Searching Symmetric Networks with Utilitarian–Postman Paths

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Abstract

For any network Q, one may consider the zero-sum search game $\Gamma(Q)$ in which the (minimizing) Searcher picks a unit speed path $S(t)$ in Q, the Hider picks a point H in Q, and the payoff is the meeting time $T = \min\{t : S(t) = H\}$. We show first that if Q is symmetric (edge and vertex transitive), then it is optimal for the Hider to pick H uniformly in Q, so that the Searcher must follow a Utilitarian Postman path (one which minimizes the time to reach a random point). We then show that if Q is symmetric of odd degree, with n vertices and m unit length edges, the value V of $\Gamma(Q)$ satisfies $V \geq \frac{m}{2} + \frac{n^2 - 2n}{8m}$, with equality if and only if Q has a path $v_1, v_2, \ldots, v_{n-1}$ of distinct vertices, such that the edge set $Q - \cup_{i=1}^{(n-2)/2} (v_{2i}, v_{2i+1})$ is connected.

1 Introduction

Search games on networks were introduced by Rufus Isaacs in the final chapter of his classic 1965 book, Differential Games. After devoting most of his book to perfect information games such as pursuit-evasion, he let darkness descend on the search space Q, so that the players were not aware of each other's actions. In the resulting 'search games', or 'hide-and-seek games', the Hider simply picks a hiding point H in Q, while the Searcher picks a unit speed path $S(t)$ that he hopes will minimize the time $T = \min\{t : S(t) = H\}$ taken to find him. Isaacs' earlier problems were modeled by ordinary differential equations, and for this reason it was natural to extend the requirement of *initial conditions* to this final chapter on search games. He did this by specifying a *searcher starting point* $S_0 = S(0)$. So for most of the intervening 40 years, search games on networks have been studied with this condition, as a zero-sum game we call $\Gamma(Q, S_0)$, with a given point S_0 in a finite network Q. A large literature has developed around this problem, including four books ([17], [15], [6], [23]) and many articles (e.g. [9], [16], [19], [22], [26]). A wider class of problems is studied in [24]. Only very recently ([12], [13], [2], [3]) has the initial condition been removed, and the resulting game $\Gamma(Q)$ (with $S(0)$ chosen by the Searcher) been studied. This is the game we study in this paper, for symmetric networks Q.

It is very natural that the removal of the initial condition S_0 should lead to the class of networks studied here: the symmetric (edge and vertex transitive) networks. For once a point S_0 is singled out on such a network, the symmetry vanishes. That, we believe, is why symmetric networks have thus far not appeared at all in the search game literature.

So how should one search a symmetric network? It turns out that we must first answer the easier question of how one should hide in it - this turns out to be the uniform distribution, very different from the case of $\Gamma(Q)$ for a tree Q, where one optimally hides only on the leaves [13]. We call networks of this type, where the uniform distribution is the Hider's optimal mixed strategy, *easily hideable*. Only one easily hideable network has been previously identified [3], the 'circle with spikes' drawn here in Figure 1. The first main result of the paper (Theorem 11) is that *symmetric networks are easily hideable*. Once we know this, the question for the Hider is how to optimally search for a point hidden uniformly in a network.

We call this problem the *Utilitarian Postman Problem*. Unlike the Chinese Postman, who wants to finish delivering his mail to the last customer as soon as possible, the Utilitarian Postman wants to minimize the *mean* time to deliver the mail to *all* his customers. This is a new network search problem, introduced simultaneously in this paper and in [3]. We will call a path with this property a *Utilitarian Postman (UP) path.* It turns out that UP paths may indeed take longer to complete (i.e., not be Chinese Postman paths) and that UP paths may have to traverse an edge of Q arbitrarily many times – not just twice.

If a symmetric network Q has even degree (valency), it is Eulerian, and it is trivial to see that an optimal Searcher mixed strategy is simply to traverse some Eulerian circuit equiprobably in either direction. So for Eulerian networks Q (those with even degree) the value V is simply half the total length of Q. So suppose the degree of a symmetric network Q is odd, and that it has n vertices and m unit length edges (symmetry implies equal lengths). The second main result (Theorem 18) of this paper says that for such networks $V \geq \frac{m}{2} + \frac{n^2 - 2n}{8m}$, and that equality holds if and only if Q has a property we call Half Hypo Hamiltonian Connected (HHHC). A network Q is HHHC if it has a path P of distinct vertices $v_1, v_2, \ldots, v_{n-1}$ (all but one of them), such that the network Q', obtained by removing the even indexed edges (v_2, v_3), (v_4, v_5), $\ldots, (v_{n-2}, v_{n-1})$ of P, is connected. While this may seem a strange condition, we know of no odd degree symmetric network (since edges here have length 1, we are really dealing with graphs) that does not have this property. In particular, all complete graphs, complete bipartite graphs, hypercube graphs, graphs of large valency, are HHHC. And yes, the Petersen graph is HHHC. If Q is HHHC, the optimal Hider mixed strategy is a randomized version (over the automorphism group of Q) of a Utilitarian Postman search path EP consisting of an Eulerian path E in Q' followed by a hypo-Hamiltonian (all but one vertex) path P of Q.

This article is organized as follows. Section 2 gives a formal definition of the search game $\Gamma(Q)$. Section 3 shows that a Utilitarian Postman path can be assumed to be *combinatorial,* that is, it can be expressed as a sequence of edges which are traversed, without turning, at maximum speed. Section 4 uses this fact to show that symmetric networks are easily hideable. These two results, that the Searcher can be assumed to follow combinatorial paths in the network Q, and that the Hider is uniformly distributed, mean that we can then simply take Q to be a graph - so from that point we revert to graph, rather than network, terminology and practice. In Section 5 we look for Utilitarian Postman paths on graphs of odd degree (not necessarily symmetric), and derive the HHHC condition. In Section 6 we combine the work of Sections 4 and 5 to determine $V(Q)$ for symmetric graphs, and discuss some open problems regarding our HHHC condition.

2 The Game $\Gamma(Q)$

Let Q be a finite network, with each edge e assigned a length $\lambda(e)$. We consider λ more generally as a measure on Q, with $\lambda(Q) = \mu$ denoting the total length of Q. We call λ the uniform distribution on Q. We define a distance d on Q in terms of λ, with $d(x, y)$ denoting the length of the shortest path between x and y. A pure Search strategy S is a unit speed covering path of Q, that is, S belongs to the Searcher pure strategy set $\mathcal{S}$ given by

$$\mathcal{S} = \{S : [0, L] \text{ onto } Q, \text{ some } L, \text{ and } d(S(t), S(t')) \leq |t - t'|, 0 \leq t, t' \leq L\}.$$

For each $S \in \mathcal{S}$, its length $L = L_S$ is the smallest l with $S([0, l]) = Q$. A pure Hider strategy H is simply a point of Q. The payoff function T is given by the meeting time

$$T(S, H) = \min\{t : S(t) = H\}.$$

When one or both of the arguments of T are mixed strategies, we interpret T as the *expected* meeting time. We will use lower case letters, s and h, to indicate mixed strategies. The existence of an optimal mixed Searcher strategy, an ε-optimal Hider mixed strategy, and a Value

$$V(Q) = \min_s \sup_h T(s, h) = \sup_h \min_s T(s, h), \tag{1}$$

for mixed Searcher and Hider strategies s, h, follows from minimax theorems of Gal [15], and Alpern and Gal [6, Appendix A], or [5].

The following definition was introduced in [2] but not analyzed until [3].

Definition 1 *A network Q is called* ***easily hideable*** *if the uniform distribution λ is an optimal mixed strategy. A Searcher path $S \in \mathcal{S}$ is called a* ***Utilitarian Postman (UP) path*** *if it minimizes the expected time $T(S, \lambda)$ to find a uniformly distributed point. This minimum time is called the* ***UP time****, and denoted $\hat{\mu}$.*

It is easy to show [3] that any network Q with an Eulerian path is easily hideable, that the Eulerian path is a UP path, and in this case $\hat{\mu} = \mu$, the total length of Q. We have also shown (same paper) that the 'circle-with-spikes' network drawn below in Figure 1, with all edges of unit length, is easily hideable. Up to symmetry, there are three UP paths, ABDDC, ABCBD and ABCCD, with $\hat{\mu} = (1 + 3 + 5 + 9)/8 = 9/4$. To see this note that the mean times to reach points in the four edges for the first time are $1/2, 3/2, 5/2, 9/2$;

and 9/4 is the average. Up to now, these are the only networks which have been shown to be easily hideable. (This paper extends the known examples to include all symmetric networks.)

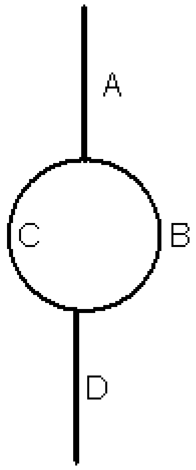


Figure 1: An easily hideable network

Note that both the UP paths for this network are CP paths, as defined in the following.

Definition 2 *A path $S \in \mathcal{S}$ of minimum length $L = L_S$ is called a* ***Chinese Postman (CP) path****. It's length is denoted $\tilde{\mu}$. If $V(Q) = \tilde{\mu}/2$, we say that Q is* ***simply searchable****. For such networks, traversing a CP path equiprobably in either direction is an optimal mixed strategy for the Searcher.*

Since the Hider always has the uniform strategy λ available, it follows that $V \geq \hat{\mu}$. Similarly, the Searcher can always randomize equiprobably between a CP path $S(t)$ and its reverse path $S^*(t) = S(L - t)$ to reach any point in Q in average time no more than $L/2 = \tilde{\mu}/2$, and hence $V \leq \tilde{\mu}/2$. Summarizing, we have

$$\hat{\mu}(Q) \leq V(Q) \leq \tilde{\mu}(Q)/2, \text{ for any network } Q, \tag{2}$$

where equality holds on the left iff Q is easily hideable and on the right iff Q is simply searchable.

We can think of a CP path as one which gets the postman (who delivers mail to all of Q) done as soon as possible, assuming he can begin at time 0 anywhere in Q. A Utilitarian Postman is one who is more concerned with the public, and instead wishes to deliver the mail as soon as possible, on average (to a random point on Q). These two aims may have a common solution (as in the network drawn in Figure 1), but this is not true for all networks. Consider the network analyzed in [2] and [3] (with identical explanatory text), drawn below in Figure 2, where all the edges have unit length.

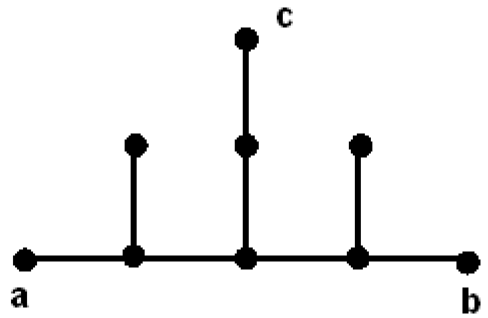


Figure 2. A network with distinct
UP and CP paths

It is easy to see that, up to isomorphism and reversal, there are two CP paths and the minimum covering time is 12. Both paths may be assumed to end at b and start at either a (call this P_a) or c call this P_c. In each time interval $J_i = [i-1, i]$, $i = 1, \ldots, 12$, either a new edge is searched (1) or an edge is retraced (0), as indicated in the following table

P_a	110111001101
P_c	111101001101

If the hider distribution is uniform and he is found in time interval J_i, the conditional expected capture time is $i - 1/2$. Hence the expected capture time is

$$T = \frac{1}{8} \sum_{x_i=1} (i - 1/2)$$

The only difference in these sequences is that the 0 at position 3 in P_a has moved to position five in P_c. So clearly against a uniform hiding strategy P_c has a smaller expected meeting time:

$$T(P_a, \lambda) - T(P_c, \lambda) = \frac{1}{8}((5 - 1/2) - (3 - 1/2)) = \frac{1}{4}.$$

So the only CP path which is a UP path is P_c. Now suppose the network is modified so that the edge directly below c has length $1 - \varepsilon$ instead of 1. For ε sufficiently small, $T(P_c, \lambda)$ will still be less than $T(P_a, \lambda)$, but the only CP path will be P_a. (For the modified network, P_a has length $12-2\varepsilon$, whereas P_c has length $12-\varepsilon$.) Thus none of the CP paths will be a UP path. This type of analysis will be used more formally later in Section 5 to determine UP paths in graphs of odd degree.

3 Combinatorial Paths

The reader will have observed that the UP and CP paths described above for the networks of Figures 1 and 2 have two special properties not prescribed simply by belonging to the set $\mathcal{S}$: they start at a vertex and never turn around inside an edge. More formally, they can be described as follows.

Definition 3 *A path in $S \in \mathcal{S}$ is called* ***combinatorial*** *if it starts at a vertex v_0 of Q and traverses each edge e_i in minimum time $\lambda(e_i)$, that is, at unit speed without turning.*

It is well known that to minimize (over S) the expected meeting time $T(S,h)$, for some distributions h on some networks Q, the optimal search strategy S will have to turn many, even infinitely many, times. For example, this is required if h is the triangular distribution (density t for $t \leq 1/2$, $1-t$ for $t \geq 1/2$) on the unit interval, and $S(0)$ is the center [11]. Optimal paths which turn within edges are also found in the network consisting of two vertices connected by three unit length edges, with the Searcher starting at one of the vertices [21].

However the following result shows that combinatorial paths are sufficient to search optimally for a uniformly distributed (strategy λ) Hider.

Theorem 4 *Every network Q has a UP path which is combinatorial.*

Proof. Let $S : [0,L] \to Q$ be a UP path on Q which starts in some edge a of length $\lambda(a) = \alpha$, where we normalize $\lambda(Q) = 1$. Label the ends of a as A and B (if a is a loop this is covered by previous paper, see Theorem 6 below), and parameterize the points in a as the interval $I = [0,\alpha]$, with 0 corresponding to A. We consider several cases. In all of them the search S begins with a partial search of a for time x; then searches from A some subnetwork Q_1 for time k; then searches all of a (A to B or B to A) in time α; and finally searches the remaining network $Q_2 = S[x+k+\alpha, L]$. Let $r = \lambda(Q_1)$ and $s = \lambda(Q_2)$ denote the probabilities the Hider is Q_1 and Q_2. The mean time required to find a point hidden uniformly in Q_1 (using the search S) is denoted c; for Q_2, it is denoted t.

We first show that given any UP path, there is a corresponding one that starts at a vertex.

Suppose $S(0) = x \in (0, \alpha), S(x) = A, S(x+k) = A, S(x+k+1) = B$. That is, S starts at an interior point x of the edge a; goes (without loss of generality) to the vertex A, then tours Q_1 from A; then covers a from A to B, then searches Q_2. For such an S, the expected capture time $f(x) = T(S, \lambda)$ is given by the sum of four terms, corresponding to the searches of a, Q_1, search of rest of a, Q_2. Each term is the product of (right) the probability that the Hider is found in that search and (left) the corresponding conditional expected capture time.

$$\begin{aligned} f(x) &= \frac{x}{2}x + (x+c)r + (2x+k+(\alpha-x)/2)(\alpha-x) + (x+k+\alpha+t)s \\ &= -x^2 + x(r-k+1+s) + \left(\alpha\left(\frac{1}{2}\alpha+k\right) + cr + s(\alpha+k+t)\right). \end{aligned}$$

Since this is strictly concave in x, there is no interior minimum. So there cannot be any UP path of this type.

Next suppose that S begins as above, but searches Q_1 from A to B, then edge a from B to A, then Q_2 from A. We now have

$$\begin{aligned} f(x) &= \frac{x}{2}x + (x+c)r + (x+k+(\alpha-x)/2)(\alpha-x) + (x+k+\alpha+t)s \\ &= x(r-k+s) + \text{constant} \end{aligned}$$

This can only have an interior minimum if $r-k+s$ is 0, in which case the paths starting at A and B are also UP paths. (An interior start is possible, for example in the Eulerian network consisting of two vertices connected by four edges.) So we have shown that if there is a UP path starting in the interior of an edge, it can be modified to one starting at a vertex.

Now we have to show that every edge a is traversed from one end to the other. Suppose a UP starts at A; goes to x in the edge a; back to A; searches Q_1 from A to B; then searches a from B to A; then Q_2 from A. As above the mean capture time $T(S, \lambda)$ is given by

$$\begin{aligned} f(x) &= \frac{x}{2}x + 0 + (2x+c)r \\ &\quad + (2x+k+(\alpha-x)/2)(\alpha-x) + (2x+k+\alpha+t)s \\ &= -x^2 + (\alpha-k+2r+2s)x + \left[cr + s(\alpha+k+t) + \alpha\left(\frac{1}{2}\alpha+k\right)\right]. \end{aligned}$$

This is also concave, so there cannot be an interior maximum, and no UP path of this type can exist. (If S starts at some other vertex, the proof is the same, but there is an extra constant term in the expected time calculation which corresponds to the portion Q_0 of the network which is explored before reaching A.)

Finally, we consider a UP path which starts as above, tours Q_1 from A; traverses a from A to B; searches Q_2 from B. In this case,

$$\begin{aligned} f(x) &= \frac{x}{2}x + 0 + (2x+c)r + 0 + (2x+k+x+(\alpha-x)/2)(\alpha-x) \\ &\quad + (2x+k+\alpha+t)s \\ &= -2x^2 + (2L-k+2r+2s)x + cr + s(\alpha+k+t) + \alpha\left(\frac{1}{2}\alpha+k\right). \end{aligned}$$

Again, this is concave, so no UP path can 'turn' inside an edge. ■

The reader will note that there are many expected time calculations in the above proof that are irrelevant to the final result. This suggests an alternative proof might be possible. Indeed, we present in the Appendix a rather more elegant, non-computational proof. However, the calculations given here give some insights that may help the reader in the sequel.

In our proof of Theorem 4, we only used the assumption that the hider was uniformly distributed *on the edge* a, and showed that an optimal search strategy for such a hider distribution could not start or turn in the interior of a. Thus we have actually proved a somewhat stronger result.

Theorem 5 *Let h be a distribution (Hider mixed strategy) on a network Q such that on some edge a, h is uniform (with some density). Then there is a pure Searcher strategy S which is optimal against h (minimizes $T(S,h)$) such that S traverses the edge a at unit speed from one end to the other.*

This result is very similar to an earlier result of the authors, Theorem 5 of [3] (or Lemma 4 of [2]), in which an analogous result is obtained for an Eulerian circuit a which intersects $Q-a$ at a *single* point. (This would apply to our edge a if it were a loop.)

Theorem 6 *Let H be a network which is the union of two networks H_1 and H_2, which have a single point v in common. Let h be a (Hider) distribution on H which is uniform on H_2. If H_2 is Eulerian, there is an optimal continuous search path $S \in \mathcal{S}$ on H which searches H_2 in an Eulerian circuit starting at v, during some time interval of the search of H.*

In the corresponding Chinese Postman Problem, the optimal (CP) path is not only combinatorial, but it also has the property that it traverses each edge either once or twice. We now give an example which shows that a Utilitarian Postman path does not necessarily have this property.

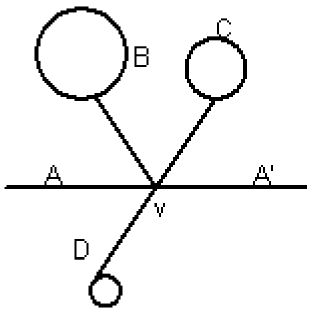


Figure 3: Network R

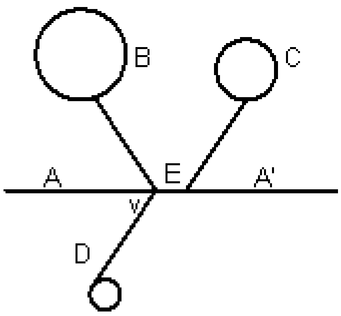


Figure 4: Network $R(\varepsilon)$

In Figure 3 we have drawn a network R which has a central vertex v to which two lines A and A' of length a, and three 'lollipops' B, C, D, are attached. The lollipops have unit length lines to which circles of lengths $b > c > d$ are attached. If the length a of the two lines is sufficiently large, a UP path must traverse each only once, so it must start and end at the ends of A and A' (either way). It is then easy to see that the lollipops must be searched in order of decreasing density of search, where the density is the measure (total length) divided by the search time. (This is a general property of least expected time search, a more general property is described in [7].) Since the function $(1+x)/(2+x)$ is increasing in x, we have

$$\frac{1+b}{2+b} > \frac{1+c}{2+c} > \frac{1+d}{2+d},$$

so the the UP path must be, up to symmetry of A and A', the path $S^* = ABCDA'$ (here B stands for a full tour of B from v). Any other combinatorial path S on R must have $T(S,\lambda) - T(S^*,\lambda) > \delta$, for some fixed $\delta > 0$. Next consider the related network $R(\varepsilon)$ drawn in Figure 4, which has an additional edge E of small length ε. The path $S^*(\varepsilon) = ABECEDEA'$ reaches no point in $R(\varepsilon)$ more than time 3ε later than the corresponding point of R is reached by S^*, so for ε sufficiently small satisfies $T(S^*(\varepsilon),\lambda) - T(S^*,\lambda) < \delta$ and so $T(S^*(\varepsilon),\lambda) < T(S^*,\lambda)$. It follows that $S^*(\varepsilon)$ must be a UP path on $R(\varepsilon)$, because any better path could be used on R to find a path at least as good as S^*, which is not possible. Thus for small ε, any UP path on the network $R(\varepsilon)$ must traverse the edge E three times. By adding additional lollipops of decreasing density on alternating sides of E, we can similarly force a Utilitarian Postman to cross the bridge E as many times as we like!

4 Symmetric Networks are Easily Hideable

Let $\mathcal{A} = \mathcal{A}(G)$ denote the group of isomorphisms (distance-preserving homeomorphisms) of a network G. Excluding the circle (so that there are vertices of degree $\neq 2$), this is a finite group. If for any vertices v_1 and v_2 there is an $\alpha \in \mathcal{A}$ with $\alpha(v_1) = v_2$, we say that G is *vertex-transitive.* If for all pairs of edges (v_1, v_2) and (w_1, w_2) there exist isomorphisms $\alpha, \beta \in \mathcal{A}$ with

$$\begin{aligned} \alpha(v_1) &= w_1 \text{ and } \alpha(v_2) = w_2, \\ \beta(v_1) &= w_2 \text{ and } \beta(v_2) = w_1, \end{aligned}$$

we say that G is *arc-transitive*; if only one of these isomorphisms need exist, it is *edge-transitive.* (Either condition implies that all edges of G have the same length, which we take to be 1.) Finally, we say that G is *symmetric* if it is edge-transitive and vertex-transitive. We will show that such networks are easily hideable.

For any hider strategy h, let $[h]$ denote the equiprobable mixture of αh for $\alpha \in \mathcal{A}$. This process of symmetrizing a strategy does not change its optimality, as shown in [4, Theorem 3] (it applies equally to Searcher strategies, but we don't use that part). The idea behind that proof is simple: Consider any pair of zero-sum games (Γ_1, Γ_2) and the game Γ in which Γ_1 is played unless either plays requests a change to Γ_2. It is clear that $V(\Gamma) = V(\Gamma_2)$, since if not, one of the Players would have preferred to play Γ_2. Now let Γ be determined by the pair $(\Gamma(Q), \Gamma_{sym}(Q))$ as above, where in $\Gamma_{sym}(Q)$ both players must symmetrize their strategies (that is, after picking a pure strategy by some means, the player must replace it by a randomly chosen isomorphism of it). Now observe that if either player chooses to symmetrize his strategy, it is as if both players have chosen to do this, because $T([S], H) = T(S, [H])$. So, as above, either player can switch the game being played to $\Gamma_{sym}(Q)$. This idea is also exploited in [1]. The full proof of the following is in [4], Theorem 3.

Lemma 7 *If h is an $(\varepsilon-)$ optimal Hider mixed strategy, then so is $[h]$.*

We can use this result to obtain $\varepsilon-$ optimal hider strategies on a symmetric graph G which have the following property.

Definition 8 *A measure (mixed hider strategy) h on a network Q is called* ***interval-symmetric*** *if it has the same distribution on each edge and this distribution is symmetric about the center of the edge.*

Lemma 9 *Let S be any combinatorial path in a network G, and let h_1 and h_2 be interval-symmetric distributions (mixed Hider strategies) on G without atoms at vertices. Then*

$$T(S,h_1) = T(S,h_2).$$

If h is any interval-symmetric distribution on G with atoms at the vertices (all vertices would have same positive measure) then

$$T(S,h) \le T(S,h_1),$$

with equality only if G is the interval graph $K_2 = [0,1]$.

Proof. Since $h_1(a) = h_2(a) (= 1/m)$ for any edge a of G, it is sufficient to show that

$$\int_a T(S,x)\ dh_1(x) = \int_a T(S,x)\ dh_2(x). \tag{3}$$

Since S is a combinatorial path, it will cover the edge a for the first time in some time interval $[i, i+1]$ so it is sufficient to show that

$$i + \int_0^1 x\ dh_1(x) = i + \int_0^1 x\ dh_2(x). \tag{4}$$

We establish this by showing that $\int_0^1 x\ dh(x) = 1/2m$ for any interval-symmetric distribution (on $[0,1]$ now). First assume that h has no atom at $1/2$ (middle of a).

$$\begin{aligned}
\int_0^1 x\ dh(x) &= \int_0^{1/2} x\ dh(x) + \int_{1/2}^1 x\ dh(x) \\
&= \int_0^{1/2} x\ dh(x) + \int_0^{1/2} (1-x)\ dh(1-x) \\
&= \int_0^{1/2} x\ dh(x) + \int_0^{1/2} (1-x)\ dh(x) \\
&= \int_0^{1/2} 1\ dh(x) = h(a)/2 = 1/2m.
\end{aligned}$$

The result clearly holds as well if h is concentrated on $1/2$, and the mixed case follows by decomposition.

To establish the inequality in the final sentence, write $h = \alpha h_V + (1-\alpha) h_E$, where h_V is supported on the vertex set and h_E has no atoms at the vertices. Since $T(S,h) = \alpha\ T(S,h_V) + (1-\alpha)\ T(S,h_E)$, it is enough to show that

$T(S,h_V) \leq T(S,h_E)$ with equality only when S is an Hamiltonian path. Consider any edge (A,B) which is first traversed by S in the time interval $[k,k+1]$. Let A_x and B_x be symmetric points in this edge at distance $x \leq 1/2$ from A and B. Then, since S may have reached one or more of the A,B before time k,

$$\begin{array}{c} S(k)=A \\ S(k+x)=A_x \\ S(k+1-x)=B_x \\ S(k+1)=B \end{array} \quad \text{so} \quad \begin{array}{c} T(S,A)\leq k \\ T(S,A_x)=k+x \\ T(S,B_x)=k+1-x \\ T(S,B)\leq k+1 \end{array},$$

and so the mean time for S to find A,B is less than or equal to that $(k+1/2)$ to find A_x,B_x, with strict inequality if either A or B has been reached before time k. This means that whenever S reaches a vertex v, there is at most one unsearched edge incident to v. So, aside from the first and last vertex, all vertices must have degree 2. Deleting such vertices, we are left with a single edge, as claimed. ■

To understand the significance of the following result, consider first the star network consisting of a central vertex v_0 with $m>2$ unit length edges (v_0,v_i) attached. Clearly this is edge-transitive, because for any $i,j>0$ there is an isomorphism taking (v_0,v_i) into (v_0,v_j). However the only optimal hiding strategy is to hide with equal probability at the non-central ends of the edges (leaves). This distribution is not interval-symmetric. Note that this network is not vertex transitive, as the central vertex is special.

Lemma 10 *Let G be a symmetric network of odd degree. Then for any $\varepsilon>0$ there is a an interval-symmetric $\varepsilon-$optimal hiding strategy h.*

Proof. Let h be any $\varepsilon-$optimal hiding strategy. The existence of such a strategy follows from the justification given earlier for the Value (1). Tutte [25] has proved that a symmetric network of odd degree is arc-transitive. Consequently $[h]$ has the same distribution on every ordered edge (v,v') and in particular the distribution on an edge (v_1,v_2) is the same as that on the reversed edge (v_2,v_1). ■

We have now accumulated all the tools we need to prove the main result of this section, that the uniform hiding strategy λ is optimal (easily hideable). Note that on the interval graph $K_2=[0,1]$ one (of the many) optimal hiding strategies is to simply hide equiprobably at either end. This is the strategy one would obtain from the analysis of K_2, viewed as a tree [13]. The second part of the result below demonstrates that, aside from this case, the hider should definitely avoid the vertices of Q.

Theorem 11 *A symmetric network is Q easily hideable.*

Furthermore, if there is an optimal hider mixed strategy h_V which gives positive probability to any vertex of Q, then Q must be the interval graph $K_2 = [0,1]$.

Proof. Let G be a symmetric network. If it has even degree (valency), it is Eulerian and hence easily hideable [3]. So assume that G has odd degree. Then for any $\varepsilon > 0$ there is by Lemma 10 an interval-symmetric $\varepsilon-$optimal hiding strategy h_ε. By Theorem 4 there exists a combinatorial Utilitarian Postman path S on G. Since S is combinatorial, and since both h_ε and the uniform distribution λ are interval-symmetric, with the λ having no atoms at the vertices, Lemma 9 gives

$$T(S, h_\varepsilon) \leq T(S, \lambda). \tag{5}$$

Since h_ε is $\varepsilon-$optimal, we have

$$V - \varepsilon < T(S, h_\varepsilon). \tag{6}$$

Since S is a UP path ('combinatorial' not needed here), we have $T(S, \lambda) = \hat{\mu}$. So combining (6) with (5) gives

$$V - \varepsilon < \hat{\mu}.$$

Including the left inequality of (2) we have

$$V - \varepsilon < \hat{\mu} \leq V.$$

But since this holds for all positive ε, we have

$$\hat{\mu} = V,$$

which is the definition of easily hideable (equivalent to λ being an optimal mixed strategy).

To establish the second part of the Theorem, let $h_\varepsilon = [h_V]$, which is an interval-symmetric strategy with positive measure on every vertex, and $T(S, h_\varepsilon) = T(S, \lambda)$, by optimality. Hence the claim follows from the last part of Lemma 9. ∎

5 UP value $\hat{\mu}$ for graphs of odd degree

In this section we find combinatorial UP paths for a network of odd degree (all vertices have odd degree, though not necessarily the same). We obtain a lower bound τ on the Utilitarian Postman time $\hat{\mu}$ and a necessary and sufficient condition for $\hat{\mu} = \tau$. Since symmetric networks have equal length edges, we take this length to be 1. Since we are looking for UP paths in this section, we can forget the network structure and simply consider the network as a combinatorial *graph*. Finally, as we are considering a uniformly distributed Hider, we assume the meeting time for a Hider in an edge (v_i, v_j) is the midpoint of the (integer) times taken by the path to reach v_i and v_j. So for the remainder of this section take the search space to be a graph G with n vertices of odd degrees and m unit length edges. Clearly n has to be even if all degrees are odd.

We now formalize the analysis we used to determine the UP time $\hat{\mu}$ for the tree drawn if Figure 1, and the reader is invited to review that analysis before reading this extension. To each covering path $S = (e_1, e_2, \ldots, e_L)$ of G, we associate its *exploration set* $\Omega = \Omega_S$ and *exploration sequence* $\alpha = \alpha(S)$ as follows:

$$\begin{aligned} \Omega_S &= \{j : e_i \neq e_j, i < j\}, \text{ and} \\ \alpha &= \alpha(S) = (\alpha_1, \ldots, \alpha_L), \text{ with } \alpha_j = 1 \text{ for } j \in \Omega_S \text{ and } 0 \text{ otherwise.} \end{aligned} \tag{7}$$

Intuitively, the 1's correspond to edges traversed for the first time; the 0's to repeated edges. The sequence α always ends in a 1, has m 1's and some number $r = L - m$ of 0's. A particularly important exploration sequence is $\alpha^* = \alpha^*_{m,n}$, with the number $r^* = (n-2)/2$ of repeated edges as small as possible (this will be shown later), which starts with $m - r^*$ 1's and then has r^* repeated pairs $0, 1$. For example

$$\alpha^*_{6,6} = (1, 1, 1, 1, 0, 1, 0, 1). \tag{8}$$

If a random point is in edge e_t, $t \in \Omega_S$, it will be found on average in time $t - 1/2$, so given that a random point is equally likely to be in any edge, we have

$$T(S, \lambda) = \frac{1}{m} \sum_{i \in \Omega_S} (i - 1/2) = \frac{1}{m} \sum_{i:\alpha_i(S)=1} (i - 1/2). \tag{9}$$

More generally, we define for any $0-1$ sequence α with m 1's, its expected capture time

$$T(\alpha) = \frac{1}{m} \sum_{i:\alpha_i=1} (i - 1/2). \tag{10}$$

In particular, we define the time $\tau = \tau_{m,n}$ by the formula

$$\tau = \tau_{m,n} = T\left(\alpha^*_{m,n}\right) \tag{11}$$

$$= \frac{1}{m}\left(\sum_{t=1}^{m-(n-2)/2}(t-1/2) + \sum_{i=1}^{(n-2)/2}\left[(m-(n-2)/2+2i)-1/2\right]\right) \tag{12}$$

$$= \frac{m}{2} + \frac{n^2-2n}{8m} \tag{13}$$

The next result shows that τ is a lower bound for the Utilitarian Postman time $\hat{\mu}$ for an odd degree graph.

Lemma 12 *For any covering path S of a graph with odd degrees, we have $T(S,\lambda) \equiv T(\alpha(S)) \geq T(\alpha^*) \equiv \tau$, with equality if and only if $\alpha(S) = \alpha^*$. Consequently $\hat{\mu} \geq \tau$ and any S with $\alpha(S) = \alpha^*$ is a UP path.*

Proof. Let $\alpha = \alpha(S)$ be the exploration sequence of covering path S. Let k denote the number of 'runs of zeros' in α, that is of strings consecutive 0's. (For example $(1,1,0,0,0,1,0,0,1,1,0,1)$ has three runs of zeros.) If S is not a circuit, every vertex of G except for the starting and ending vertices of S must have an odd number of its incident edges repeated in the path S. So at least one of these must correspond to the initial or final zero in one of the k runs. Hence we must have $k \geq (n-2)/2 \equiv r^*$, the number of runs of (single) zeros in α^*. If S is a circuit, then *every* vertex has this property, and $k \geq n/2 > r^*$. So in both cases $k \geq r^*$. Let α' be the sequence obtained from α by replacing each run of zero's by a single zero. (In the above example, we get the sequence $(1,1,0,1,0,1,1,0,1)$.) Note that α' has k zeros. Since this either leaves a 1 in the same position or brings it forward, it is clear from (10) that we have $T(\alpha') \leq T(\alpha)$. Next, let α'' denote the sequence obtained from α' by moving each 0, starting with the rightmost one and moving left, as far as possible to the right, subject to being to the left of the final 1 and not being adjacent to any other 0. (In the example, this gives $(1,1,1,0,1,0,1,0,1) = \alpha^*_{6,8}$.) Note that α'' still has k 0's. Since the 0's are moved to the right, 1's either stay still or are moved to the left. Hence $T(\alpha'') \leq T(\alpha')$, and hence $T(\alpha'') \leq T(\alpha)$. Observe that $\alpha'' = \alpha^*_{m,n''}$, where $n'' = 2k+2$. (In the example, $k = 3$ and $n'' = 8$.) Since $n = 2r+2$ and $k \geq r$, we have $n'' \geq n$ (note that n'' does not represent the number of vertices in any graph we have described). It follows from (13) that $\tau_{m,n} = T\left(\alpha^*_{m,n}\right)$ is increasing in n, we have that

$$T(S) = T(\alpha) \geq T(\alpha') \geq T(\alpha'') \geq T\left(\alpha^*_{m,n}\right) = \tau_{m,n} = \tau. \tag{14}$$

Next, we consider when equality holds in (14). Note that $T(\alpha') = T(\alpha)$ if and only if all the runs of 0's in α have length 1 (single 0); $T(\alpha') = T(\alpha'')$ holds if and only if all the 0's in α are as far to the right as possible (in positions $L-1, L-3, \ldots$, where L is the length of α); $T(\alpha'') = T\left(\alpha^*_{m,n''}\right) = T\left(\alpha^*_{m,n}\right)$ (that is, the last equality) hold if and only if $n'' = n$ (or $k = r^*$). So the equality $T(S) = \tau$ holds if and only if $\alpha(S) = \alpha^*$. ■

It turns out that there is a combinatorial characterization of graphs G for which there exists an S with $\alpha(S) = \alpha^*$. Recall that the even number of vertices is denoted by n.

Definition 13 *A **hypo Hamiltonian (HH) path** is a path $v_1, v_2, \ldots, v_{n-1}$ of distinct vertices.*

Definition 14 *A graph G is called **Half-Hypo-Hamiltonian Connected (HHHC)** if it has a (HH) path $H = v_1, v_2, \ldots, v_{n-1}$ of distinct vertices such that the removal of the even indexed edges of H leaves a connected set of edges, that is, if*

$$G - \cup_{i=1}^{(n-2)/2} (v_{2i}, v_{2i+1}) \text{ is connected.} \tag{15}$$

Lemma 15 *Let G be any odd degree graph satisfying $\hat{\mu} = \tau$. Then G is HHHC.*

Proof. According to Lemma 12, every UP path S in G has an the exploration sequence α^*. Let $e_1, \ldots, e_{2r}$ be the last $2r$ edges of S, so $e_1, e_3, \ldots, e_{2r-1}$ are the $r = r^* = (n-2)/2$ repeated edges in S, those corresponding to the 0's. Consider the graph G' in which these r edges are doubled. Clearly S is an Eulerian path, from vertex v_1 to vertex v_n (by relabeling) in G'. So only v_1 and v_n have odd degree in G'. Since S has the minimum possible number of repeated edges, these r repeated edges cannot have any incident vertices in common. So their $2r = n-2$ incident vertices (which will have even degree in G') include all but the first and last vertices of S. So by relabeling, we may call the (distinct) last $n-1$ vertices of S: $v_2, v_3, \ldots, v_n$, where the first vertex of S is v_1. Define S^- to be the path S from v_1 to v_2; define $S^+ = e_1, e_3, \ldots, e_{2r}$ be the remaining path from v_2 to v_n.

Consider the graph $G'' = G - \{e_2, e_4, \ldots, e_{2r}\}$.. Since S covers all the edges of G, the path S^- must cover all the edges of G which are not covered by S^+ and also cover all edges of G which are covered twice by S (that is, $\{e_1, e_3, e_{2r-1}\}$). Consequently the image of S is exactly G''. Since S^+ is a hypo Hamiltonian path, we are done. ■

Lemma 16 *Let G be an odd degree graph which is HHHC. Then there is a covering path S of G with $\alpha(S) = \alpha^*$. Hence S is UP, and $V(G) = \hat{\mu} = \tau$. Furthermore, if $P_1 = v_1, v_2, \ldots, v_{n-1}$ is the HH in G with $G' = G - \cup_{i=1}^{(n-2)/2}(v_{2i}, v_{2i+1})$ connected, then we may take S to be any Eulerian path in G' from the remaining vertex v_n to the initial vertex v_1, followed by P_1.*

Proof. Let $a = v_n$ denote the unique vertex of G not contained in P_1, and let $b = v_1$ be the starting vertex of P_1. The graph G' has even degree except at vertices a and b, as all other vertices are incident to exactly one of the edges (v_{2i}, v_{2i+1}), and their degree has been reduced by one, to some even number. Hence there is an Eulerian path P_2 in G' which starts at a and ends at b. The concatenated path $S = P_2P_1$ (that is, P_2 followed by P_1) covers all the edges of G once, except for the $r^* = (n-2)/2$ edges (v_{2i-1}, v_{2i}), which are covered twice (once in P_1 and once in P_2). The path P_2P_1 has length $L = m + r^*$, and may be described as a sequence of edges $e_1, e_2, \ldots, e_L$. The edges covered twice are covered for the second time as edges $e_{L-1}, e_{L-3}, \ldots, e_{L-(n-2)}$. Hence $\alpha(S) = \alpha^*$, as required. ■

The proof is illustrated below for complete graph K_4 on $n = 4$ vertices. The left picture shows the HH path P_1 of length $n - 2 = 2$. The middle picture shows that if the second edge of P_2 is removed, what remains is connected. In this graph, only vertices a and b have odd degree (3), so there is an Eulerian path in this graph (length 5) which starts at a and ends at b. The concatenated path P_2P_1 on the original graph (with edges 1 and 2 of P_1 now numbered 1+5=6 and 2+5=7) is shown on the right. One can see that the only repeated edge (numbered e_5 and e_6) is covered for the second time at the next to last position, 6. Hence the path on the right is a UP path.

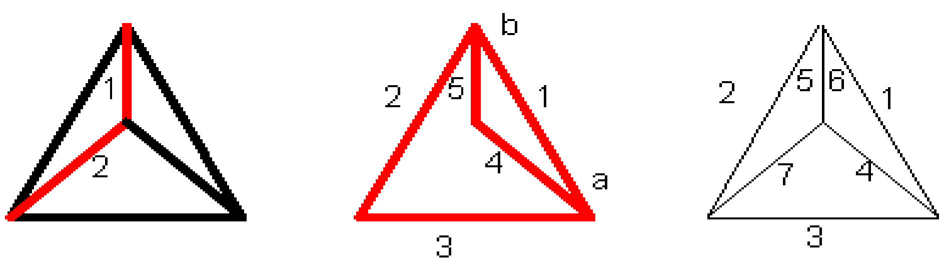


Figure 5. HH path P_1, Eulerian path P_2 in $Q - H_{even}$, UP path P_2P_1

A more interesting example is the Petersen graph, shown below in Figure 6. A hypo Hamiltonian path is drawn on the left in red. When its even edges are removed, what remains is the red graph on the right, which is clearly connected. So the UP path starts with an path P_2 in the red graph on the right, followed by the hypo

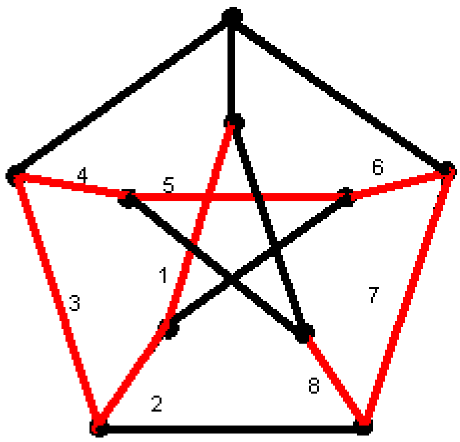
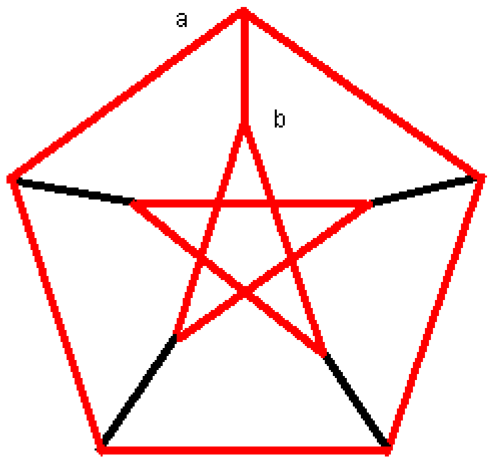


Figure 6: HH path P_1



Eulerian path P_2

Combining the three previous results gives our main result on UP paths on odd degree networks.

Theorem 17 *Let G be a graph with n vertices, all of odd degree, and m unit length edges. Then $\hat{\mu} \geq \frac{m}{2} + \frac{n^2 - 2n}{8m}$, with equality if and only if G is HHHC.*

6 UP Paths on Symmetric Graphs

We can now combine our results on the optimality of uniform hiding in a symmetric network (Theorem 11) with our complementary results (Theorems 16,17) on optimal (UP) pure strategy search for a uniformly distributed Hider on a network of odd degree. All we need to do is generalize the latter work to mixed strategies to obtain our main result. We revert to our original use of 'network', as this is the context in which our general game $\Gamma(Q)$ was defined.

Theorem 18 *Consider the search game $\Gamma(Q)$ on a symmetric network Q with n vertices and m unit length edges. Then*

1. *Q is easily hideable: $V(Q) = \hat{\mu}$, and the uniform distribution λ is an optimal mixed Hider strategy.*

2. *If Q has even degree, $V(Q) = m/2$. Any Eulerian circuit, traversed equiprobably in either direction, is an optimal mixed Searcher strategy.*

3. *If Q has odd degree, then $V(Q) \geq \frac{m}{2} + \frac{n^2 - 2n}{8m}$, with equality if and only if Q is HHHC.*

4. *Suppose Q is HHHC and has odd degree. Let $P_1 = v_1, v_2, \ldots, v_{n-1}$ be any (HH) path, with $Q' = Q - \cup_{i=1}^{(n-2)/2}(v_{2i}, v_{2i+1})$ connected. Then an optimal Searcher mixed strategy is given by $[P_2 P_1]$, where P_2 is any Eulerian path in Q', from the remaining vertex v_n to v_1, and $[\ \ ]$ denotes averaging with respect to the isomorphism group of Q.*

Proof. Part (1) is a restatement of Theorem 11. Part (2), mentioned earlier, is trivial. Part (3) is follows immediately from Theorems 11 and 16. Next consider part (4). Theorem 16 shows that P_2P_1 is a UP path on Q. However it is not in general true that for a UP path P on an easily hideable network, $[P]$ is an optimal mixed strategy. For example, the network in Figure 1 has $V = 9/2$ and UP path ABDDC. But against the randomized Searcher strategy $[ABDDC]$, the Hider can obtain $T([ABDDC], x) = 3 = (3/2 + 9/2)/2$ by taking x to be the middle of B. However Q is symmetric, so for any $H \in Q$, $[H] = \lambda$. It follows that for any Hider pure strategy H, $T([P_2P_1], H) = T(P_2P_1, [H]) = T(P_2P_1, \lambda) = \hat{\mu}$, since P_2P_1 is a UP path. Hence $[P_2P_1]$ is an optimal strategy for the Searcher. ■

The full strength of our main result (Theorem 18) requires that the odd degree symmetric graph Q has the HHHC property. This may seem an unlikely property. However, in fact we know of no odd degree symmetric graph that is not HHHC. So we make the following.

Conjecture 19 *Every odd degree symmetric graph is HHHC.*

It is useful to note that if the HH path P in the definition of HHHC was required to be *Hamiltonian* (rather than hypo-Hamiltonian), then the conjecture would certainly be false. (In this case P would include n vertices and have an odd number $n-1$ of edges, and the condition would be that the removal of the *odd* edges of P leaves G connected - for hypo-Hamiltonian paths, which have even length $n-2$, the odd edges in one direction are the even edges in the other.) Brian Alspach [8] has observed that the Petersen graph G of Figure 6 would be a counterexample. The Petersen graph *does* have a Hamiltonian path (put the edge (a, b) before the path P_1 of Figure 4) but is known to have no Hamiltonian circuit. Suppose that G has has a Hamiltonian P such that the graph G', obtained by removing the odd edges of P, is connected. Then since every vertex of G' has even degree 2, G' has an Eulerian circuit C. But since G' has degree 2, this means that C enters each vertex once and leaves once - it is a Hamiltonian circuit. But no such circuit exists on the Petersen graph, so this is impossible.

In the positive direction of the Conjecture, we have already shown (Figures 5 and 6) that the Petersen graph and K_4 are HHHC. Much work has already gone into establishing that many classes of symmetric graphs must have two (or more) edge-disjoint Hamiltonian paths or circuits, at least for n sufficiently large. Either of these is of course a much stronger property than HHHC. Some of the following are proved by filling in the remaining cases for small n; others by a direct argument.

Theorem 20 *All odd degree symmetric graphs of the following types are HHHC:*

1. *The complete graphs K_n, $n \geq 2$ (odd degree for n even)*
2. *The complete bipartite graphs $K_{n,n}$, $n \geq 1$ (odd degree for n odd)*
3. *The $d-$dimensional hypercube graphs $[0,1]^d$, $d \geq 1$ (odd degree for d odd)*
4. *Graphs G with degree $\geq n/2$, for $n \geq 79$.*

Proof. In the following, let P_{even} denote the even edges of the path P.

1. Number the vertices $1, 2, \ldots, n$, and take $P = [1, 2, \ldots, n-1]$. $Q - P$ is connected, as all edges (v_i, v_n) are not in P.
2. Label the vertices v_i, w_i, $i = 1, \ldots, n$, with the edges (v_i, w_j). Let $P = v_1, w_2, v_3, w_4, \ldots, w_{n-1}, v_n, w_1, v_2, w_3, v_4, \ldots, w_{n-2}, v_{n-1}$. The Hamiltonian path $v_1, w_1, v_2, w_2, \ldots, w_{n-1}, v_n, w_n$ is disjoint from P_{even}.
3. It is known that $[0,1]^d$ contains $d/2$ edge disjoint Hamiltonian paths [10]. So for $d \geq 5$, there are two edge disjoint Hamiltonian paths, a stronger condition than HHHC. For $d = 3$ (the edge graph of a cube), the HH path P drawn below leaves a connected graph when its three

even numbered edges are removed.

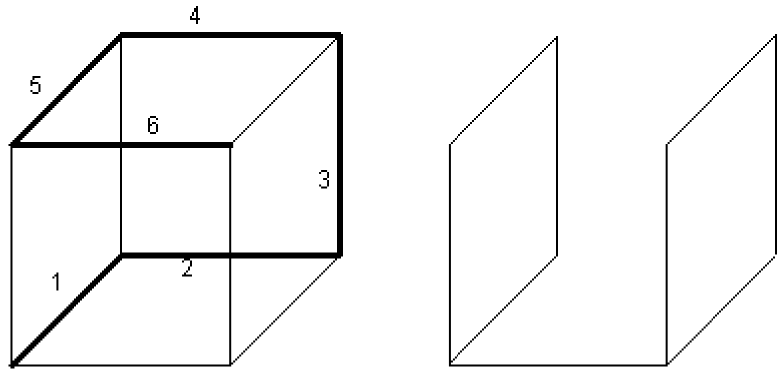


Figure 7. HH path P on cube, cube with P_{even} deleted.

4. Nash–Williams [20] proved that such a G has $\lfloor 5(n - \lfloor n/2 \rfloor + 5)/112 \rfloor$ edge-disjoint Hamiltonian circuits, so two of them for $n \geq 79$.

■

We have checked other small symmetric graphs and found them to be HHHC. But this is not a graph theory paper so we have merely given a list of families of symmetric graphs where the HHHC property holds and whose value is therefore given by $\frac{m}{2} + \frac{n^2 - 2n}{8m}$. Even if there are odd degree symmetric graphs for which this condition fails, we have still found an optimal Hider mixed strategy and reduced the Searcher problem to the construction of a UP path. We believe this Utilitarian Postman Problem may prove a fruitful area of future research in operations research and graph theory.

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