Network Search Games With Immobile Hider, Without a Designated Searcher Starting Point

Centre for Discrete and Applicable Mathematics
Research Report
CDAM 2006-03

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December 2005 (Revised March 2006)

Abstract

In the (zero-sum) search game $\Gamma(G,x)$ proposed by Isaacs, the Hider picks a point H in the network G and the Searcher picks a unit speed path $S(t)$ in G with $S(0)=x$. The payoff to the maximizing Hider is the time $T=T(S,H)=\min\{t:S(t)=H\}$ required for the Searcher to find the Hider. An extensive theory of such games has been developed in the literature. This paper considers the related games $\Gamma(G)$, where the requirement $S(0)=x$ is dropped, and the Searcher is allowed to choose his starting point. This game has been solved by Dagan and Gal for the important case where G is a tree, and by Alpern for trees with Eulerian networks attached. Here, we extend those results to a wider class of networks, employing theory initiated by Reijnierse and Potters and completed by Gal, for the fixed-start games $\Gamma(G,x)$.

Our results may be more easily interpreted as determining the best worst-case method of searching a network from an arbitrary starting point.

keywords: search game, network, zero-sum, chinese postman, Eulerian

1 Introduction

There are two related search games that can be played on a compact network G, in which a unit speed Searcher seeks to minimize the time required to 'find' a stationary Hider. These are zero sum games where the payoff is the time $T = \min\{t : S(t) = H\}$ for a Searcher following the path $S(t)$ in G to reach a Hider who chooses to stay at the point $H \in G$. In the fixed-start game $\Gamma(G,x)$, the Searcher is constrained to start at the designated point $x = S(0)$; in the arbitrary start game $\Gamma(G)$, he may start anywhere in G. Compactness arguments ([14], [4]) establish that these games always have optimal mixed searcher strategies and (minimax) values which we denote respectively as $V_x = V(G,x)$ and $V = V(G)$. Both versions of the game can be attributed to Rufus Isaacs, who introduced them in the final chapter of his classical work on differential games [19]. While an extensive general theory has been developed for fixed-start search games (see [14] and [4]), arbitrary-start games have only recently begun to be studied, by Dagan [8], Dagan and Gal [9] and Alpern [2]. (There is also some early work when G is the circle for related games where the Hider is also mobile ([26],[1]), but very different techniques are required for those games.)

This paper develops a theory of arbitrary-start search games. The optimal search strategies we find here for these games can be more easily interpreted as best worst-case methods for searching a network. As such, they can be applied to many search problems where there is no active antagonist.

Related work on search games can be found in: Anderson and Aramendia [6]; Beck and Newman [7]; Demaine, Fekete and Gal [10]; Garnaev [16]; Kikuta [20]; Pavlovic [22]; Ruckle [24]; and von Stengel and Werchner [25].

2 Definitions and Results

In this article we will assume the network G is connected and has a finite number of nodes and arcs. Each arc a of G has a length, denoted by $\lambda(a)$, and more generally λ is taken as Lebesgue measure on G. The distance $d(x,y)$ denotes the usual network distance between points of G corresponding to the length λ.

The number

$$\rho(G) = \min_{x \in G} \max_{y \in G} d(x,y) \qquad (1)$$

is called the *radius* of G and any point x achieving this value is called a *center* of G. (If G is a tree, the center is in fact unique, but we shall not need to use this fact.) Algorithms for finding the center are given by Hakimi [17], Hassan and Tamir [18], Dvir and Handler [11], and Megiddo and Tamir [21]. The *diameter* of G is defined by $d(G) = \max_{x,y \in G} d(x,y)$. Note that $\rho(G) \le d(G) \le 2\rho(G)$.

A *path* S in G is a continuous function from some finite time interval $[0,\tau]$ into G. In our games $\Gamma(G)$ and $\Gamma(G,x)$ the Searcher will only use (as pure strategies) unit (maximal) speed paths in G, that is, paths belonging respectively to the sets

$$\begin{aligned}\mathcal{S} &= \{d((S(t)), S(t')) \leq |t-t'|\} \text{ and} \\ \mathcal{S}_x &= \{S \in \mathcal{S} : S(0) = x\}.\end{aligned}$$

Such a path S is called *closed* if $S(0) = S(\tau)$, and called a *tour* if additionally its range is G.

Let $\mu = \lambda(G)$ denote the total length (sum of arc lengths) of G. A *Chinese Postman (CP) tour* is a tour of minimum length $\bar{\mu}$, and a *CP path S* is a covering (range G) path of minimum length of $\tilde{\mu}$. For trees, a CP path (and hence $\tilde{\mu}$) is determined by picking two points $a, b \in G$ (leaves) at maximum distance $d(G)$, and going from (say) a to b while traversing all arcs on the geodesic L between a and b once, and all other arcs twice (so $\tilde{\mu} = 2\lambda(G) - d(G) = 2\mu - d(G)$).

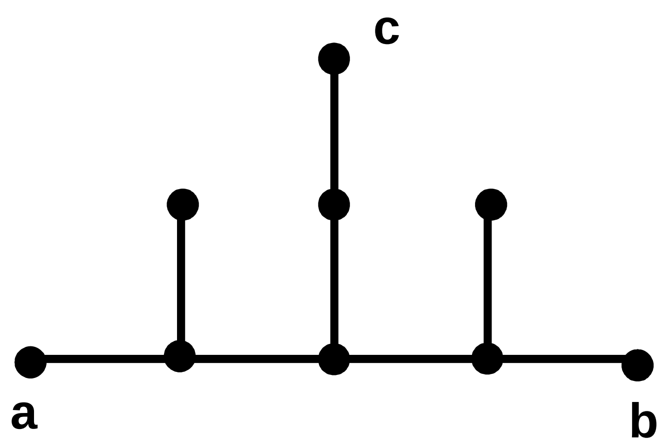

Figure 1: CP paths on a tree.

Consider the network drawn in Figure 1, where all arcs have unit length. The diameter $d(G)$ is four and, up to isomorphism and direction, there are two CP paths, both end at b and start at either a (call this P_a) or c call this P_c. They have length $\bar{\mu} = 2\lambda(G) - d(G)) = 2 \cdot 8 - 4 = 12$.

More generally, the Chinese Postman Problem and constructions, have been analyzed by Edmonds and Johnson [12] and Eiselt et al [13]. These constructions produce 'combinatorial' paths which start at a node and traverse the arcs in succession, without turning in the interior of an arc.

Our main result (Theorem 2) has a condition involving the difference between the length of a CP tour and a CP path, denoted by

$$\triangle = \triangle(G) = \bar{\mu}(G) - \tilde{\mu}(G) \geq 0.$$

Note that $\triangle \leq d(G) \leq 2\rho(G)$, as we can convert any covering path to a covering tour by extending it by the shortest route from the end of the path to the beginning, which cannot be longer than $d(G)$.

A *random CP tour* is a CP tour which is traversed equiprobably in either direction. A *random CP path* is an equiprobably mixture of a CP path $\tilde{S}(t)$ and its reverse path $\tilde{S}'(t) = \tilde{S}(\tilde{\mu} - t)$. If $\tilde{S}(t) = x$, then $\tilde{S}'(\tilde{\mu} - t) = x$, so a randomized CP path finds all $x \in G$ in average time no more than $\tilde{\mu}/2$. Hence for any network G we have

$$V(G) \leq \tilde{\mu}(G)/2. \tag{2}$$

If equality holds in (2) we say that G is *simply searchable.* In this case any random CP path is clearly optimal. The main results of this article give sufficient conditions for simple searchability and show that it is not a topological property.

3 Fixed–Start Games

Although we are mainly concerned with arbitrary start search games, we will need the following important fixed-start result of Gal ([15],[4]) (extending work of Reijnierse and Potters [23]) characterizing networks G for which $V(G, x) = \bar{\mu}/2$. The condition is that G is *Weakly Eulerian,* meaning it contains a finite number of disjoint Eulerian ($\bar{\mu}(E_i) = \mu(E_i)$) networks E_i which, when each is shrunk to a point, leaves a tree.

Theorem 1 (Gal) *If G is a Weakly Eulerian network, then for any starting point $x \in G$, $V(G, x) = \bar{\mu}/2$, any random CP tour starting at x is an optimal searcher strategy in $\Gamma(G, x)$, and there is an optimal hiding strategy $\bar{h}_x$ in $\Gamma(G, x)$ which is uniform on each Eulerian subnetwork. Conversely, if $V(G, x) = \bar{\mu}/2$ for some x, then G is Weakly Eulerian.*

An algorithm for constructing the Hider distribution $\bar{h}_x$ is described in [15] and [4].

4 Simply Searchable Networks

In this section we present a result, Theorem 2, which gives a sufficient condition for a network to be simply searchable. The condition involves both the notion of weakly Eulerian and the following notion of *Eulerian deletion*: If $G = B \cup E$ is a network in which a maximal disjoint finite family E_i of Eulerian networks (whose union is E) are each attached to a connected network B at a single point, we say that B is the *Eulerian deletion* of G, denoted by $B = E'(G)$.

The following result gives an inequality which is sufficient for a Weakly Eulerian network to be simply searchable.

Theorem 2 *If G is a Weakly Eulerian network satisfying*

$$\triangle(G) \geq 2\ \rho(E'(G)), \tag{3}$$

then G is simply searchable. Furthermore, if c is a center of $E'(G)$, then the strategy $\bar{h}_c$ of Theorem 1 is optimal for the Hider.

The proof of Theorem 2 will be given in Section 6. Before then, we wish to discuss several specific results which follow from this. To make the list more complete, we also include some simple cases which follow simply from the definition of simple searchability.

Corollary 3 *A network G is simply searchable if it satisfies any of the following:*

1. *G has an Eulerian path.*
2. *G is Eulerian.*
3. *G consists of a tree A to which disjoint Weakly Eulerian networks are each attached at single points of A, and $\rho(A) = \rho(G)$.*
4. *G is 'Partly Eulerian' (this means $E'(G)$ is a tree).*
5. *G is a tree.*

The first two results follow directly from the definition of simply searchable, the remaining three are true corollaries.

Proof. Note that all the conditions except 1 imply that G is Weakly Eulerian, and 1 is proved directly.

1. Note that $V(G) \geq \mu/2$ because the Hider can hide uniformly, while the Searcher can ensure that $V(G) \leq \tilde{\mu}/2 = \mu/2$ by adopting a random CP path, so $V(G) = \tilde{\mu}/2$.
2. We have $E'(G)$ is a singleton with 0 radius and $\bar{\mu} = \tilde{\mu}$. (Or use condition 1.)

3. and 4. Write $G = A \cup F$, where A is the tree and F is the union of the attached networks. Let r denote the radius of A, let x and y denote points of A at maximum distance $d(x, y) = 2r$, and let L denote a simple path from x to y (of length $2r$). Fix any CP tour S of G and let $\bar{G}$ denote the Eulerian network obtained by doubling any arc of G doubly traversed by S. Observe that $\lambda\left(\bar{G}\right) = \bar{\mu}$ and that every arc in L is doubly traversed by S. Let $\tilde{G}$ be the network obtained from $\bar{G}$ by removing the doubled arcs of the path L, so that $\lambda\left(\tilde{G}\right) = \bar{\mu} - \lambda(L) = \bar{\mu} - 2r$. Note that since $\bar{G}$ is Eulerian it follows that every node of $\tilde{G}$ has even degree except for the

ends x and y of L. Hence there is an Eulerian path (of length $\lambda\left(\tilde{G}\right)$) in $\tilde{G}$ from x to y. Such a path is a covering path of G, so $\tilde{\mu} \leq \lambda\left(\tilde{G}\right) = \bar{\mu} - 2r$. Consequently we have

$$\triangle = \bar{\mu} - \tilde{\mu} \geq 2r = 2 \quad \rho(A).$$

Under assumption 3, $A = E'(G)$, so (3) holds. Under assumption 4, we have $\rho(A) = \rho(G)$, and since $A \subset E'(G) \subset G$ this implies that $\rho(A) = \rho(E'(G))$, and (3) holds.

5. This is a special case of 3 or 4.

■

Condition 5 is due to Dagan and Gal [9], condition 4 is due to Alpern [2], and condition 3 is due to Gal and also is implicit in Alpern's proof of 4. Conditions 1 and 2 are fairly obvious in any case.

Also included in Theorem 2 are networks not previously covered by any of these conditions, such as the network G drawn in Figure 2, for which $E'(G)$ is simply G with the top circle removed, and for which (3) is satisfied because

$$\bar{\mu} = 20, \; \tilde{\mu} = 14, \; \triangle = 6, \; \rho(E'(G)) = 3.$$

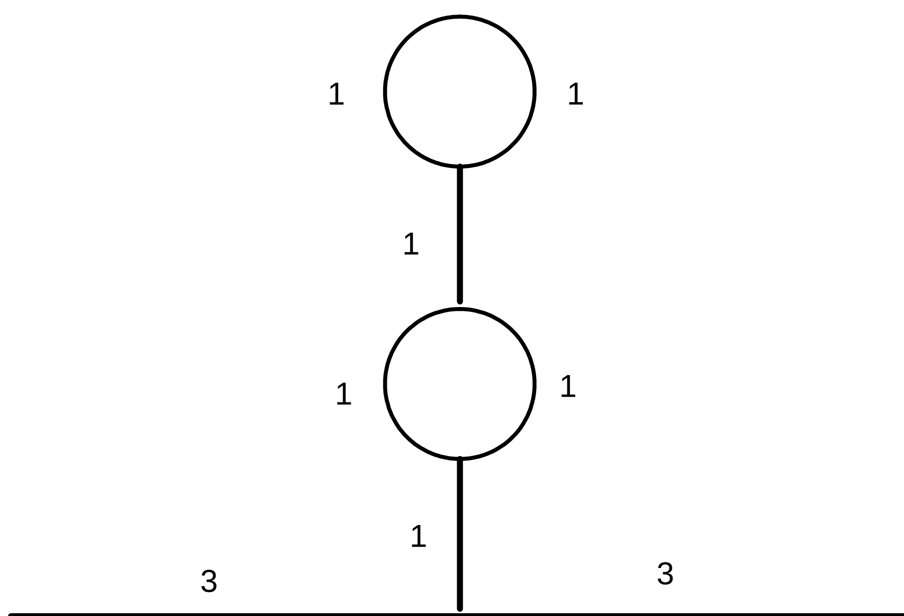

Figure 2: A simply searchable network

5 An Optimal Search Problem

In this section we consider a one-sided search problem, which asks how to optimally search a network in which a hider is distributed uniformly on an Eulerian subnetwork. The main conclusion is that the subnetwork should be searched without interruption. In order to obtain this result, we will have to consider a wider family of search strategies $\mathcal{S}'$ (larger than $\mathcal{S}$), in which the subnetwork may be searched in a discontinuous manner. *Note that except for this section, all strategies will be assumed to be in S (continuous).*

Our main result, which will be essential in the rest of the paper, is the following Theorem which extends a similar result of [2] without explicitly referring to the underlying 'alternating search theory' of [5] which was used in the earlier proof.

Theorem 4 *Let H be a network which is the union of two networks H_1 and H_2, which have a single point e in common. Let h be a (Hider) distribution on H which is uniform on H_2. If H_2 is Eulerian, there is an optimal continuous search path $S \in \mathcal{S}$ on H which searches H_2 in an Eulerian circuit starting at e, during some time interval of the search of H. In particular, we can assume that $S(0) \in H_1$.*

To prove Theorem 4, we introduce a new search problem Γ' based on H, in which the searcher must, as usual, search H_1 in a continuous unit speed path, but whenever he reaches e he can search H_2 in a discontinuous path as long as he searches any set $A \subset H_2$ in time at least $\lambda(A)$. Denote the set of such paths $\mathcal{S}'$. We can assume these paths in H_2 start and end at the point e. Let V' be the expected capture time for an optimal search. Clearly $V' \leq V$, because the searcher has a wider class of strategies ($\mathcal{S} \subset \mathcal{S}'$) in the problem Γ'.

Consider the network H drawn in Figure 3. It has four arcs of length 2. To enable us to easily indicate paths on H (especially discontinuous ones), we have put directions on the arcs. Thus c^{-1} goes from the left of the figure to the intersection point. In this notation, $P = \left(bcc^{-1}a^{-1}d\right) \in \mathcal{S}'$ is a search path for the problem $\Gamma'(H)$. Here H_1 is the bottom line and H_2 is the circle on top. The connection point e (not labelled) is the center of the line.

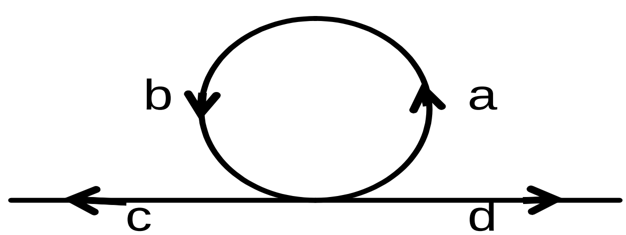

Figure 3: Network H with arcs directed

To establish Theorem 4, we first establish a similar result for the problem Γ', which in the example of Figure 3 says that if P is optimal so is the strategy

$P^* = ba^{-1}cc^{-1}d$ which puts the second search of H_2 (that is, a^{-1}) immediately after the first search (b), and hence searches H_2 in a single go.

Lemma 5 *Let H and h be as in Theorem 4. There exists an optimal search path S' of H in Γ' which searches H_2 exhaustively during a single time interval of length $\lambda(H_2)$ at some point of the search.*

Once we have obtained this Lemma, it is an easy matter to replace the discontinuous path (say P^* in the example) by one which searches H_2 in a (continuous) Eulerian path ($P^{**} = abcc^{-1}d$).

Proof of Theorem 4. Let S' be the path given by Lemma 5, with $T(S', h) = V' \leq V$. Define S'' to be the same as S' except that in the intervals where S' is searching H_2, S'' follows an Eulerian circuit of H_2 starting at e. This is possible because an Eulerian circuit of H_2 has length $\lambda(H_2)$ by definition. Since h is uniform on H_2, the order in which it is searched (assuming no overlap) is irrelevant to the search time, so we have $T(S'', h) = T(S', h) = V' \leq V$. But $S'' \in \mathcal{S}$ (is a continuous path) so $V' = V$ and S'' satisfies the requirements of the Theorem. ■

We now complete this section with a proof of Lemma 5.

Proof of Lemma 5. Let P be an optimal strategy for H in the problem Γ' with disjoint equal length time-intervals $I_1 = [t_1, t_1 + L]$ (earlier) and $I_2 = [t_2, t_2 + L]$ (later) in which H_2 is searched, such that some points of $H_1 - e$ are searched in between. Let $d = t_2 - (t_1 + L)$ denote the distance between I_1 and I_2. Let $I_\alpha, I_\beta, I_\gamma$ denote the time intervals before, between, and after I_1 and I_2. Thus we may write P as a concatenation $P = P_\alpha P_1 P_\beta P_2 P_\gamma$, where P_i denotes the restriction of P to the interval I_i. Define H^α, H^1, etc. to be those points of H found by P in the corresponding time intervals. Define $P^+ = P_\alpha P_1 P_2 P_\beta P_\gamma$ and $P^- = P_\alpha P_\beta P_1 P_2 P_\gamma$. Observe that all points of $H^\alpha \cup H^\gamma$ are reached in the same time by strategies P, P^+, and P^-. Compared with P, points in H^β are reached in time L earlier under P^- and time L later by P^+; points in $H^1 \cup H^2$ are met on average at time $d/2$ later under P^- (those in H^1 are met d later), and on average at time $d/2$ earlier under P^+. Hence the expected capture time for the equiprobable mixture of P^+ and P^- is the same as that for P. Hence if P is optimal, all three paths P, P^- and P^+ must have the same (optimal) expected capture time, as neither P^+ nor P^- can have a smaller time.

Next suppose that the time intervals I_1 and I_2 (with lengths L_i) are maximal (path enters H_1 immediately before and after them) and do not have necessarily have the same length, say I_1 is longer. Let I_1' denote the right hand part of I_1 of the same length L_2 as I_2. Then applying the above construction to the intervals I_1' and I_2 we see that the optimal strategy P^+ has (compared with P) an extra maximal interval of length $L_1 + L_2$ and one less maximal interval of length L_2.

Now assume that the Lemma is false, and let $k < \lambda(H_2)$ denote the supremum of the length of individual time intervals spent entirely in H_2 among all optimal paths in $\mathcal{S}'$. Starting with any optimal path $P \in \mathcal{S}'$ we may successively combine (as above) maximal intervals of search in H_2 until, after a finite number of iterations, we obtain an optimal path in $\mathcal{S}'$ which searches in H_2 on some time interval of length greater than k. Since this contradicts the definition of k, we are done. ■

6 Proof of Theorem 2

We are now in a position to prove Theorem 2, which gives a simple condition for the simple searchability of a network.

The following result is implicit in the Dagan-Gal proof that a tree is simply searchable.

Lemma 6 *Let G be any network. If h is an optimal hider mixed strategy in $\Gamma(G,y)$ and S is a pure search strategy with $S(0)=x$, then $T(S,h)\geq V_y - d(x,y)$. Consequently for all $x,y\in G$ we have $|V_x - V_y|\leq d(x,y)$.*

Proof. Let S' denote the pure search strategy with $S'(0)=y$, which begins by going directly to x and then follows S. Since S' reaches any point of G at most time $d(x,y)$ later than S does, we have

$$\begin{aligned} V_y &\leq T(S',h)\leq d(x,y)+T(S,h)\,, \text{ or} \\ T(S,h) &\geq V_y - d(x,y)\,. \end{aligned}$$

Since this is true for all S with $S(0)=x$, we have

$$V_x \geq V_y - d(x,y)\,, \text{ and so by symmetry } |V_x - V_y|\leq d(x,y)\,.$$

■

Lemma 7 *If G is Weakly Eulerian, then*

$$V(G)\geq \bar{\mu}/2-\rho(B)\,, \tag{4}$$

where $B=E'(G)$ is the Eulerian deletion of G.

Proof. Let c be a center of B and let $\bar{h}_c$ be the Hider strategy of Theorem 1, which is uniform on each Eulerian component E_i of G. Since G is Weakly Eulerian, Theorem 1 implies that

$$V_c(G)=\bar{\mu}/2.$$

By the last sentence of Theorem 4, there is a pure search strategy S starting at some point $x=S(0)\in B$ which is an optimal reply to $\bar{h}_c$. Since $\bar{h}_c$ is also a valid hiding strategy for the arbitrary start game $\Gamma(G)$, we have by Lemma 6 with $y=c$, that

$$V(G)\geq T\left(S,\bar{h}_c\right)\geq V_c(G)-d(x,c)\geq \bar{\mu}/2-\rho(B)\,, \tag{5}$$

since $x\in B$ and c is a center of B. ■

Lemma 8 *For any Weakly Eulerian network G, we have*

$$\bar{\mu}(G) - 2\ \rho(E'(G)) \leq 2\ V(G) \leq \tilde{\mu}(G). \tag{6}$$

Proof. The left hand inequality is the same as (5), while the right hand inequality is simply (2). ■

The rest is now easy.

Proof of Theorem 2. If $\Delta = \bar{\mu}(G) - \tilde{\mu}(G) \geq 2\ \rho(E'(G))$ then

$$\bar{\mu}(G) - 2\ \rho(E'(G)) \geq \tilde{\mu}(G)$$

so the left and right (and hence the middle) terms in (6) are equal. So $V(G) = \tilde{\mu}(G)/2$ and hence G is simply searchable (by definition), as claimed. Also, by (5), we have for some (and hence every) optimal reply S to $\bar{h}_c$, that

$$T\left(S, \bar{h}_c\right) \geq \bar{\mu}(G)/2 - \ \rho(E'(G)) \geq \tilde{\mu}(G)/2 = V(G),$$

and so $\bar{h}_c$ is an optimal Hider strategy. ■

7 Easily Hidable Networks

In the previous sections, we discussed networks G which were simple to search, that is, where an optimal search strategy in $\Gamma(G)$ consisted of a random CP path. In this section, we consider networks G which are easy to hide in, because an optimal hiding strategy is simply the uniform distribution λ.

A Chinese Postman path is one in which the Postman finishes his work as soon as possible. It is good for him, but not necessarily for the customers to whom he delivers. A Utilitarian Postman wants to deliver to his customers as early as possible, on average. That is, a *Utilitarian Postman (UP)* path $\hat{S}$ is one in which the average delivery time is minimized, and this minimum is denoted by $\hat{\mu}$,

$$\hat{\mu} = T\left(\hat{S}, \lambda\right) = \int_G T\left(\hat{S}, y\right)\ d\lambda(y) = \min_S \int_G T(S, y)\ d\lambda(y).$$

In other words, a UP path is any path which is optimal against λ. Since search paths have unit speed, clearly $\hat{\mu} \geq \mu/2$.

To illustrate the distinction between UP and CP paths, consider again the network drawn in Figure 1. In each time interval $J_i = [i-1, i]$, $i = 1, \ldots, 12$, the paths P_a and P_c either search a new arc (1) or retrace an arc which has already been searched (0), as indicated in the following table

$$\begin{array}{ll} P_a & 110111001101 \\ P_c & 111101001101 \end{array} \tag{7}$$

If the hider distribution is uniform and he is found in time interval J_i, the expected capture time is $i - 1/2$. Hence the expected capture time is

$$T = \frac{1}{8} \sum_{x_i = 1} (i - 1/2). \tag{8}$$

The only difference in the sequences is that the 0 at position 3 in P_a has moved to position five in P_c. So clearly against a uniform hiding strategy P_c has a smaller expected meeting time:

$$T(P_a, \lambda) - T(P_c, \lambda) = \frac{1}{8}((5 - 1/2) - (3 - 1/2)) = \frac{1}{4}.$$

So the only CP path which can be (it is) a UP path is P_c (the reverse path of P_c has a larger T). Now suppose the network is modified so that the arc directly below c has length $1 - \varepsilon$ instead of 1. For ε sufficiently small, $T(P_c, \lambda)$ will still be less than $T(P_a, \lambda)$, but the only CP path will be P_a. (For the modified network, P_a has length $12 - 2\varepsilon$, whereas P_c has length $12 - \varepsilon$.) Thus none of the CP paths will be a UP path.

Any mixture $\hat{s}$ of UP paths is called a *Random Utilitarian Postman (RUP)* strategy, and clearly for such Searcher strategies we also have

$$\hat{\mu} = T(\hat{s}, \lambda). \tag{9}$$

Since the uniform hiding strategy $h = \lambda$ is always available, we clearly have for all networks G that

$$V(G) \geq \hat{\mu}(G). \tag{10}$$

If equality holds in (10) we say that G is *easily hideable*. This is equivalent to saying that the uniform strategy λ is optimal for the hider. It is clear that if G has an Eulerian path it is easily hideable, as $V(G) = \mu/2 = \hat{\mu}$.

An immediate consequence of the definition of easily hideable is the following.

Lemma 9 *If there is a mixed search strategy s on G such that*

$$\max_{y \in G} T(s, y) = \hat{\mu}, \tag{11}$$

then G is easily hideable.

Proof. The condition (11) implies that $V \leq \hat{\mu}$, so the result follows from (10). ■

A useful method for showing that a network is easily hideable is the following easy result.

Lemma 10 *Let $G = G_1 \cup G_2$ be a network with $\lambda(G_1 \cap G_2) = 0$. (In applications, G_1 and G_2 will intersect only at nodes of G.) Let s_1 and s_2 denote RUP Searcher strategies on G such that for some constants $t_{i,j}$, $i, j = 1, 2$, we have $T(s_i, y) = t_{i,j}$ for all $y \in G_j$. If*

$$t_{i,i} \leq \hat{\mu} \quad \textit{for } i = 1, 2 \tag{12}$$

then G is easily hideable.

Proof. Let $\mu_i = \lambda(G_i)/\lambda(G)$, $i = 1,2$. Since both s_i are RUP strategies, we have

$$T(s_i, \lambda) = \mu_1 t_{i,1} + \mu_2 t_{i,2} = \hat{\mu},\ i = 1,2.$$

It follows that the three points $(t_{1,1}, t_{1,2})$, $(t_{2,1}, t_{2,2})$ and $(\hat{\mu}, \hat{\mu})$ are all on the line $\mu_1 x + \mu_2 y = \hat{\mu}$, and by (12) that $(\hat{\mu}, \hat{\mu})$ is between the other two, and hence can be written as

$$(\hat{\mu}, \hat{\mu}) = p(t_{1,1}, t_{1,2}) + (1-p)(t_{2,1}, t_{2,2}). \tag{13}$$

Consequently the RUP strategy $s = ps_1 + (1-p)s_2$ satisfies

$$T(s, y) = pt_{1,j} + (1-p)t_{2,j} = \hat{\mu},\ \text{for } y \in G_j,\ j = 1,2. \tag{14}$$

Since $G_1 \cup G_2 = G$, the result follows from the previous Lemma. ■

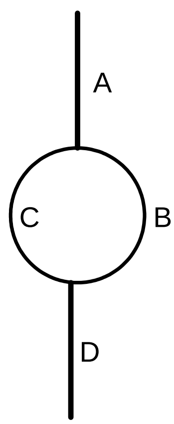

Figure 4: An easily hidable network

Example 11 *Consider the network G drawn in Figure 4, consisting of four arcs (A, B, C, D) each of length 2. Note that if we draw it in the plane with the origin at the center of the circle BC, it has both horizontal and vertical symmetry. For any pure strategy S, let $[S]$ denote the mixed strategy given by an equiprobable average of the four symmetrical versions of S (optimality properties of symmetric strategies of this type are discussed in Alpern and Asic [3]). Let $G_1 = A \cup D$ and $G_2 = C \cup B$. We have $\hat{\mu} = 9/2$, given by two UP paths, both starting at the top, as $S_1 = A, B, D, D, C$ and $S_2 = A, B, C, C, D$. The path S_1 reaches the midpoints of A, B, D, C at times 1,3,5,9 and S_2 reaches the midpoints of A, B, C, D at these times. So both achieve the minimum average time of $(1+3+5+9)/4 = 9/2$. Taking $s_i = [S_i]$, the times $t_{i,j}$ in the above Lemma are given by the matrix*

$$\{t_{i,j}\} = \begin{pmatrix} \frac{1+5}{2} & \frac{3+9}{2} \\ \frac{1+9}{2} & \frac{3+5}{2} \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 5 & 4 \end{pmatrix}.$$

Since both $t_{1,1} = 3$ and $t_{2,2} = 4$ are less than $9/2 = 4.5$, condition (12) is satisfied, and the network G is easily hideable, with $V(G) = \hat{\mu} = 9/2$.

We have already observed that if G has a Eulerian path it is both simply searchable and easily hideable. The converse is also true, and will in fact be useful in the following section.

Lemma 12 *If a network G is both simply searchable and easily hideable it has a Eulerian path. Consequently any simply searchable network with more than two nodes of odd degree is not easily hideable. In particular, any partly Eulerian network with more than two nodes of odd degree, or any tree with more than two terminal nodes, is not easily hideable.*

Proof. Suppose G is simply searchable and has no Eulerian path. To prove the first sentence, we have to show that the uniform hider strategy λ is not optimal. Let S be a CP path. Its length is $\tilde{\mu} > \mu$ by assumption. Hence some arc A of G is traversed twice, during distinct time intervals. For any point y in the interior of A, we have $S(t_1) = y = S(t_2)$, $t_1 < t_2$. Hence $T(S, y) \leq t_1$ and $T(S', y) \leq \tilde{\mu} - t_2$, where S' is the reverse path to S. If s denotes the random CP strategy consisting equiprobably of S and S', then we have for $y \in A$, $T(s, y) \leq \frac{1}{2}(t_1) + \frac{1}{2}(\tilde{\mu} - t_2) < \frac{\tilde{\mu}}{2} = V(G)$. On the other hand, $T(s, x) \leq \frac{\tilde{\mu}}{2}$ for all $x \in G$, so it follows that

$$T(s, \lambda) < \frac{\tilde{\mu}}{2} = V(G).$$

Hence λ is not optimal, and G is not easily hideable.

The second sentence follows from the fact that a network with an Eulerian path has 0 or 2 nodes of odd degree, and the third from Corollary 3 and the observation that terminal nodes of trees have degree 1 together with the Dagan-Gal result [9] that trees are simply searchable. ■

8 Simple Searchability is not a topological property

Gal's result (Theorem 1) gives a topological (that is, combinatorial) characterization of networks with starting points for which a random Chinese postman tour is an optimal search strategy, namely that the network is Weakly Eulerian. The definition of Weakly Eulerian depends only on the graph theoretic structure of the network, not on its arc-lengths, so is a topological invariant. We give a simple example which shows that no such topological characterization is possible for simple searchability. Consider the following family of networks $\hat{G}(r)$, $r \geq 0$, consisting of four edges A,B,C,D of fixed length 2, and (for $r > 0$) two additional edges of varying length r. Observe that these are all Weakly Eulerian.

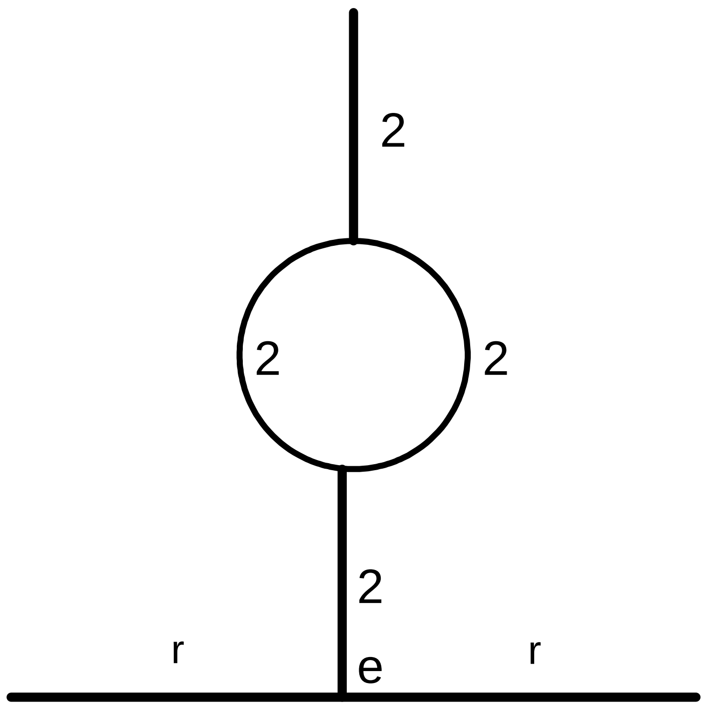

Figure 5: The family of networks $\hat{G}_r$

For $r > 0$, the networks $\hat{G}_r$ are topologically the same. However we shall show that for r sufficiently large these are simply searchable, while for r sufficiently small they are not.

First consider the network $\hat{G}_0$. This is the network of Figure 4, which was shown to be easily hideable. This network has four nodes of odd degree, and consequently it has no Eulerian path. By Lemma 12, this implies it is not simply searchable, that $V\left(\hat{G}_0\right) < \tilde{\mu}/2 - \varepsilon = 5 - \varepsilon$, for some $\varepsilon > 0$. (In fact, we showed that $V\left(\hat{G}_0\right) = 9/2$, but we will not need this exact value for our argument.) Now it is easy to see that for any $r > 0$,

$$V\left(\hat{G}_r\right) \leq V\left(\hat{G}_0\right) + 4r. \tag{15}$$

Simply take any optimal mixed searcher strategy in $\hat{G}_0$ and modify it by taking time $4r$ to search the additional rays the first time the node e is reached. Then no point of $\hat{G}_r$ will be reached more than time $4r$ later than its corresponding point in $\hat{G}_0$ is reached under the optimal strategy for $\hat{G}_0$ (points in $\hat{G}_0$ correspond to themselves; those in the new rays correspond to e). Thus (15) holds, and for r

sufficiently small ($r < \varepsilon/4$) we have $V\left(\hat{G}_r\right) < 5$. However $\tilde{\mu}\left(\hat{G}_r\right) > \tilde{\mu}\left(\hat{G}_0\right) = 10$, so for sufficiently small r we have

$$V\left(\hat{G}_r\right) < \tilde{\mu}\left(\hat{G}_r\right)/2$$

and hence the network $\hat{G}_r$ is not simply searchable.

For larger r write G_r as a network of the type in Corollary 3, condition 3, taking A to be the bottom line of length $2r$. We have $\rho(A) = r$, and since for any $x \in G_r$ we have $d(e, x) \leq \max[3, r]$ it follows that

$$r = \rho(A) \leq \rho(G_r) \leq \max[3, r]. \tag{16}$$

So for $r \geq 3$ we have $\rho(A) = \rho(G_r)$, which by condition 3 of Corollary 3 ensures that G_r is simply searchable. Taking together the results for r small and r large, we have proved the following.

Theorem 13 *The simple searchability of a network is not a topological property.*

9 Conclusions

If an object is known to be hidden in a known network, how should one look for it, if constrained to follow a path in the network. This problem has been extensively studied for the case when the search path must start at a given location. The present paper considers how the Searcher can improve the search if the initial location constraint is removed. This line of research was initiated by Dagan [8] and Dagan and Gal [9], who solved the problem for trees. Here, we extended their work (and that of Alpern [2]) to more general networks. An important concept introduced for this purpose is that of the *Eulerian deletion* of a network, as well as the comparison of the time taken to search a network *and return to the starting point* versus the time taken to search the network. These concepts, when combined, gave a sufficient condition (Theorem 2) for the optimal search strategy to be simply an equiprobable mixture of a minimum time (Chinese Postman) covering path of the network, and its reverse path. We also gave a sufficient conditions (Lemmas 9 and 10) for the uniform hiding strategy to be optimal for the Hider, that is, to be the worst case scenario for the search problem.

It is interesting to note that while the fixed-start and arbitrary-start search problems are very different in character, the results presented here for the later version use in a crucial way the important theorem of Gal (our Theorem 1) for the former version.

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