An operator corona theorem for a class of subspaces of H^∞

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Abstract: Let E, E_* be separable Hilbert spaces. If S is an open subset of $\mathbb{T}$, then $A_S(\mathscr{L}(E, E_*))$ denotes the space of all functions $f : \mathbb{D} \cup S \to \mathscr{L}(E, E_*)$ that are holomorphic in $\mathbb{D}$, and bounded and continuous on $\mathbb{D} \cup S$. In this article we prove the following main results:

1. A theorem concerning the approximation of $f \in A_S(\mathscr{L}(E, E_*))$ by a function F that is holomorphic in a neighbourhood of $\mathbb{D} \cup S$ and such that the error $F - f$ is uniformly bounded in the disk $\mathbb{D}$.

2. The corona theorem for $A_S(\mathscr{L}(E, E_*))$ when $\dim(E) < \infty$: If there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $f(z)^* f(z) \geq \delta^2 I$, then there exists a $g \in A_S(\mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D} \cup S$, $g(z)f(z) = I$.

3. The problem of complementing to an isomorphism for $A_S(\mathscr{L}(E, E_*))$ when $\dim(E) < \infty$ (Tolokonnikov's lemma): $f \in A_S(\mathscr{L}(E, E_*))$ has a left inverse $g \in A_S(\mathscr{L}(E_*, E))$ iff it is a 'part' of an invertible element F in $A_S(\mathscr{L}(E_*))$.

4. A corona theorem for $A(\mathscr{L}(E, E_*))$ when $\dim(E) = \infty$, and the corona data function f is a 'small' perturbation of a 'nice' function f_0.

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1 Notation and introduction

$\mathbb{D}$ denotes the open unit disk centered at 0 in the complex plane $\mathbb{C}$, that is, $\mathbb{D} = \{z \mid |z| < 1\}$, and $\mathbb{T}$ denotes the boundary of $\mathbb{D}$, that is, $\mathbb{T} = \{z \mid |z| = 1\}$. We also use the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let X be a Banach space. If Ω is a domain in $\mathbb{C}$, then by $H^\infty(\Omega, X)$ we mean the space of all bounded holomorphic functions in Ω, equipped with the supremum norm:

$$\|f\|_\infty = \sup_{z\in\Omega} \|f\|, \quad f \in H^\infty(\Omega, X).$$

If $\Omega = \mathbb{D}$, then we denote the space $H^\infty(\mathbb{D}, X)$ simply by $H^\infty(X)$. For preliminaries on vector- and operator-valued holomorphic functions, we refer the reader to Dieudonné [3] (Chapter IX), or Nikolski [7] (§3.11).

Definition. Let X be a Banach space. If $S \subset \mathbb{T}$, then $A_S(X)$ denotes the set of functions $f : \mathbb{D}\cup S \to X$ that are holomorphic in $\mathbb{D}$, and continuous and bounded on $\mathbb{D}\cup S$. The space $A_S(X)$ is equipped with the supremum norm defined by $\|f\|_\infty = \sup_{z\in\mathbb{D}\cup S} \|f(z)\|$, $f \in A_S(X)$.

The motivation for using 'A' in the notation above is that the symbol A is used to denote the disk algebra ($S = \mathbb{T}$ and $X = \mathbb{C}$). If $S = \emptyset$, then we get the other extreme $H^\infty(X)$.

Theorem 1.1 *Let X be a Banach space and $S \subset \mathbb{T}$. Then $A_S(X)$ is a Banach space.*

Proof The completeness can be shown as follows. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then for each $z \in \mathbb{D} \cup S$, the sequence $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and so by the completeness of X, it has a limit, say $f(z)$. These pointwise limits give rise to a X-valued function f defined on $\mathbb{D} \cup S$. We claim that f belongs to $A_S(X)$. f is the uniform limit of the f_n's on $\mathbb{D}\cup S$. In particular, in each compact subset of $\mathbb{D}$, the sequence $(f_n)_{n\in\mathbb{N}}$ of holomorphic functions converges uniformly to f, and so f is holomorphic in $\mathbb{D}$ (see Theorem 9.12.1 on page 229 of Dieudonné [3]). Continuity and boundedness on $\mathbb{D}\cup S$ follows from the fact that the convergence is uniform. ∎

Note that if X is a Banach algebra (for instance, if $X = \mathscr{L}(E)$, where E is a Hilbert space), then $A_S(X)$, with pointwise multiplication, is also a Banach algebra.

In this article, we mostly consider the case $X = \mathscr{L}(E, E_*)$, where E, E_* are separable Hilbert spaces. The space $\mathscr{L}(E, E_*)$ is equipped with the uniform topology induced by the operator norm. These function classes $A_S(\mathscr{L}(E, E_*))$ arise in control theory as they are natural choices for the spaces of transfer functions of infinite-dimensional systems that are not exponentially stable [10].

In this paper we will prove an operator corona theorem and Tolokonnikov's lemma for the space $A_S(\mathscr{L}(E, E_*))$. In order to do this, we will use the corresponding theorems for H^∞ and a certain approximation result, which we prove first in §2. Subsequently we prove the operator corona theorem (§3) and Tolokonnikov's lemma (§4) for the space $A_S(\mathscr{L}(E,E_*))$, when $\dim(E)<\infty$ and S is an open subset of $\mathbb{T}$. Finally in the last section §5, we prove an operator corona theorem for the space $A(\mathscr{L}(E,E_*))$ (that is, $S=\mathbb{T}$), when $\dim(E)=\infty$, under some additional assumptions on the corona data function f.

2 An approximation result

In order to prove the corona theorem and Tolokonnikov's lemma for our class A_S, we will use the H^∞ versions of these theorems together with a key approximation result (Corollary 2.3 below). This result is a consequence of the following lemma, which we prove following Gamelin [4] (§1 of Chapter II) and Gamelin and Garnett [5].

Lemma 2.1 *Let Ω be a bounded domain with a smooth boundary Γ, containing zero, and such that $\Omega\subset r\Omega$ for all r sufficiently close to 1. Suppose that C is a closed subarc in Γ, and let I be a neighbourhood of C in Γ.*

Let X be a Banach space, and suppose that $f:\Omega\to X$ is bounded and holomorphic in Ω, and that f extends continuously to I.

Then given any $\epsilon>0$, there exists a neighbourhood O of C in $\mathbb{C}$ and a holomorphic function $F:\Omega\cup O\to X$ such that for all $z\in\Omega$, $\|F(z)-f(z)\|<\epsilon$.

Proof We extend f across I to the open sector V as shown in Figure 1. f is constant along rays (joining 0 to a point z_0 in I), in the region across I, with the value along the ray being the one at the corresponding boundary point (z_0) on I. The extension is again denoted by f.

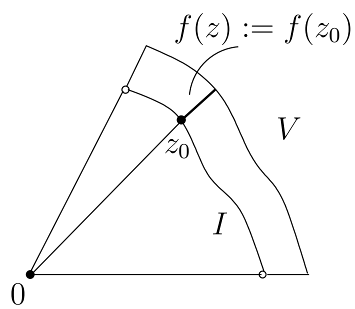

Figure 1: Continuous extension of f across I.

Let $\varphi\in\mathscr{D}(\mathbb{R}^2)$ be a test function such that $0\le\varphi\le1$, $\varphi=1$ on a neighbourhood U of C (in $\mathbb{C}$) and 0 off a slightly larger neighbourhood W; see Figure 2.

Define $h:\Omega\cup V\to X$ by

$$\begin{aligned} h(\zeta) &= \frac{1}{\pi}\iint_{\Omega\cup V}\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)-f(\zeta)}{z-\zeta}dxdy \\ &= \frac{1}{\pi}\iint_{\Omega\cup V}\frac{1}{z-\zeta}\frac{\partial\varphi}{\partial\overline{z}}f(z)dxdy-\frac{1}{\pi}\iint_{\Omega\cup V}\frac{\partial\varphi}{\partial\overline{z}}\frac{1}{z-\zeta}dxdyf(\zeta),\quad \zeta\in\Omega\cup V.\end{aligned}$$

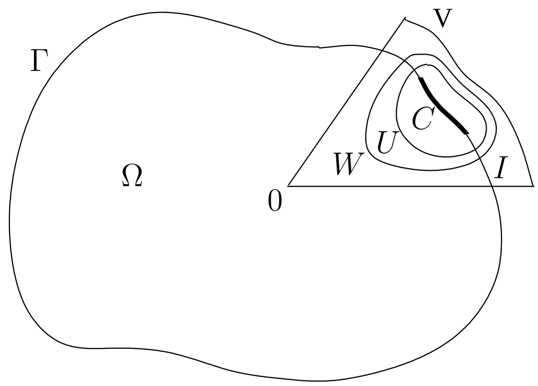

Figure 2: Support of the cut-off function φ is contained in W.

Claim: h is well-defined and continuous.

Observe that the convolution of the locally integrable function $\frac{1}{z}$ with a bounded function with compact support is well-defined and continuous. As the functions $z \mapsto \frac{\partial\varphi}{\partial\bar{z}} f(z)$ and $z \mapsto \frac{\partial\varphi}{\partial\bar{z}}$ are both bounded and have compact support, it follows that

$$\zeta \mapsto \frac{1}{\pi}\iint_{\Omega\cup V} \frac{1}{z-\zeta}\frac{\partial\varphi}{\partial\bar{z}} f(z)dxdy \quad \text{and} \quad \frac{1}{\pi}\iint_{\Omega\cup V} \frac{\partial\varphi}{\partial\bar{z}}\frac{1}{z-\zeta}dxdy$$

are continuous on $\Omega\cup V$. Finally as $\zeta \mapsto f(\zeta)$ is also continuous, it follows that h is continuous.

Claim: h is holomorphic in Ω.

For all ζ, $\zeta+t$ in Ω with $t \neq 0$, we have

$$\begin{aligned}
&\frac{h(\zeta+t)-h(\zeta)}{t} \\
= \; & \frac{1}{\pi t}\iint \frac{\partial\varphi}{\partial\bar{z}}\left[\frac{f(z)-f(\zeta+t)}{z-(\zeta+t)} - \frac{f(z)-f(\zeta)}{z-\zeta}\right]dxdy \\
= \; & \frac{1}{\pi}\iint \frac{\partial\varphi}{\partial\bar{z}}\frac{f(z)-f(\zeta)}{(z-\zeta)(z-(\zeta+t))}dxdy - \frac{1}{\pi}\iint \frac{\partial\varphi}{\partial\bar{z}}\frac{1}{z-\zeta-t}dxdy\frac{f(\zeta+t)-f(\zeta)}{t}.
\end{aligned}$$

As f is holomorphic in Ω, it follows that $\lim_{t\to 0}\frac{f(\zeta+t)-f(\zeta)}{t}$ exists. Since the convolution integrals vary continuously with t, we deduce that h is holomorphic in Ω, as explained below:

1. Indeed first of all the map $\zeta \mapsto \iint \frac{\partial\varphi}{\partial\bar{z}}\frac{1}{z-\zeta}dxdy$ is continuous in $\Omega\cup V$.

2. Let $\zeta\in\Omega$. Then the function $z \mapsto \frac{f(z)-f(\zeta)}{z-\zeta}$ is bounded on $\Omega\cup V$, which can be

seen as follows: for z in Ω close to ζ, we know $\lim_{z\to\zeta}\dfrac{f(z)-f(\zeta)}{z-\zeta}$ exists, and if z is away from ζ, then as f and $\dfrac{1}{\cdot-\zeta}$ are both bounded, we are done. The map

$$t \mapsto \iint \frac{1}{z-\zeta-t}\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)-f(\zeta)}{z-\zeta}dxdy$$

is the convolution of the locally integrable map $z \mapsto -\dfrac{1}{\zeta+z}$ and the compactly supported bounded function $z \mapsto \dfrac{\partial\varphi}{\partial\overline{z}}\dfrac{f(z)-f(\zeta)}{z-\zeta}$, and so it is continuous (and in particular at 0).

Claim: h is holomorphic in U (or on a neighbourhood of C in $\mathbb{C}$ where $\varphi=1$).

We recall Green's formula, which says that if D is a domain with a smooth boundary γ and g is a continuously differentiable on $D\cup\gamma$, then

$$g(\zeta)=\frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-\zeta}dz-\frac{1}{\pi}\iint_D\frac{1}{z-\zeta}\frac{\partial g}{\partial\overline{z}}dxdy, \quad \zeta\in D.$$

We will apply this Green's formula below to simplify the second integral in (1) below, with the g above replaced by φ. Let γ be a smooth curve in $\Omega\cup V$ that contains W (or the support of φ) in its interior. We have

$$\begin{aligned}
h(\zeta) &= \frac{1}{\pi}\iint\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)-f(\zeta)}{z-\zeta}dxdy \\
&= \frac{1}{\pi}\iint\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)}{z-\zeta}dxdy-\frac{1}{\pi}\iint\frac{\partial\varphi}{\partial\overline{z}}\frac{1}{z-\zeta}dxdyf(\zeta) && (1)\\
&= \frac{1}{\pi}\iint\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)}{z-\zeta}dxdy+\varphi(\zeta)f(\zeta), && (2)
\end{aligned}$$

since $\varphi=0$ on γ. As $\varphi=1$ on U, if $\zeta\in U$, we have

$$(h-f)(\zeta)=\frac{1}{\pi}\iint\frac{\partial\varphi}{\partial\overline{z}}\frac{f(z)}{z-\zeta}dxdy,$$

and so differentiating under the integral sign (note that as $\zeta\in U$, it follows that for all z close enough to U, $\dfrac{\partial\varphi}{\partial\overline{z}}=0$ since $\varphi=1$ in U), we obtain that $\dfrac{\partial(h-f)}{\partial\overline{\zeta}}=0$. Consequently $h-f$ is holomorphic in U.

Finally we are ready to construct F with the properties stated in the lemma. If $0<r<1$, then

$$h_r(z):=h(rz), \quad z\in\frac{1}{r}\Omega=:\Omega_r$$

is holomorphic on Ω_r. Choose r_0 close enough to 1 such that

$$\|h_{r_0}|_{\Omega} - h\|_\infty < \epsilon. \tag{3}$$

That this is possible can be seen as follows: First of all note that (2) is valid for all $\zeta \in \Omega \cup V$. Then

1. Observe that the map h_1 given by

$$\zeta \mapsto \frac{1}{\pi}\iint_{\Omega\cup V} \frac{\partial \varphi}{\partial \overline{z}} \frac{f(z)}{z-\zeta} dxdy = \frac{1}{\pi}\iint_{\mathbb{R}^2} \frac{\partial \varphi}{\partial \overline{z}} \frac{f(z)}{z-\zeta} dxdy$$

   is continuous on $\mathbb{C}$ (since it is the convolution of a function with compact support and the locally integrable function $\frac{1}{z}$). In particular, it is continuous on the compact set $\overline{\Omega}$, and hence uniformly continuous there. Hence we can choose a $r_1 \in (0,1)$ close enough to 1 such that

$$\sup_{z\in\Omega} \|h_1(z) - h_1(r_1 z)\| < \frac{\epsilon}{2}.$$

2. Note that the map h_2 given by $\zeta \mapsto \varphi(\zeta) f(\zeta)$ can be extended continuously to $\mathbb{C}$ since φ has compact support contained in $\Omega \cup V$ where f is continuous. Thus we can choose a $r_2 \in (0,1)$ close enough to 1 such that

$$\sup_{z\in\Omega} \|h_2(z) - h_2(r_2 z)\| < \frac{\epsilon}{2}.$$

By letting $r_0 = \max\{r_1, r_2\}$, we obtain (3).

Define $F = f - h + h_{r_0}$ on $(\Omega \cup V) \cap \Omega_{r_0}$. Then $\|F - f\|_\infty = \|h_{r_0} - h\|_\infty < \epsilon$. Moreover, F is holomorphic in $(\Omega \cup U) \cap \Omega_{r_0} = \Omega \cup (U \cap \Omega_{r_0})$. Indeed this is because f, h, h_{r_0} are all holomorphic in Ω, $f - h$ is holomorphic in U, and h_{r_0} is holomorphic in Ω_{r_0}. ■

Using the result above, we now prove our main result of this section, concerning uniform holomorphic approximation of functions in A_S.

Theorem 2.2 *Let X be a Banach space, S an open subset of $\mathbb{T}$, and $f \in A_S(X)$. Then given any $\epsilon > 0$, there exists a neighbourhood O of S in $\mathbb{C}$ and a holomorphic function $F : \mathbb{D} \cup O \to X$ such that for all $z \in \mathbb{D}$, $\|F(z) - f(z)\| < \epsilon$.*

Proof Let I_n, $n \in \mathbb{N}$, be pairwise disjoint open intervals such that

$$S = \bigcup_{n=1}^{\infty} I_n.$$

Each I_n can be written as a union of closed intervals as follows:

$$I_n = \left(\bigcup_{m=1}^{\infty} Q_{nm}\right) \bigcup \left(\bigcup_{m=1}^{\infty} \widetilde{Q}_{nm}\right),$$

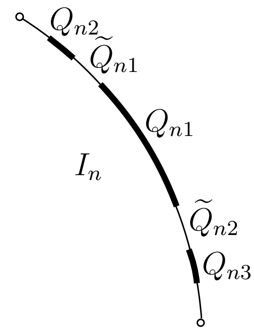

Figure 3: The interlaced closed intervals.

where Q_{n1}, Q_{n2}, Q_{n3}, ... are pairwise disjoint closed intervals, $\widetilde{Q}_{n1}$, $\widetilde{Q}_{n2}$, $\widetilde{Q}_{n3}$, ... are pairwise disjoint closed intervals, each $\widetilde{Q}_{nk}$ joins the endpoints of two of the Q_{nl}'s, and each Q_{nk} joins the endpoints of two of the $\widetilde{Q}_{nl}$'s; see Figure 3.

We can renumber these sets so that

$$S = \left(\bigcup_{n=1}^{\infty} Q_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} \widetilde{Q}_n\right),$$

where Q_1, Q_2, Q_3, ... are pairwise disjoint closed intervals, $\widetilde{Q}_1$, $\widetilde{Q}_2$, $\widetilde{Q}_3$, ... are pairwise disjoint closed intervals, each $\widetilde{Q}_n$ joins the endpoints of two of the Q_k's, and each Q_n joins the endpoints of two of the $\tilde{Q}_k$'s.

STEP 1. We construct open sets O_n (in $\mathbb{C}$) and functions φ_n with the following properties:

1. O_n is an open bounded neighbourhood of Q_n, $\overline{O}_n \cap \overline{O}_m = \emptyset$ if $n \neq m$, $\Omega_n := \Omega_{n-1} \cup O_n$ has a smooth boundary, and for r sufficiently close to 1, $\Omega_n \subset r\Omega_n$.

2. $\varphi_n : \Omega_n \to X$ is holomorphic and bounded in Ω_n, φ_n extends continuously to S and the boundary of O_n, and $\|\varphi_n|_{\Omega_{n-1}} - \varphi_{n-1}\|_\infty < \dfrac{\epsilon}{2^{n+1}}$.

We do this construction inductively as follows. Let $O_0 := \emptyset$, $\Omega_0 = O_0 \cup \mathbb{D}$, and let $\varphi_0 : \Omega_0 \to X$ be f. Assuming that we have already constructed $O_0, \dots, O_{k-1}$ and $\varphi_0, \dots \varphi_{k-1}$, the existence of O_k and φ_k follows from Lemma 2.1 above, applied to φ_{k-1} and the closed subarc $C = Q_k$ (by suitably shrinking O from the Lemma 2.1).

We observe that

$$\sum_{k=1}^{\infty} (\varphi_k - \varphi_{k-1}) + f$$

converges uniformly on compact subsets of $\mathbb{D}$ to a function Φ which is bounded and holomorphic in $\mathbb{D}$. Also for each n,

$$\Phi = \sum_{k=1}^{\infty} (\varphi_k - \varphi_{k-1}) + f = \sum_{k=n+1}^{\infty} (\varphi_k - \varphi_{k-1}) + \varphi_n,$$

and so φ extends analytically to each Ω_n. Finally, we also observe that

$$\|\Phi|_{\mathbb{D}} - f\|_\infty < \frac{\epsilon}{2}.$$

STEP 2. Let

$$\Omega = \mathbb{D} \bigcup \left(\bigcup_{n=1}^{\infty} O_n \right),$$

and consider $\Phi : \Omega \to X$. We can shrink the O_n's so that Φ is bounded on Ω and Ω has a smooth boundary Γ, and so that Φ has a continuous extension to a neighbourhood of $\bigcup_{n=1}^{\infty} \widetilde{Q}_n$. Repeating the argument in STEP 1 above with Φ instead of f, we can find a neighbourhood Ω_1 of $\Omega \cup \bigcup_{n=1}^{\infty} \widetilde{Q}_n$ and a $\Psi : \Omega_1 \to X$ which is holomorphic in Ω_1 and such that

$$\text{for all } z \in \Omega, \quad \|\Psi(z) - \Phi(z)\| < \frac{\epsilon}{2}.$$

Finally, we obtain that for all $x \in \mathbb{D}$, $\|\Psi(z) - f(z)\| < \epsilon$, which completes the proof. ■

The scalar version (case when $X = \mathbb{C}$) of the main result of this section (Theorem 2.2) was obtained first by Stray [11].

In order to prove the operator corona theorem and Tolokonnikov's lemma for A_S, we will use the following corollary of the Theorem 2.2.

Corollary 2.3 *Let E, E_* be Hilbert spaces, S an open subset of $\mathbb{T}$, and $f \in A_S(\mathscr{L}(E, E_*))$. Then given any $\epsilon > 0$, there exists a neighbourhood O of S in $\mathbb{C}$ and a holomorphic $F : \mathbb{D} \cup O \to \mathscr{L}(E, E_*)$ such that for all $z \in \mathbb{D}$, $\|F(z) - f(z)\| < \epsilon$.*

One can apply Theorem 2.2 to various subspaces of $\mathscr{L}(E, E_*)$ to get different versions of the approximation result. In particular, we will use the version with the Hilbert-Schmidt class in section 5 in order to prove the corona theorem for $A(\mathscr{L}(E, E_*))$.

3 An operator corona theorem

Throughout this section, we assume that E, E_* are separable Hilbert spaces and that $\dim(E) < \infty$.

The operator corona theorem for $H^\infty(\mathscr{L}(E, E_*))$ says that the existence of a left inverse for a function $f \in H^\infty(\mathscr{L}(E, E_*))$ is equivalent to the condition that

$$\forall z \in \mathbb{D}, \quad f(z)^* f(z) \geq \delta^2 I.$$

This was proved by Vasyunin, and the proof can be found in Tolokonnikov [13]. If E is not finite-dimensional, then this equivalence is not true, and this was shown in Treil [14]. We refer the reader to §9.2 of the book by Nikolski [7] for an account of these results.

In this section we will prove a similar equivalence for functions in $A_S(\mathscr{L}(E, E_*))$. In order to prove this (Theorem 3.2 below), we will use the approximation result from the previous section (Corollary 2.3) and the following corona theorem in the H^∞ case.

Proposition 3.1 (Vasyunin-Fuhrmann) *Let E, E_* be separable Hilbert spaces and $\dim(E) < \infty$. Suppose that Ω is a simply connected domain not equal to the plane $\mathbb{C}$, and $f \in H^\infty(\Omega, \mathscr{L}(E, E_*))$. If there exists a $\delta > 0$ such that for all $z \in \Omega$, $I \geq f(z)^* f(z) \geq \delta^2 I$, then there exists a $g \in H^\infty(\Omega, \mathscr{L}(E_*, E))$ such that*

$$\forall z \in \Omega, \;\; g(z)f(z) = I \textit{ and } \|g\|_\infty < \sqrt{\dim(E)}C(\delta^{\dim(E)}). \tag{4}$$

Proof For $\Omega = \mathbb{D}$, this is precisely the statement of the Vasyunin-Fuhrmann theorem (see §11 in Appendix on page 293 of Nikolski [6]). The general case can be seen as follows. By the Riemann mapping theorem, there exists a one-to-one holomorphic map φ from Ω onto $\mathbb{D}$. Thus $\varphi^{-1} : \mathbb{D} \to \Omega$ is also holomorphic. Hence $f_0 := f \circ \varphi^{-1} \in H^\infty(\mathscr{L}(E_*, E))$, and $I \geq f_0(z)^* f_0(z) \geq \delta^2 I$ for all $z \in \mathbb{D}$. From the Vasyunin-Fuhrmann theorem in the case of $\mathbb{D}$, it follows that there exists a $g_0 \in H^\infty(\mathscr{L}(E_*, E))$ such that

$$\forall z \in \mathbb{D}, \quad g_0(z)f_0(z) = I, \quad \text{and} \quad \sup_{z \in \mathbb{D}} \|g_0(z)\| \leq \sqrt{\dim(E)}C(\delta^{\dim(E)}). \tag{5}$$

Defining $g := g_0 \circ \varphi$, we see that $g \in H^\infty(\Omega, \mathscr{L}(E_*, E))$, and (4) then follows from (5). ∎

We are now ready to prove our new operator corona theorem for $A_S(\mathscr{L}(E, E_*))$.

Theorem 3.2 *Let E, E_* be separable Hilbert spaces and $\dim(E) < \infty$. Suppose that S is an open subset of $\mathbb{T}$, and that $f \in A_S(\mathscr{L}(E, E_*))$. Then the following are equivalent:*

1. *There exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $f(z)^* f(z) \geq \delta^2 I$.*
2. *There exists a $g \in A_S(\mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D} \cup S$, $g(z)f(z) = I$.*

Proof 2 $\Rightarrow$ 1. For $x \in E$ and $z \in \mathbb{D} \cup S$,

$$\|x\| = \|Ix\| = \|g(z)f(z)x\| \leq \|g(z)\|\|f(z)x\| \leq \|g\|_\infty \|f(z)x\|,$$

and so with $\delta := \dfrac{1}{\|g\|_\infty}$, we have

$$\langle f(z)^* f(z)x, x\rangle = \langle f(z)x, f(z)x\rangle = \|f(z)x\|^2 \geq \delta^2 \|x\|^2 = \langle \delta^2 Ix, x\rangle,$$

that is, $f(z)^* f(z) \geq \delta^2 I$.

1 $\Rightarrow$ 2. Given $f \in A_S(\mathscr{L}(E, E_*))$ and $\epsilon_1 > 0$, by Corollary 2.3 there exists a neighbourhood O of S and a $\mathscr{L}(E, E_*)$-valued holomorphic function f_e defined on $\Omega := \mathbb{D} \cup O$ such that $\|f_e|_{\mathbb{D}} - f\|_\infty < \epsilon_1$.

As f_e is continuous in O, given any $\epsilon_2 > 0$, we can shrink O suitably so as to ensure that for the new $\Omega = \mathbb{D} \cup O$, we have that for all $z \in \Omega \setminus \mathbb{D}$, there exists a $z_* \in S$ such that $\|f_e(z) - f_e(z_*)\| < \epsilon_2$.

For all $z \in \mathbb{D} \cup S$ and $x \in E$,

$$\begin{aligned}
\|f_e(z)x\| &= \|f(z)x + (f_e(z) - f(z))x\| \geq \|f(z)x\| - \|(f_e(z) - f(z))x\| \\
&\geq \|f(z)x\| - \|f_e(z) - f(z)\|\|x\| \geq \|f(z)x\| - \|f_e|_{\mathbb{D}} - f\|_\infty \|x\| \\
&> \delta\|x\| - \frac{\delta}{2}\|x\| \quad \text{(ensured by choosing } \epsilon_1 < \frac{\delta}{2}) \\
&= \frac{\delta}{2}\|x\| \qquad\qquad (6) \\
&> \frac{\delta}{4}\|x\|.
\end{aligned}$$

For all $z \in \Omega \setminus \mathbb{D}$ and $x \in E$,

$$\begin{aligned}
\|f_e(z)x\| &= \|f_e(z_*)x + (f_e(z) - f_e(z_*))x\| \geq \|f_e(z_*)x\| - \|(f_e(z_*) - f_e(z))x\| \\
&\geq \frac{\delta}{2}\|x\| - \|f_e(z_*) - f_e(z)\|\|x\| \quad \text{(using (6))} \\
&\geq \frac{\delta}{2}\|x\| - \frac{\delta}{4}\|x\| \quad \text{(ensured by choosing } \epsilon_2 < \frac{\delta}{4}) \\
&= \frac{\delta}{4}\|x\|.
\end{aligned}$$

Consequently for all $z \in \Omega$, and all $x \in E$, $\|f_e(z)x\| \geq \frac{\delta}{4}\|x\|$.

For all $z \in \mathbb{D} \cup S$, $\|f_e(z)\| \leq \|f_e(z) - f(z)\| + \|f(z)\| < \epsilon_1 + \|f\|_\infty$, and for all $z \in \Omega \setminus \mathbb{D}$, $\|f_e(z)\| \leq \|f_e(z) - f_e(z_*)\| + \|f_e(z_*)\| < \epsilon_2 + \epsilon_1 + \|f\|_\infty$. With

$$M(\delta, \|f\|_\infty) := \frac{1}{\frac{\delta}{2} + \frac{\delta}{4} + \|f\|_\infty} < \frac{1}{\epsilon_1 + \epsilon_2 + \|f\|_\infty},$$

we note that for all $z \in \Omega$, $1 \geq \|M(\delta, \|f\|_\infty) f_e(z)\|$, and for all $x \in E$,

$$\|M(\delta, \|f\|_\infty) f_e(z)x\| \geq M(\delta, \|f\|_\infty)\frac{\delta}{4}\|x\|.$$

Hence by Proposition 3.1, it follows that there exists a $g_e \in H^\infty(\Omega, \mathscr{L}(E_*, E))$ such that for all $z \in \Omega$, $g_e(z) f_e(z) = I$, and

$$\sup_{z\in\Omega} \|g_e\| \leq M(\delta, \|f\|_\infty)\sqrt{\dim(E)}C\left(\left(M(\delta, \|f\|_\infty)\frac{\delta}{4}\right)^{\dim(E)}\right) =: \Theta(\delta).$$

Consequently for all $z \in \mathbb{D} \cup S$, $g_e(z) f_e(z) = I$, and so $g_e(z) f(z) = I - g_e(z)[f_e(z) - f(z)]$. By ensuring that the chosen ϵ_1 is smaller that $\frac{1}{2\Theta(\delta)}$, we see that $I - g_e[f_e - f]$ is invertible as an element of the Banach algebra $A_S(\mathscr{L}(E))$. Defining $g(z) = (I - g_e(z)[f_e(z) - f(z)])^{-1} g_e(z)$, $z \in \mathbb{D} \cup S$, we have that $g \in A_S(\mathscr{L}(E_*, E))$ and $g(z)f(z) = I$ for all $z \in \mathbb{D} \cup S$. ■

Remark 1. *Bound on the left inverse.* Note that in Theorem 3.2 we have also proved that it is possible to choose the left inverse g of f such that it satisfies the following estimate:

$$\|g\|_\infty \leq 2\Theta(\delta) = 2M(\delta, \|f\|_\infty)\sqrt{\dim(E)}C\left(\left(M(\delta, \|f\|_\infty)\frac{\delta}{4}\right)^{\dim(E)}\right),$$

where C is as in Proposition 3.1.

Remark 2. *Scalar case.* The result in Theorem 3.2 in the case when $E = E_* = \mathbb{C}$ (the scalar case) was shown in Theorem 2 of Détraz [2] using algebraic tools. Arne Stray gave another proof in the scalar case, and the proof of Theorem 3.2 follows his approach and the proof by Rosay [9] which can be found in [10].

Remark 3. *Application to control theory.* Coprimeness plays an important role in the factorization approach to solving stabilization problems in control theory (see Vidyasagar [16]). Using the corona theorem 3.2, we can give a necessary and sufficient condition for a matrix pair to be right coprime in A_S. We recall the definition of coprimeness below:

Definition. Let U, Y be Hilbert spaces, and let S be a subset of $\mathbb{T}$. Suppose that $N \in A_S(\mathscr{L}(U, Y))$ and $D \in A_S(\mathscr{L}(U))$. The pair (N, D) is called *right coprime* (with respect to A_S) if there exists a $P \in A_S(\mathscr{L}(U, Y))$ and a $Q \in A_S(\mathscr{L}(U))$ such that the following Bézout identity holds: $PN + QD = I$. A *left coprime* pair of matrices is defined analogously.

We have the following consequence of Theorem 3.2.

Corollary 3.3 *Let U, Y be separable Hilbert spaces with $\dim(U) < \infty$. Suppose that S is an open subset of $\mathbb{T}$, and that $N \in A_S(\mathscr{L}(U, Y))$, $D \in A_S(\mathscr{L}(U))$. Then the following are equivalent:*

1. *The pair (N, D) is right coprime.*
2. *There exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $N(z)^*N(z) + D(z)^*D(z) \geq \delta I$.*

4 Complementing to an isomorphism

In this section, we will prove the equivalence of the operator corona problem with the problem of completing an embedding to an isomorphism. We note that items 1 or 2 of Theorem 3.2 imply that $f(z)$ is one-to-one for each z, and so $\dim(E) \leq \dim(E_*)$. Without loss of generality, we may assume that $E \subset E_*$. The problem of complementing to an isomorphism is now that of describing those functions $f \in A_S(\mathscr{L}(E, E_*))$ for which there exists an invertible $F \in A_S(\mathscr{L}(E_*))$ such that $F|_E = f$. In the case of H^∞, this was shown by Tolokonnikov, and we will use this (Proposition 4.1 below), together with the approximation result (Corollary 2.3) in order to prove the corresponding version for A_S (Theorem 4.2).

Proposition 4.1 (Tolokonnikov's lemma) *Let $E \subset E_*$ be separable Hilbert spaces and $\dim(E) < \infty$. Suppose that Ω is a simply connected domain not equal to $\mathbb{C}$, and that $f \in H^\infty(\Omega, \mathscr{L}(E, E_*))$. Then the following statements are equivalent:*

1. *There exists a $g \in H^\infty(\Omega, \mathscr{L}(E_*, E))$ such that for all $z \in \Omega$, $g(z)f(z) = I$.*
2. *There exists an invertible $F \in H^\infty(\Omega, \mathscr{L}(E_*))$ such that $F(z)|_E = f(z)$ for all $z \in \Omega$.*

Furthermore, the F can be so chosen that it satisfies $\|F^{-1}\|_\infty \leq \|g\|_\infty(1 + \|f\|_\infty) + 1$.

Proof For $\Omega = \mathbb{D}$, this is precisely the statement of Tolokonnikov's lemma (see §10 in Appendix on page 293 of Nikolski [6], and also the remarks following the statement of the theorem in the same section). The general case is a trivial consequence using the Riemann mapping theorem by proceeding in the same manner as with the proof of Proposition 3.1. ■

We now give the main result in this section on complementing to an isomorphism.

Theorem 4.2 *Let $E \subset E_*$ be separable Hilbert spaces and $\dim(E) < \infty$. Suppose that S is an open subset of $\mathbb{T}$, and that $f \in A_S(\mathscr{L}(E, E_*))$. Then the following are equivalent:*

1. *There exists $g \in A_S(\mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D} \cup S$, $g(z)f(z) = I$.*
2. *There exists an invertible $F \in A_S(\mathscr{L}(E_*))$ such that for all $z \in \mathbb{D} \cup S$, $F(z)|_E = f(z)$.*

Proof Given $f \in A_S(\mathscr{L}(E, E_*))$ and $\epsilon_1 > 0$, by Corollary 2.3 there exists a neighbourhood O of S and a $\mathscr{L}(E, E_*)$-valued holomorphic function f_e defined on $\Omega := \mathbb{D} \cup O$ such that $\|f_e|_\mathbb{D} - f\|_\infty < \epsilon_1$.

As f_e is continuous in O, given any $\epsilon_2 > 0$, we can shrink O suitably so as to ensure that for the new $\Omega = \mathbb{D} \cup O$, we have that for all $z \in \Omega \setminus \mathbb{D}$, there exists a $z_* \in S$ such that $\|f_e(z) - f_e(z_*)\| < \epsilon_2$.

Proceeding as in the proof of Theorem 3.2, we obtain

$$\forall z \in \Omega, \text{ and } \forall x \in E, \quad \|f_e(z)x\| \geq \frac{\delta}{4}\|x\|, \text{ and } \|f_e(z)\| \leq \epsilon_1 + \epsilon_2 + \|f\|_\infty.$$

By Proposition 3.1 (applied to $M(\delta, \|f\|_\infty)f_e$, where $M(\delta, \|f\|_\infty) := \dfrac{1}{\frac{\delta}{2} + \frac{\delta}{4} + \|f\|_\infty}$), it follows that there exists a $g_e \in H^\infty(\Omega, \mathscr{L}(E_*, E))$ such that for all $z \in \Omega$, $g_e(z)f_e(z) = I$, and

$$\|g_e\|_\infty \leq M(\delta, \|f\|_\infty)\sqrt{\dim(E)}C\left(\left(M(\delta, \|f\|_\infty)\frac{\delta}{4}\right)^{\dim(E)}\right) =: \Theta_1(\delta, \|f\|_\infty),$$

where C is as in Proposition 3.1.

By Proposition 4.1, there exists an invertible $F_e \in H^\infty(\Omega, \mathscr{L}(E_*))$ such that for all $z \in \Omega$, $F_e(z)|_E = f_e(z)$, and

$$\begin{aligned} \|F_e^{-1}\|_\infty &\leq \|g_e\|_\infty(1 + \|f_e\|_\infty) + 1 \\ &\leq \Theta_1(\delta, \|f\|_\infty)(1 + \epsilon_1 + \epsilon_2 + \|f\|_\infty) + 1 \\ &\leq \|f\|_\infty \left(1 + \frac{\delta}{2} + \frac{\delta}{4} + \|f\|_\infty\right) + 1 =: \Theta_2(\delta, \|f\|_\infty). \end{aligned}$$

Let $P \in \mathscr{L}(E_*, E)$ denote the projection onto E. Consider the function $H : \mathbb{D} \cup S \to \mathscr{L}(E_*)$ defined by

$$H(z) = F_e(z)^{-1}(f(z) - f_e(z))P \in \mathscr{L}(E_*), \quad z \in \mathbb{D} \cup S.$$

It is clear that $H \in A_S(\mathscr{L}(E_*))$. Furthermore, we have that for all $z \in \mathbb{D} \cup S$,

$$\|H(z)\| \leq \|F_e(z)^{-1}\| \|f(z) - f_e(z)\| \|P\| \leq \Theta_2(\delta, \|f\|_\infty) \cdot \epsilon_1 \cdot 1 < \frac{1}{2}$$

provided that we choose $\epsilon_1 < \dfrac{1}{\Theta_2(\delta, \|f\|_\infty)}$ at the outset. So $I + H$ is invertible in $A_S(\mathscr{L}(E_*))$. Define $F : \mathbb{D} \cup S \to \mathscr{L}(E_*)$ by

$$F(z) = F_e(z)(I + H(z)), \quad z \in \mathbb{D} \cup S.$$

Then we have that $F \in A_S(\mathscr{L}(E_*)$ is invertible, and if $x \in E$, then

$$\begin{aligned} F(z)x &= F_e(z)(I + H(z))x = F_e(z)x + F_e(z)H(z)x = f_e(z)x + (f(z) - f_e(z))Px \\ &= f_e(z)x + (f(z) - f_e(z))x = f(z)x, \end{aligned}$$

and so $F|_E = f$. This completes the proof. ■

Combining Theorems 3.2 and 4.2, we have the following.

Corollary 4.3 *Let $E \subset E_*$ be separable Hilbert spaces and* $\dim(E) < \infty$. *Suppose that S is an open subset of $\mathbb{T}$, and that $f \in A_S(\mathscr{L}(E, E_*))$. Then the following are equivalent:*

1. *There exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $f(z)^* f(z) \geq \delta^2 I$.*
2. *There exists $g \in A_S(\mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D} \cup S$, $g(z)f(z) = I$.*
3. *There exists an invertible $F \in A_S(\mathscr{L}(E_*))$ such that for all $z \in \mathbb{D} \cup S$, $F(z)|_E = f(z)$.*

5 Corona theorem for $A(\mathscr{L}(E, E_*))$, when $\dim(E) = \infty$

The counterexample by Treil [14] shows that the operator corona theorem does not hold if E is an infinite dimensional Hilbert space. Nevertheless, it can hold under further assumptions on the corona data function f. Recently, Treil [15] proved an operator corona theorem for $f \in H^\infty(\mathscr{L}(E, E_*))$ under some extra assumptions on f: if f is a 'small' perturbation of a 'nice' function f_0, then the operator corona theorem holds for such functions. In this last section, we use Treil's positive result when $\dim(E) = \infty$ in order to prove a similar result for $A(\mathscr{L}(E, E_*))$ (which is the analogue of the disk algebra case).

We first recall Treil's result from [15]. In the following, $\mathscr{S}_2(E, E_*)$ denotes the space of Hilbert-Schmidt operators, equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathscr{S}_2}$, and this forms a Banach space. If $T \in \mathscr{S}_2(E, E_*)$, then $\|T\| \leq \|T\|_{\mathscr{S}_2}$ (see for instance Pietsch [8]).

Proposition 5.1 (Treil) *Let E, E_* be separable Hilbert spaces. Suppose that Ω is a simply connected domain not equal to $\mathbb{C}$. Let $f \in H^\infty(\Omega, \mathscr{L}(E, E_*))$ be such that it satisfies* **one** *of the following assumptions:*

A1. There exists a $C \in \mathscr{L}(E, E_*)$ and there exists an $f_1 \in H^\infty(\Omega, \mathscr{S}_2(E, E_*))$ such that $f(z) = C + f_1(z)$ for all $z \in \Omega$.

A2. There exists a left invertible $f_0 \in H^\infty(\Omega, \mathscr{L}(E, E_*))$ and an $f_1 \in H^\infty(\Omega, \mathscr{S}_2(E, E_*))$ such that $f(z) = f_0(z) + f_1(z)$ for all $z \in \Omega$.

Then the following are equivalent:

1. *There exists a $\delta > 0$ such that for all $z \in \Omega$, $f(z)^* f(z) \geq \delta^2 I$.*
2. *There exists a $g \in H^\infty(\Omega, \mathscr{L}(E_*, E))$ such that for all $z \in \Omega$, $g(z)f(z) = I$.*

In order to prove our version of the above result for $A(\mathscr{L}(E, E_*))$ (Theorem 5.3 below), we will also need the following theorem by Arveson–Sz.-Nagy–Foias (Arveson [1] and Sz.-Nagy–Foias [12]). We use $H^2(E)$ to denote the vector-valued Hardy class H^2 with values in E. Recall that the Toeplitz operator $T_h \in \mathscr{L}(H^2(E), H^2(E_*))$ corresponding to a function $h \in L^\infty(\mathscr{L}(E, E_*))$ is defined by

$$T_h \varphi = P_+(h\varphi), \quad \varphi \in H^2(E),$$

where $P_+ : L^2(E_*) \to H^2(E_*)$ denotes the orthogonal projection operator onto $H^2(E_*)$.

Theorem 5.2 *Let E, E_* be separable Hilbert spaces and let $f \in H^\infty(\mathscr{L}(E, E_*))$. Then the following are equivalent:*

1. *There exists a $g \in H^\infty(\mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D}$, $g(z)f(z) = I$, and $\|g\|_\infty \leq \delta^{-1}$.*

2. *If $\overline{f}$ denotes the function defined by $\overline{f}(z) = f(\overline{z})$, $z \in \mathbb{T}$, then*

$$\inf_{\substack{\varphi \in H^2(E),\\ \|\varphi\|=1}} \|T_{\overline{f}}\varphi\| \geq \delta > 0,$$

where $T_{\overline{f}}$ denotes the the Toeplitz operator corresponding to $\overline{f} \in L^\infty(E, E_*)$.

We are now ready to prove the following theorem.

Theorem 5.3 *Let E, E_* be separable Hilbert spaces. Suppose that $f \in A(\mathscr{L}(E, E_*))$ is such that it satisfies* **one** *of the following assumptions:*

S1. There exists a $C \in \mathscr{L}(E, E_*)$ and $f_1 \in A(\mathscr{S}_2(E, E_*))$ such that $f(z) = C + f_1(z)$ for all $z \in \overline{\mathbb{D}}$.

S2. There exists a left invertible $f_0 \in A(\mathscr{L}(E, E_*))$ and a $f_1 \in A(\mathscr{S}_2(E, E_*))$ such that for all $z \in \overline{\mathbb{D}}$, $f(z) = f_0(z) + f_1(z)$.

Then the following are equivalent:

1. *There exists a $\delta > 0$ such that for all $z \in \overline{\mathbb{D}}$, $f(z)^* f(z) \geq \delta^2 I$.*
2. *There exists a $g \in A(\mathscr{L}(E_*, E))$ such that for all $z \in \overline{\mathbb{D}}$, $g(z)f(z) = I$.*

Proof The proof is divided into two main steps.

STEP 1. We consider the two cases:

$\underline{1}^\circ$ Suppose S1 holds.

By the approximation result in Theorem 2.2, given any $\epsilon_1 > 0$, there exists a neighbourhood O of $\mathbb{T}$ in $\mathbb{C}$ and a function $f_1^e : \mathbb{D} \cup O \to \mathscr{S}_2(E, E_*)$ such that $\|f_1^e(z) - f_1(z)\|_{\mathscr{S}_2} < \epsilon_1$ for all $z \in \mathbb{D}$.

Given $\epsilon_2 > 0$, we can then choose a $r \in (0, 1)$ such that $\frac{1}{r}\mathbb{D} =: \mathbb{D}_r$ is contained in $\Omega := \mathbb{D} \cup O$ and such that for all $z \in \overline{\mathbb{D}_r} \setminus \mathbb{D}$, $\|f_1^e(\frac{z}{r}) - f_1^e(z)\|_{\mathscr{S}_2} < \epsilon_2$.

As $\sup\limits_{z \in \mathbb{D}_r} \|f_1^e(z)\|_{\mathscr{S}_2} \leq \sup\limits_{z \in \mathbb{D}} \|f_1(z)\|_{\mathscr{S}_2} + \epsilon_1 + \epsilon_2$, A1 in Proposition 5.1 holds.

Define f^e by $f^e(z) = C + f_1^e(z)$, $z \in \Omega$. Clearly $f^e \in H^\infty(\mathbb{D}_r, \mathscr{L}(E, E_*))$. Moreover, for all $z \in \overline{\mathbb{D}}$ and $x \in E$, we have

$$\begin{aligned}
\|f^e(z)x\| &\geq \|f(z)x\| - \|(f^e(z) - f(z))x\| \geq \delta\|x\| - \|(f_1^e(z) - f_1(z))x\| \\
&\geq \delta\|x\| - \|f_1^e(z) - f_1(z)\|\|x\| \geq \delta\|x\| - \|f_1^e(z) - f_1(z)\|_{\mathscr{S}_2}\|x\| \\
&> \delta\|x\| - \frac{\delta}{2}\|x\| \quad \text{(ensured by choosing } \epsilon_1 < \frac{\delta}{2}) \\
&= \frac{\delta}{2}\|x\| \\
&> \frac{\delta}{4}\|x\|.
\end{aligned}$$

Furthermore, for all $z \in \mathbb{D}_r \setminus \overline{\mathbb{D}}$ and $x \in E$,

$$\begin{aligned}
\|f^e(z)x\| &\geq \|f^e(z_*)x\| - \|(f^e(z_*) - f(z))x\| \quad (\text{where } z_* := \frac{z}{|z|} \in \mathbb{T}) \\
&\geq \frac{\delta}{2}\|x\| - \|f_1^e(z_*) - f_1^e(z)\|_{\mathscr{S}_2}\|x\| \\
&> \frac{\delta}{4}\|x\| \quad (\text{ensured by choosing } \epsilon_2 < \frac{\delta}{4}).
\end{aligned}$$

Consequently, for all $z \in \mathbb{D}_r$ and all $x \in E$, $\|f^e(z)x\| \geq \dfrac{\delta}{4}\|x\|$.

$\underline{2}^\circ$ Suppose that S2 holds.

By Corollary 2.3, given $\epsilon_1 > 0$, there exists a neighbourhood O of $\mathbb{T}$ in $\mathbb{C}$ and a function $f_1^e : \mathbb{D} \cup O \to \mathscr{L}(E, E_*)$ such that for all $z \in \mathbb{D}$, $\|f_0^e(z) - f_0(z)\| < \epsilon_1$.

Given any ϵ_2, we can choose a $r \in (0,1)$ such that $\frac{1}{r}\mathbb{D} =: \mathbb{D}_r$ is contained in $\mathbb{D} \cup O$ and such that for all $z \in \overline{\mathbb{D}_r} \setminus \mathbb{D}$, $\|f_0^e(\frac{z}{r}) - f_0^e(z)\| < \epsilon_2$.

Let $g_0 \in A(\mathscr{L}(E_*, E))$ be a left inverse of f_0. Then for all $z \in \mathbb{D}$, $g_0(z)f_0(z) = I$. By the Arveson–Sz.-Nagy–Foias Theorem 5.2, it follows that

$$\inf_{\substack{\varphi \in H^2(E), \\ \|\varphi\|=1}} \|T_{\overline{f_0}}\varphi\| \geq \frac{1}{\|g_0\|_\infty} > 0.$$

If $f_{0,r}^e$ is defined by $f_{0,r}^e(z) = f_0^e\left(\frac{z}{r}\right)$, $z \in \mathbb{D}$, then we have that $f_{0,r}^e \in H^\infty(\mathscr{L}(E, E_*))$. For $\varphi \in H^2(E)$ and $\|\varphi\| = 1$, we have

$$\begin{aligned}
\|T_{\overline{f_{0,r}^e}}\varphi\| &= \|T_{\overline{f_0} + \overline{f_{0,r}^e - f_0}}\varphi\| = \|T_{\overline{f_0}}\varphi + T_{\overline{f_{0,r}^e - f_0}}\varphi\| \geq \|T_{\overline{f_0}}\varphi\| - \|T_{\overline{f_{0,r}^e - f_0}}\varphi\| \\
&\geq \|T_{\overline{f_0}}\varphi\| - \|T_{\overline{f_{0,r}^e - f_0}}\|\|\varphi\| = \|T_{\overline{f_0}}\varphi\| - \|T_{\overline{f_{0,r}^e - f_0}}\| \\
&\geq \|T_{\overline{f_0}}\varphi\| - \|\overline{f_{0,r}^e - f_0}\|_\infty = \|T_{\overline{f_0}}\varphi\| - \sup_{z \in \mathbb{T}} \|f_{0,r}^e(\overline{z}) - f_0(\overline{z})\| \\
&= \|T_{\overline{f_0}}\varphi\| - \sup_{z \in \mathbb{T}} \left\| f_0^e\left(\frac{\overline{z}}{r}\right) - f_0(\overline{z}) \right\| \\
&\geq \|T_{\overline{f_0}}\varphi\| - \sup_{z \in \mathbb{T}} \left\| f_0^e\left(\frac{\overline{z}}{r}\right) - f_0^e(\overline{z}) \right\| - \sup_{z \in \mathbb{T}} \|f_0^e(\overline{z}) - f_0(\overline{z})\| \\
&> \|T_{\overline{f_0}}\varphi\| - \epsilon_2 - \epsilon_1.
\end{aligned}$$

Thus

$$\inf_{\substack{\varphi \in H^2(E), \\ \|\varphi\|=1}} \|T_{\overline{f_{0,r}^e}}\varphi\| > \frac{1}{2\|g_0\|_\infty} > 0,$$

if we choose $\epsilon_1, \epsilon_2 < \dfrac{1}{4\|g_0\|_\infty}$ at the outset. By the Arveson–Sz.-Nagy–Foias Theorem 5.2, it follows that there exists a $g_{0,r}^e \in H^\infty(\mathscr{L}(E_*, E))$ such that $g_{0,r}^e(z)f_{0,r}^e(z) =$

I for all $z \in \mathbb{D}$. Defining g_0^e by $g_0^e(z) = g_{0,r}^e(rz)$, $z \in \mathbb{D}_r$, we see that $g_0^e \in H^\infty(\mathbb{D}_r, \mathscr{L}(E_*, E))$ and $g_0^e(z) f_0^e(z) = I$ for all $z \in \mathbb{D}_r$, that is,

$$f_0^e \in H^\infty(\mathbb{D}_r, \mathscr{L}(E, E_*)) \text{ has a left inverse } g_0^e \in H^\infty(\mathbb{D}_r, \mathscr{L}(E_*, E)). \tag{7}$$

By the approximation result in Theorem 2.2, given $\epsilon_1 > 0$, we can refine the above neighbourhood O of $\mathbb{T}$ in $\mathbb{C}$ and a find a function $f_1^e : \mathbb{D} \cup O \to \mathscr{S}_2(E, E_*)$ such that for all $z \in \mathbb{D}$, $\|f_1^e(z) - f_1(z)\|_{\mathscr{S}_2} < \epsilon_1$.

We can then make the above choice of $r \in (0,1)$ small enough such that $\frac{1}{r}\mathbb{D} =: \mathbb{D}_r$ is contained in $\Omega := \mathbb{D} \cup O$ and such that for all $z \in \overline{\mathbb{D}_r} \setminus \mathbb{D}$, $\|f_1^e(\frac{z}{r}) - f_1^e(z)\|_{\mathscr{S}_2} < \epsilon_2$.

We have

$$\sup_{z \in \mathbb{D}_r} \|f_1^e(z)\|_{\mathscr{S}_2} \leq \sup_{z \in \mathbb{D}} \|f_1(z)\|_{\mathscr{S}_2} + \epsilon_1 + \epsilon_2. \tag{8}$$

Defining f^e by $f^e(z) = f_0^e(z) + f_1^e(z)$, $z \in \mathbb{D}_r$, we see from (7) and (8) that A2 from Proposition 5.1 holds.

Moreover, for all $z \in \overline{\mathbb{D}}$ and $x \in E$, we have

$$\begin{aligned}
\|f^e(z)x\| &\geq \|f(z)x\| - \|(f^e(z) - f(z))x\| \\
&\geq \delta\|x\| - \|(f_0^e(z) - f_0(z))x\| - \|(f_1^e(z) - f_1(z))x\| \\
&\geq \delta\|x\| - \|f_0^e(z) - f_0(z)\|\|x\| - \|f_1^e(z) - f_1(z)\|\|x\| \\
&\geq \delta\|x\| - \|f_0^e(z) - f_0(z)\|\|x\| - \|f_1^e(z) - f_1(z)\|_{\mathscr{S}_2}\|x\| \\
&> \delta\|x\| - \frac{\delta}{2}\|x\| \quad (\text{ensured by choosing } \epsilon_1 < \frac{\delta}{4}) \\
&= \frac{\delta}{2}\|x\| \\
&> \frac{\delta}{4}\|x\|.
\end{aligned}$$

Furthermore, for all $z \in \mathbb{D}_r \setminus \overline{\mathbb{D}}$ and $x \in E$,

$$\begin{aligned}
\|f^e(z)x\| &\geq \|f^e(z_*)x\| - \|(f^e(z_*) - f(z))x\| \quad (\text{where } z_* := \frac{z}{|z|} \in \mathbb{T}) \\
&\geq \frac{\delta}{2}\|x\| - \|f_0^e(z_*) - f_0^e(z)\|\|x\| - \|f_1^e(z_*) - f_1^e(z)\|_{\mathscr{S}_2}\|x\| \\
&> \frac{\delta}{4}\|x\| \quad (\text{ensured by choosing } \epsilon_2 < \frac{\delta}{8}).
\end{aligned}$$

Consequently, for all $z \in \mathbb{D}_r$ and all $x \in E$, $\|f^e(z)x\| \geq \dfrac{\delta}{4}\|x\|$.

Step 2. By Treil's result (Proposition 5.1), there exists a $g^e \in H^\infty(\mathbb{D}_r, \mathscr{L}(E_*, E))$ such that for all $z \in \mathbb{D}_r$, $g^e(z) f^e(z) = I$. If g_r^e, f_r^e are defined by

$$g_r^e(z) = g^e\left(\frac{z}{r}\right) \quad \text{and} \quad f_r^e(z) = f^e\left(\frac{z}{r}\right), \quad z \in \mathbb{D},$$

then we have for all $z \in \mathbb{D}$, $g_r^e(z) f_r^e(z) = I$. By the Arveson–Sz.-Nagy–Foias Theorem 5.2, g_r^e can be chosen so as to satisfy

$$\|g_r^e\|_\infty \leq \frac{1}{\zeta}, \text{ where } \zeta := \inf_{\substack{\varphi \in H^2(E),\\ \|\varphi\|=1}} \|T_{\overline{f_r^e}} \varphi\|.$$

If $\varphi \in H^2(E)$ and $\|\varphi\| = 1$, then

$$\begin{aligned}
\|T_{\overline{f_r^e}} \varphi\| &= \|T_{\overline{f} + \overline{f_r^e - f}} \varphi\| = \|T_{\overline{f}} \varphi + T_{\overline{f_r^e - f}} \varphi\| \geq \|T_{\overline{f}} \varphi\| - \|T_{\overline{f_r^e - f}} \varphi\| \\
&\geq \|T_{\overline{f}} \varphi\| - \|T_{\overline{f_r^e - f}}\| \|\varphi\| = \|T_{\overline{f}} \varphi\| - \|T_{\overline{f_r^e - f}}\| \\
&\geq \|T_{\overline{f}} \varphi\| - \|\overline{f_r^e - f}\|_\infty = \|T_{\overline{f}} \varphi\| - \sup_{z \in \mathbb{T}} \|f_r^e(\overline{z}) - f(\overline{z})\| \\
&= \|T_{\overline{f}} \varphi\| - \sup_{z \in \mathbb{T}} \left\| f^e\left(\frac{\overline{z}}{r}\right) - f(\overline{z}) \right\| \\
&\geq \|T_{\overline{f}} \varphi\| - \sup_{z \in \mathbb{T}} \left\| f^e\left(\frac{\overline{z}}{r}\right) - f^e(\overline{z}) \right\| - \sup_{z \in \mathbb{T}} \|f^e(\overline{z}) - f(\overline{z})\| \\
&> \|T_{\overline{f}} \varphi\| - 2\epsilon_2 - 2\epsilon_1.
\end{aligned}$$

If $\epsilon_1, \epsilon_2 < \frac{\tau(f)}{8}$, where

$$\tau(f) := \inf_{\substack{\varphi \in H^2(E),\\ \|\varphi\|=1}} \|T_{\overline{f}} \varphi\|,$$

then $\zeta > \frac{\tau(f)}{2}$, and $\|g^e\|_\infty = \|g_r^e\|_\infty \leq \frac{1}{\zeta} \leq \frac{2}{\tau(f)}$. So

$$\forall z \in \overline{\mathbb{D}}, \quad \|g^e(z)[f^e(z) - f(z)]\| \leq \frac{2}{\tau(f)} \epsilon_1 < \frac{1}{4}.$$

Consequently $I - g^e[f^e - f]$ is invertible in $A(\mathscr{L}(E))$. Thus, if we define

$$g = (I - g^e[f^e - f])^{-1} g^e,$$

then $g \in A(\mathscr{L}(E_*, E))$ and moreover, for all $z \in \overline{\mathbb{D}}$, $g(z) f(z) = I$. ∎

One would like to know whether Theorem 5.3 holds for $A_S(\mathscr{L}(E, E_*))$ in the general case when S is an arbitrary open subset of $\mathbb{T}$. The proof given above does not generalize to arbitrary S, since Theorem 5.2 (used in Steps 1 and 2) does not apply.

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