then for $\widetilde{\Omega} = \Omega \cup O$ the condition (S2) holds (because $f, h, h_r \in C^n(O_1)$). The notation $O \Subset O_1$ here means that $\operatorname{clos} O \subset \operatorname{int} O$.

Since F is holomorphic in $O \Supset C$, for every point $\zeta \in C$ there exists a neighborhood $V_\zeta \subset O$ of ζ such that

$$\sum_{k=0}^{n} \frac{1}{k!} \| F(\zeta) - F(z) \|_{\ell^2} < \varepsilon/3 \qquad \forall z \in V_\zeta.$$

Taking into account (4.3) we conclude from here that if we replace O by $\cup_{\zeta\in C} V_\zeta$, then the condition (S3) will be satisfied.

And it is a trivial exercise to show that we can make O smaller so that the condition (S4) is satisfied. □

Using the result above, we now prove the following result concerning uniform holomorphic approximation of functions in $\partial^{-n} A_S(\ell^2)$. In Lemma 4.2, we produced an approximate extension of a function across a *compact* arc, but in the following theorem we construct an approximate extension across an *open* arc.

In order to do this, we decompose the open arc into disjoint open intervals, and furthermore, we will write each open interval as a union of closed intervals, and these closed intervals will serve as the compact arcs of Lemma 4.2: this lemma will then be used recursively in order to construct the desired extension.

Theorem 4.3. *Let S be an open subset of $\mathbb{T}$, $n \geq 0$, and $f \in \partial^{-n} A_S(\ell^2)$. Then given any $\varepsilon > 0$ and $N \geq n$, there exists a domain $\Omega = \mathbb{D} \cup O$, where O is an open neighborhood of S in $\mathbb{C}$ and a function $F \in \partial^{-n} H^\infty(\Omega; \ell^2)$ such that*

(1) $\|F|_{\mathbb{D}} - f\|_{\partial^{-n} H^\infty} < \varepsilon$.
(2) $\left| \|F\|_{\partial^{-n} H^\infty(\Omega)} - \|f\|_{\partial^{-n} H^\infty(\mathbb{D})} \right| < \varepsilon$.
(3) *The boundary $\partial\Omega$ has a C^N-smooth polar parametrization $r = \rho(\theta)$, and moreover $\|\rho - 1\|_{C^N} < \varepsilon$.*

Proof. Any open set on $\mathbb{T}$ can be represented as a countable union of disjoint open intervals (arcs). Each open interval can be represented as a countable union of *closed* intervals, so we can represent the open set S as $S = \cup_{n=1}^\infty Q_n$, where Q_1, Q_2, Q_3, … are closed intervals.

Applying inductively Lemma 4.2 we construct an increasing sequence of domains Ω_k (in $\mathbb{C}$) and functions $\varphi_k \in \partial^{-n} H^\infty(\Omega_k; \ell^2)$ with the following properties:

(1) $\Omega_0 = \mathbb{D}$, $\varphi_0 = f$.
(2) $Q_j \subset \Omega_k$ for $j = 1, 2, \ldots, k$.
(3) The boundary of Ω_k has a C^N-smooth polar representation $r = \rho_k(\theta)$, and moreover $\|\rho_k - \rho_{k-1}\|_{C^N} < \varepsilon 2^{-k}$;
(4) $\varphi_k \in \partial^{-n} H^\infty(\Omega_k, \ell^2)$ and its derivatives $\varphi^{(j)}$, $j = 0, 1, \ldots, n$ extend continuously to the radial projection S_k of the set S onto $\partial\Omega_k$, $S_k := \{\rho_k(\theta) e^{i\theta} : \theta \in S\}$.
(5) $\|\varphi_k|_{\Omega_{k-1}} - \varphi_{k-1}\|_{\partial^{-n} H^\infty(\Omega_{k-1})} < \varepsilon 2^{-k}$.
(6) $\left| \|\varphi_k\|_{\partial^{-n} H^\infty(\Omega_k)} - \|\varphi_{k-1}\|_{\partial^{-n} H^\infty(\Omega_{k-1})} \right| < \varepsilon 2^{-k}$

As we mentioned above, we start with $\Omega_0 = \mathbb{D}$, $\varphi_0 = f$. Suppose Ω_{k-1}, φ_{k-1} are constructed. To get Ω_k, φ_k we apply Lemma 4.2 to the pair Ω_{k-1}, φ_{k-1} with $2^{-k}\varepsilon$ for ε. For the arc C we take the radial projection C_k of Q_k onto $\partial\Omega_{k-1}$, $C_k = \rho_{k-1}(\theta) e^{i\theta}$, and for K the radial projection S_{k-1} of S, $S_{k-1} := \{\rho_{k-1}(\theta) e^{i\theta} : \theta \in S\}$.

We need the above assumption (4) to be able to successfully apply Lemma 4.2. Condition (4) implies that the sequence φ_j converges uniformly on each Ω_k, so $F = \lim_j \varphi_j$ is an analytic function on $\Omega := \cup_k \Omega_k$.

Conditions (5) and (6) imply the conclusions (1) and (2) of the theorem. Condition (3) on φ_k implies the smoothness of $\partial\Omega$ (conclusion (3) of the theorem). □

The above Theorem 4.3, for the case $n = 0$ and complex valued functions, can be found in Stray [10] and Gamelin and Garnett [2]. We will use Theorem 4.3 in Subsection 4.3, in order to prove the estimates in the corona theorem for $\partial^{-n} A_S$.

4.2. **Estimates in the algebra $\partial^{-n} H^\infty(\Omega)$.** We will prove the corona theorem with bounds for $\partial^{-n} H^\infty(\Omega)$ by using the corresponding result for $\partial^{-n} H^\infty$ obtained earlier, via a conformal map taking $\mathbb{D}$ to Ω. We will need the following result by Specht (see Theorem V and the remark following it, on pages 185–186 of [9]), which gives bounds on the derivatives of a conformal map from $\mathbb{D}$ to Ω, when the boundary of Ω is smooth and "close" to $\mathbb{T}$.

Proposition 4.4. *Let C be a closed Jordan curve which satisfies the following assumptions:*

(A1) *Every ray from the origin intersects the curve in exactly one point, and there exists an $\varepsilon' \in (0, 1)$ such that C lies in the ring $\{w \in \mathbb{C} \mid 1 \le |w| < 1 + \varepsilon'\}$.*

(A2) *Let the polar parameterization of C be given by $\theta(1 + \rho(\theta))e^{i\theta}$, $\theta \in [0, 2\pi]$, where $\rho(\theta)$ is nonnegative, and $\rho \in C^n$. Define $\kappa(\theta) = \frac{\rho'(\theta)}{1+\rho(\theta)}$, $\theta \in [0, 2\pi]$. Let $|\kappa'(\theta)| < \varepsilon'/\pi$ and $|\omega^{(k)}(\theta)| < \varepsilon'/\pi$, $k \in \{2, \dots, n-1\}$, where $\omega(\theta) = -\arctan(\kappa(\theta))$ (the principal value of the arctangent is chosen here).*

(A3) *For all $\theta_0 \in [0, 2\pi]$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\omega^{(n-1)}(\theta) - \omega^{(n-1)}(\theta_0)}{\sin(\frac{\theta-\theta_0}{2})}\right| d\theta \le \varepsilon'.$$

Let φ be any conformal map φ mapping $\mathbb{D}$ onto the interior Ω of C in such a manner that $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then $\varphi^{(n)}(z)$ exists for $z \in \operatorname{clos}(\mathbb{D})$, and there exist absolute constants $J_1, \dots, J_n$ (that is, numbers which depend only on n, but not on ε' or the curve C), such that $|\varphi'(z) - 1| \le J_1\varepsilon'$ and $|\varphi^{(k)}(z)| \le J_k\varepsilon'$, $k \in \{2, \dots, n\}$.

Remark 4.5. The assumptions (A1)–(A3) of above proposition are satisfied if $\|\rho\|_{C^{n+1}} < \varepsilon$ for appropriately small ε, with $\varepsilon' = \varepsilon'(\varepsilon)$, $\varepsilon'(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The conclusion of the proposition implies the conformal map φ belongs to $\partial^{-n}A$ is close to the map $z \mapsto z$ in the norm of $\partial^{-n}H^\infty$, $\|\varphi - z\|_{\partial^{-n}H^\infty} < \gamma(\varepsilon')$, $\gamma(\varepsilon') \to 0$ as $\varepsilon' \to 0$.

We now prove the following:

Theorem 4.6. *Let n be a nonnegative integer. Let Ω be the simply connected open set with boundary a closed Jordan curve satisfying the assumptions (A1), (A2), (A3) from Proposition 4.4, where ε' is such that $J_1\varepsilon' < \frac{1}{2}$. Let $\mathcal{A} = \partial^{-n}H^\infty(\Omega)$.*

Then for all $f = (f_1, f_2, \dots, f_k, \dots) \in \mathcal{A}(\ell^2)$ satisfying

$$0 < \delta \le \left| f(z) \right|_{\ell^2} \quad \text{for all } z \in \Omega, \quad \text{and} \quad \|f\|_{\mathcal{A}(\ell^2)} \le 1,$$

there exists a $g = (g_1, g_2, \dots, g_k, \dots) \in \mathcal{A}(\ell^2)$ such that

$$\sum_k g_k(z)f_k(z) = 1 \text{ for all } z \in \Omega, \quad \text{and} \quad \|g\|_{\mathcal{A}(\ell^2)} \le (1 + \alpha(\varepsilon'))C(\partial^{-n}H^\infty, \delta),$$

where $\alpha(\varepsilon') \to 0$ as $\varepsilon' \to 0$.

Proof. Let $\varphi : \mathbb{D} \to \Omega$ be a holomorphic map such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. By Proposition 4.4, according to Remark 4.5, the conformal map φ is close to the identity map z.

Differentiating $f \circ \varphi$ we get that the $\partial^{-n} H^\infty$ norms of f and $f \circ \varphi$ are close:

$$\big| \|f\|_{\partial^{-n}H^\infty(\Omega;\ell^2)} - \|f \circ \varphi\|_{\partial^{-n}H^\infty(\mathbb{D};\ell^2)} \big| \le \alpha_1 \|f\|_{\partial^{-n}H^\infty(\Omega;\ell^2)} \le \alpha_1, \tag{4.4}$$

where $\alpha_1 = \alpha_1(\varepsilon') \to 0$ as $\varepsilon' \to 0$. The estimate (4.4) implies that $\|f \circ \varphi\|_{\partial^{-n}H^\infty(\mathbb{D};\ell^2)} \le 1+\alpha_1$, so the "normalized" vector-function $(1+\alpha_1)^{-1} f \circ \varphi$ has the $\partial^{-n}H^\infty$-norm at most 1, and satisfies

$$\frac{1}{1+\alpha_1} \big| f \circ \varphi(z) \big|_{\ell^2} \ge \frac{\delta}{1+\alpha_1} =: \widetilde{\delta}, \qquad \forall z \in \mathbb{D}.$$

Applying to this function the definition of $C(\partial^{-n}H^\infty, \delta)$, we get by solving the Bezout equation for $(1+\alpha_1)^{-1} f \circ \varphi$ and then scaling everything back, that there exists $\widetilde{g} \in \partial^{-n}H^\infty(\mathbb{D};\ell^2)$ such that

$$(\widetilde{g} \cdot (f \circ \varphi))(z) := \sum_k \widetilde{g}_k(z)(f_k \circ \varphi)(z) = 1 \qquad \forall z \in \mathbb{D},$$

and

$$\|\widetilde{g}\|_{\partial^{-n}H^\infty(\mathbb{D};\ell^2)} < (1+\alpha_1)^{-1} C(\partial^{-n}H^\infty, \widetilde{\delta}) + \varepsilon' \le C(\partial^{-n}H^\infty, \widetilde{\delta}) + \varepsilon'.$$

Recalling the continuity of $\delta \mapsto C(\mathcal{A}, \delta)$, see Lemma 3.1, and noticing that $\widetilde{\delta} = \widetilde{\delta}(\varepsilon') \to \delta$ as $\varepsilon' \to 0$, we can get from the last estimate that

$$\|\widetilde{g}\|_{\partial^{-n}H^\infty(\mathbb{D};\ell^2)} < (1+\alpha_2) C(\partial^{-n}H^\infty, \delta).$$

where $\alpha_2 = \alpha_2(\varepsilon') \to 0$ as $\varepsilon' \to 0$.

Finally defining $g \in \partial^{-n}H^\infty(\Omega, \ell^2)$ by $g := \widetilde{g} \circ \varphi^{-1}$ we get the solution of the Bezout equation $g \cdot f \equiv 1$. Using (4.4) again with g replacing f, we can see that the norms of g and $\widetilde{g} = g \circ \varphi$ cannot differ too much, so we get the desired estimate on the norm of g. □

4.3. **Estimates for $\partial^{-n} A_S$.** Using Theorem 4.3 and Theorem 4.6 from the previous two subsections, we are now ready to prove the estimates in the corona theorem for $\partial^{-n} A_S$.

Theorem 4.7. *For an open subset $S \subset T$ and $n \ge 0$ we have $C(\partial^{-n}H^\infty, \delta) = C(\partial^{-n}A_S, \delta)$.*

Proof. Let $\mathcal{A} = \partial^{-n}A_S$,and let $f = (f_1, f_2, \dots, f_k, \dots) \in \mathcal{A}(\ell^2)$ satisfy

$$0 < \delta \le \big| f(z) \big|_{\ell^2} \quad \text{for all } z \in \mathbb{D}, \qquad \text{and} \qquad \|f\|_{\mathcal{A}(\ell^2)} \le 1.$$

Let $\varepsilon > 0$, be a small number to be specified later. Applying Theorem 4.3 (with this ε and $N = n+1$) to the function f we get a domain $\Omega \supset \mathbb{D} \cup S$ such that its boundary admits a C^{n+1} polar parameterization $z = (1+\rho(\theta))e^{i\theta}$, and $\|\rho\|_{C^{n+1}} < \varepsilon$. We also get a function $F \in \partial^{-n}H^\infty(\Omega;\ell^2)$ such that the estimates (1) and (2) from the conclusion of Theorem 4.3 are satisfied. Estimate (2) implies that

$$\|F\|_{\partial^{-n}H^\infty(\Omega)} \le 1+\varepsilon \tag{4.5}$$

and that

$$\big| F(z) \big|_{\ell^2} \ge \delta - \varepsilon \qquad \forall z \in \Omega \tag{4.6}$$

Let us assume for a moment that ε and F are fixed. Note that if we make Ω smaller, the above estimates (4.5), (4.6) will still hold. Also, if we make Ω smaller by replacing ρ by $\gamma\rho$, $0 < \gamma < 1$, the inclusion $\mathbb{D} \cup S \subset \Omega$ will still hold for this smaller Ω.

In light of Remark 4.5, if we pick sufficiently small γ the boundary of the "shrunk" Ω will satisfy the assumption assumptions (A1), (A2), (A3) of Proposition 4.4, and, moreover ε' can be made as small as we want.

Applying Theorem 4.6 to the rescaled function $(1+\varepsilon)^{-1}F$ and then scaling everything back we get that there exists a $\widetilde{g} \in \partial^{-n}H^\infty(\Omega;\ell^2)$ such that

$$\widetilde{g}\cdot F := \sum_k \widetilde{g}_k(z)F_k(z) = 1 \qquad \forall z \in \Omega,$$

and

$$\|\widetilde{g}\|_{\partial^{-n}H^\infty(\Omega)} \le (1+\varepsilon)^{-1}(1+\alpha(\varepsilon'))C(\partial^{-n}H^\infty,\widetilde{\delta}),$$

where $\widetilde{\delta} := (\delta-\varepsilon)/(1+\varepsilon)$. Since we consider only small ε, we can assume that $\widetilde{\delta} \ge \delta/2$. If we make the other parameter ε' sufficiently small, we get from here the estimate

$$\|\widetilde{g}\|_{\partial^{-n}H^\infty(\Omega)} \le C(\partial^{-n}H^\infty,\widetilde{\delta}).$$

Define the scalar function $h \in \partial^n A_S(\mathbb{D})$ by $h := \widetilde{g}\cdot f$ (both f and $\widetilde{g}$ are clearly in $\partial^{-n}A_S$). Note that

$$\|h-1\|_{\partial^{-n}A_S} = \|\widetilde{g}\cdot(f-F)\|_{\partial^{-n}A_S} \le \|\widetilde{g}\|_{\partial^{-n}A_S}\varepsilon \le C\varepsilon,$$

where $C = C(\partial^{-n}H^\infty,\delta/2)$. Therefore, for sufficiently small ε, the function h is invertible in $\partial^{-n}A_S$ and

$$\|h^{-1}\|_{\partial^{-n}A_S} \le \frac{1}{1-C\varepsilon}.$$

The function $g := h^{-1}\widetilde{g}$ clearly belongs to $\partial^{-n}A_S$, solves the Bezout equation $g\cdot f \equiv 1$, and its norm can be estimated as

$$\|g\|_{\partial^{-n}A_S} \le \|h^{-1}\|_{\partial^{-n}A_S}\|\widetilde{g}\|_{\partial^{-n}A_S} \le \frac{C(\partial^{-n}H^\infty,\widetilde{\delta})}{1-C\varepsilon},$$

where recall that $\widetilde{\delta} := (\delta-\varepsilon)/(1+\varepsilon)$.

Using the continuity of the function $\delta \mapsto C(\partial^{-n}H^\infty,\delta)$, see Lemma 3.1 above, we get that by picking sufficiently small ε in the beginning, we can make this bound as close to $C(\partial^{-n}H^\infty,\delta)$ as we want. □

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