Irrational transfer function classes, coprime factorization and stabilization

Amol J. Sasane
Department of Mathematics, London School of Economics,
Houghton Street, London WC2A 2AE, United Kingdom.
E-mail: `A.J.Sasane@lse.ac.uk`

CDAM Research Report CDAM-LSE-2005-10

May, 2005

Abstract. Classes of irrational function classes, denoted by A_S, that lie between the extreme cases of the disk algebra A and the Hardy space $H^\infty(\mathbb{D})$, are considered. The corona theorem holds for A_S, and the following properties are shown: A_S is an integral domain, but not a Bézout domain, A_S is a Hermite ring with stable rank 1, and the Banach algebra A_S has topological stable rank 2. Consequences to the coprime factorization of transfer functions and stabilizing controller synthesis using a factorization approach are discussed.

Keywords. Function algebras, coprime factorization, stabilization, infinite-dimensional systems.

1 Introduction

In the factorization approach to control system analysis and synthesis, one starts with a frequency domain description of the system in terms of its transfer function, and expresses the transfer function as a ratio of two stable transfer functions. Many important control problems can then be formulated and solved with this approach. The book by Vidyasagar [36] is a classical reference and the recent papers by Quadrat [23], [24], [25] give a modern comprehensive treatment of the factorization approach.

As opposed to finite-dimensional systems, the transfer functions of infinite-dimensional systems are irrational, and there are many different useful classes of transfer functions; see Section 7.5 from Curtain and Zwart [8]. In order to use a factorization approach for solving control problems, we would like to factor the unstable transfer function as a ratio of transfer functions from a certain stable subclass. In the case of infinite-dimensional systems, there are many different notions of internal stability: weak stability, strong stability, exponential stability and so on. So it is natural to expect a wide range of function classes for stable transfer functions. Among the classical Banach algebras considered for the purposes of systems theory, we mention the disk algebra, the Callier-Desoer class (see [2]), the Hardy space $H^\infty(\mathbb{D})$ and the Nevanlinna class (see [7]).

In this article, we consider a family of function classes A_S lying between the extremal classes of the disk algebra A and the Hardy space $H^\infty(\mathbb{D})$. A_S consists of functions that are analytic in the open unit disk, and bounded and continuous on the open unit disk together with a subset S of the unit circle. Many transfer functions have this property and so it is useful to be able to develop a factorization approach based on this class of stable transfer functions A_S. The properties that play an important role in the factorization approach (see [36], [23], [24], [25]) are listed below, and it is known that the disk algebra A and $H^\infty(\mathbb{D})$ have these useful properties:

P1. The corona theorem.

P2. The Hermite property.

P3. Stable rank $= 1$.

P4. Topological stable rank $= 2$.

We prove that the properties P2, P3 and P4 also hold for the infinitely many intermediate spaces A_S, where S is an arbitrary subset of the unit circle. (The property P1 is also true for A_S, and this was already known to be true, but we give new bounds in Section 2.)

We will use the following standard notation:

$$\begin{array}{rcll} \mathbb{D} & = & \{z \in \mathbb{C} \mid |z| < 1\} & \text{(open unit disk)} \\ \overline{\mathbb{D}} & = & \{z \in \mathbb{C} \mid |z| \leq 1\} & \text{(closed unit disk)} \\ \mathbb{T} & = & \{z \in \mathbb{C} \mid |z| = 1\} & \text{(unit circle).} \end{array}$$

For convenience, we work with the unit disk, but the function classes and their corresponding results can be translated to the half-plane case by the usual linear fractional transformation

$$\mu : \overline{\mathbb{D}} \to \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\} \cup \{\infty\}$$

given by

$$z \mapsto s = \frac{1-z}{1+z}, \tag{1}$$

that takes $\mathbb{D}$ to the open right half-plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, and $\mathbb{T}$ to the imaginary axis $i\mathbb{R}$ with the point at ∞. The map μ is one-to-one, onto, analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Its inverse μ^{-1} is analytic in $\mathbb{C}_+$ and continuous on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\} \cup \{\infty\}$.

We now introduce the transfer function class A_S.

Definition. Let S be a subset of $\mathbb{T}$. Let

$$A_S = \{f : \mathbb{D} \cup S \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{D} \text{ and } f \text{ is continuous and bounded on } \mathbb{D} \cup S\},$$

equipped with the supremum norm $\|\cdot\|_\infty$: if $f \in A_S$, then

$$\|f\|_\infty := \sup_{z \in \mathbb{D} \cup S} |f(z)|.$$

We note that if $S = \mathbb{T}$, then $A_\mathbb{T}$ is the usual disk algebra, often denoted simply by A:

$$A = A_\mathbb{T} = \{f : \overline{\mathbb{D}} \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{D} \text{ and } f \text{ is continuous and bounded on } \overline{\mathbb{D}}\},$$

while if $S = \emptyset$, then one obtains the Hardy space with $p = \infty$, usually denoted by $H^\infty(\mathbb{D})$:

$$A_\emptyset = H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and bounded in } \mathbb{D}\}.$$

If S_1, S_2 are two subsets of $\mathbb{T}$ such that $S_1 \subset S_2$, then we have $A_{S_2} \subset A_{S_1}$. In this manner, we obtain the family of function algebras, $\mathscr{F} = \{A_S \mid S \subset \mathbb{T}\}$, partially ordered with respect to set inclusion. The extremes are the classical spaces:

$$A = A_\mathbb{T} \subset A_S \subset A_\emptyset = H^\infty(\mathbb{D}).$$

Thus we classify transfer functions by points on the boundary of the disk[1] to which there exists a continuous extension. This is a natural thing to do, since the transfer function of state linear systems

$$G(s) = C(sI - A)^{-1}B + D, \quad s \in \mathbb{C} \setminus \sigma(A)$$

is continuous on the imaginary axis at all points in the resolvent set of the operator A.

The spaces A_S considered here have been studied earlier from a pure mathematics point of view in Détraz [10], [11], [12].

The Callier-Desoer class $\hat{\mathscr{A}}(0)$ is an important class of irrational transfer functions (see Chapter 7 of Curtain and Zwart [8]), and it is shown in Theorem 1.1 below that $\hat{\mathscr{A}}(0)$ is contained in A_S. First we recall the definition of $\hat{\mathscr{A}}(0)$.

Definition. The *Callier-Desoer class* is the set of functions $f : \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\} \to \mathbb{C}$ such that

$$f(s) = \int_0^{+\infty} e^{-st} f_\text{a}(t)dt + \sum_{n=1}^{+\infty} f_n e^{-st_n}, \quad s \in \mathbb{C} \text{ with } \text{Re}(s) \geq 0,$$

for some $f_\text{a} \in L^1((0,\infty);\mathbb{C})$ and some complex sequence $(f_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{+\infty} |f_n| < +\infty.$$

The Callier-Desoer class is denoted by $\hat{\mathscr{A}}(0)$.

For properties of the class $\hat{\mathscr{A}}(0)$, see Callier and Desoer [2] or §A.7.4, pages 661-668 of Curtain and Zwart [8].

[1]equivalently on the imaginary axis, when passing over to the half-plane

Theorem 1.1 *Let $f \in \hat{\mathscr{A}}(0)$ and $f^{\mathrm{d}} : \overline{\mathbb{D}} \setminus \{-1\} \to \mathbb{C}$ be defined by*

$$f^{\mathrm{d}}(z) = (f \circ \mu)(z), \quad z \in \overline{\mathbb{D}} \setminus \{-1\}.$$

Then $f^{\mathrm{d}} \in A_{\mathbb{T}\setminus\{-1\}}$.

Proof Analyticity in $\mathbb{D}$ and boundedness on $\overline{\mathbb{D}} \setminus \{-1\}$ follow from parts c. and b., respectively, of Lemma A.7.47 on page 663 of Curtain and Zwart [8]. Continuity on $\overline{\mathbb{D}}\setminus\{-1\}$ can be seen as follows. As $f_{\mathrm{a}} \in L^1((0,\infty);\mathbb{C})$, the map

$$s \mapsto \int_0^{+\infty} e^{-st} f_{\mathrm{a}}(t)dt$$

is continuous on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$. Moreover, as

$$\left| \sum_{n=1}^{+\infty} f_n e^{-st_n} - \sum_{n=1}^{N} f_n e^{-st_n} \right| \leq \sum_{n=N}^{+\infty} |f_n|,$$

it follows that the convergence of the partial sums is uniform. Since each finite sum is continuous on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$, we obtain continuity of the limit function

$$s \mapsto \sum_{n=1}^{+\infty} f_n e^{-st_n}$$

on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$. ■

Just as with the extremal cases of the disk algebra A and the Hardy space $H^\infty(\mathbb{D})$, which are Banach algebras, it turns out that each function class A_S is a Banach algebra, and we prove this below, after we recall the notion of a Banach algebra.

Definitions. A *complex algebra* is a vector space R over $\mathbb{C}$ in which an associative and distributive multiplication is defined, that is,

$$x(yz) = (xy)z, \quad (x+y)z = xz + yz, \quad x(y+z) = xy + xz$$

for all $x, y, z \in R$, and which is related to scalar multiplication so that

$$\alpha(xy) = x(\alpha y) = (\alpha x)y$$

for all $x, y \in R$ and all scalars α.

A *Banach algebra* is a complex algebra R which is also a Banach space under a norm satisfying

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in R$.

Theorem 1.2 *Let $S \subset \mathbb{T}$. A_S is a Banach algebra.*

Proof The completeness can be shown as follows. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then for each $z \in \mathbb{D} \cup S$, the sequence $(f(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and so by the completeness of $\mathbb{C}$, it has a limit, say $f(z)$. These pointwise limits give rise to a complex valued function f defined on $\mathbb{D} \cup S$. We claim that f belongs to A_S. Clearly f is the uniform limit of the f_n's on $\mathbb{D} \cup S$. As the uniform limit of analytic functions is analytic, it follows that f is analytic in $\mathbb{D}$. Continuity and boundedness in $\mathbb{D} \cup S$ follow from the fact that the set of bounded continuous functions from a topological space X (in our case $\mathbb{D} \cup S$) to $\mathbb{C}$ with the supremum norm is a Banach space; see for instance, Example 1 on page 32 and the Remark on page 55 of Yosida [37]. ■

A_S is commutative, and has 1_{A_S} as the identity element, where 1_{A_S} denotes the constant function taking value 1 everywhere on $\mathbb{D} \cup S$. It satisfies $\|1_{A_S}\|_\infty = 1$.

We now give examples to show that these Banach algebras A_S arise quite naturally when considering transfer functions of infinite-dimensional linear systems.

Example. *(Pure delay)* Delay differential equations can be viewed as infinite-dimensional systems (see Section 2.4 of Curtain and Zwart [8]), and they have irrational transfer functions that are quotients of polynomials in s and e^{-st_n} with t_n's being the positive delays. The simplest example is provided by the pure delay system, namely:

$$y(t) = u(t-1), \quad t \geq 0,$$

which has the transfer function $G(s) = e^{-s}$. The composition of G with the map μ defined by (1), gives the function $G^{\mathrm{d}} : \overline{\mathbb{D}} \setminus \{-1\} \to \mathbb{C}$ given by

$$G^{\mathrm{d}}(z) = \exp\left(\frac{z-1}{z+1}\right), \quad z \in \overline{\mathbb{D}} \setminus \{-1\}.$$

We note that the map G^{d} belongs to A_S, where $S = \mathbb{T} \setminus \{-1\}$. Indeed, G^{d} is analytic in $\mathbb{D}$ since $s \mapsto e^{-s}$ is entire, and the boundedness of G^{d} follows from the fact that

$$|e^{-s}| = e^{-\mathrm{Re}(s)} \leq 1 \text{ for all } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) \geq 0.$$

Also, as $\omega \mapsto e^{-i\omega}$ does not have limits at $\pm\infty$, it follows that $G^{\mathrm{d}} \notin A = A_{\mathbb{T}}$. Of course, $G^{\mathrm{d}} \in H^\infty(\mathbb{D})$, but this class seems too large for the case under consideration. So we arrive quite naturally at the class

$$A_{\mathbb{T}\setminus\{-1\}} = \{f : \overline{\mathbb{D}}\setminus\{-1\} \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{D} \text{ and continuous and bounded on } \overline{\mathbb{D}}\setminus\{-1\}\}.$$

◊

In control design, the concept of stability chosen for consideration depends on the application at hand. Hence the properties demanded from the class of stable transfer functions depends on the type of systems being considered.

The following example shows that the classes A_S might be particularly useful when considering systems that have generators A that are not exponentially stable, but are stable only in some weaker sense. Indeed, it is typical that the spectrum of A has accumulation points on the extended imaginary axis when A is strongly stable, and so one can expect a loss of continuity at these points on the extended imaginary axis for the transfer function.

Example. Let $\ell_2(\mathbb{N})$ denote the Hilbert space of square summable sequences, and let the standard orthonormal basis for $\ell_2(\mathbb{N})$ be denoted by $\{e_n \mid n \in \mathbb{N}\}$. Consider the system

$$\begin{aligned} \frac{dx}{dt}(t) &= A_0 x(t) + Bu(t) \\ y(t) &= B^* x(t) \end{aligned}$$

on $\ell_2(\mathbb{N})$, where $A_0 : D(A_0)\,(\subset \ell_2(\mathbb{N})) \to \ell_2(\mathbb{N})$ is the operator given by

$$A_0 = \left[\begin{array}{cc|cc|cc|c} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ \hline & & 0 & 2 & & & \\ & & -2 & 0 & & & \\ \hline & & & & 0 & 3 & \\ & & & & -3 & 0 & \\ \hline & & & & & & \ddots \end{array}\right], \tag{2}$$

with

$$D(A_0) = \left\{ x \in \ell_2(\mathbb{N}) \;\middle|\; \sum_{n=1}^{\infty} \left(|n\langle x, e_{2n-1}\rangle|^2 + |n\langle x, e_{2n}\rangle|^2 \right) < \infty \right\}, \tag{3}$$

and $B : \mathbb{C} \to \ell_2(\mathbb{N})$ is the bounded linear operator given by

$$B = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{3} \\ 0 \\ \vdots \end{bmatrix}. \tag{4}$$

The following result was shown in Curtain and Sasane [5].

Theorem 1.3 *Let A_0 be given by (2) and (3), and B be given by (4). Then the following hold:*

1. *A_0 is a Riesz spectral operator with the eigenvalues $\pm ni$, $n \in \mathbb{N}$, and the corresponding (orthogonal) Riesz basis of eigenvectors $\frac{1}{\sqrt{2}}(e_n \pm ie_{n+1})$,*

2. *A_0 is the infinitesimal generator of a strongly continuous contraction semigroup on $\ell_2(\mathbb{N})$,*

3. *A_0 has compact resolvent,*

4. *(A_0, B) is approximately controllable, and (A_0^*, B^*) is approximately observable,*

5. *$A_0 - BB^*$ and $A_0^* - BB^*$ generate strongly stable semigroups on $\ell_2(\mathbb{N})$,*

6. *the state linear system given by the triple $(A_0 - BB^*, B, B^*)$ has a Hankel operator that is bounded, but not compact.*

From Hartman's theorem (see for instance Corollary 4.10 on page 46 of Partington [22]), we see that the transfer function $G(s) = B^*(sI - A_0 + BB^*)^{-1}B$ cannot be continuous at infinity. Hence the corresponding function on the disk, namely $G^{\mathrm{d}} = G \circ \mu$, where μ is given by (1), does not belong to the disk algebra A. As $\|BB^*\| \leq \frac{1}{4}$, from Theorem 3.6 on page 209 of Kato [16], it follows that

$$\sigma(A_0 - BB^*) \subset \bigcup_{m\in\mathbb{Z}} \left\{ s \in \mathbb{C} \;\middle|\; |s - mi| \leq \frac{1}{4} \right\}.$$

Thus if μ denotes the map given by (1), then with

$$S := \bigcup_{m\in\mathbb{Z}} \mu^{-1} \left\{ s \in \mathbb{C} \;\middle|\; \left| s - \left(m + \frac{1}{2}\right) i \right| < \frac{1}{4} \right\},$$

we have that $G^{\mathrm{d}} \in A_S$. $\Diamond$

Example. Consider a well-posed linear system Σ with the generating operators A, B, C and transfer function G, such that $0 \in \mathbb{C} \setminus \sigma(A)$. For the theory of well-posed linear systems, we refer the reader to Staffans [31]. The *reciprocal system* of the well-posed linear system Σ, introduced by Curtain (see for example [3]), is the well-posed linear system with the bounded generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$, $G(0)$ and transfer function

$$G_-(s) = G(0) - CA^{-1}\left(sI - A^{-1}\right)^{-1} A^{-1}B = G\left(\frac{1}{s}\right).$$

Reciprocal systems are useful in the analysis of control systems, since the operators A^{-1}, $A^{-1}B$, $-CA^{-1}$, $G(0)$ are all bounded: indeed, one can pass from the original system to its reciprocal, solve the transformed control problem for it, and then return back to the original system (see for example, [4], [6], [21]).

We note that G_- is bounded and analytic in the open right half-plane $\mathbb{C}_+$ and continuous (and even analytic) in a neighbourhood 0. Hence the corresponding function G_-^{d} on the unit disk belongs to the space A_S, where S is a suitably small arc around the point $\{1\}$. $\Diamond$

The properties P1, P2, P3, P4 are proved in Sections 2, 3, 4, 5, respectively. Applications of these properties to coprime factorization and stabilization are given in Section 6.

2 The corona theorem

In Section 6 we will give a test for coprimeness of a matrix pair (N, D) in Theorem 6.1. This test for coprimeness is obtained by using a necessary and sufficient condition for the Bézout identity to hold in the algebra A_S, which is given in Theorem 2.4, called the corona theorem for A_S.

The first part of Theorem 2.4, that is, the statement in Theorem 2.4 up to (17), is not new, and can be found in Theorem 2 of Détraz [12], and for closed subsets S of $\mathbb{T}$, it was shown in the Corollary on page 514 of Stray [32]. Nevertheless, for the sake of completeness, a proof of Theorem 2.4 is given here, using Carleson's corona theorem and an approximation result. This proof was given by Rosay [29]. We also show the existence of solutions with bounds (see (19) and the remark following Theorem 2.4).

Theorem 2.4 is a generalization of Carleson's corona theorem for $H^\infty(\mathbb{D})$, and the proof of Theorem 2.4 given here uses the full strength of Carleson's theorem. So we do not obtain a new proof of the Carleson corona theorem when $S = \mathbb{T}$!

The classical Carleson's corona theorem is the following, and for a simplified proof of this theorem, we refer the reader to Narasimhan and Nievergelt [19].

Theorem 2.1 (Carleson) *Let $f_1, \dots, f_n \in H^\infty(\mathbb{D})$. There exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{5}$$

iff there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1. \tag{6}$$

Furthermore, there exists a constant $C_\emptyset(n, \delta)$ such that if $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ satisfy (5), and for all $i \in \{1, \dots, n\}$,

$$\forall z \in \mathbb{D}, \quad |f_i(z)| \leq 1, \tag{7}$$

then there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ satisfying (6) with the bounds

$$\forall z \in \mathbb{D}, \quad |g_i(z)| \leq C_\emptyset(n, \delta), \tag{8}$$

for all $i \in \{1, \dots, n\}$.

This theorem also happens to be true with the disk algebra A instead of $H^\infty(\mathbb{D})$. A nonconstructive proof (relying on Zorn's lemma) using elementary theory of Banach algebras can be found in Rudin [30] (see Theorem 18.18 on page 365), which gives the result below without statement following (10) about the existence of a universal constant. Theorem 2.4 below, applied to the case $S = \mathbb{T}$, also yields the existence of such a universal constant, and hence we get solutions satisfying the estimates given in (12).

Theorem 2.2 *Let $f_1, \dots, f_n \in A$. There exists a $\delta > 0$ such that*

$$\forall z \in \overline{\mathbb{D}}, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{9}$$

iff there exist $g_1, \dots, g_n \in A$ such that

$$\forall z \in \overline{\mathbb{D}}, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1. \tag{10}$$

Furthermore, there exists a constant $C_{\mathbb{T}}(n, \delta)$ such that if $f_1, \dots, f_n \in A$ satisfy (9), and for all $i \in \{1, \dots, n\}$,

$$\forall z \in \overline{\mathbb{D}}, \quad |f_i(z)| \leq 1, \tag{11}$$

then there exist $g_1, \dots, g_n \in A$ satisfying (10) with the bounds

$$\forall z \in \overline{\mathbb{D}}, \quad |g_i(z)| \leq C_{\mathbb{T}}(n, \delta), \tag{12}$$

for all $i \in \{1, \dots, n\}$.

Theorem 2.4 gives the same results as the above two cases, in the more general case when S is between the two extreme cases: $\emptyset \subset S \subset \mathbb{T}$. This can be proved using Carleson's corona theorem for $H^\infty(\mathbb{D})$ and the following approximation result.

Lemma 2.3 *Let $S \subset \mathbb{T}$. If $f_1, \dots, f_n \in A_S$, then given any $\epsilon_1 > 0$ and any $\epsilon_2 > 0$, there exists an open connected set Ω containing $\mathbb{D} \cup S$ (which depends on ϵ_1 and ϵ_2 in general), and there exist analytic functions $f_i^{\mathrm{e}} : \Omega \to \mathbb{C}$, $i \in \{1, \dots, n\}$ such that*

$$\forall z \in \mathbb{D} \cup S, \quad \forall i \in \{1, \dots, n\}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon_1, \text{ and} \tag{13}$$

$$\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } \forall i \in \{1, \dots, n\}, \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \epsilon_2. \tag{14}$$

Proof Let $i \in \{1, \dots, n\}$. From the result in Corollary 1.3 on page 38 of Davie et al. [9], it follows that there exists an open set Ω_i' containing $\mathbb{D} \cup S$, and an analytic $f_i^{\mathrm{e}} : \Omega_i' \to \mathbb{C}$ such that

$$\forall z \in \mathbb{D}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon_1. \tag{15}$$

Let

$$\Omega' = \bigcap_{i=1}^{n} \Omega_i',$$

and replace f_i^{e}'s by their restrictions to Ω'. By using the continuity of f_i on $\mathbb{D} \cup S$, and also that of f_i^{e}, (15) yields (13).

Let $i \in \{1, \dots, n\}$. For each $z_* \in S$, there exists an $r_{z_*}^i > 0$ such that the open ball with center z_* and radius $r_{z_*}^i$ is contained in Ω', that is, $B(z_*, r_{z_*}^i) \subset \Omega'$, and moreover,

$$\forall z \in B(z_*, r_{z_*}^i), \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \epsilon_2.$$

Define $r_{z_*} = \min\{r^1_{z_*}, \dots, r^n_{z_*}\}$, and let

$$\Omega = \mathbb{D} \cup \left(\bigcup_{z_* \in S} B(z_*, r_{z_*}) \right).$$

Then Ω is an open connected set containing $\mathbb{D} \cup S$, and the restriction of f_i^{e}'s to Ω satisfy (13) and (14). ■

The following result is the corona theorem for the algebra A_S.

Theorem 2.4 *Let $S \subset \mathbb{T}$ and suppose that $f_1, \dots, f_n \in A_S$. There exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{16}$$

iff there exist $g_1, \dots, g_n \in A_S$ such that

$$\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1. \tag{17}$$

Furthermore, there exists a constant $C_S(n, \delta)$ such that if $f_1, \dots, f_n \in A_S$ satisfy (16), and for all $i \in \{1, \dots, n\}$,

$$\forall z \in \mathbb{D} \cup S, \quad |f_i(z)| \leq 1, \tag{18}$$

then there exist $g_1, \dots, g_n \in A_S$ satisfying (17) with the bounds

$$\forall z \in \mathbb{D} \cup S, \quad |g_i(z)| \leq C_S(n, \delta), \tag{19}$$

for all $i \in \{1, \dots, n\}$.

Proof The necessity of the condition (16) for (17) to hold is obvious, since

$$1 = \left| \sum_{i=1}^{n} f_i(z) g_i(z) \right| \leq \max\{\|g_1\|_\infty, \dots, \|g_n\|_\infty\} \sum_{i=1}^{n} |f_i(z)|,$$

and we prove the sufficiency below.

Assume that (18) holds, as this can always be ensured by multiplication by a suitable constant (and replacing the δ). The proof is long, and so we have divided it into a sequence of steps.

Step 1. Let

$$\epsilon = \min\left\{ \frac{1}{2nM_\delta C_\emptyset\left(n, \frac{\delta}{4} M_\delta\right)}, \frac{\delta}{2n} \right\}, \tag{20}$$

where $C_\emptyset(\cdot, \cdot)$ denotes a universal constant in Carleson's Theorem 2.1 above, and

$$M_\delta = \frac{1}{\frac{\delta}{4n} + \frac{\delta}{2n} + 1}.$$

Then from Lemma 2.3, there exists an open connected neighbourhood Ω of $\mathbb{D} \cup S$ and analytic functions $f_i^{\mathrm{e}} : \Omega \to \mathbb{C}$, $i \in \{1, \dots, n\}$, such that

$$\forall z \in \mathbb{D} \cup S, \quad \forall i \in \{1, \dots, n\}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon, \text{ and} \tag{21}$$

$$\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } \forall i \in \{1, \dots, n\}, \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \frac{\delta}{4n}. \tag{22}$$

Then for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned}
\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i(z) - (f_i(z) - f_i^{\mathrm{e}}(z))| \\
&\geq \sum_{i=1}^{n} (|f_i(z)| - |f_i(z) - f_i^{\mathrm{e}}(z)|) \\
&\geq \delta - n \cdot \frac{\delta}{2n} \quad \text{(using (16), (21) and (20))} \\
&= \frac{\delta}{2} \qquad (23) \\
&> \frac{\delta}{4}. \qquad (24)
\end{aligned}$$

Furthermore, for $z \in \Omega \setminus \mathbb{D}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z_*) - (f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z))| \quad \text{(where } z_* \text{ is as in (22))} \\
&\geq \sum_{i=1}^{n} (|f_i^{\mathrm{e}}(z_*)| - |f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z)|) \\
&\geq \frac{\delta}{2} - n \cdot \frac{\delta}{4n} \quad \text{(using (23) and (22))} \\
&= \frac{\delta}{4}. \qquad (25)
\end{aligned}$$

From (24) and (25), we obtain

$$\forall z \in \Omega, \quad \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| \geq \frac{\delta}{4}. \tag{26}$$

STEP 2. For all $z \in \mathbb{D} \cup S$,

$$\begin{aligned}
|f_i^{\mathrm{e}}(z)| &\leq |f_i^{\mathrm{e}}(z) - f_i(z)| + |f_i(z)| \\
&< \epsilon + 1 \\
&\leq \frac{\delta}{2n} + 1 \qquad (27) \\
&< \frac{\delta}{4n} + \frac{\delta}{2n} + 1 \\
&= \frac{1}{M_\delta}.
\end{aligned}$$

Furthermore, for all $z \in \Omega \setminus \mathbb{D}$,

$$\begin{aligned} |f_i^{\mathrm{e}}(z)| &\le |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| + |f_i^{\mathrm{e}}(z_*)| \quad \text{(where } z_* \text{ is as in (22))} \\ &< \frac{\delta}{4n} + \frac{\delta}{2n} + 1 \quad \text{(using (22) and (27))} \\ &= \frac{1}{M_\delta}. \end{aligned}$$

Hence for all $z \in \Omega$,

$$|M_\delta f_i^{\mathrm{e}}(z)| \le 1. \tag{28}$$

STEP 3. By the Riemann mapping theorem (see for instance Theorem 14.8 on page 283 of Rudin [30]), there exists a one-to-one analytic map φ from Ω onto $\mathbb{D}$. Thus $\varphi^{-1} : \mathbb{D} \to \Omega$ is also analytic. For each $i \in \{1, \dots, n\}$, the maps $M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1} \in H^\infty(\mathbb{D})$ and moreover, from (26) and (28) we obtain

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} |(M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| \ge \frac{\delta}{4} M_\delta,$$

and

$$\forall z \in \mathbb{D}, \quad |(M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| \le 1.$$

Thus by Carleson's corona theorem (Theorem 2.1), it follows that there exist $\tilde{g}_1, \dots, \tilde{g}_n \in H^\infty(\mathbb{D})$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} (M_\delta f_i^{\mathrm{e}} \circ \varphi^{-1})(z) \tilde{g}_i(z) = 1,$$

and moreover we can choose the $\tilde{g}_i$'s such that

$$\forall z \in \mathbb{D}, \quad |\tilde{g}_i(z)| \le C_\emptyset \left(n, \frac{\delta}{4} M_\delta \right),$$

for all $i \in \{1, \dots, n\}$. Now define

$$g_i^{\mathrm{e}} = M_\delta \tilde{g}_i \circ \varphi, \quad i \in \{1, \dots, n\}.$$

Then we have that each g_i^{e} is analytic in Ω, and

$$\forall z \in \Omega, \quad \sum_{i=1}^{n} f_i^{\mathrm{e}}(z) g_i^{\mathrm{e}}(z) = 1 \quad \text{and} \quad |g_i^{\mathrm{e}}(z)| \le M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta \right). \tag{29}$$

STEP 4. Let $h : \mathbb{D} \cup S \to \mathbb{C}$ be defined by

$$h(z) = \sum_{i=1}^{n} f_i(z) g_i^{\mathrm{e}}(z), \quad z \in \mathbb{D} \cup S. \tag{30}$$

Then $h \in A_S$. Furthermore, for all $\mathbb{D} \cup S$,

$$
\begin{aligned}
|h(z)| &= \left| \sum_{i=1}^{n} f_i(z) g_i^{\mathrm{e}}(z) \right| = \left| 1 - \sum_{i=1}^{n} (f_i^{\mathrm{e}}(z) - f_i(z)) g_i^{\mathrm{e}}(z) \right| \\
&\geq 1 - \left| \sum_{i=1}^{n} (f_i^{\mathrm{e}}(z) - f_i(z)) g_i^{\mathrm{e}}(z) \right| \\
&\geq 1 - \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z) - f_i(z)| |g_i^{\mathrm{e}}(z)| \\
&\geq 1 - n \cdot \frac{1}{2nM_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right)} \cdot M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right) \qquad \text{(using (21), (20) and (29))} \\
&= \frac{1}{2}. \qquad (31)
\end{aligned}
$$

Now define $g_i : \mathbb{D} \cup S \to \mathbb{C}$, $i \in \{1, \dots, n\}$ by

$$
g_i(z) = \frac{g_i^{\mathrm{e}}(z)}{h(z)}, \quad z \in \mathbb{D} \cup S.
$$

Then the g_i's belong to A_S, and from (30) we obtain

$$
\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1.
$$

Moreover, for all $i \in \{1, \dots, n\}$,

$$
\forall z \in \mathbb{D} \cup S, \quad |g_i(z)| \leq 2M_\delta C_\emptyset \left(n, \frac{\delta}{4} M_\delta\right) =: C_S(n, \delta). \qquad (32)
$$

This completes the proof of the theorem. ■

Remark. In Garnett [14] (see page 327), the following bound was given for the universal constant $C_\emptyset(n, \delta)$ in (8):

$$
C_\emptyset(n, \delta) \leq C \cdot \left(\frac{n^{\frac{3}{2}}}{\delta^2} + \frac{n^2}{\delta^4} \right). \qquad (33)
$$

In Theorem 2.4, for $C_S(n, \delta)$ in (19), the following bound was obtained (see (32) in the proof):

$$
C_S(n, \delta) \leq C' \cdot \left(n^{\frac{3}{2}} \left(\frac{3}{n} + \frac{4}{\delta} \right)^2 + n^2 \left(\frac{3}{n} + \frac{4}{\delta} \right)^4 \right). \qquad (34)
$$

For a fixed n, the right hand sides of (33) and (34) are of the same order in δ for $\delta \downarrow 0$.

Before we derive consequences of Theorem 2.4, we recall the following terminology from the elementary theory of Banach algebras.

Definitions. Let R be a commutative Banach algebra with identity 1_R. A *complex homomorphism* is a nonzero homomorphism $\varphi : R \to \mathbb{C}$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad \varphi(\alpha x) = \alpha\varphi(x), \quad \varphi(xy) = \varphi(x)\varphi(y)$$

for all $x, y \in R$ and all scalars α. It can be verified that for every complex homomorphism φ, there holds that $\varphi(1_R) = 1$, and that φ is a continuous linear functional with norm at most equal to 1:

$$\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)| \leq 1.$$

Let $\mathfrak{M}(R)$ denote the set of complex homomorphisms of R. Then $\mathfrak{M}(R)$ is a subset of R^*, the set of all bounded linear functionals from R to $\mathbb{C}$, and in fact it is contained in the unit ball of R^*. R^* can be equipped with the weak-star topology. Recall that a set $G \subset R^*$ is open in the weak-star topology iff for every $g \in G$, there are finitely many points $x_1, \dots, x_n \in X$ and positive reals $\epsilon_1, \dots, \epsilon_n$ such that

$$\bigcap_{i=1}^{n} \{f \in R^* \mid |f(x_i) - g(x_i)| < \epsilon_i\} \subset G.$$

$\mathfrak{M}(R)$ equipped with the induced weak-star topology from R^* is a topological space, and this topology on $\mathfrak{M}(R)$ is called the *Gelfand topology*.

A subset I of R is called an *ideal* if I is a subspace of R (as a vector space), and $xy \in I$ for all $x \in R$ and $y \in I$. A *maximal ideal* is a proper ideal (that is, $\neq R$) which is not contained in any larger proper ideal.

There is a one-to-one correspondence between homomorphisms of R onto $\mathbb{C}$ and maximal ideals M in R. The correspondence is defined by $M = \ker(\varphi)$. Owing to this correspondence, the set $\mathfrak{M}(R)$ of all complex homomorphisms of R is called the *space of maximal ideals of R*.

With each element $x \in R$, we associate a complex-valued function $\hat{x}$ on $\mathfrak{M}(R)$ as follows:

$$\hat{x}(\varphi) = \varphi(x), \quad \varphi \in \mathfrak{M}(R).$$

$\hat{x}$ is called the *Gelfand transform* of x.

We now recall the following known result.

Lemma 2.5 *Let R be a commutative complex Banach algebra with identity 1_R and let $\mathfrak{M}(R)$ be the space of maximal ideals of R, and let $\mathfrak{M}_0 \subset \mathfrak{M}(R)$. Then the following are equivalent:*

1. *$\mathfrak{M}_0$ is dense (in the Gelfand topology) in $\mathfrak{M}(R)$.*
2. *Let $x_1, \dots, x_n \in R$. There exist $y_1, \dots, y_n \in R$ such that $x_1y_1 + \dots + x_ny_n = 1_R$ iff there exists a $\delta > 0$ such that for all $\varphi \in \mathfrak{M}_0$, $|\hat{x}_1(\varphi)| + \dots + |\hat{x}_n(\varphi)| \geq \delta$.*

3. Let $\Lambda \in R^{n\times m}$. Then there is a $V \in R^{m\times n}$ such that $V\Lambda = I$ iff there exists a $\delta > 0$ such that for all $\varphi \in \mathfrak{M}_0$, $\hat{\Lambda}(\varphi)^*\hat{\Lambda}(\varphi) \geq \delta I$.

Proof This is precisely Lemma 4.1.4 on page 124 of Mikkola [18], and the details can be found in Lemmas 28 and 34 on pages 339-340 of Vidyasagar [36], and pages 201-203 of Duren [13]. ■

In 2 and 3, we can also write 'if' instead of 'iff', as the converse can be shown to be true for any $\mathfrak{M}_0 \subset \mathfrak{M}(R)$.

Let $S \subset \mathbb{T}$. Then for each $z_0 \in \mathbb{D} \cup S$, the evaluation map $f \mapsto f(z_0)$ is a complex homomorphism from A_S onto $\mathbb{C}$. With this identification of the set $\mathbb{D} \cup S$ as a subset of $\mathfrak{M}(A_S)$, we now obtain the following theorem.

Corollary 2.6 *Let $S \subset \mathbb{T}$.*

1. $\mathbb{D} \cup S$ is dense (in the Gelfand topology) in $\mathfrak{M}(A_S)$.

2. Let $\Lambda \in A_S^{n\times m}$. Then there is a $V \in A_S^{m\times n}$ such that $V\Lambda = I$ iff there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $\Lambda(z)^*\Lambda(z) \geq \delta I$.

Proof This follows from Theorem 2.4 and Lemma 2.5. ■

Note that 1 in the above Corollary 2.6 says that the *corona*[2] $\mathfrak{M}(A_S) \setminus \overline{\mathbb{D} \cup S}$ is empty.

In Section 6, we will apply the result given in item 2 of Corollary 2.6 in order to characterize matrix coprime pairs in A_S.

3 The Hermite property

In Section 6, we will consider unstable transfer functions that can be expressed as a quotient of two elements from A_S. We first prove here that A_S is an integral domain, so that we can consider its field of fractions.

Definition. An *integral domain* is a commutative ring with an identity element, such that the product of two nonzero elements is nonzero: that is, if $x, y \in R$ and $xy = 0$, then $x = 0$ or $y = 0$.

Theorem 3.1 *Let $S \subset \mathbb{T}$. A_S is an integral domain.*

Proof Let $f, g \in A_S$ and $f(z)g(z) = 0$ for all $z \in \mathbb{D} \cup S$. If $f \not\equiv 0$, then there exists a $z_0 \in \mathbb{D} \cup S$ such that $f(z_0) \neq 0$. As f is continuous, it follows that there exists a $r > 0$ such that for all $z \in B(z_0, r) \cap (\mathbb{D} \cup S)$, $f(z) \neq 0$. It follows that for all $z \in B(z_0, r) \cap (\mathbb{D} \cup S)$, $g(z) = 0$, and so by the corollary on page 209 of Rudin [30], it follows that for all $z \in \mathbb{D}$, $g(z) = 0$. As g is continuous in $\mathbb{D} \cup S$, it follows that $g \equiv 0$. ■

[2] This terminology is motivated by the fact that the word 'corona' is used to describe a ring of light seen around the sun during an eclipse.

In Section 6, we will show that not every transfer function obtained as a ratio of elements of A_S has a coprime factorization in A_S. We will prove this claim by using the result in Theorem 3.2 below, which says that A_S is not a Bézout domain.

Definition. R is said to be a *Bézout domain* if every finitely generated ideal in R is principal.

The fact that A_S is a Bézout domain is unlike the situation with the ring $H(\mathbb{D})$ of analytic functions (see Theorem 15.15 of Rudin [30]), but is similar to the extremal cases of $A_\emptyset = H^\infty(\mathbb{D})$ (see von Renteln [26]) and of $A_\mathbb{T} = A$ (see Vidyasagar et al. [35]).

Theorem 3.2 *Let $S \subset \mathbb{T}$. A_S is not a Bézout domain.*

Proof In Logemann [17], it was shown that if R is subring of $H^\infty(\mathbb{C}_+)$ that contains the Laplace transform of functions in $L^1((0,\infty);\mathbb{C})$, then R contains a finitely generated ideal which is not principal. (In fact, on page 249 of [17], an explicit construction of such a finitely generated, non-principal ideal is given in terms of Blaschke products.) The disk algebra A contains the Laplace transforms of integrable functions (see §A.6.2 on page 636 of Curtain and Zwart [8]), and $A \subset A_S$. Consequently, A_S contains finitely generated ideals that are not principal. Hence A_S is not a Bézout domain. ∎

In Section 6, we will show that Theorem 3.2 has the consequence that not every transfer function has a coprime factorization. However, we will also show that if a transfer function does have a right (or left) coprime factorization then it also has a left (respectively, right) coprime factorization. This is a consequence of Theorem 3.4, which we prove next. We begin by giving a few preliminaries.

Definitions. Let R be a ring. A square matrix $U \in R^{m\times m}$ is said to be *unimodular* if it is invertible in $R^{m\times m}$. Let $X \in R^{m\times n}$ with $m < n$. X is said to be *complementable* if there exists a unimodular matrix $U \in R^{n\times n}$ that contains X as a submatrix. A row

$$\left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right] \in R^{1\times n}$$

is called a *unimodular row* if the ideal generated by $x_1, \dots, x_n$ is equal to the ring R. A ring R is called *Hermite* if every unimodular row is complementable.

Let $S \subset \mathbb{T}$. If $f_1, \dots, f_n \in A_S$, and

$$f := \left[\begin{array}{ccc} f_1 & \dots & f_n \end{array}\right],$$

then

$$\|f\|_\infty := \sup_{z\in\mathbb{D}\cup S} \left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}}.$$

If $P \in A_S^{p\times m}$, then

$$\|P\|_\infty = \sup_{z\in\mathbb{D}\cup S} \|P(z)\|_{\mathscr{L}(\mathbb{C}^m,\mathbb{C}^p)}, \tag{35}$$

The case $m = p$ is of particular interest. Indeed, $A_S^{m\times m}$ equipped with the norm $\|\cdot\|_\infty$ is a Banach algebra with the unit I. The set of invertible elements in $A_S^{m\times m}$ is denoted by $\mathscr{G}(A_S^{m\times m})$.

In order to prove Theorem 3.4, we will need the following key result.

Theorem 3.3 *If $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ and there exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta, \tag{36}$$

then

$$\exists \Lambda = \begin{bmatrix} f \\ F \end{bmatrix} \in \mathscr{G}(H^\infty(\mathbb{D})^{n\times n}), \text{ where } f = \begin{bmatrix} f_1 & \dots & f_n \end{bmatrix}, \text{ and } F \in H^\infty(\mathbb{D})^{(n-1)\times n}. \tag{37}$$

Furthermore, if $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ are such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1,$$

then Λ satisfying (37) can be chosen such that $\|\Lambda^{-1}\|_\infty \leq \|g\|_\infty(1 + \|f\|_\infty) + 1$, where $g := \begin{bmatrix} g_1 & \dots & g_n \end{bmatrix}$.

Proof By Carleson's corona theorem, we know that under the condition (36), there exist $g_1, \dots, g_n$ in $H^\infty(\mathbb{D})$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} f_i(z) g_i(z) = 1.$$

Then the result follows from Tolokonnikov's lemma (see for example, Appendix 3, §10 on page 293 of Nikol'skiĭ [20]). ∎

We are now ready to prove the following theorem. This result was known in the case of $A_\emptyset = H^\infty(\mathbb{D})$ (this follows from Tolokonnikov's lemma; see §10 in Appendix 3 of Nikol'skiĭ [20]), and also in the case of the disk algebra $A = A_\mathbb{T}$ (see Corollary 71 and Example 72 on pages 346-347 of Vidyasagar [36]).

Theorem 3.4 *Let $S \subset \mathbb{T}$. A_S is a Hermite ring.*

Proof Let $f_1, \dots, f_n \in A_S$ be such that the ideal generated by $f_1, \dots, f_n$ is the full ring A_S. Then there exists a $\delta > 0$ such that

$$\forall z \in \mathbb{D} \cup S, \quad \sum_{i=1}^{n} |f_i(z)| \geq \delta > 0.$$

Without loss of generality, we can also assume that

$$\forall z \in \mathbb{D} \cup S, \quad \left(\sum_{i=1}^{n} |f_i(z)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2}.$$

(Indeed, the f_i's and δ can be scaled without altering the hypothesis that the ideal generated by $f_1, \dots, f_n$ is the full ring A_S.)

Let

$$\epsilon_1 = \min \left\{ \frac{\delta}{2n}, \frac{1}{2M(\delta, n)}, \frac{1}{4\sqrt{n}} \right\} \text{ and } \epsilon_2 = \min \left\{ \frac{\delta}{4n}, \frac{1}{4\sqrt{n}} \right\},$$

where

$$M(\delta, n) = 2\sqrt{n} C_{\emptyset} \left(n, \frac{\delta}{4} \right) + 1.$$

Then from Lemma 2.3, there exists an open connected neighbourhood Ω of $\mathbb{D} \cup S$ and analytic functions $f_i^{\mathrm{e}} : \Omega \to \mathbb{C}$, $i \in \{1, \dots, n\}$, such that

$$\forall z \in \mathbb{D} \cup S, \quad \forall i \in \{1, \dots, n\}, \quad |f_i(z) - f_i^{\mathrm{e}}(z)| < \epsilon_1, \text{ and} \tag{38}$$

$$\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } \forall i \in \{1, \dots, n\}, \quad |f_i^{\mathrm{e}}(z) - f_i^{\mathrm{e}}(z_*)| < \epsilon_2. \tag{39}$$

Then for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned} \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i(z) - (f_i(z) - f_i^{\mathrm{e}}(z))| \\ &\geq \sum_{i=1}^{n} (|f_i(z)| - |f_i(z) - f_i^{\mathrm{e}}(z)|) \\ &> \delta - n \cdot \frac{\delta}{2n} \\ &= \frac{\delta}{2} \\ &> \frac{\delta}{4}, \end{aligned}$$

and for all $z \in \Omega \setminus \mathbb{D}$, we have

$$\begin{aligned} \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| &= \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z_*) - (f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z))| \quad \text{(where } z_* \text{ is as in (39))} \\ &\geq \sum_{i=1}^{n} (|f_i^{\mathrm{e}}(z_*)| - |f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z)|) \\ &> \frac{\delta}{2} - n \cdot \frac{\delta}{4n} \\ &= \frac{\delta}{4}. \end{aligned}$$

Consequently,

$$\forall z \in \Omega, \quad \sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)| > \frac{\delta}{4} > 0. \tag{40}$$

Furthermore for all $z \in \mathbb{D} \cup S$,

$$|f_i^{\mathrm{e}}(z)| \leq |f_i(z)| + |f_i^{\mathrm{e}}(z) - f_i(z)| \leq |f_i(z)| + \epsilon_1 < |f_i(z)| + \epsilon_1 + \epsilon_2, \tag{41}$$

and so for all $z \in \mathbb{D} \cup S$,

$$\begin{aligned} \left(\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}} + (\epsilon_1 + \epsilon_2)\sqrt{n} \\ &\leq \sup_{z \in \mathbb{D} \cup S} \left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}} + (\epsilon_1 + \epsilon_2)\sqrt{n} \\ &\leq \frac{1}{2} + (\epsilon_1 + \epsilon_2)\sqrt{n}. \end{aligned} \tag{42}$$

On the other hand, if $z \in \Omega \setminus \mathbb{D}$, and if z_* is as in (39), then

$$|f_i^{\mathrm{e}}(z)| \leq |f_i(z_*)| + |f_i(z_*) - f_i^{\mathrm{e}}(z_*)| + |f_i^{\mathrm{e}}(z_*) - f_i^{\mathrm{e}}(z)| \leq |f_i(z_*)| + \epsilon_1 + \epsilon_2, \tag{43}$$

and so for all $z \in \Omega \setminus \mathbb{D}$,

$$\begin{aligned} \left(\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n} |f_i(z_*)|^2\right)^{\frac{1}{2}} + (\epsilon_1 + \epsilon_2)\sqrt{n} \\ &\leq \sup_{z \in \mathbb{D} \cup S} \left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}} + (\epsilon_1 + \epsilon_2)\sqrt{n} \\ &\leq \frac{1}{2} + (\epsilon_1 + \epsilon_2)\sqrt{n}. \end{aligned} \tag{44}$$

From (42) and (44), it follows that

$$\sup_{z \in \Omega} \left(\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)|^2\right)^{\frac{1}{2}} \leq \frac{1}{2} + (\epsilon_1 + \epsilon_2)\sqrt{n} \leq \frac{1}{2} + \left(\frac{1}{4\sqrt{n}} + \frac{1}{4\sqrt{n}}\right)\sqrt{n} \leq 1. \tag{45}$$

By the Riemann mapping theorem, there exists a one-to-one analytic map φ from Ω onto $\mathbb{D}$. For each $i \in \{1, \dots, n\}$, the maps $f_i^{\mathrm{e}} \circ \varphi^{-1} \in H^\infty(\mathbb{D})$ satisfy

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} |(f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| > \frac{\delta}{4} > 0 \quad \text{(using (40)), and}$$

$$\forall z \in \mathbb{D}, \quad |(f_i^{\mathrm{e}} \circ \varphi^{-1})(z)| \leq \sup_{z \in \Omega} \left(\sum_{i=1}^{n} |f_i^{\mathrm{e}}(z)|^2\right)^{\frac{1}{2}} \leq 1 \quad \text{(using (45)).}$$

So by Carleson's corona theorem, it follows that there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{i=1}^{n} (f_i^{\mathrm{e}} \circ \varphi^{-1})(z) g_i(z) = 1,$$

and for each $i \in \{1, \dots, n\}$,

$$\forall z \in \mathbb{D}, \quad |g_i(z)| \leq C_\emptyset \left(n, \frac{\delta}{4} \right).$$

Let

$$f^{\mathrm{e}} := \left[\begin{array}{ccc} f_1^{\mathrm{e}} & \dots & f_n^{\mathrm{e}} \end{array} \right].$$

By Theorem 3.3, there exists $\Lambda \in \mathscr{G}(H^\infty(\mathbb{D})^{n\times n})$ such that

$$\Lambda = \left[\begin{array}{c} f^{\mathrm{e}} \circ \varphi^{-1} \\ F \end{array} \right],$$

where $F \in H^\infty(\mathbb{D})^{(n-1)\times n}$, and if

$$g := \left[\begin{array}{ccc} g_1 & \dots & g_n \end{array} \right],$$

then

$$\begin{array}{rcl} \|\Lambda^{-1}\|_\infty & \leq & \|g\|_\infty (1 + \|f^{\mathrm{e}} \circ \varphi^{-1}\|_\infty) + 1 \\ & < & \sqrt{n} C_\emptyset \left(n, \frac{\delta}{4} \right) (1+1) + 1 = M(\delta, n). \end{array}$$

For $z \in \mathbb{D}$, $\Lambda(z)\Lambda^{-1}(z) = I$, and so

$$\forall z \in \Omega, \quad \left[\begin{array}{c} f^{\mathrm{e}}(z) \\ (F \circ \varphi)(z) \end{array} \right] (\Lambda^{-1} \circ \varphi)(z) = I.$$

In particular,

$$\forall z \in \mathbb{D} \cup S, \quad \left[\begin{array}{c} f(z) \\ (F \circ \varphi)(z) \end{array} \right] (\Lambda^{-1} \circ \varphi)(z) = I - \left[\begin{array}{c} f(z) - f^{\mathrm{e}}(z) \\ (F \circ \varphi)(z) \end{array} \right] (\Lambda^{-1} \circ \varphi)(z).$$

As

$$\left\| \left[\begin{array}{c} f - f^{\mathrm{e}} \\ F \circ \varphi \end{array} \right] (\Lambda^{-1} \circ \varphi) \right\|_\infty \leq \left\| \left[\begin{array}{c} f - f^{\mathrm{e}} \\ F \circ \varphi \end{array} \right] \right\|_\infty \|\Lambda^{-1} \circ \varphi\|_\infty \leq \epsilon_1 M(\delta, n) \leq \frac{1}{2},$$

it follows that (see for example, Theorem 18.3 on page 357 of Rudin [30])

$$I - \left[\begin{array}{c} f - f^{\mathrm{e}} \\ F \circ \varphi \end{array} \right] (\Lambda^{-1} \circ \varphi) \in \mathscr{G}(A_S^{n\times n}).$$

We have

$$F \circ \varphi \in A_S^{(n-1)\times n} \quad \text{and} \quad (\Lambda^{-1} \circ \varphi)\left(I - \left[\begin{array}{c} f - f^{\mathrm{e}} \\ F \circ \varphi \end{array}\right](\Lambda^{-1} \circ \varphi)\right) \in \mathscr{G}(A_S^{n\times n}),$$

and for all $z \in \mathbb{D} \cup S$,

$$\left[\begin{array}{c} f(z) \\ (F \circ \varphi)(z) \end{array}\right](\Lambda^{-1} \circ \varphi)(z)\left(I - \left[\begin{array}{c} f(z) - f^{\mathrm{e}}(z) \\ (F \circ \varphi)(z) \end{array}\right](\Lambda^{-1} \circ \varphi)(z)\right) = I.$$

This completes the proof of this theorem. ■

In Section 6, we use this Hermite property of A_S to show that if an unstable transfer function has either a left or a right coprime factorization, then it has both.

4 Stable rank

In this section we prove that just as with A and $H^\infty(\mathbb{D})$, the stable rank of each A_S equals 1. In Section 6, we will apply this result to conclude that stabilizability is equivalent to strong stabilizability for transfer functions that are obtained as a ratio of elements from A_S. This means that if a plant is stabilizable (which means that there exists a controller, possibly *unstable*, that stabilizes the closed loop interconnection), then in fact it can be stabilized by a *stable* controller.

We begin by recalling the definition of stable rank.

Definitions. Let $n \in \mathbb{N}$. Then the set of unimodular rows in $R^{1\times n}$ is denoted by $\mathscr{U}_n(R)$. A row

$$\left[\begin{array}{ccc} a_1 & \dots & a_{n+1} \end{array}\right] \in \mathscr{U}_{n+1}(R)$$

is said to be *stable* if there exists a row

$$\left[\begin{array}{ccc} b_1 & \dots & b_n \end{array}\right] \in R^{1\times n}$$

such that

$$\left[\begin{array}{ccc} a_1 + a_{n+1}b_1 & \dots & a_n + a_{n+1}b_n \end{array}\right] \in \mathscr{U}_n(R).$$

If there exists an $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable, then the *stable rank* of R, is the smallest $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable.

If there does not exist an $n \in \mathbb{N}$ such that every vector of $\mathscr{U}_{n+1}(R)$ is stable, then the *stable rank* of R, is defined to be $+\infty$.

It turns out that the stable rank of $H^\infty(\mathbb{D})$ is equal to 1, and this was shown in Theorem 1 on page 131 of Treil [34]:

Theorem 4.1 (Treil) *Let $f_1, f_2 \in H^\infty(\mathbb{D})$. If there exists a $\delta > 0$ such that*

$$\forall z \in \mathbb{D}, \quad |f_1(z)| + |f_2(z)| \geq \delta,$$

and

$$\forall z \in \mathbb{D}, \quad |f_1(z)| \leq 1 \text{ and } |f_2(z)| \leq 1,$$

then there exists a $g \in H^\infty(\mathbb{D})$ *such that* $h := f_1 + f_2 g \in \mathscr{G}(H^\infty(\mathbb{D}))$*, and moreover*

$$\forall z \in \mathbb{D}, \quad |g(z)| \leq D_\emptyset(\delta) \text{ and } |h^{-1}(z)| \leq D_\emptyset(\delta),$$

where $D_\emptyset(\delta)$ *denotes a constant depending only on* δ*.*

Also, the stable rank of the disc algebra is equal to 1, and this was proved in Theorem 1 of Jones et al. [15]. We prove below that in fact the stable rank of each A_S is equal to 1.

Theorem 4.2 *Let* $S \subset \mathbb{T}$*. The stable rank of* A_S *is equal to* 1*.*

Proof Let

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \in A_S^{1\times 2}$$

be a unimodular row. Without loss of generality, we may assume that

$$\forall z \in \mathbb{D} \cup S, \quad |f_1(z)| \leq \frac{1}{2} \quad \text{and} \quad |f_2(z)| \leq \frac{1}{2}.$$

Then there exists a $\delta > 0$ such that for all $z \in \mathbb{D} \cup S$, $|f_1(z)| + |f_2(z)| \geq \delta$. Let

$$\epsilon_1 = \min\left\{\frac{\delta}{4}, \frac{1}{4}, \frac{1}{2\left(1 + D_\emptyset\left(\frac{\delta}{2}\right)\right)^2}\right\} \quad \text{and} \quad \epsilon_2 = \min\left\{\frac{\delta}{4}, \frac{1}{4}\right\}.$$

Proceeding as in the proof of Theorem 3.4, using Lemma 2.3, we obtain the existence of an open connected set Ω containing $\mathbb{D} \cup S$ and analytic functions f_1^{e}, f_2^{e} defined on Ω that satisfy

$$\begin{aligned}
&\forall z \in \mathbb{D} \cup S, \quad |f_1(z) - f_1^{\mathrm{e}}(z)| < \epsilon_1 \text{ and } |f_2(z) - f_2^{\mathrm{e}}(z)| < \epsilon_1,\\
&\forall z \in \Omega \setminus \mathbb{D}, \quad \exists z_* \in S \text{ such that } |f_1^{\mathrm{e}}(z) - f_1^{\mathrm{e}}(z_*)| < \epsilon_2 \text{ and } |f_2^{\mathrm{e}}(z) - f_2^{\mathrm{e}}(z_*)| < \epsilon_2,\\
&\forall z \in \Omega, \quad |f_1^{\epsilon}(z)| + |f_2^{\epsilon}(z)| > \frac{\delta}{4} \quad \text{(see (40))}\\
&\forall z \in \Omega, \quad |f_1^{\mathrm{e}}(z)| \leq 1 \text{ and } |f_2^{\mathrm{e}}(z)| \leq 1, \quad \text{(see (41) and (43)).}
\end{aligned}$$

If $\varphi : \Omega \to \mathbb{D}$ denotes a one-to-one analytic map from Ω onto $\mathbb{D}$, then from Treil's theorem (Theorem 4.1), it follows that there exists a $g \in H^\infty(\mathbb{D})$ such that

$$h := f_1^{\mathrm{e}} \circ \varphi^{-1} + (f_2^{\mathrm{e}} \circ \varphi^{-1}) \cdot g \in \mathscr{G}(H^\infty(\mathbb{D})), \text{ with } |g(z)| \leq D_\emptyset\left(\frac{\delta}{4}\right) \text{ and } |h^{-1}(z)| \leq D_\emptyset\left(\frac{\delta}{4}\right).$$

So for all $z \in \mathbb{D}$,

$$|h(z)| \leq |(f_1^{\mathrm{e}} \circ \varphi^{-1})(z)| + |(f_2^{\mathrm{e}} \circ \varphi^{-1})(z)||g(z)| \leq 1 + D_\emptyset\left(\frac{\delta}{4}\right).$$

For all $z \in \Omega$, $(f_1^{\rm e}(z) + f_2^{\rm e}(z)(g \circ \varphi)(z))(h \circ \varphi)(z) = 1$. In particular, for all $z \in \mathbb{D} \cup S$,

$$(f_1(z)+f_2(z)(g\circ\varphi)(z))(h\circ\varphi)(z) = 1-((f_1^{\rm e}(z)-f_1(z))+(f_2^{\rm e}(z)-f_2(z))(g\circ\varphi)(z))(h\circ\varphi)(z) =: \Phi(z).$$

For all $z \in \mathbb{D} \cup S$,

$$|\Phi(z)| = |1 - ((f_1^{\rm e}(z) - f_1(z)) + (f_2^{\rm e}(z) - f_2(z))(g \circ \varphi)(z))(h \circ \varphi)(z)| \geq 1-\epsilon_1 \left(1 + D_{\emptyset}\left(\frac{\delta}{4}\right)\right)^2 \geq \frac{1}{2}.$$

Hence $\Phi \in \mathscr{G}(A_S)$, and so $f_1 + f_2 \cdot (g \circ \varphi) \in \mathscr{G}(A_S)$. As $g \circ \varphi \in A_S$, this completes the proof. ■

5 Topological stable rank

In this section we prove that just as with A and $H^\infty(\mathbb{D})$, the topological stable rank of each A_S is equal to 2. In Section 6, we will apply this theorem to show that every unstabilizable plant is as close as we want to a stabilizable plant.

First we recall the notion of topological stable rank.

Definition. Let R be a Banach algebra. If there exists an $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$ in the product topology, then the *topological stable rank* of R is the smallest $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$.

If there does not exist an $n \in \mathbb{N}$ such that $\mathscr{U}_n(R)$ is dense in $R^{1\times n}$ in the product topology, then the *topological stable rank* of R is defined to be $+\infty$.

We recall the following two known results.

Theorem 5.1 *The topological stable rank of $H^\infty(\mathbb{D})$ is equal to* 2.

Proof This was shown in Suárez [33]. ■

Theorem 5.2 *The following hold:*

1. *The topological stable rank of A is equal to* 2.
2. $\overline{\mathscr{G}(A)} = \{0\} \cup \{f \in A \mid \mathscr{Z}(f) \subset \mathbb{T}\}$, *where the notation $\mathscr{Z}(f)$ is used to denote the set of zeros of $f \in A$: $\mathscr{Z}(f) = \{z \in \overline{\mathbb{D}} \mid f(z) = 0\}$.*

Proof Item 1 was established in Rieffel [27]. The claim in item 2, giving the characterization of $\overline{\mathscr{G}(A)}$, was shown in the example on page 154 following the proof of Proposition 1 in Robertson [28]. ■

Using the Theorems 5.1 and 5.2 above, we prove that the topological stable rank of A_S is equal to 2, for arbitrary $S \subset \mathbb{T}$.

Theorem 5.3 *Let $S \subset \mathbb{T}$. The topological stable rank of A_S is equal to 2.*

Proof Let

$$\left[\begin{array}{cc} f_1 & f_2 \end{array}\right] \in A_S^{1\times 2}.$$

Let $\epsilon > 0$. Using Lemma 2.3, we obtain the existence of an open connected set Ω containing $\mathbb{D} \cup S$ and analytic functions f_1^{e}, f_2^{e} defined on Ω that satisfy

$$\forall z \in \mathbb{D} \cup S, \quad |f_1(z) - f_1^{\mathrm{e}}(z)| < \frac{\epsilon}{2} \quad \text{and} \quad |f_2(z) - f_2^{\mathrm{e}}(z)| < \frac{\epsilon}{2}.$$

Let $\varphi : \Omega \to \mathbb{D}$ denote a one-to-one analytic map from Ω onto $\mathbb{D}$. Then

$$\left[\begin{array}{cc} f_1^{\mathrm{e}} \circ \varphi^{-1} & f_2^{\mathrm{e}} \circ \varphi^{-1} \end{array}\right] \in H^\infty(\mathbb{D})^{1\times 2},$$

and since the topological stable rank of $H^\infty(\mathbb{D})$ is equal to 2, it follows that there exist g_1, g_2 in $H^\infty(\mathbb{D})$ such that

$$\left[\begin{array}{cc} g_1 & g_2 \end{array}\right] \in \mathscr{U}_2(H^\infty(\mathbb{D})),$$

and

$$\forall z \in \mathbb{D}, \quad |(f_1^{\mathrm{e}} \circ \varphi^{-1})(z) - g_1(z)| < \frac{\epsilon}{2} \quad \text{and} \quad |(f_2^{\mathrm{e}} \circ \varphi^{-1})(z) - g_2(z)| < \frac{\epsilon}{2}.$$

As

$$\left[\begin{array}{cc} g_1 & g_2 \end{array}\right] \in \mathscr{U}_2(H^\infty(\mathbb{D})),$$

it follows that there exists a $\delta > 0$ such that

$$\forall z \in \mathbb{D}, \quad |g_1(z)| + |g_2(z)| \geq \delta.$$

Hence

$$\forall z \in \mathbb{D} \cup S, \quad |(g_1 \circ \varphi)(z)| + |(g_2 \circ \varphi)(z)| \geq \delta > 0,$$

and by Theorem 2.4, it follows that

$$\left[\begin{array}{cc} g_1 \circ \varphi & g_2 \circ \varphi \end{array}\right] \in \mathscr{U}_2(A_S).$$

Moreover,

$$\forall z \in \mathbb{D} \cup S, \quad |f_1(z) - (g_1 \circ \varphi)(z)| < \epsilon \quad \text{and} \quad |f_2(z) - (g_2 \circ \varphi)(z)| < \epsilon.$$

So it follows that the topological stable rank of A_S is at most equal to 2.

Next we show that the topological stable rank cannot be 1, that is, $\mathscr{G}(A_S)$ is not dense in A_S. In order to do this, we first mention that since the topological stable rank of A is equal to 2, $\mathscr{G}(A)$ is not dense in A. Indeed from item 2 in Theorem 5.2 above, it follows that if $f \in A$ is not identically zero, and has a zero in $\mathbb{D}$, then f does not lie in the closure of $\mathscr{G}(A)$. Consequently, the polynomial function p in A, defined by

$$p(z) = z, \quad z \in \overline{\mathbb{D}},$$

does not belong to $\overline{\mathscr{G}(A)}$. Clearly $p \in A_S$. We now prove that $p \notin \overline{\mathscr{G}(A_S)}$. Assume, on the contrary, that $p \in \overline{\mathscr{G}(A_S)}$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{G}(A_S)$ that converges to p uniformly on $\mathbb{D} \cup S$. Let $r \in (0,1)$, and define $q(z) = rz$, $z \in \overline{\mathbb{D}}$, and for each $n \in \mathbb{N}$,

$$g_n(z) = f_n(rz), \quad z \in \overline{\mathbb{D}}.$$

Then q and the g_n's all belong to A and the sequence $(g_n)_{n\in\mathbb{N}}$ converges to q in A. As the f_n's belong to $\mathscr{G}(A_S)$, from item 2 of Corollary 2.6, it follows that for each $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that

$$\forall z \in \mathbb{D} \cup S, \quad |f_n(z)| \geq \delta_n.$$

Consequently

$$\forall z \in \overline{\mathbb{D}}, \quad |g_n(z)| \geq \delta_n > 0,$$

and by again from item 2 of Corollary 2.6 (now with $S = \mathbb{T}$!), it follows that $g_n \in \mathscr{G}(A)$. But q is not identically zero, and $q(0) = 0$. This contradicts the fact that any nonzero element of A having a zero in $\mathbb{D}$ does not belong to $\overline{\mathscr{G}(A)}$. This completes the proof. ∎

6 Coprime factorization and stabilization

Finally, in this section we proceed to give consequences for systems theory of the results established in the previous sections. The outline is as follows.

1. Using the corona theorem for A_S, we give an necessary and sufficient condition for a matrix pair to be right coprime.

2. We consider unstable transfer functions which we write as a ratio of elements from A_S. Not all such unstable transfer functions will have a coprime factorization. However, using the Hermite property of A_S we get the fact that a transfer function has a doubly coprime factorization iff it has a right (or a left) coprime factorization. Thus, using the result from Vidyasagar [36], we get a parameterization of all stabilizing controllers, analogous to the famous Youla parameterization.

3. Using the fact that the stable rank of A_S is equal to 1, we prove that plants which are stabilizable are in fact strongly stabilizable, that is, the stabilizing controller can be chosen to be stable.

4. Finally, we use the property that the topological stable rank of A_S is 2 to show that any transfer function is as close as we like to a transfer function that is stabilizable.

We begin by applying the result given in 2 of Corollary 2.6 in order to characterize matrix coprime pairs in A_S.

Definitions. Let $S \subset \mathbb{T}$. Matrices with entries in A_S will be denoted by $\mathrm{Mat}(A_S)$. If $N, D \in \mathrm{Mat}(A_S)$, then the pair (N, D) is called *right coprime* (with respect to A_S) if there exist $X, Y \in \mathrm{Mat}(A_S)$ such that the matrix Bézout identity holds:

$$XN + YD = I.$$

A *left coprime* pair of matrices is defined analogously.

The following result gives a test for coprimeness of a matrix pair.

Theorem 6.1 *Let $S \subset \mathbb{T}$. Let $N \in A_S^{m\times p}$ and $D \in A_S^{p\times p}$. The pair (N, D) is right coprime iff there exits a $\delta > 0$ such that*

$$\forall z \in \mathbb{D} \cup S, \quad N(z)^* N(z) + D(z)^* D(z) \geq \delta I.$$

Proof This follows from Corollary 2.6 (see also Lemma 34 on page 340 of Vidyasagar [36]). ■

We now consider unstable transfer functions that can be expressed as a quotient of two elements from A_S. Having shown that A_S is an integral domain in Theorem 3.1, we can consider its field of fractions. We recall this notion below.

Definitions. If R is an integral domain, then a *fraction* is a symbol $\frac{N}{D}$, where $N, D \in R$ and $D \neq 0$. Define the relation $\sim$ on the set of all fractions as follows:

$$\frac{N_1}{D_1} \sim \frac{N_2}{D_2} \quad \text{if} \quad N_1 D_2 = N_2 D_1.$$

The relation $\sim$ is an equivalence relation on the set of all fractions. The equivalence class of $\frac{N}{D}$ is denoted by $\left[\frac{N}{D}\right]$. The *field of fractions*, denoted by $\mathbb{F}(R)$, is the set

$$\mathbb{F}(R) = \left\{ \left[\frac{N}{D}\right] \;\middle|\; N, D \in R \text{ and } D \neq 0 \right\},$$

of equivalence classes of the relation $\sim$, with addition and multiplication defined as follows:

$$\left[\frac{N_1}{D_1}\right] + \left[\frac{N_2}{D_2}\right] = \left[\frac{N_1 D_2 + N_2 D_1}{D_1 D_2}\right] \quad \text{and} \quad \left[\frac{N_1}{D_1}\right]\left[\frac{N_2}{D_2}\right] = \left[\frac{N_1 N_2}{D_1 D_2}\right].$$

$\mathbb{F}(R)$ is then a field with these operations. Let $S \subset \mathbb{T}$. Matrices with entries in $\mathbb{F}(A_S)$ will be denoted by $\mathrm{Mat}(\mathbb{F}(A_S))$.

If $P \in \mathrm{Mat}(\mathbb{F}(A_S))$, then P is said to have a *right coprime factorization* if there exists a pair (N, D) with $N, D \in \mathrm{Mat}(A_S)$ such that D is a square matrix, $\det(D) \neq 0$, $P = ND^{-1}$, and (N, D) is right coprime. A *left coprime factorization* is defined analogously. A transfer function having a right coprime factorization and a left coprime factorization is said to have a *doubly coprime factorization*.

Using the result from Theorem 3.2 which says that A_S is not a Bézout domain, we obtain the following result, which says that not every element from $\mathbb{F}(A_S)$ possesses a coprime factorization.

Corollary 6.2 *Let $S \subset \mathbb{T}$. There exist $P \in \mathbb{F}(A_S)$ that do not have a coprime factorization.*

Proof This is a consequence of Lemma 7 on page 332 of Vidyasagar [36] and Theorem 3.2. ∎

Thus, given an arbitrary $P \in \text{Mat}(\mathbb{F}(A_S))$, the existence of a right coprime factorization for P is not automatic. However, if P does have a right coprime factorization, then *all* right coprime factorizations of P can be characterized, and we give this characterization in the next result. A similar characterization can also be obtained for left coprime factorizations.

Theorem 6.3 *Let $S \subset \mathbb{T}$. If $P \in \text{Mat}(\mathbb{F}(A_S))$ has a right coprime factorization (N, D), then (N', D') is a right coprime factorization of P iff there exists a unimodular matrix U such that $N' = NU$ and $D' = DU$.*

Proof This follows from Lemma 2 on page 331 of Vidyasagar [36]. ∎

Coprime factorization plays an important role in stabilizing a plant using a factorization approach, where by 'stabilization', we mean the following.

Definitions. Let $S \subset \mathbb{T}$. Let $P, C \in \text{Mat}(\mathbb{F}(A_S))$. The pair (P, C) is said to be *stable* if

$$\mathscr{H}(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+PC)^{-1} \\ C(I+PC)^{-1} & (I+PC)^{-1} \end{bmatrix} \tag{46}$$

is well defined, and belongs to $\text{Mat}(A_S)$. We define

$$\mathscr{S}(P) = \{C \in \text{Mat}(\mathbb{F}(A_S)) \mid (P, C) \text{ is a stable pair}\}.$$

$P \in \mathbb{F}(A_S)^{p \times m}$ is said to be *stabilizable* if $\mathscr{S}(P) \neq \emptyset$.

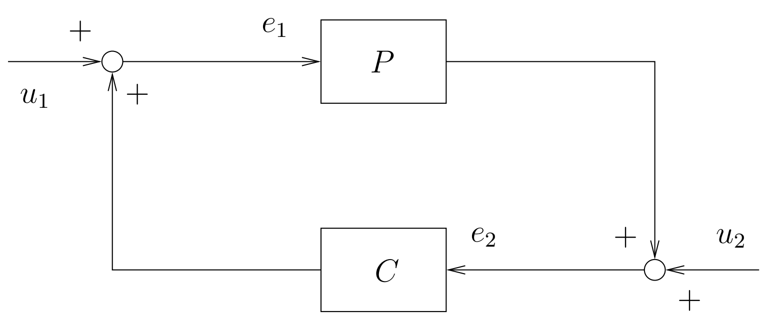

Figure 1: Closed loop interconnection of the plant P and the controller C.

As shown in Figure 1, $\mathscr{H}(P, C)$ in (46) is the transfer function of

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The stabilization problem for a plant is solved completely once a transfer function has a doubly coprime factorization.

Theorem 6.4 *Let $S \subset \mathbb{T}$. Let $P \in \mathrm{Mat}(\mathbb{F}(A_S))$ have a right coprime factorization (N_r, D_r) and a left coprime factorization (D_l, N_l). Let $X_r, Y_r, X_l, Y_l \in \mathrm{Mat}(A_S)$ be such that*

$$X_r N_r + Y_r D_r = I \quad \text{and} \quad N_l X_l + D_l Y_l = I.$$

Then

$$\begin{aligned}
\mathscr{S}(P) &= \{(Y_r - QN_l)^{-1}(X_r + QD_l) \mid Q \in \mathrm{Mat}(A_S) \text{ and } \det(Y_r - QN_l) \neq 0\} \\
&= \{(X_l + D_r Q)(Y_l - N_r Q)^{-1} \mid Q \in \mathrm{Mat}(A_S) \text{ and } \det(Y_l - N_r Q) \neq 0\}.
\end{aligned}$$

Proof This follows from Theorem 12 on page 364 of Vidyasagar [36]. ■

We know that not every $P \in \mathrm{Mat}(\mathbb{F}(A_S))$ has a coprime factorization. Thus in light of Theorem 6.4, the natural question then arises: if P has a right (or a left) coprime factorization, then does it have a left (respectively right) coprime factorization? It turns out that $P \in \mathrm{Mat}(\mathbb{F}(A_S))$ has one iff it has the other, which we prove below in Corollary 6.5. This is a consequence of Theorem 3.4.

Corollary 6.5 *Let $S \subset \mathbb{T}$ and suppose that $P \in \mathrm{Mat}(\mathbb{F}(A_S))$. Then:*

1. *If P has a right coprime factorization, then P has a left coprime factorization.*
2. *If P has a left coprime factorization, then P has a right coprime factorization.*

Proof This follows from Theorem 3.4 and Theorem 66 on page 347 of Vidyasagar [36]. ■

Thus the above result says that if P possesses either a left or a right coprime factorization, then it possesses a doubly coprime factorization.

Next, using the fact that the stable rank of A_S is equal to 1, we show the equivalence of stabilizability and strong stabilizability.

Definition. Let $S \subset \mathbb{T}$. $P \in \mathbb{F}(A_S)^{p\times m}$ is said to be *strongly stabilizable* if $\mathscr{S}(P) \cap A_S^{m\times p} \neq \emptyset$.

We have the following result.

Theorem 6.6 *Let $S \subset \mathbb{T}$ and suppose that $P \in \mathrm{Mat}(\mathbb{F}(A_S))$. The following are equivalent:*

1. *P is stabilizable.*
2. *P is strongly stabilizable.*

Proof This follows from Corollary 6.6 on page 2280 of Quadrat [25] and Theorem 4.2. ■

Finally, using the fact that the topological stable rank of A_S is equal to 2, we show that every unstabilizable SISO plant defined by a transfer function $P \in \mathbb{F}(A_S)$ is as 'close' as we want to a stabilizable plant, in the following sense.

Theorem 6.7 *Let $S \subset \mathbb{T}$ and suppose that $P = \frac{N}{D} \in \mathbb{F}(A_S)$, with $N, D \in A_S$ and $D \neq 0$. Given any $\epsilon > 0$, there exist $N_\epsilon \in A_S$ and $D_\epsilon \in A_S \setminus \{0\}$ such that*

$$\|N - N_\epsilon\|_\infty < \epsilon \quad \text{and} \quad \|D - D_\epsilon\|_\infty < \epsilon,$$

and moreover (N_ϵ, D_ϵ) are coprime.

Proof This follows from Proposition 7.4 on page 2281 of Quadrat [25] and Theorem 5.3. ■

Acknowledgements. I would like to thank Professor Jean-Pierre Rosay (University of Wisconsin-Madison, U.S.A.) for showing me a proof of Theorem 2.4 and for useful references. I am grateful to Prof. Ruth Curtain (University of Groningen, The Netherlands) for helpful comments.

References

[1] B. Bollobás. *Linear Analysis: an Introductory Course*. Cambridge University Press, 1990.

[2] F.M. Callier and C.A. Desoer. An algebra of transfer functions for distributed linear time-invariant systems. *IEEE Transactions on Circuits and Systems*, 25:651-663, 1978.

[3] R.F. Curtain. Regular linear systems and their reciprocals: applications to Riccati equations. *Systems and Control Letters*, 49:81-89, 2003.

[4] R.F. Curtain. Riccati equations for stable well-posed linear systems: the generic case. *SIAM Journal on Control and Optimization*, 42:1681-1702, 2003.

[5] R.F. Curtain and A.J. Sasane. Compactness and nuclearity of the Hankel operator and internal stability of infinite-dimensional state linear systems. *International Journal of Control*, 74:1260-1270, 2001.

[6] R.F. Curtain and A.J. Sasane. Hankel norm approximation for well-posed linear systems. *Systems and Control Letters*, 48:407-414, 2003.

[7] R.F. Curtain, G. Weiss and M. Weiss. Stabilization of irrational transfer functions with internal loop. In *Systems, Approximation, Singular Integral Operators and Related Topics*, A.A. Borichev and N.K. Nikol'skiĭ (editors), in series Advances in Operator Theory and Applications, 129:179-208, Birkhäuser, 2000.

[8] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Systems Theory*. Springer, 1995.

[9] A.M. Davie, T.W. Gamelin and J. Garnett. Distance estimates and pointwise bounded density. *Transactions of the American Mathematical Society*, 175:37-68, 1973.

[10] J. Détraz. Algébras de restrictions du disque. (French) *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B*, 268:A853–A856, 1969.

[11] J. Détraz. Algèbres de restrictions du disque. (French) *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B*, 269:A688–A691, 1969.

[12] J. Détraz. Étude du spectre d'algèbres de fonctions analytiques sur le disque unité. (French) *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B*, 269:A833-A835, 1969.

[13] P.L. Duren. *The Theory of H^p Spaces*. Academic Press, 1970.

[14] J.B. Garnett. *Bounded Analytic Functions*. Academic Press, 1981.

[15] P.W. Jones, D. Marshall and T. Wolff. Stable rank of the disc algebra. *Proceedings of the American Mathematical Society*, 96:603-604, 1986.

[16] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1976.

[17] H. Logemann. Finitely generated ideals in certain algebras of transfer functions for infinite-dimensional systems. *International Journal of Control*, 45:247-250, 1987.

[18] K.M. Mikkola. *Infinite Dimensional Linear Systems, Optimal Control and Riccati Equations*. Ph.D. thesis, Helsinki University of Technology, Finland, 2002. Available at `http://www.math.hut.fi/reports`.

[19] R. Narasimhan and Y. Nievergelt. *Complex Analysis in One Variable*. 2nd Edition, Birkhäuser, 2001.

[20] N.K. Nikol'skiĭ. *Treatise on the Shift Operator: Spectral Function Theory*. Springer, 1986.

[21] M.R. Opmeer and R.F. Curtain. New Riccati equations for well-posed linear systems. *Systems and Control Letters*, 52:339-347, 2004.

[22] J.R. Partington. *An Introduction to Hankel Operators*. Cambridge University Press, 1988.

[23] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part I: (Weakly) doubly coprime factorizations. *SIAM Journal on Control and Optimization*, 42:266-299, 2004.

[24] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part II: Internal Stabilization. *SIAM Journal on Control and Optimization*, 42:300-320, 2004.

[25] A. Quadrat. On a general structure of the stabilizing controllers based on stable range. *SIAM Journal on Control and Optimization*, 42:2264-2285, 2004.

[26] M. von Renteln. Hauptideale und äussere funktionen im ring H^∞. (German) *Archiv der Mathematik*, 28:519-524, 1977.

[27] M.A. Rieffel. Dimension and stable rank in the K-theory of C^*-algebras. *Proceedings of the London Mathematical Society*, 46:301-333, 1983.

[28] G. Robertson. On the density of the invertible group in C^*-algebras. *Proceedings of the Edinburgh Mathematical Society*, 20:153-157, 1976.

[29] J.-P. Rosay. *Private Communication*. Department of Mathematics, University of Wisconsin-Madison, U.S.A., 2005.

[30] W. Rudin. *Real and Complex Analysis*. 3rd Edition, McGraw-Hill, 1987.

[31] O.J. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, 2005.

[32] A. Stray. An approximation theorem for subalgebras of H^∞. *Pacific Journal of Mathematics*, 35:511-515, 1970.

[33] D. Suárez. Trivial Gleason parts and the topological stable rank of H^∞. *American Journal of Mathematics*, 118:879-904, 1996.

[34] S. Treil. The stable rank of the algebra H^∞ equals 1. *Journal of Functional Analysis*, 109:130-154, 1992.

[35] M. Vidyasagar, H. Schneider and B.A. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Transactions on Automatic Control*, 27:880-894, 1982.

[36] M. Vidyasagar. *Control System Synthesis: a Factorization Approach*. MIT Press, 1985.

[37] K. Yosida. *Functional Analysis*. Springer, 1980.