Mathematical Models of Mutual Mate Choice

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abstract:

In this review, we present several variations of the Alpern-Reyniers two-sided matching model, with particular application to its biological interpretation as a mate selection game. In this context, the model describes equilibrium behavior in a dynamic game where unmated males and females of various types in a given cohort group are randomly matched in a succession of periods. If they 'accept' each other, they mate permanently and leave the cohort. The models differ in the utility u(x,y) they assign to individuals of type x who mates with one of type y. The two main models assume that (i) individuals prefer mates of similar type, u(x,y)= -|x-y|; or that (ii) they have a common preference for high types, u(x,y)= y. Other applications of the matching model, for example to job search, are only described briefly

1 INTRODUCTION

Who we are as individuals, and also as a species, has been greatly influenced by the mate choices of our personal and collective ancestors. This may in part explain the extensive literature, in both the biological and social sciences, on the subject of mate selection. However, as observed by Bergstrom and Real [6], relatively little of this literature is concerned with the problem of *mutual* mate choice, where mating requires a positive choice from both partners (in many species, the males compete while the females alone choose, so selection is one-sided).

This article surveys the work carried out by Steve Alpern and Diane Reyniers [2,3] at the London School of Economics, towards modelling the problem of mutual mate choice as a dynamic nonatomic game. (Some recent extensions of the theory by Steve Alpern and Ioanna Katrantzi [1] are also described.) The game is played in successive periods (of a 'mating season') by a cohort of males and females, who are randomly paired in each period and mate by mutual consent based on their own preferences over traits of their potential partners. These games tend to have positive assortative mating in their equilibrium outcomes, a widely observed empirical finding in both humans and animals.

Although the models we discuss are formal and based on rigorous mathematics, our discussion here will be less formal and for the most part proofs will be omitted. Our brief in this exercise did not include surveying related work, but it should be mentioned that the papers of McNamara and Collins [19] and Johnstone [15] are particularly significant, and will be briefly mentioned at appropriate places.

1.1 A 'grand programme'

Much of the work in this area has its background and its inspiration in a 'grand programme' in which empirical data (e.g. observed patterns of *a posteriori* couples and positive assortative mating) could be used to infer processes (game structures) and preferences (utilities of individuals for potential mates) which may have led to such patterns. For example the work of Kalick and Hamilton [17] seeks to determine the mate preferences which have led to the observed high correlation in facial attractiveness - in particular an analysis of the so called 'matching hypothesis' which asserts that individuals prefer similarly attractive mates. In fact there is no established theory that we are aware of that can effectively carry out this 'grand programme'. Even if the process (game) is known apart from the preferences (payoffs), there seems to be no theory parallel to that of 'revealed preferences' (developed by Samuelson [25, Chapter VI] and others to infer utilities from actions in an economic setting) which determines the payoffs from observations of (equilibrium) play. However this 'grand programme' is still at the heart of many models, including ours, in which equilibrium behavior is determined in various formal models of unknown or partially known processes.

1.2 The Kalick-Hamilton ‘dating simulation model’

The game theoretic models of the authors have as their immediate inspiration the simulation model developed by Kalick and Hamilton [17] to analyze the matching hypothesis stated earlier. They created (in a computer simulation model) an initial cohort of male and female individuals of various attractiveness levels, and randomly paired them (‘dates’) in a sequence of periods. If both individuals on a date accepted the other, they formed a ‘mated couple’ and left for the remainder of the simulation. In one simulation, all individuals were programmed to accept similar mates (close attractiveness levels) with higher probability than dissimilar ones. As expected, this led to high inter-couple attractiveness levels at the end of the run (when everyone was paired). In another run, everyone was programmed simply to accept dates with high attractiveness levels with high probability. This simulation, more controversially (at least in the social science literature - see [4],[18]), also led to high correlations. In fact both correlations were near empirically observed ones.

1.3 The Alpern-Reyniers game

The Alpern-Reyniers game $\Gamma_n\left(u_x, u_y, F_1, G_1\right)$ (or its sex-symmetric version $\Gamma_n\left(u_x, F_1\right)$) introduced in [2] took the dynamic aspects of the Kalick-Hamilton simulation model [17](fixed initial cohort, random pairing of unmated individuals in each period, mutual acceptance for mating) and made it into a game. The players are the males x in a *type set* X and females in a *type set* Y, where X and Y are taken to be a common fixed interval, either $[0,1]$ or $[-1,1]$ (used when the symmetry around the central type 0 is important for the analysis). Thus we consider only a single parameter of differentiation of individuals, though there is nothing in our model that would restrict us from considering higher dimensions. Sometimes a discrete type set $\{1,2,\ldots,n\}$ is adopted for even greater simplicity. It is convenient to think of X and Y as simply the set of males and females; however in the continuous model males are *distributed* over X according to an initial continuous cumulative distribution F_1 (with females over Y according to G_1). In any case we will still talk of ”*a* male x” rather than "of type x”. This game differs from the Kalick-Hamilton simulation in the process by which individuals choose to accept or reject their date. In their simulation, the choice was programmed into the model (probabilistically). In our game the choice depends on the chooser’s preferences, which are modelled by specifying their utility function. The utility that a male x gets from mating with a female y is denoted $u_x\left(y\right)$, the female’s utility is $w_y\left(x\right)$ (in symmetric models we just use u). Their actual choice (when considering a particular date) then depends on many things. They must take into account the choices of other individuals (of both the same and opposite sex), because those choices will affect the distribution over type in the next period, and this will affect their estimation of how well they will do (the expected utility of their eventual mate) if they reject their current date and enter the next period unmated. All these considerations are combined into our notion of an equilibrium strategy.

For simplicity, we now restrict our attention to games where there is complete symmetry between the sexes, so that $X = Y$, $F_1 = G_1$, and $u_x(y)$ is the utility any person of type x (male or female) gets when mated with a type y. A sex-symmetric strategy (for either sex, in a symmetric model) is a function $s = (s_1, s_2, \ldots, s_{n-1})$ such that for each $t = 1, \ldots, n-1$ (in the final period t any date is accepted) $s_t(x) \subset Y$ is the set of all types $y \in Y$ that an individual of type x will accept. If everyone adopts the strategy s then the distributions F_t of types in period are determined as well as the expected payoffs $v_t(x)$ to an individual of type x who enters period t unmated. In particular,

$$v_n(x) = \int_Y u_x(y)\, dF_n(y) \tag{1}$$

is simply the expected utility to x of a random unmated individual in the final period n. The equilibrium condition is that

$$y \in s_t(x) \Leftrightarrow u_x(y) \geq v_{t+1}(x). \tag{2}$$

That is, at equilibrium x accepts y if and only this mating would give him at least as much utility as he gets on average if he enters the next period unmated.

1.4 Particular preferences

Section 2 gives the analysis of the game $\Gamma(n)$ for *similarity* (or *homotypic*) preferences $u_x(y) = -|x-y|$, where the cost of a mating between types x and y (for each player) is the distance $|x-y|$ between their types. This section is based mainly on the original article of Alpern and Reyniers [2], together with some recent results of Alpern and Katrantzi [1]. Section 3 gives the analysis of Alpern and Reyniers [3] of the game $\Gamma(n)$ with common (or maximizing) preferences $u_x(y) = y$, where there is a common ranking (all males have a common ranking of the females, and visa versa) and higher ranking goes with higher types. Readers should note the similar paper of Johnstone [15] which is more computational and also offers more biological insight.

It is worth noting that both of the preference types (utility functions) we discuss (similarity and common) are equivalent to one of the form $u_x^{z(x)}(y) = -|y - z(x)|$, where the cost of a mating with y to an individual of type x is the distance to x's ideal mate, $z(x)$. For similarity preferences we have $z(x) = x$ (an identical mate is the ideal), and for common preferences the common ideal mate is the one with maximum type $z = 1$, and $u_x^1(y) = -(1-y) = y - 1$, which is affine equivalent (same preference structure) to y.

1.5 Alternative interpretations of the models

As the models presented here are formal and mathematical, they may be easily interpreted in other situations. The two groups X and Y need not represent males and females of a biological species who wish to mate with preferred individuals. They could equally represent buyers and sellers (mating equals a sale), employers and employees (the so-called *job search problem*, as described by McNamara and Collins [19]), or schools and students. The last two matching problems are generally one to many, so some modifications are needed. It should also be observed that, as long as symmetric assumptions are made between the sexes, the model can be interpreted as matching *within* a single group X. Examples of this might be finding car pool partners (where preferences would by of similarity type with respect to home address) or finding tennis partners (perhaps the ideal for x is $x+\varepsilon$, if one likes to have a slightly better partner). However centrally organized systems are very distinct from our decentralized models, where meeting (dating) is random. Indeed internet auctions which bring buyers and sellers together (or dating sites) will require very different models than those discussed here.

1.6 Related work

In the economics literature, some of the work on 'matching theory' is related to mate selection models. Articles of this type mentioned in our surveyed papers but not discussed individually here are: [7], [8], [9], and [13][26]. In the biology literature, the related articles not discussed individually here but mentioned in our surveyed papers are the following: [11], [16], [20].

2 SIMILARITY PREFERENCES

In this section we discuss the mating game Γ_n for similarity (homotypic) preferences, following Alpern and Reyniers [2], except for the analysis of I_2 and I_3 which comes from Alpern and Katrantzi [1]. This means that the common cost (negative utility) to an x and a y who mate is simply $|x-y|$, and players are minimizers. (These preferences are a particular case of what Eriksson and Strimling [14] describe as *romantic*, or *symmetric* preferences, as x likes y just as much as y likes x.) Similarity preferences seem to be fairly common across various traits and species, and may have some explanatory value in positive assortative mating. For example, psychologists Russell and Bartrip [24] assert that, "Substantial investigations ... going back to the last century have shown that what people want in a mate is someone like themselves." For animals, it may seem impossible that they choose mates who look like themselves (i.e. the trait in question may be an observable surface feature), as they may lack this type of self knowledge (humans have mirrors). However, they do see their parents, whose appearance has a positive correlation with their own. So choosing someone 'like mom', may be revealing a form of similarity preference. There is evidence of this; for example Cooke and Davies [10] found that Snow Geese tend to choose a mate who has similar coloration to their parents. Partridge [21] observed the importance of adaptation to local conditions, so that offspring of parents from nearby (type is location) places, who also are brought up in that area, will be well suited to the environment.

Returning to our mathematical model, the game Γ_n defined in the Introduction, we see that with similarity preferences there is no difference between types which are high or low in the type interval X - the significant property is how far from the center they are. Hence for this preference it is useful to take $X = [-1, 1]$ in order to exploit the symmetry around 0. This means we need only consider the behavior of positive types $x \geq 0$, and may assume that types $-x$ behave like x. (Of course there may also be spatially asymmetric solutions, but these have not yet been explored.) Recall that a strategy $s = (s_1, s_2, \ldots, s_{n-1})$ specifies the set of y's that an x will accept at time t. Writing the equilibrium condition (2) in minimizing form with $c_x(y) = -u_x(y) = |x - y|$ gives

$$y \in s_t(x) \Leftrightarrow c_x(y) \leq v_{t+1}(x). \tag{3}$$

so that $s_t(x) = [x - v_{t+1}(x), x + v_t(x)] \cap [-1, 1]$. Given that equilibrium strategies must be of this form, it is reasonable to identify a strategy $s_t(x)$ with a maximum distance at which x will accept a y at time t. So we interpret $s_t(x)$ as a number rather than a set. That is, x accepts y at time t if and only if $|x - y| \leq s_t(x)$. Recall from the Introduction that $v_t(x) = v_t(s, x)$ (the latter denoting its dependence on the strategy s) denotes the minimum expected cost that an individual of type x can obtain if unmated at the beginning of period t, assuming everyone else is adopting strategy s. The equilibrium condition can now be written simply as

$$\begin{gathered} s_t(x) = v_{t+1}(x), \text{ for} \\ 1 \leq t \leq n-1, \quad -1 \leq x \leq 1. \end{gathered} \tag{4}$$

If the cumulative distribution of the population (same for males and females) in the final period is denoted by F (that is, $F(y)$ is the fraction of the final period population with type less than y), then the expected payoff to a type x entering this period unmated is given by the value function

$$v_n(x) = \int_{-1}^{1} |x - y| \, dF(y). \tag{5}$$

This function has some nice properties:

Theorem 1 ([2], Theorem 1) *If F denotes the final period cumulative distribution function over types, then the final period value function v_n corresponding to a strategy profile s has the following properties:*

1. *v_n is a symmetric (even) convex function*
2. *v_n has a unique minimum at* 0
3. $v_n(-1) = v_n(1) = 1$
4. $v_n'(x) = 2F(x) - 1$, *and in particular*
5. $v_n'(-1) = -1$, $v_n'(0) = 0$, $v_n'(1) = 1$.

2.1 Equilibrium for $n = 2$

In this section we determine equilibrium properties of the symmetric two-period game Γ_2 for an initial distribution which is uniform on the interval $X = [-1, 1]$. For notational simplicity we let $s(x) = s_1(x)$ denote the first period acceptance strategy and $v(x) = v_2(x)$ denote the value of entering the second and final period. For equilibrium we have the single equation

$$s(x) = v(x), \tag{6}$$

as a special case of (4). So Theorem 1 ensures that an equilibrium strategy $s(x)$ must be increasing on $[0, 1]$. That is, types closer to the central type 0 are choosier (smaller acceptance distance s) than those further from 0. This implies that if $0 < x_1 < x_2$ then x_1 accepts x_2 implies x_2 accepts x_1. The following simple consequence of Theorem 1 will be useful.

Corollary 2 ([2], Theorem 2) *At equilibrium, if $x_0 < x_1 < x_2$ and x_0 accepts x_2, then x_1 also accepts x_2. That is, if s is an equilibrium strategy ($s = v$) then*

$$x_0 + s(x_0) \geq x_2 \text{ implies } x_1 + s(x_1) \geq x_2.$$

We now consider the process by which a first period strategy s determines the second period population distribution F and hence the second period value function, via (5). Given a strategy s, what fraction of the type x's will be mated in period 1 (and hence not enter period 2). A type x will be mated if and only if he meets a type y in the 'mutual acceptance set' $M(x)$ defined by x accepts y ($|x - y| \leq s(x)$) and y accepts x ($|x - y| \leq s(y)$), that is

$$M(x) = \{y : |x - y| \leq \min[s(x), s(y)]\}.$$

If we define

$$\begin{aligned} R(x) &= R_s(x) = x + s(x), \\ L(x) &= L_s(x) = x - s(x), \end{aligned}$$

then $R(x)$ and $L(x)$ are the rightmost and leftmost types that x accepts in strategy s. Hence In Figure 1 the mutually acceptable pairs (x, y) constitute the light set, the set where only one accepts the other is medium grey, and the set where neither accepts the other is dark. For a given horizontal value x, the vertical set $M(x)$ is the light portion of the vertical line above x.

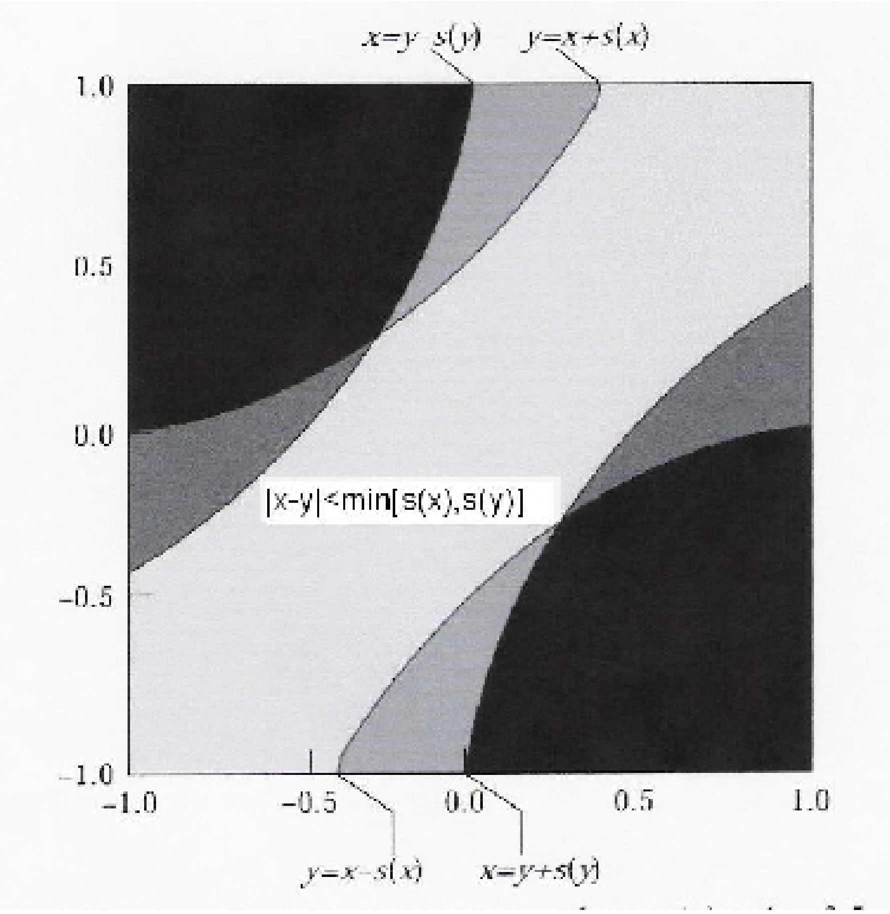

Figure 1: Acceptance patterns, $s(x) = 1 - (1/2)\cos(x/2)$

Given a strategy s, the probability $p(x) = p_s(x)$ that a type x is mated in period 1 is simply the probability that a random y will belong to the mutual mating interval $M(x) = M_s(x)$, that is, the length of $M(x)$ divided by the length of $X = [-1, 1]$, or

$$p(x) = \text{length}\,[M(x)]/2. \tag{7}$$

We seek to determine $p(x)$ for $x \geq 0$, as our symmetry assumption implies $p(-x) = p(x)$. For any strategy s, define two particular types $b = b_s$ and $a = a_s$, as follows

$$\begin{aligned} a &= \max\{x : -x \text{ accepts } x\} = \max\{x : -x + s(x) = x\}, \text{or} \qquad (8) \\ s(a) &= 2a \text{ (this equation has a unique solution).} \end{aligned}$$

$$\begin{aligned} b &= \min\{x : \ x \text{ accepts } 1\} \qquad (9) \\ &= \min\{x : x + s(x) = 1\}, \text{ or} \\ s(b) &= 1 - b \text{ (this equation has a unique solution)} \end{aligned}$$

For example, if $s(x) = 1 - (1/2)\cos(x/2)$ (as in Figure 2), we have $a \approx 1/4$ and $b \approx 1/2$.

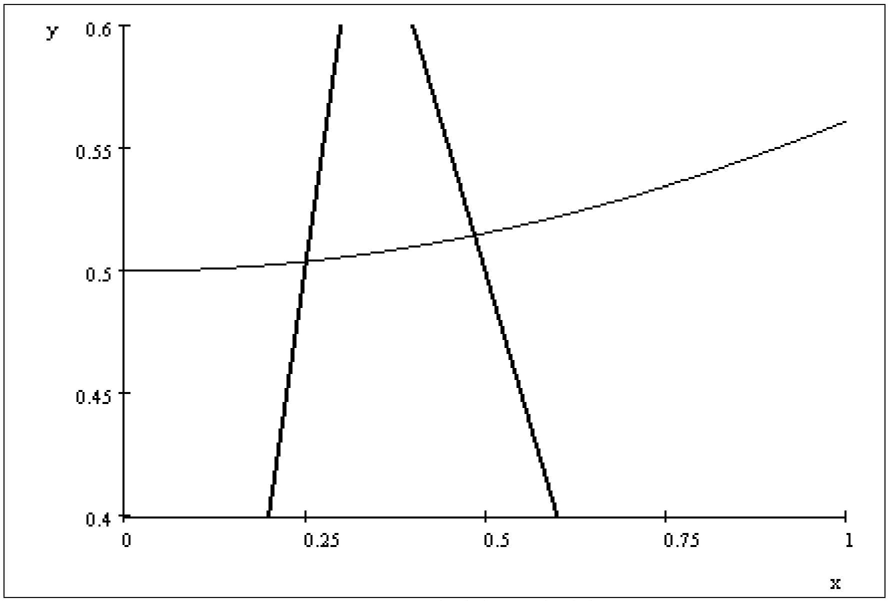

Figure 2: Plots of $1 - (1/2)\cos(x/2), 1 - x, 2x$

In general, define three intervals

$$I_1 = [0, a], \ I_2 = [a, b], \ I_3 = [b, 1].$$

For $x \in [0, 1]$, we have $M(x) = M_s(x)$ is given by

$$M(x) = \begin{cases} [L(x), R(x)], & \text{for } x \in I_1, \\ [R^{-1}(x), R(x)], & \text{for } x \in I_2, \\ [R^{-1}(x), 1], & \text{for } x \in I_3. \end{cases} \qquad (10)$$

and so by (7), $p(x) = p_s(x)$ is given by

$$p(x) = \begin{cases} s(x), & \text{for } x \in I_1, \\ \dfrac{x + s(x) - R^{-1}(x)}{2}, & \text{for } x \in I_2, \\ \dfrac{1 - R^{-1}(x)}{2}, & \text{for } x \in I_3. \end{cases} \qquad (11)$$

It is useful to note that types in I_3 are 'universal acceptors' in the sense that if a type y accepts a type x in I_3, then the x also accepts the y. So the mating behavior is determined entirely by the definition of the strategy s on the interval $I_1 \cup I_2 = [0, a]$ That is

Proposition 3 *If two strategies s and $\hat{s}$ agree for $|x| \leq a_s = a_{\hat{s}}$, then $p_s(x) = p_{\hat{s}}(x)$ and consequently $v_s(x) = v_{\hat{s}}(x)$ for all $x \in X$.*

At equilibrium we have $s = v$, so if we differentiate part 4 of Theorem 1, we see that any equilibrium strategy s must satisfy the differential equation

$$s''(x) = v''(x) = 2F'(x). \tag{12}$$

If the first period density function is the uniform density $1/2$ on $[-1, 1]$, and any types x leave the population with probability $p(x)$ then the normalized density function F' in the final period is given by

$$F'(x) = \frac{(1 - p(x))/2}{c}, \tag{13}$$

where

$$c = c_s = \int_{-1}^{1} ((1 - p(x))/2)\ dx \tag{14}$$

is the total population (of either sex) in the final period. Thus we have

$$\begin{aligned} s''(x) &= 2F'(x) \qquad\qquad (15)\\ &= \frac{1 - p_s(x)}{c} \\ &= C(1 - p_s(x)),\ \text{taking } C = 1/c. \\ s'(0) &= 0 \text{ (because } s(x) = s(-x)) \end{aligned}$$

Applying this differential equation to the interval I_1 gives

$$\begin{aligned} s''(x) &= C(1 - s(x)) \\ s'(0) &= 0. \end{aligned}$$

Consequently we have the following.

Theorem 4 ([2], p. 77) *Let s be an equilibrium strategy for the two period game Γ_2 with an initial uniform distribution. Then for all $x \in I_1(s)$, that is for all x with $s(x) \leq 2x$, the strategy s must be of the form*

$$s_1(x) = 1 - (1 - s(0)) \cos\left(\sqrt{C}\ \ x\right). \tag{16}$$

Numerical data from a discrete approximation indicates that $s(0) \approx 0.55$ and $C \approx 1.96$ (that is, $c \approx .51$, so about half the population is mated in the first period).

For the interval I_2 differential equation for the equilibrium solution is given by (15) and (11) as

$$s''(x) = C\left(1 - \frac{x + s(x) - R^{-1}(x)}{2}\right), a \leq x \leq b. \tag{17}$$

Here $s(a)$ and $s'(a)$ are known from the solution of s on $I_1 = [0, a]$, and $R^{-1}(x)$ is known from that solution as well, as $R(a) = R_s(a) = a + s(a) = 3a$ is assumed to be greater than b and $R_s(-a) = -a + 2a = a$, so $R_s([-a, a]) \supset [a, b] = I_2$.

For the interval I_3, the differential equation for the equilibrium solution is given by (15) and (11) as

$$\begin{aligned} s''(x) &= C\left(1 - \frac{1 - R^{-1}(x)}{2}\right) \\ &= C\left(\frac{R^{-1}(x) - 1}{2}\right), \end{aligned} \tag{18}$$

with $s(b)$ and $s'(b)$ known from the solution for $s(x)$ on I_2 given by (17) (as functions of $s(0)$). Note that $s(x)$ does not appear on the right hand side of (18) so it can be solved by integration. We also know that $s'(1) = v'(1) = 1 = v(1) = s(1)$ from Theorem 1 and the $n = 2$ period equilibrium equation(6).

2.2 A discrete uniform initial distribution

Given the analytical problems associated with establishing even existence for an equilibrium in a continuous type model, an analysis with a uniform distribution over types $X = \{-m, -m+1, \ldots, 0, 1, 2, \ldots m\}$ was carried out. First assume that $n = 2$. A type k corresponds to the type $x = k/m$ in the previous section, so comparisons can be made, particularly as m gets large. Denoting discrete strategies by $s(k)$, $-m \leq k \leq m$, observe that a strategy which accepts anyone whose distance from k is less than or equal to say 2.3 is equivalent to one where the given distance is 2.4, or their common floor $\lfloor 2.3 \rfloor = \lfloor 2.4 \rfloor = 2$. So we may restrict σ to taking integer values. The equilibrium equation in this case is simply

$$\sigma(k) = \lfloor v_\sigma(k) \rfloor, -m \leq k \leq m. \tag{19}$$

Here $v_\sigma(k)$ is the expected distance of a second period individual to k if strategy σ is adopted in the first period. The population is simply a $2m+1$ vector giving the fraction of types of each k. The set of strategies is finite. For *any* strategy σ we may mirror the equilibrium equation (19) to define a new strategy $T(\sigma)$ by the equation

$$(T\sigma)(k) = \lfloor v_\sigma(k) \rfloor, -m \leq k \leq m. \tag{20}$$

Any fixed point of (20) satisfies (19) and is an equilibrium. So if the iterates of T converge (are eventually constant, as the domain is finite), they converge to a fixed point. The iterates of the strategy $\sigma \equiv 1$ (accept your type or your neighbor's) converge to the discrete strategy plotted in Figure 3, together with the continuous solution $s_1(x)$ on I_1. The discrete values were used to estimate the initial condition $s_1(0)$ and C, the explicit solution for I_1 (that is, before the strategy meets the line $2x$) is taken from (16). The solution on I_2 is obtained by number solution of the ODE (17). A close fit was expected only for $x \leq a \approx .295$

(up to intersection with line $2x$) but in fact goes further.

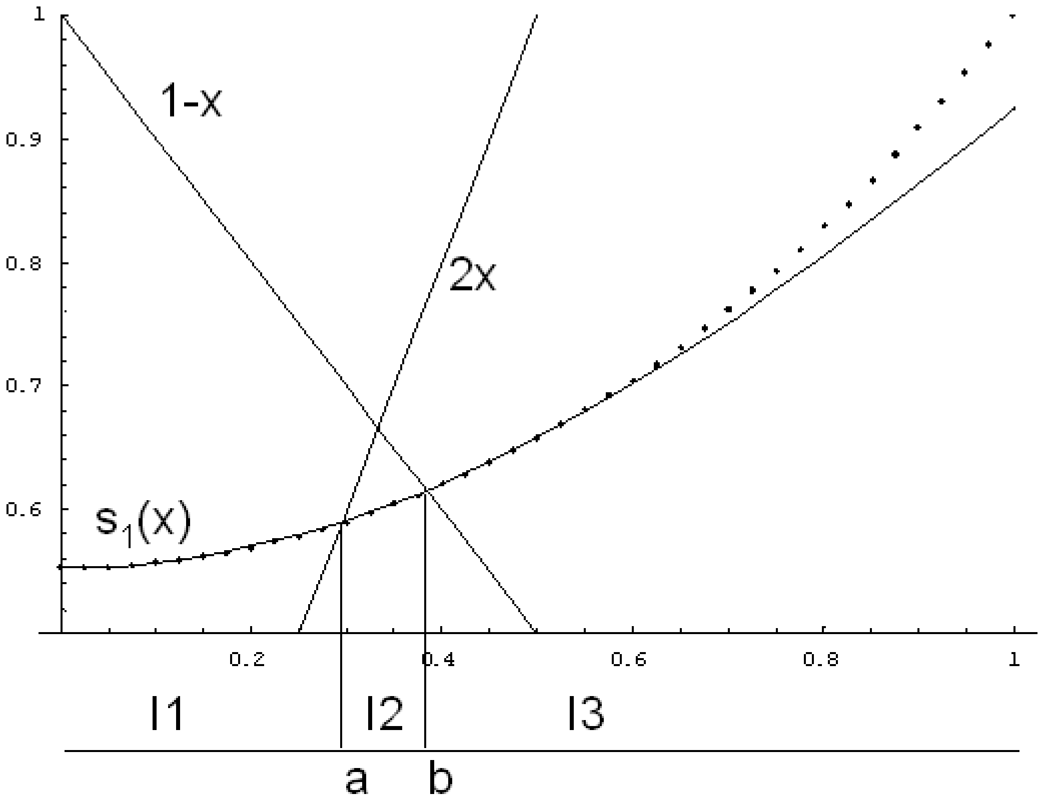

Figure 3: Continuous and discrete equilibria

The iterative method used to calculate an equilibrium strategy for the two period discrete problem can also be used to for any $n-$period problem.

2.3 Steady-state models

The previous models discussed in this article all had a 'cohort' of individuals who left the unmated population without replacement once they were mated, which took at most n periods. In the steady-state model the population statistics (distribution over types) remain constant over time. There is no limit on the number of periods an individual can wait until accepting a mate. However, there is some pressure not to wait for an ideal mate because a waiting penalty $r > 1$ is imposed in that the cost to x of mating y after k periods is given by $r^k |x - y|$. In addition to r, the other exogenous variable of the model is the population Q (a distribution over types) which arrives in each period. A symmetric stationary strategy $s(x)$, as before, gives the maximum distance that x will accept a mate, which is the same (stationarity) in each period in this steady-state model. At the end of each period there is a residual unmated population G (again, a distribution over types). The next period thus begins with a population $G+Q$. If for some type x, the stationary probability of mating is given by p and the expected distance to a mate is δ, then the expected cost $V = V(x)$ of the eventual mate is given by

$$\begin{aligned} V &= p\delta + r(1-p)p\delta + r^2(1-p)^2 p\delta + \cdots \\ &= \frac{p\delta}{1-(1-p)r}, \text{ if } (1-p)r < 1. \end{aligned}$$

We seek a pair (s, G), where s is a steady state symmetric strategy and G is a residual distribution, for which the population dynamics are constant over time and each type x is acting optimally. This motivates the following.

Definition 5 *A pair (s, G) is a steady-state equilibrium (SSE) for the r, Q game if it satisfies the following:*

1. *(steady-state condition) If the strategy s is adopted against the initial population distribution $G+Q$, then the distribution of unmated individuals at the end of the period with be G.*
2. *(equilibrium condition) $s(x) = r\ V(x)$, for all types x.*

As an example, consider the game with $Q = (.2, .2, .2, .2, .2)$ is the uniform distribution of five types $x = 1, 2, 3, 4, 5$. Take $r = 1.6$. An SSE is given by the strategy $s = (4, 1, 1, 1, 4)$ in which central types accept only neighbors and extreme types accept everyone; and a residual population $G = (2/15, 2/15, 2/15, 2/15, 2/15)$. In this case the residual population mirrors the given influx population Q in being uniform. This is not necessarily the case, as is indicated by keeping the same Q and taking $r = 1.5$. Then an SSE is given by the strategy $(2, 1, 1, 1, 2)$ and residual population (to three places) of

$$G = (0.293, 0.136, 0.186, 0.136, 0.293).$$

Note that while this is still symmetric (it has to be, as the strategy is symmetric), it is no longer uniform - some types are more likely to be mated than others. A steady-state model with common preferences (as described in the next section) was developed earlier by McNamara and Collins [19].

3 COMMON PREFERENCES

This section, covering the work of Alpern and Reyniers (2005), assumes that males have *common preferences* over females; and females over males. (This work is parallel to that of Johnstone (1997) which we discuss briefly in the next section.) This assumption means that a type x male can be defined as one which is preferred (by all females) to a fraction x of the male population.

Thus higher number types (in an interval of types that we now take as $[0,1]$) are universally preferred to lower numbered types. These preferences are also known as *maximizing preferences*, or *type preferences,* although the latter term is not very helpful in our context. In a mating of a male x and a female y, the male's utility is y and the female's is x. In the notation of Section 1.3, we have

$$u_x(y) = y, \quad w_y(x) = x,$$

and since these are utilities rather than costs, all players are maximizers. These utilities determine a particular version of the dynamic mating game Γ_n defined in Section 1.3.

The main finding is that, at equilibrium, each period $t = 1, 2, \ldots, n$ is associated with a pair of acceptance levels a_t and b_t, both decreasing in t (player's get 'less picky' over time) with the following property: the only couples formed in period t consist of males over b_t and females over a_t, and all such pairing result in couples (mating). This mating pattern differs from both the similarity preference version of Γ_n (previous section), where mating patterns vary continuously with type; and also the steady-state common preference model of McNamara and Collins [19], where the type spaces were partitioned into bands of mutual acceptance. The mating pattern of Alpern and Reyniers [3] gives analytical support to that posited by Parker [20], in that individuals with high (fitness) levels pair off with each other first, leaving the lower fitness individuals to pair off with each other later. Johnstone [15] has similar findings, supported by computational work in a game model similar to ours. The conclusion that acceptance levels are decreasing in time (individuals get less picky) is equivalent to a model in which acceptance levels are constant over time but perceptance of type is increasing over time, as empirically observed by Pennebaker et. al. [22] in their investigations prompted by the country and western song "Don't the girls get prettier at closing time", and in animal breeding seasons by Real [23].

Let $v_m^1(x)$ and $v_m^2(y)$ denote the expected utility (type level of eventual mate) of a male of type x (female of type y) who enters period m unmated, assuming a particular strategy (not necessarily symmetry between sexes) is being followed. Since at equilibrium a male of type x in period m should accept any $y \in \left[v_m^1(x), 1\right]$, this acceptance set can be denoted simply by its left endpoint. Hence we define a male strategy $f = (f_1, f_2, \ldots, f_{n-1})$ (since $f_n = 0$) to specify acceptance of any $y \geq f_m(x)$ by a male of type x in period m; taking g to denote similar female strategies. Thus the equilibrium condition in this model is that for all periods $m = 1, 2, \ldots, m-1$ and all $x, y \in [0,1]$ we have

$$\begin{aligned} f_m(x) &= v_{m+1}^1(x), \\ g_m(y) &= v_{m+1}^2(y). \end{aligned} \tag{21}$$

Note the $v_n^1(x)$ is simply the mean female type y in the final period, and as such does not depend on x. Thus at equilibrium in period $n-1$ all males (of any type x) simply accept a female of type y if and only if $y \geq v_n^1$.

That is, accept anyone with a higher type than the mean type you will be mated with if you go into the next period unmated.

If everything is symmetric between the sexes, including equilibrium strategies, the equilibrium condition is simply

$$f_m(x) = v_{m+1}(x).$$

3.1 Two period asymmetric model

To illustrate the general features of a common preference mutual choice mating model, we consider a two period game Γ_2 with a common uniform distribution over the type set $[0,1]$ for both males and females. Since v_n^j is a constant (see above), equilibrium strategies for period $n-1$ are also constant. So suppose that in the first period males accept females $y \geq a$ and females accept males $x \geq b$.

Hence the total population of each sex in period 2 is $1-(1-a)(1-b) = b+a-ab$. The 'low' males (below b) have average fitness $b/2$, and they form an interval of length b of full density 1, the 'high' males have average fitness $(1+b)/2$, form an interval of length $1-b$, and have density a (since a fraction $1-a$ of them have been mated in period 1). Hence period 2 mean male fitness is

$$\begin{aligned} v_2^2 &= \frac{(b/2)(b)\,1 + ((1+b)/2)(1-b)\,a}{b+a-ab} \\ &= \frac{1}{2}\frac{-b^2-a+ab^2}{-b-a+ab}. \end{aligned}$$

Thus the second equilibrium condition of (21) is

$$b = \frac{1}{2}\frac{-b^2-a+ab^2}{-b-a+ab}. \tag{22}$$

By symmetry, the first equilibrium condition of (21) is

$$a = \frac{1}{2}\frac{-a^2-b+ba^2}{-a-b+ba}. \tag{23}$$

These two curves are shown together with the line of symmetry $b=a$ in Figure 4, the top equation (for b) drawn with a thicker line. (For the moment, ignore the dotted line as it is only relevant in the next paragraph.) The only intersection corresponds to a symmetric equilibrium

$$a = b = \frac{3}{2} - \frac{1}{2}\sqrt{5} = .381\,97. \tag{24}$$

The fact that the two curves meet only on the line $a=b$ is a special case of Theorem 6 for the uniform distribution.

Theorem 6 ([3], Theorem 4) *If (f,g) is an equilibrium for the two period symmetric game $\Gamma_2(F) = \Gamma_2(F,F)$, then $f = g$.*

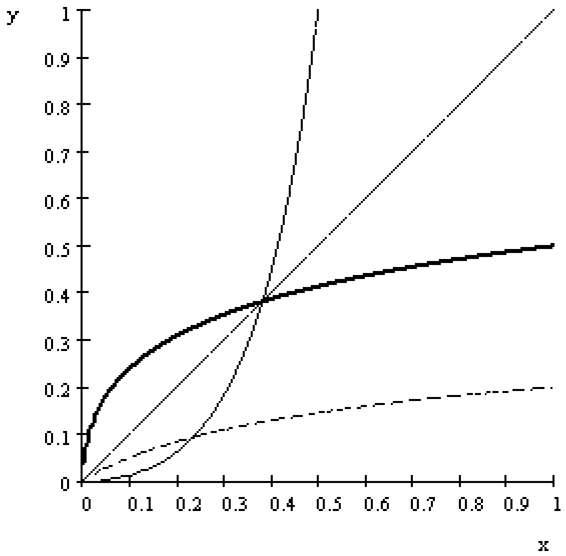

Figure 4: Male and female equilibrium equations

Suppose we modify the female utility function so that the utility of mating with male x in period 2 is reduced by c, $0 < c < 1/2$, that is, to $x - c$. There are no changes to first period or male utility functions. The equilibrium is now the solution to the original male equilibrium equation (23) and the female equation (22) with c subtracted from the right hand side. The unique solution is given by period one male and female acceptance levels a and b, where

$$\begin{aligned} a &= \frac{b}{1-2c}, \text{ and} \\ b &= \frac{1}{2}\left(3 - 4c - \sqrt{(-8c+5)}\right). \end{aligned}$$

In period 1 the females become less choosy (b is decreasing in c), even to the extent of accepting any male when $c = 1/2$. Males are choosier than females ($a > b$), but less choosy than in the original model (a is also decreasing in c). The explanation can be seen in Figure 4, the dotted line represents $c = .3$, which intersects with the unchanged male (thin) at a smaller a value and a smaller b value. To understand the limiting case $c = 1/2$, where females are universal acceptors, suppose males accept females above a in period 1. Then in period 2 females are uniformly distributed on $[0, a]$, with mean $a/2$. Hence the equilibrium equation for the males is $a = a/2$, with solution $a = 0$. Hence males are also universal acceptors.

3.2 Existence and properties of equilibria

The main results obtained on equilibria are the following. The first result says that any equilibrium must be of the kind described in the beginning of this section, namely.

Theorem 7 ([3], Theorem 1) *Every equilibrium (f, g) in the common preference game $\Gamma_n(F_1, G_1)$ (with given reservation values w_{n+1} and z_{n+1} determining minimum acceptance levels in the final period -usually taken to be zero) is characterized by two sequences*

$$\begin{aligned} 0 &= w_{n+2} \leq w_{n+1} < w_n < \cdots < w_2 < w_1 = 1 \textit{ and} \\ 0 &= z_{n+2} \leq z_{n+1} < z_n < \cdots < z_2 < z_1 = 1, \textit{ such that} \end{aligned}$$

$$x \textit{ and } y \textit{ are mutually acceptable in period } i \textit{ iff } x > w_{i+1},\ y > z_{i+1}.$$

Furthermore the strategy pair (f, g) is determined by these sequences according to the following rules. In period $i = 1, \ldots, n$, a male x will accept a female y iff $y \geq f_i(x)$, where.

$$f_i(x) = \begin{cases} z_k, & \textit{if } x \in (w_{k+1}, w_k),\ k \geq i+2 \\ z_{i+1}, & \textit{if } x > w_{i+2}, \end{cases} \tag{25}$$

and a female y will accept a male x iff $x \geq g_i(y)$, where

$$g_i(y) = \begin{cases} w_k, & \textit{if } y \in (z_{k+1}, z_k),\ k \geq i+2 \\ w_{i+1}, & \textit{if } y > z_{i+2}. \end{cases} \tag{26}$$

The following figure shows the computed values (using an iterative algorithm) for the common acceptance levels $w_k = z_k$, $k = 1, \ldots, n-1$, $n = 1, 2, \ldots, 10$, for the symmetric game $\Gamma(n)$ with the uniform initial distribution. For example the equilibrium acceptance level .381 97 of (24) shown in Figure 3 is denoted as the single dot at that height above $n = 2$.

13

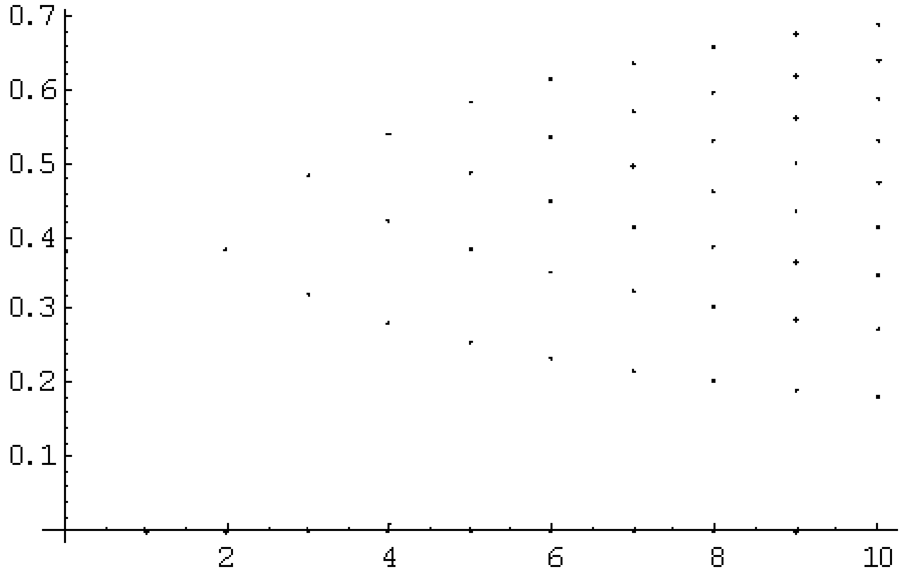

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Figure 5: Equilibrium acceptance levels in period $k = 1, \ldots, n-1$ for $n \leq 10$.

The second result says that such equilibria always exist. It uses the Brouwer Fixed Point Theorem.

Theorem 8 *Suppose the initial population distributions F_1 and F_2, of males and females, are continuous. Then the common preference game $\Gamma_n(F_1, F_2)$ has an equilibrium strategy pair.*

Furthermore, if the initial distributions are the same, then we can say more.

Theorem 9 *Every symmetric common preference game $\Gamma_n(F) = \Gamma_n(F, F)$, with F continuous, has a symmetric equilibrium.*

3.3 Multiple and Mixed Equilibria

The theorems in the previous subsection leave open certain questions regarding mixed strategies and multiple equilibria. To show the variety of equilibria that can occur in common preference games, we consider a symmetric two period game with initial an distribution $\hat{F}$ equally divided (1/3 each) between fitness levels $0, .1$, and $.3$. At equilibrium a 0 will never be accepted in period 1, while a .3 will always be accepted. So the only two possible equilibrium strategies in period 1 are ‘high’ (accepting only .3), or ‘middle’ (accepting only .1 or .3). We show that both of these are equilibria. We know that only symmetric equilibria can occur in a two period symmetric game, so we assume common strategies for males and females.

For the 'high' strategy only two .3 individuals can be mated in the first period. This yields a period 2 population with mass 1/3 of 0, 1/3 of .1, and 2/9 of .3, with a total population of 8/9. The mean type of the period 2 population is

$$v_2 = \frac{\left(\frac{1}{3}\right) 0 + \left(\frac{1}{3}\right) .1 + \left(\frac{2}{9}\right) .3}{\frac{8}{9}} = .1125. \tag{27}$$

Since

$$.1 < .1125 < .3, \tag{28}$$

it is consistent (equilibrium behavior) for all individuals to reject a .1 in period 1 (receiving on average $.1125$ in the next period), but to accept a .3. Thus the strategy of accepting only .3's in period 1 is an equilibrium.

For the 'middle' strategy any pairing not involving a type 0 in period 1 will result in a mating. The population in period 2 will have a population with mass 1/3 of 0's, 1/9 of .1's and 1/9 of .3's, with a total population of 5/9. The population mean in period 2 will be

$$v_2 = \frac{\left(\frac{1}{3}\right) 0 + \left(\frac{1}{9}\right) .1 + \left(\frac{1}{9}\right) .3}{\frac{5}{9}} = 0.08. \tag{29}$$

Since

$$0 < .08 < .1, \tag{30}$$

it is consistent to accept a .1 in period 1, rather than getting an average of .08 in period 2. Hence this is also an equilibrium. We could say that the equilibrium strategy is $f(x) = .08$, but this is equivalent to $f(x) = .1$, because there are no types between .08 and .1.

Let f_p be the mixed strategy which always accepts a .3, and accepts a .1 with probability p. If everyone adopts f_p (with independent randomization) the normalized population at the beginning of the second (and last) period has

$$\frac{3}{8-2p-p^2} \text{ level 0's, } \frac{3-p-p^2}{8-2p-p^2} \text{ level .1's, and } \frac{2-p}{8-2p-p^2} \text{ level .3's,}$$

with a final period mean fitness level of

$$v_2(p) = \frac{1}{10}\frac{3-p-p^2}{8-2p-p^2} + \frac{3}{10}\frac{2-p}{8-2p-p^2} = \left(\frac{1}{10}\right)\frac{-9+4p+p^2}{-8+2p+p^2}. \tag{31}$$

The mean fitness $v_2(p)$ is strictly decreasing, with $v_2(1/2) = .1$. Consequently when the strategy $p = 1/2$ is adopted, a player is indifferent between accepting a player with fitness .1 in period 1 and going into the next period and getting on average .1. Thus the mixed strategy with $p = 1/2$ is an equilibrium strategy. It is worth noting that the number of equilibrium strategies is odd (3). The equilibrium definition (21) rules out mixed strategy equilibria for continuous distributions F (or F_1, F_2), as the only acceptance indifference for a male x in period m is of a female $y = v_{m+1}(x)$. For continuous F_2 (or F) the probability of encountering such a y is zero.

Is there a way of seeing the three equilibria described above from a common viewpoint, that shows the three values v_2 of .1125, .8, and .1 in a single diagram. One method is to consider 'relative fitness'. For any individual, let the relative fitness r, $0 \leq r \leq 1$, denote the fraction of the population whose fitness (type) is below that of the individual. For individuals of the same type (as in the discrete distribution in this subsection) we arbitrarily distinguish an order among them. So individuals of type .3 will have different r's between 1/3 and 2/3. Their absolute fitness (type) $x = A(r)$ will be the common value 0.1. Suppose a strategy now denotes the lowest relative fitness r that an individual will accept. Let $m(r)$ denote the mean second period absolute fitness for strategy r. Any r with $m(r) = A(r)$ will be an equilibrium. If r corresponds to an interval on which A is constant, it is interpreted as a mixed strategy. The figure below plots $\Psi(r)$ ($A(r)$ with vertical lines at discontinuities) and $m(r)$, and the three intersections at $r = 1/3$, $1/2$, and $2/3$ correspond to the three equilibria. A rigorous definition of the closed correspondence Ψ is given in the original paper.

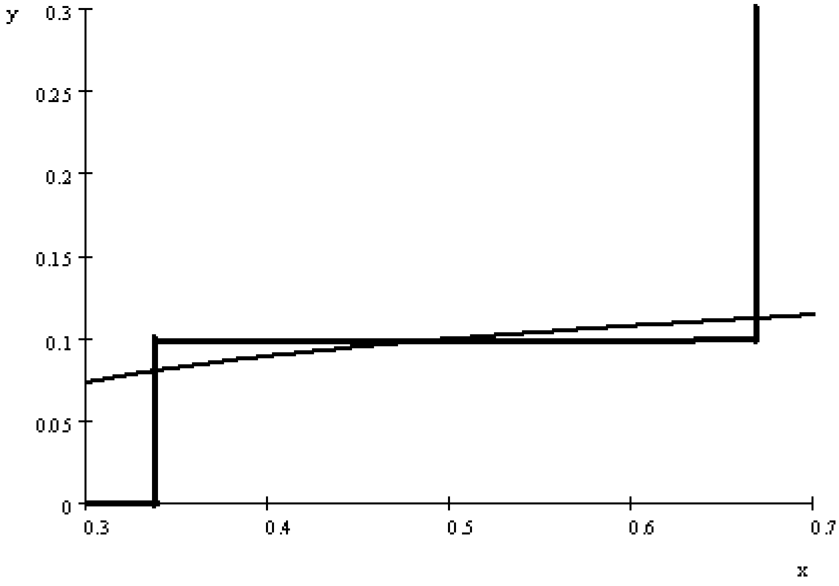

Figure 6: Plot of $m(r)$ and correspondence $\Psi(r)$

3.4 Learning

In the Alpern-Reyniers game Γ_n, it is implicitly assumed that individuals know their own type (a strategy is a function of type). We now consider whether the model can be adapted to allow individuals to learn about their type over time, based on knowledge accumulated by learning who has accepted and who has rejected them. Of course this knowledge is only useful if they remain unmated after they acquire it. Note that with similarity preferences, it is essential that individuals know their own type from the start. However this requirement is not so obvious for common preferences. But for common preferences the nature of the equilibrium only requires that individuals know the current period (not their type) to play the equilibrium strategy of Theorem 7. Thus in our models, any learning that takes place cannot be about one's own type.

So the uncertainty that one reduces by learning must be about the population distribution. A simple example analyzed in [3] is the following. There are three fitness levels, 0, M (medium), and H, in that order. Nature chooses equally likely between the distributions $D_1 = (1/4, 1/2, 1/4)$ and $D_2 = (1/4, 1/4, 1/2)$. The prior probabilities of D_1 and D_2 are each 1/2, but the posterior probabilities p for a pair of individuals of types (x, y) with $x, y \in \{M, H\}$ (if either type is 0, there will not be a mating in any case) are as follows:

$$\begin{aligned} p\left(I \setminus (H,H)\right) &= 1/5,\, p\left(II \setminus (H,H)\right) = 4/5, \\ p\left(I \setminus (M,M)\right) &= 4/5,\, p\left(II \setminus (M,M)\right) = 1/5, \\ p\left(I \setminus (M,H)\right) &= 1/2,\, p\left(II \setminus (M,H)\right) = 1/2, \end{aligned} \tag{32}$$

A strategy $[x, y, z], x, y, z \in \{M, H\}$ gives the minimum acceptance level (the same for both), in the three cases that the paired individuals have unordered types (H, H), (M, M), and (H, M), respectively. For example, the strategy $[M, H, M]$ says that if two type M's meet (middle entry) each should accept only an H, and so two M's will not mate.

Theorem 10 *Define* $\lambda = H/M$. *Then*

$$\begin{aligned} [H,H,H] \text{ is an equilibrium} &\iff 38/17 \leq \lambda \\ [M,M,M] \text{ is an equilibrium} &\iff \lambda \leq 29/9 \\ [H,M,H] \text{ is an equilibrium} &\iff 15/7\lambda \leq 38/17 \end{aligned}$$

Note that in some cases there are multiple equilibria. An interesting feature of this model is that one gets information from observing one's own type, as well as that of the individual with whom one is paired.

References

[1] Alpern, S. and Katrantzi, I. (2005). A dynamic mating game with similarity preferences. Preprint

[2] Alpern, S., and Reyniers, D. J. (1999). Strategic mating with homotypic preferences. *J. Theo. Biol.* **198**, 71-88.

[3] Alpern, S., and Reyniers, D. J. (2005). Strategic mating common preferences. *J. Theo. Biol.* , in press.

[4] Aron, A. (1988). The matching hypothesis reconsidered again: Comment on Kalick and Hamilton. *Journal of Personality and Social Psychology* 54, 441-446.

[5] Bateson, P. (1983). Mate Choice. Cambridge University Press.

[6] Bergstrom, C. T., and Real, L. A. (2000). Towards a theory of mutual mate choice: Lessons from two-sided matching. *Evolutionary Ecology Research* **2**, 493-508.

[7] Bloch, F., and Ryder, H. (2000). Two-sided search, marriages, and matchmakers. *International Economic Review* **41**, 93-114.

[8] Burdett, K., and Coles, M. (1997), Marriage and class. *Quarterly Journal of Economics* **112**, 141-168.

[9] Burdett, K., and Coles, M. (1999). Long term partnership formation: employment and marriage. *Economic Journal*, 307-335.

[10] Cooke, F. and Davies, J. C. (1983). Mating in Snow Geese. In: Mate Choice (Bateson, P., ed.), op. cit.

[11] Cronin, H. (1991). The ant and the peacock. Cambridge University Press.

[12] Crowley, P.H. , Travers, S. E., Linton, M. C., Cohn, S. L., Sih, A., Sargent, R. C. (1991). Mate density, predation risk and the seasonal sequence of mate choices – a dynamic game. *Am Nat* **137**, 567-596.

[13] Eeckhout, J. (2000). Bilateral search and vertical heterogeneity. *International Economic Review* **40**, 869-888.

[14] Eriksson, K., and Strimling, P. (2004) How unstable are matchings from decentralized mate search? Preprint.

[15] Johnstone, R. A. (1997). The tactics of mutual mate choice and competitive search. *Behav. Ecol. Sociobiol.* **40**, 51-59.

[16] Johnstone, R. A., Reynolds, J. D., and Deutsch, J. C. (1996). Mutual mate choice and sex differences in choosiness. *Evolution* **50**, 1382-1391.

[17] Kalick, S. Michael, and Hamilton, Thomas E. (1986) The matching hypothesis reexamined. *Journal of Personality and Social Psychology* **51**, 673-682.

[18] Kalick, S. Michael, and Hamilton, Thomas E. (1988). Closer look at a matching simulation: reply to Aron. *Journal of Personality and Social Psychology* **54,** 447-451.

[19] McNamara, J. M., and Collins, E. J. (1990) The job search problem as an employer-candidate game. *Journal of Applied Probability* **28**, 815-827.

[20] Parker, G. A., (1983) Mate quality and mating decisions. In *Mate Choice*, ed. P. Bateson, Cambridge University Press, 141-164.

[21] Patridge, L. (1983). Non-random mating and offspring fitness. In Mate Choice, op. cit..

[22] Pennebaker, J. W., Dyer, M. A., Caulkins, R. S., Litowitz, D. L., Ackreman, P. L., Anderson, D. B. and McGraw, K. M. (1979) Don't the girls get prettier at closing time: A country and western application to psychology. *Personality and Social Psychology Bulletin* **5**, 122-125.

[23] Real, L. (1990). Search theory and mate choice. I. Models of single sex discrimination. *Am. Nat.* **136**, 376-404.

[24] Russel, R. J. H., and Bartrip, J. (1989). Homo sociobiogicus not found. *Behav. Brain Sci* **12**, 32-33.

[25] Samuelson, P. (1948). Foundations of Economic Analysis, Harvard University Press, Cambridge, Mass.

[26] Shimer, R., and Smith, L. (2000). Assortative Matching and Search. *Econometrica* **68**, no. 2, 342-370.

[27] Zohar, Ada, and Guttman, Ruth (1989) Mate preference is not mate selection. *Behavioral and Brain Sciences* **12,**38-39.