How to Search a Tree to which Eulerian Networks are Attached

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Abstract

We call a network *partly Eulerian* if consists of a tree (of length a and radius r) to which a finite number of disjoint Eulerian networks (of total length b) are attached, each at a single point. We show that for such networks, a search strategy consisting equiprobably of a minimal (Chinese Postman) covering path and its reverse path is optimal, in the sense that it minimizes (at $a + b/2 - r$) the expected time to find a point hidden according to the worst case distribution. This generalizes a similar result of Dagan and Gal for search games on trees.

1 Introduction

Let G be a finite network with arc length measure λ and distance function d. We consider two zero-sum search games originating in R. Isaacs' book [7] in which a unit speed Searcher tries to minimize the time (payoff) T required to find a stationary ($T-$maximizing) Hider. The Hider simply picks a point y in G. In the *fixed-start* game $\Gamma(G, x)$ the Searcher picks a unit speed path in G starting at the designated point x, while in the *arbitrary-start* game $\Gamma(G)$ he may start anywhere. Compactness arguments [5] [2] establish that these games always have (minimax) values which we denote respectively as $V_x = V(G, x)$ and $V = V(G)$. While much work has been done on fixed-start search games [5], arbitrary-start games have only very recently been studied by Dagan and Gal [4] for the case where G is a tree. (There is also some early some early work when G is the circle for related games where the Hider is also mobile ([9],[1]), but very different techniques are required for those games.) This paper generalizes the work of [4] by solving arbitrary-start games when G is a *partly Eulerian* network (a tree to which disjoint Eulerian circuits are attached). Our results may be interpreted as finding the best worst-case method to search such a network.

2 Definitions

A path S in G is a continuous function from some finite time interval $[0,\tau]$ into G. In our games $\Gamma(G)$ and $\Gamma(G,x)$ the Searcher will only use unit speed paths in G belonging respectively to the sets

$$\begin{aligned} \mathcal{S} &= \{S : d((S(t)), S(t')) \leq |t-t'|\} \text{ and} \\ \mathcal{S}_x &= \{S \in \mathcal{S} : S(0) = x\}. \end{aligned}$$

Such a path is called *closed* if $S(0) = S(\tau)$, and called a *tour* if additionally its range is G. If the Searcher chooses a path S and the Hider a point $y \in G$, then the payoff is the capture time $T(S,y) = \min\{t : S(t) = y\}$. If T has mixed strategies has arguments (strategies), it will be interpreted as the *expected* time.

Let $\mu = \lambda(G)$ denote the total length (sum of arc lengths) of G. A Chinese Postman (CP) tour is a tour of minimum length $\bar{\mu}$, and a CP path is a covering (range G) path of minimum length $\tilde{\mu}$. We call G Eulerian if $\bar{\mu} = \mu$ and any tour with length μ is called an Eulerian tour. We say that G is *weakly Eulerian* if it contains a finite number of disjoint Eulerian networks which, when each is shrunk to a point, leaves a tree.

3 Fixed-Start Games

Although we are mainly concerned with arbitrary-start search games, we will need the following important result of Gal ([6],[2])(extending work of Reijniers and Potters [8]) on fixed-start games, characterizing networks for which *random CP tours* (CP tours traversed equally likely in either direction) are optimal for the Searcher.

Theorem 1 (Gal) *If G is weakly Eulerian then for any starting point x, $V(G,x) = \bar{\mu}/2$. An optimal Searcher strategy is a random CP tour starting at x, and there is an optimal Hider strategy which is uniform on every Eulerian subnetwork. Conversely, if $V(G,x) = \bar{\mu}/2$ for some x, then G is weakly Eulerian.*

4 Partly Eulerian Networks

We say that G is *partly Eulerian* if it is the union of a tree A and a finite number of disjoint Eulerian networks E_i, such that each E_i intersects A in a single point. We denote as a, b and r the length $\lambda(A) = a$ of the tree part A, the length $\lambda(B) = b$ of the Eulerian part $B = \cup E_i$, and the radius $r = \min_{x\in A} \max_{y\in A} d(x,y)$ of A. The minimizing point is called the *center* c of A. The numbers for $\bar{\mu}$ and $\tilde{\mu}$ are easily calculated as follows. (The inequality is later shown to be an equality in Theorem 6)

Lemma 2 *If G is a partly Eulerian network we have*

$$\bar{\mu} = 2a + b, \text{ and} \tag{1}$$

$$\tilde{\mu} \leq 2a + b - 2r. \tag{2}$$

Proof. The equality holds because doubling all the arcs of A gives a network having all nodes of even degree (so $\bar{\mu} \leq 2a+b$), and furthermore any tour of G must traverse every arc of A at least twice (because every non-leaf node of A is a cut point), so $\bar{\mu} \geq 2a+b$. To obtain the inequality, let x and y be distinct nodes of A at distance r from the center c of the tree A. Let G^* be the network of total length $2a-2r+b$ in which every arc of G is doubled, except those in B or those on the simple path in A of length $2r$ from x to y. Then all nodes of G^* except x and y have degree 2, so there is an Eulerian path from x to y in G^*. This path has length $2a-2r+b$ and may be interpreted as a covering path of G. Hence $\tilde{\mu} \leq 2a+b-2r$ ■

A *random CP path* is an equiprobable randomization of a CP path S and its reverse CP path $S'(t) = S(\tilde{\mu}-t)$ (going from $S(\tau=\tilde{\mu})$ to $S(0)$). If a point $x \in G$ is found by S at time t_x, then it is found by S' no later than $\tilde{\mu}-t_x$, hence a random CP path finds every point of G in average time no more than $\tilde{\mu}/2$. Since this is a mixed strategy in $\Gamma(G)$, we have

$$V(G) \leq \tilde{\mu}/2. \tag{3}$$

We say that G is *simply searchable* if $V(G) = \tilde{\mu}/2$, in which case a random CP path is an optimal strategy in $\Gamma(G)$. We show that a partly Eulerian network is simply searchable, generalizing a similar result of Dagan and Gal [4] for trees.

5 Lower bounds on $V(G)$

The following result is implicit in the Dagan-Gal proof that a tree is simply searchable.

Lemma 3 *Let G be any network. If h is an optimal hider mixed strategy in $\Gamma(G,y)$ and S is a pure search strategy with $S(0)=x$, then $T(S,h) \geq V_y - d(x,y)$. Consequently for all $x,y \in G$ we have $|V_x - V_y| \leq d(x,y)$.*

Proof. Let S' denote the pure search strategy with $S'(0)=y$, which begins by going directly to x and then follows S. Since S' reaches any point of G at most time $d(x,y)$ later than S does, we have

$$\begin{aligned} V_y &\leq T(S',h) \leq d(x,y) + T(S,h), \text{ or} \\ T(S,h) &\geq V_y - d(x,y). \end{aligned}$$

Since this is true for all S with $S(0)=x$, we have

$$V_x \geq V_y - d(x,y), \text{ and so by symmetry } |V_x - V_y| \leq d(x,y).$$

■

Lemma 4 *Let $G = A \cup B$ be a partly Eulerian network with tree part A and Eulerian part B. Let h be a mixed hider strategy (probability measure) on G which is uniform (possibly with different densities) on each component E_i of the Eulerian part B of G. Then there is a pure search strategy which is optimal against h and starts in A.*

Proof. Suppose S is an optimal strategy against h with $S(0) \notin A$. (Optimal strategies against h exist by the usual compactness argument, and if they all start in A we are done). Let $E_1 \subset B$ be the Eulerian component in which the search strategy S begins, and let e denote its attachment point to A. We show that there is another optimal search strategy $\hat{S}$ against h which starts at e.

Let $C = (G - E_1) \cup \{e\}$ be the complement of the interior of E_1. We may break up the search S into a search piece S^1 in E_1 followed by a search piece S^2 in C, a search piece S^3 in E_1, and so on, ending either in E_1 or C. Let t_i denote the time the i'th search ends (at e, except possibly for the last search S^k). Let p_i denote the probability that the hider will be found in the i'th search, let $L_i = t_i - t_{i-1}$ ($t_0 \equiv 0$) denote length of the i'th search, ρ_i its density and let c_i denote the offset of the center of mass of the i'th search piece from its starting time. That is, if $F(t)$ is the probability that S finds the hider by time t, we have

$$\begin{aligned} p_i &= F(t_i) - F(t_{i-1}), \\ L_i &= t_i - t_{i-1} \quad (t_0 \equiv 0), \\ c_i &= \frac{1}{p_i} \int_{t_{i-1}}^{t_i} (t - t_{i-1})\ dF(t) \\ \rho_i &= p_i / L_i. \end{aligned}$$

In these terms we can calculate the expected time w to find the hider as

$$w = T(S,h) = p_1 c_1 + p_2 (L_1 + c_2) + p_3 (L_1 + L_2 + c_3) + \cdots . \tag{4}$$

Note that except for S^1 and S^k, all the search pieces S^i start and end at the node e, and so could be interchanged to still describe a valid search path. Observe that if we modify the search S to a search $\hat{S}$ which is the same as S except that S^i and S^{i+1} are interchanged (some i between 2 and $k-2$, we get

$$\begin{aligned} T(S,h) - T\left(\hat{S},h\right) &= (p_i (t_{i-1} + c_i) + p_{i+1} (t_i + c_{i+1})) \\ &\quad - (p_{i+1} (t_{i-1} + c_{i+1}) + p_i (t_{i-1} + L_{i+1} + c_i)) \\ &= L_i p_{i+1} - L_{i+1} p_i, \text{or} \\ T(S,h) &\le T\left(\hat{S},h\right) \text{ if and only if } \rho_i \ge \rho_{i+1}. \end{aligned}$$

So transposing the order between an earlier lower density search and a later higher density search cannot increase the expected capture time. (5)

Consequently, the optimality of S implies that higher density search pieces are carried out before lower density ones. Thus

$$\rho_2 \geq \rho_3 \geq \cdots \geq \rho_{k-2} \geq \rho_{k-1}. \tag{6}$$

This optimality condition for alternating search problems is essentially that of Proposition 3 of Alpern and Howard [3].

The next part of the proof can be interpreted as a new problem, better for the searcher, in which he can always search the remainder of E_1 according to the density δ (the density of the measure h on E_1) whenever he arrives at e, without retracing any parts of E_1 he has already searched. Observe that for all searches S^i in E_1 (that is, i odd) we have

$$\begin{aligned} L_i &\geq p_i/\delta, \text{ and} \\ c_i &\geq p_i/2\delta. \end{aligned}$$

Hence if $\bar{w}$ represents the sum (4) with L_i replaced by$\bar{L}_i = p_i/\delta$ and c_i by $\bar{c}_i = p_i/2\delta$ and hence $\bar{\delta}_i = \delta$ for all odd i, we have

$$\bar{w} = T\left(\bar{S}\right) \leq T\left(S, h\right) \leq w.$$

We call the shortened searches of E_1, searches $\bar{S}^i$, for odd i. Of course we don't know if the search pattern $\bar{S}$, with these distributions of capture times, can be realized by a continuous unit speed path in G.

In the sum for $\bar{w}$, all the odd numbered densities $\bar{\rho}_i$ will be equal to $p_i/L_i = \delta$, and the even numbered ones are unchanged, $\bar{\rho}_i' = \rho_i$, i even. So the density sequence in the sum for $\bar{w}$ will look like

$$\delta, \rho_2, \delta, \rho_4, \delta, \rho_6, \delta, \ldots, \rho_k \text{ (if } k \text{ is even), or} \tag{7}$$

$$\delta, \rho_2, \delta, \rho_4, \delta, \rho_6, \delta, \ldots, \rho_{k-1}, \delta \text{ (if } k \text{ is odd).} \tag{8}$$

The modification of S to $\bar{S}$ left the searches of C unchanged while making the searches of E_1 maximally efficient, and did not change the order of the searches. We next modify $\bar{S}$ to a search $\hat{S}$ by changing the *order* of the search pieces,without increasing the expected meeting time, using the observation (5). Note that the densities ρ_{2j} are decreasing because of (6) so we can make transpositions of adjacent searches which do not increase the resulting expected capture time sum $\hat{w}$, and put all the searches in E_1 in a consecutive sequence. For example, if δ lies between ρ_6 and ρ_4, the sequence $\hat{S}$ given by

$$S^2,\ S^4,\ \overbrace{\bar{S}^1,\ \bar{S}^3, \ldots,\ \bar{S}^{k-1}},\ S^6,\ S^8, \ldots,\ S^k \text{ (if } k \text{ is even)}$$

with density sequence

$$\rho_2, \rho_4, \delta, \delta, \delta, \ldots \delta, \rho_6, \rho_8, \rho_{10}, \ldots$$

has an expected capture time no larger than that of $\bar{S}$, that is

$$T\left(\hat{S}, h\right) \leq T\left(\bar{S}, h\right) \leq T\left(S, h\right).$$

We now show that the search procedure $\hat{S}$ can be carried out by a unit speed path in G. This requires us to establish that the total duration of the searches in E_1 is exactly the length of E_1, since E_1 is Eulerian, and can be searched in a time equal to its length by a tour starting and ending at e.

By construction, the length of time spent by $\hat{S}$ in exploring E_1 is

$$\bar{L}_1 + \bar{L}_3 + \cdots = \frac{1}{\delta}\left(p_1 + p_2 + \ldots\right) = \lambda\left(E_1\right), \text{ as required}$$

Thus all the search pieces in $\hat{S}$, including in particular the first one (which is either S^2 or an Eulerian tour of E_1 starting at e), start at e_1. ■

Lemma 5 *If G is partly Eulerian, then*

$$V\left(G\right) \geq \bar{\mu}/2 - r = a + b/2 - r, \tag{9}$$

where r is the radius of the tree part A of G.

Proof. Since a partly Eulerian network is weakly Eulerian, Theorem 1 implies that for any $z \in G$ we have

$$V_z\left(G\right) = \bar{\mu}/2.$$

Let h be an optimal hider mixed strategy for the game $\Gamma\left(G, c\right)$, where c is the center of the tree part A of G. Let S be a pure search strategy starting at some point $x \in G$ which is an optimal reply to h. By the previous Lemma we may assume that $x \in A$. Since h is also a valid hiding strategy for the arbitrary-start game $\Gamma\left(G\right)$, we have by Lemma 3 with $y = c$,

$$V\left(G\right) \geq T\left(S, h\right) \geq V_c\left(G\right) - d\left(x, c\right) \geq \bar{\mu}/2 - d\left(x, c\right) \geq \bar{\mu}/2 - r. \tag{10}$$

■

6 Simplicity of partly Eulerian networks

We can now establish our main result.

Theorem 6 *A partly Eulerian network G is simply searchable. In particular, if G based on a tree A of length a, center c, and radius r, to which are attached disjoint Eulerian networks of total length b, then*

$$V\left(G\right) = \tilde{\mu}/2 = a + b/2 - r.$$

Furthermore, an equiprobable mixture of any CP path with its reverse path is an optimal searcher mixed strategy. An optimal hider strategy for the game $\Gamma\left(G, c\right)$ is also optimal for the hider in $\Gamma\left(G\right)$.

Proof. By (9), (2) and (3), we have

$$a + b/2 - r \leq V(G) \leq \tilde{\mu}/2 \leq a + b/2 - r.$$

Since the leftmost and rightmost constants are the same, all the inequalities must be equalities and we have

$$V(G) = \tilde{\mu}/2 = a + b/2 - r,$$

and the rest follows from the definitions. (Note that this argument also establishes that (2) holds as an equality.) ■

Corollary 7 *Let G be a network of total length μ.*

1. *If G is a tree of radius r, then $V(G) = \mu - r$.*
2. *If G is Eulerian, then $V(G) = \mu/2$.*

Proof. In the first case $b = 0$ and $\mu = a$. In the second case $a = r = 0$ and $\mu = b$. ■

Part 1 is due to Dagan and Gal [4]. Part 2 is easy to establish directly from the definitions.

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