Unions of Perfect Matchings in Cubic Graphs and Implications of the Berge-Fulkerson Conjecture

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Abstract

The Berge-Fulkerson Conjecture states that every cubic bridgeless graph has six perfect matchings such that every edge of the graph is in exactly two of the perfect matchings. If the Berge-Fulkerson Conjecture is true, then what can we say about the proportion of edges of a cubic bridgeless graph that can be covered by k of its perfect matchings? This is the question we address in this paper. We then give a possible method for proving, independently of the Berge-Fulkerson Conjecture, the bounds obtained.

1 Introduction

In this paper, we shall be concerned only with finite graphs without loops, although we permit multiple edges. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$.

A *perfect matching* of G is a set of edges, $M \subseteq E(G)$, such that every vertex in G is incident with exactly one edge in M.

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A *cubic graph* is one in which each vertex is incident with exactly three edges. An edge in a graph G is called a *bridge* if its removal increases the number of components of G. We shall predominantly be concerned with *cubic bridgeless graphs*.

A well known conjecture, which is attributed to Berge in [4], but which first appears published in [2], says that every cubic bridgeless graph has a collection of six perfect matchings such that each edge in the graph is covered by exactly two of the perfect matchings.

Conjecture (Berge-Fulkerson) Every cubic bridgeless graph G has a family of six perfect matchings, $M_1, ..., M_6$, such that each edge of G is contained in precisely two of the perfect matchings.

We will consider the maximum proportion of edges in G that can be covered by k of its perfect matchings, for $k \leq 6$. In order to make this more precise, we set up some notation.

Given a cubic bridgeless graph G, let $\mathcal{M}$ be the set of distinct perfect matchings in G. Fix a positive integer k. Define

$$m_k(G) = \max_{M_1,\ldots,M_k \in \mathcal{M}} \frac{|\bigcup_{i=1}^{k} M_i|}{|E(G)|}.$$

Define

$$m_k = \inf_G m_k(G),$$

where the infimum is taken over all cubic bridgeless graphs. Thus, for every cubic bridgeless graph G and every positive integer k, G has a set of k perfect matchings covering at least a proportion m_k of its edges. We study these numbers m_k.

Let P denote the Petersen graph. The following facts about P can be easily verified. We have that P has exactly six distinct perfect matchings, which we denote $M_1^P, \ldots, M_6^P$. Each edge of P is contained in exactly two of its perfect matchings and so the Berge-Fulkerson Conjecture holds for P. Furthermore, for each $1 \leq i < j \leq 6$, $M_i^P \cap M_j^P$ gives a single edge of P, and the $\binom{6}{2} = 15$ choices of $1 \leq i < j \leq 6$ give the 15 edges of P. We also have that

$$m_2(P) = \frac{3}{5},\ \ m_3(P) = \frac{4}{5},\ \ m_4(P) = \frac{14}{15},\ \ m_5(P) = 1.$$

Conjecture 1 We have $m_k = m_k(P)$ for $1 \leq k \leq 5$.

The case $k = 1$ of Conjecture 1 is trivially true. The case $k = 2$ of Conjecture 1 was proved by Kaiser, Král, and Norine in [3], and they also proved that $m_3 \geq \frac{27}{35} = m_3(P) - \frac{1}{35}$. The case $k = 5$ of Conjecture 1, which follows trivially from the Berge-Fulkerson Conjecture, is another conjecture attributed to Berge, and it remains an open problem. Indeed, it is not known whether $m_k = m_k(P) = 1$ for any $k \geq 5$. The best known result in this direction is the following: given a cubic bridgeless graph G, $m_k(G) = 1$ for $k > \log_{3/2}(|E(G)|)$. This result follows easily from the methods introduced in Section 3.

We review what is to follow. In Section 2, we prove that the Berge-Fulkerson Conjecture implies Conjecture 1. In Section 3, we describe the perfect matching polytope, which is central to the ideas we present in Section 4. In Section 4, we state a conjecture that is stronger than Conjecture 1, and we show how this stronger conjecture may aid in proving Conjecture 1. Finally, we show that this stronger conjecture also follows from the Berge-Fulkerson Conjecture.

2 The First Implication

Theorem 2.1 *The Berge-Fulkerson Conjecture implies Conjecture 1.*

Proof It is sufficient to show for every cubic bridgeless graph G, and each $1 \leq k \leq 5$, that $m_k(G) \geq m_k(P)$. Fix $1 \leq k \leq 5$.

Given G, if the Berge-Fulkerson Conjecture holds, then we can find six perfect matchings of G, $M_1, \ldots, M_6$, such that each edge of G is contained in precisely two of these perfect matchings.

Let S_k be a set of k elements chosen uniformly at random from $\{1, \ldots, 6\}$. Fix $e \in E(G)$ and let M_a and M_b be the two perfect matchings from $M_1, \ldots, M_6$ that contain e. Then

$$\begin{aligned}
\Pr(e \in \cup_{i \in S_k} M_i) &= \Pr(a \in S_k \text{ or } b \in S_k) \\
&= 1 - \Pr(a \notin S_k \text{ and } b \notin S_k) \\
&= 1 - \frac{\binom{4}{k}}{\binom{6}{k}},
\end{aligned}$$

where $\binom{a}{b}$ is defined to be zero if $a < b$. It is easy to verify that

$$1 - \frac{\binom{4}{k}}{\binom{6}{k}} = m_k(P).$$

Now

$$\begin{aligned}\mathbb{E}(|\cup_{i\in S_k} M_i|) &= \sum_{e\in E(G)} \Pr(e \in \cup_{i\in S_k} M_i) \\ &= |E(G)| m_k(P).\end{aligned}$$

Therefore, there exists some k element subset of $\{1,\ldots,6\}$, S_k^* say, satisfying

$$|\cup_{i\in S_k^*} M_i| \geq |E(G)| m_k(P),$$

hence $m_k(G) \geq m_k(P)$. $\square$

We can slightly generalise the above as follows. For $k \leq 5$, we say that a cubic bridgeless graph G is k-covered if it has k perfect matchings, $M_1, \ldots, M_k$, such that

1. $|\cup_{i=1}^k M_i| \geq m_k(P)|E(G)|$ and
2. no edge of G is in more than two of the M_i's.

Proposition 2.2 *If G is k-covered, then G is l-covered for all $l < k$.*

The proof uses the same idea as the proof of Theorem 2.1, and so is omitted.

3 The Perfect Matching Polytope

In Section 4, we suggest a method for proving Conjecture 1. As remarked earlier, it has been proven that $m_2 = m_2(P)$. Central to its proof, and to our proposed method for proving Conjecture 1, is Edmonds' Perfect Matching Polytope Theorem, [1]. We now set up some notation and define the perfect matching polytope, following [3].

Let H be any graph. For a set $X \subseteq V(H)$, we set ∂X to be the set of edges with precisely one end in X. A *cut* in H is a set of edges, $C \subseteq E(H)$, such that $C = \partial X$ for some $X \subseteq V(H)$. A *k-cut* is a cut consisting of k edges.

Let $\mathbf{w}$ be a vector in $\mathbb{R}^{E(H)}$. The entry of $\mathbf{w}$ corresponding to $e \in E(H)$ is denoted by $\mathbf{w}(e)$. For $A \subseteq E(H)$, we define the *weight* $\mathbf{w}(A)$ of A as $\sum_{e\in A} \mathbf{w}(e)$. The vector $\mathbf{w}$ is said to be a *fractional perfect matching* if it satisfies the following:

1. $0 \leq \mathbf{w}(e)$ for each $e \in E(H)$,
2. $\mathbf{w}(\partial\{v\}) = 1$ for each vertex $v \in V(H)$, and

3. $\mathbf{w}(\partial X) \geq 1$ for each $X \subseteq V(H)$ with X of odd cardinality.

The perfect matching polytope of H is the set of all fractional perfect matchings of H, and is denoted by $P(H)$.

Theorem 3.1 (Edmonds [1]) *For any graph H, the set $P(H)$ coincides with the convex hull of the characteristic vectors of all perfect matchings of H.*

Next we give a lemma, which is due to Kaiser, Král, and Norine [3]. It is central both to their proof of $m_2 = m_2(P)$ and to our proposed method of proving Conjecture 1, which we present in the next section.

If $A \subseteq E(H)$, then let χ_A denote the characteristic vector of A.

Lemma 3.2 *If $\mathbf{w}$ is a fractional perfect matching in a graph H, and $\mathbf{c} \in \mathbb{R}^{E(H)}$, then H has a perfect matching M such that*

$$\mathbf{c} \cdot \chi_M \geq \mathbf{c} \cdot \mathbf{w}.$$

Proof Let $M_1, \ldots, M_r$ be the perfect matchings in H. Then $\mathbf{w}$ is a weighted average of $\chi_{M_1}, \ldots, \chi_{M_r}$, and so $\mathbf{c}\cdot\mathbf{w}$ is a weighted average of $\mathbf{c}\cdot\chi_{M_1}, \ldots, \mathbf{c}\cdot\chi_{M_r}$. Hence not all of $\mathbf{c} \cdot \chi_{M_1}, \ldots, \mathbf{c} \cdot \chi_{M_r}$ can be smaller than $\mathbf{c} \cdot \mathbf{w}$. □

4 A Second Conjecture

In this section, we state a conjecture that is stronger than Conjecture 1, but that may help in the proof of Conjecture 1, as we shall see.

We remark that if G is a cubic graph and $X \subseteq V(H)$, then $|X|$ is odd if and only if $|\partial X|$ is odd, and so such cuts will be referred to as *odd cuts.*

Conjecture 2 Let G be a cubic bridgeless graph. For each $k \in \{2, \ldots, 5\}$, G has k perfect matchings, $M_1, ..., M_k$, satisfying:

1. no edge of G is contained in more than two of the M_i's,
2. $|\cup_{i=1}^{k} M_i| \geq m_k(P)|E(G)|$, and
3. for every odd cut C of G, if $|C| = r$ then $\sum_{i=1}^{k} |M_i \cap C| \leq 2(r-3) + k$.

We shall see later that the Berge-Fulkerson Conjecture implies Conjecture 2, but first we show why Conjecture 2 is useful for proving Conjecture 1.

Theorem 4.1 *If Conjecture 2 holds for a given* $k \in \{2, \dots, 4\}$*, then Conjecture 1 holds for* $k+1$*. If Conjecture 2 holds for* $k = 5$*, then the Berge-Fulkerson Conjecture holds.*

Proof Let G be a cubic bridgeless graph. Suppose that G has two perfect matchings, M_1 and M_2, satisfying Conjecture 2 for $k = 2$. Then set

$$\mathbf{w}(e) = \begin{cases} 0 & \text{if } e \text{ is in both perfect matchings;} \\ \frac{1}{4} & \text{if } e \text{ is in exactly one perfect matching;} \\ \frac{1}{2} & \text{if } e \text{ is in neither of the perfect matchings.} \end{cases}$$

We now check that this is a fractional perfect matching by verifying each of the three conditions given in the definition of a fractional perfect matching. The first condition is trivially true.

For the second condition, pick a vertex v and consider the three edges, e_1, e_2, and e_3, incident with v. After relabeling of indices, we must have that either e_1 is in M_1, e_2 is in M_2, and e_3 is in neither, or e_1 is in M_1 and M_2, e_2 is in neither, and e_3 is in neither. In either case, $\mathbf{w}(\partial v) = \sum_{i=1}^{3} \mathbf{w}(e_i) = 1$, so the second condition is verified.

For the third condition, pick an odd cut C and let $|C| = r$. We know that $|M_1 \cap C| + |M_2 \cap C| \leq 2r - 4$. Let a_0, a_1, and a_2 be respectively the numbers of edges of C covered by none, exactly one, and both of M_1 and M_2. Then (i) $a_0 + a_1 + a_2 = r$, and (ii) $a_1 + 2a_2 \leq 2r - 4$. Taking $\frac{1}{2}$(i) $-$ $\frac{1}{4}$(ii) yields

$$\mathbf{w}(C) = \frac{1}{2} a_0 + \frac{1}{4} a_1 \geq 1,$$

so the third condition is verified and $\mathbf{w}$ is a fractional perfect matching.

By Lemma 3.2, there exists a perfect matching, M_3, such that

$$\begin{aligned} \chi_{M_3} \cdot \chi_{(M_1 \cup M_2)^c} &\geq \mathbf{w} \cdot \chi_{(M_1 \cup M_2)^c} \\ &= \frac{1}{2} |(M_1 \cup M_2)^c|. \end{aligned}$$

Now we have that

$$\begin{aligned}|M_1 \cup M_2 \cup M_3| &= |M_1 \cup M_2| + |M_3 \cap (M_1 \cup M_2)^c| \\ &= |M_1 \cup M_2| + \chi_{M_3} \cdot \chi_{(M_1 \cup M_2)^c} \\ &\geq |M_1 \cup M_2| + \frac{1}{2}|(M_1 \cup M_2)^c| \\ &= \frac{1}{2}|M_1 \cup M_2| + \frac{1}{2}(|M_1 \cup M_2| + |(M_1 \cup M_2)^c|) \\ &\geq \frac{1}{2}(\frac{3}{5}|E(G)|) + \frac{1}{2}|E(G)| \\ &= \frac{4}{5}|E(G)| = m_3(P)|E(G)|,\end{aligned}$$

thus G satisfies the case $k = 3$ of Conjecture 1. The proofs of the remaining cases below follow a similar pattern.

Suppose that G has three perfect matchings, M_1, M_2, and M_3, satisfying Conjecture 2 for $k = 3$. Set

$$\mathbf{w}(e) = \begin{cases} 0 & \text{if } e \text{ is in exactly two of these perfect matchings;} \\ \frac{1}{3} & \text{if } e \text{ is in exactly one of these perfect matching;} \\ \frac{2}{3} & \text{if } e \text{ is in none of these perfect matchings.} \end{cases}$$

It is easy to check that this is a fractional perfect matching. By Lemma 3.2, there exists a fractional perfect matching, M_4, such that

$$\begin{aligned}\chi_{M_4} \cdot \chi_{(\cup_{i=1}^3 M_i)^c} &\geq \mathbf{w} \cdot \chi_{(\cup_{i=1}^3 M_i)^c} \\ &= \frac{2}{3}|(\cup_{i=1}^3 M_i)^c|.\end{aligned}$$

Now we have that

$$\begin{aligned}|(\cup_{i=1}^4 M_i)| &= |\cup_{i=1}^3 M_i| + |M_4 \cap (\cup_{i=1}^3 M_i)^c| \\ &= |\cup_{i=1}^3 M_i| + \chi_{M_4} \cdot \chi_{(\cup_{i=1}^3 M_i)^c} \\ &\geq |\cup_{i=1}^3 M_i| + \frac{2}{3}|(\cup_{i=1}^3 M_i)^c| \\ &= \frac{1}{3}|(\cup_{i=1}^3 M_i)^c| + \frac{2}{3}(|\cup_{i=1}^3 M_i| + |(\cup_{i=1}^3 M_i)^c|) \\ &\geq \frac{1}{3}(\frac{4}{5}|E(G)|) + \frac{2}{3}|E(G)| \\ &= \frac{14}{15}|E(G)| = m_4(P)|E(G)|,\end{aligned}$$

thus G satisfies the case $k = 4$ of Conjecture 1.

Suppose that G has four perfect matchings, $M_1, \ldots, M_4$, satisfying Conjecture 2 for $k = 4$. Set

$$\mathbf{w}(e) = \begin{cases} 0 & \text{if } e \text{ is in exactly two of these perfect matchings;} \\ \frac{1}{2} & \text{if } e \text{ is in exactly one of these perfect matching;} \\ 1 & \text{if } e \text{ is in none of these perfect matchings.} \end{cases}$$

It is easy to check that this is a fractional perfect matching. By Lemma 3.2, there exists a fractional perfect matching, M_5, such that M_5 covers all the edges of G not covered by $M_1, \ldots, M_4$, thus G satisfies the case $k = 5$ of Conjecture 1.

Finally the case $k = 5$ of Conjecture 2 implies the Berge-Fulkerson Conjecture, indeed, the edges of G that are covered only once by the five perfect matchings form the sixth. □

Finally we show that the Berge-Fulkerson Conjecture implies Conjecture 2, adding weight to the case for attempting to prove part or all of Conjecture 2 and hence Conjecture 1.

Theorem 4.2 *The Berge-Fulkerson Conjecture implies Conjecture 2.*

Proof Let G be a cubic bridgeless graph and suppose the Berge-Fulkerson Conjecture holds, so that G has perfect matchings, $M_1, \ldots, M_6$, with each edge of G in exactly two of the perfect matchings. For each $k \in \{2, \ldots, 5\}$, $\{M_1, \ldots, M_6\}$ has a subset of k perfect matchings satisfying condition 1 of Conjecture 2 (by the proof of Theorem 2.1), and clearly also satisfying condition 2. It is therefore sufficient to show that every set of k perfect matchings from $M_1, \ldots, M_6$ satisfies condition 3 of Conjecture 2.

Observe that if C is an odd cut of G and M is a perfect matching of G, then $|M \cap C| \geq 1$. Let $|C| = r$. Since $|M_i \cap C| \geq 1$ for all $i \in \{1, \ldots, 6\}$, and $\sum_{i=1}^{6} |M_i \cap C| = 2r$, then for $S \subseteq \{1, \ldots, 6\}$ with $|S| = k$, we have

$$\begin{aligned} \sum_{i \in S} |M_i \cap C| &= 2r - \sum_{i \in S^c} |M_i \cap C| \\ &\leq 2r - |S^c| \\ &= 2r - (6 - k) \\ &= 2(r - 3) + k, \end{aligned}$$

as required. □

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