Chromatic Polynomials of some Families of Graphs
I: Theorems and Conjectures

Norman Biggs

Centre for Discrete and Applicable Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
U.K.
n.l.biggs@lse.ac.uk

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Abstract

The chromatic polynomials of 'bracelets' can be studied by means of a theory based on representations of the symmetric group. This paper contains a detailed study of the theory as it relates to one type of bracelet. The underlying theory is presented rather more clearly than hitherto, and some surprising features are exhibited. The methods involve an extension of the standard theory of distance-regular graphs, and they lead to several plausible conjectures.

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I: Theorems and Conjectures

1 Introduction

The study of chromatic polynomials is partly motivated by the following simple observation: if $P(G; z)$ is the chromatic polynomial of a graph G, then the roots of $P(G; z) = 0$ provide useful information about the chromatic number of G. In this context, a number of interesting facts about the real roots have been established, such as the results of Tutte [**13**] on the 'golden root' and a number of results about 'zero-free intervals' [**14**]. Work on the complex roots started in 1972 [**2**], and progressed slowly until recently. At one time it was thought that roots with negative real parts might never occur, but Sokal [**12**] has shown that this is quite false. His examples are theta-graphs, which might be thought to be atypical in some respects, but roots with negative real parts have also been shown to occur in other families of graphs, such as the 'generalized dodecahedra'.

Here we shall focus on the chromatic polynomials of families known as bracelets, for which a theory based on representations of symmetric groups has been developed [**4**,**5**,**6**]. This theory leads to results that are well-adapted to the application of a theorem of Beraha-Kahane-Weiss [**1**], and it implies that the complex roots of a family of bracelets lie close to certain curves.

The paper contains a detailed study of the theory as it relates to one type of bracelet. The justification for dealing at length with this case is twofold. First, the underlying theory is still being developed: focussing on a particular case allows it to be presented rather more clearly than hitherto, and suggests several improvements. Secondly, this particular case exhibits some very surprising features, which are as yet unexplained. The explanation will almost certainly involve some deep algebraic relationships, and we shall begin to investigate what these relationships might be. In doing so we encounter algebraic structures that may well be of wider interest.

This paper is in two parts.

- In Sections 2-4 we set up the link between chromatic polynomials of bracelets and representations of the symmetric group. The treatment is a slightly simplified version of earlier ones, and makes the link clearer, as well as allowing some basic facts to be proved directly.
- In Sections 5-10 we develop a mechanism for doing calculations based on this relationship. This involves an extension of the standard theory of distance-regular graphs, obtained by generalizing the algebra of distance matrices.

In a sequel we shall use these techniques to study the the complex curves formed by the limit points of chromatic roots of families of bracelets.

2 The bracelets $B(r,n)$

The graph $B(r,n)$ is constructed by taking n copies of the complete graph K_r and joining each vertex v in the ith copy to the same vertex v in the $(i+1)$th copy (where, by convention $n+1=1$). We shall denote by $\mathcal{B}_r$ the family of bracelets $B(r,n)$ with $n \geq 3$.

The following formula for the chromatic polynomial of $B(r,n)$ is a special case of a more general result [**5**, Theorem 1]:

$$P(B(r,n);z) \;=\; \sum_{\pi} m_\pi(z)\;\mathrm{tr}(N^\pi(z))^n.$$

The sum is taken over all partitions π with $0 \leq |\pi| \leq r$. Note that (unconventionally) the partition of 0 with one part equal to 0 is included: we shall denote this partition by o.

The terms appearing in the formula are defined as follows. Define $m_o(z) = 1$. When $|\pi| = \ell \geq 1$, let $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_\ell$ be the parts of π, together with the appropriate number of zeros. Define

$$q_i = \pi_i + \ell - i \quad (1 \leq i \leq \ell).$$

and let $d(\pi)$ be the degree of the representation of S_ℓ associated with π. Then

$$m_\pi(z) = \frac{d(\pi)}{\ell!}(z-q_1)(z-q_2)\cdots(z-q_\ell).$$

It can be shown that $m_\pi(z)$ takes positive integer values when $z = k \geq 2\ell$ is a positive integer; in fact it is the degree of the representation of S_k associated with the partition π^k formed by adding a part of size $k-\ell$ to π.

The matrix $N^\pi(z)$ is a square matrix of size

$$\binom{r}{\ell} d(\pi),$$

with entries that are polynomials in z with integer coefficients. When z is a sufficiently large positive integer k, this matrix represents the action of a 'compatibility operator' on a subspace of the space of all k-colourings of K_r (see Sections 3 and 4). It is conjectured that that, for the families $\mathcal{B}_r$, the eigenvalues of $N^\pi(z)$ are also polynomials in z with integer coefficients. This conjecture is known to be true for all π when $r \leq 6$, and for all r in the case of certain kinds of partition π. If true in general, it would imply that the trace of $N^\pi(z)^n$ always takes the simple form

$$\mu_1\lambda_1(z)^n + \mu_2\lambda_2(z)^n + \cdots + \mu_s\lambda_s(z)^n,$$

where the $\lambda_i(z)$ $(1 \leq i \leq s)$ are polynomials and the μ_i $(1 \leq i \leq s)$ are their multiplicities as eigenvalues of $N^\pi(z)$. It would then follow from the general result stated above that $P(B(r,n);z)$ takes the form

$$\sum(\text{polynomial of degree } |\pi|) \times (\text{sum of } n\text{th powers of polynomials of degree } r-|\pi|).$$

The main aim of this paper is to provide evidence that makes the conjecture plausible.

Example 2.1: the eigenvalues when $r = 4$, $\ell = 3$

There are three partitions π with $|\pi| = 3$: [3], [111] and [21]. In Section 9 we shall explain how the eigenvalues relevant to $\mathcal{B}_4$ in all three cases can be obtained from a single 4×4 matrix. The results are as follows.

• The partition [3] is associated with the principal representation of S_3, which has degree 1, so $N^{[3]}(z)$ is a 4×4 matrix. Its eigenvalues are $7 - z$ (multiplicity 1) and $3 - z$ (multiplicity 3).

• The partition [111] is associated with the alternating representation of S_3, which has degree 1, so $N^{[111]}(z)$ is a 4×4 matrix. Its eigenvalues are $1 - z$ (multiplicity 1) and $5 - z$ (multiplicity 3).

• The partition [21] is associated with a representation of S_3 having degree 2, so $N^{[21]}(z)$ is an 8×8 matrix. Its eigenvalues are $6 - z$ (multiplicity 3), $4 - z$ (multiplicity 2), and $2 - z$ (multiplicity 3).

The polynomials $m_\pi(z)$ corresponding to these partitions π are given by the general formula above, and so we can deduce that the chromatic polynomial of $B(4, n)$ contains the following terms:

$$\frac{1}{6}z(z-1)(z-5)\Big((7-z)^n + 3(3-z)^n\Big)$$
$$+\frac{1}{6}(z-1)(z-2)(z-3)\Big((1-z)^n + 3(5-z)^n\Big)$$
$$+\frac{1}{3}z(z-2)(z-4)\Big(3(6-z)^n + 2(4-z)^n + 3(2-z)^n\Big).$$

The complete chromatic polynomial $P(B(4, n); z)$ is given in [**4**]. ◇

3 The π-standard elements

Let $\mathcal{V}_{k,r}$ denote the complex vector space with basis the set of all proper k-colourings of K_r (that is, injections from the set of r vertices to the set of k colours). Given $S \subseteq V$ and an injection ξ from S into the set of colours, we define $[S \mid \xi]$ to be the formal sum of all k-colourings of K_r that agree with ξ on S. For the purposes of calculation it is useful to have a more explicit notation: if $S = \{s_1, s_2, \ldots, s_\ell\}$ and $\xi(s_i) = a_i$, we write

$$[S \mid \xi] \;=\; [s_1, s_2, \ldots, s_\ell \mid a_1, a_2, \ldots, a_\ell].$$

The original approach to calculating chromatic polynomials of bracelets [**4**] used the elements $[S \mid \xi]$. However it was subsequently discovered [**5**] that a modification, based on constructions well-known in the theory of the symmetric group, provided a more elegant mechanism. Here we shall develop the mechanism simply as a practical tool, with only passing references to its links with the classical theory.

For the time being we consider a fixed partition π with $|\pi| = \ell$ and a fixed integer k such that $k \geq 2\ell$. In this situation we have a partition of k, denoted by π^k, with parts $k - \ell, \pi_1, \pi_2, \ldots, \pi_\ell$ in weakly decreasing order.

The vertex-set of K_r will be denoted by V and given a fixed total ordering. A proper k-colouring of K_r is simply an injection from V into a set of k colours, $c_1, c_2, \ldots, c_k$. An important part of our approach is a specific notation for the colours. This is based upon the diagram associated with π^k, which comprises rows of cells of lengths equal to the parts of π^k, left-justified so that we can refer also to the columns. There are no cells corresponding to parts that are 0. The rows will be labelled $0, 1, 2, \ldots$, and the columns $1, 2, \ldots$. Thus, when row 0 is omitted, we obtain the diagram for π. The sets of cells of these diagrams will be denoted by $[\pi^k]$ and $[\pi]$.

We shall identify the colours $c_1, c_2, \ldots, c_k$ with the cells of the diagram for π^k in the 'canonical' order, that is:

$$c_i = \begin{cases} (0, i) & \text{if } 1 \leq i \leq k - \ell; \\ (1, i - (k - \ell)) & \text{if } k - \ell < i \leq k - \ell + \pi_1; \\ (2, i - (k - \ell + \pi_1)) & \text{if } k - \ell + \pi_1 < i \leq k - \ell + \pi_1 + \pi_2; \\ \text{and so on.} & \end{cases}$$

Define the *row stabilizer* $R = R(\pi)$ and the *column stabilizer* $C = C(\pi^k)$ as follows. Let R_i be the group of all permutations of the colours appearing in row i, and C_j the group of all permutations of the colours appearing in column j. Then

$$R = R_1 \times R_2 \times \cdots , \qquad C = C_1 \times C_2 \times \cdots .$$

These groups are subgroups of the symmetric group $Sym\{c_1, c_2, \ldots, c_k\}$; it is often convenient to use i instead of c_i, in which case the symmetric group is denoted by S_k. Note that the colours in row 0 are not permuted by R, but they are permuted by C.

For any injection $\xi : S \to [\pi^k]$ define an element of $\mathcal{V}_{k,r}$ by

$$[[S \mid \xi]] \ = \sum_{g \in C, h \in R} sign(g)[S \mid gh\xi].$$

By construction $[\pi^k]$, the set of colours, has a subset $[\pi]$ of size ℓ. A *π-standard tableau* is a bijection from the set $\{1, 2, \ldots, \ell\}$ to $[\pi]$, such that the values are in increasing order along each row and down each column. (Note that this bijection is the inverse of the usual one.) We denote the π-standard tableaux by $\sigma_1, \sigma_2, \ldots, \sigma_d$: it is known that $d = d(\pi)$ is the degree of a representation of S_ℓ associated with π.

Given $S \subseteq V$ with $|S| = \ell = |\pi|$, let the elements of S in the fixed order on V be $s_1 < s_2 < \ldots < s_\ell$. For any injection $\xi : \{1, 2, \ldots, \ell\} \to [\pi^k]$ let $\xi^S : S \to [\pi^k]$ denote the injection given by

$$\xi^S(s_j) = \xi(j) \quad (1 \leq j \leq \ell).$$

In particular, when ξ is a π-standard tableau σ_i we refer to σ_i^S as a *π-standard injection* on S. We think of σ_i^S as a partial k-colouring of K_r, with the property that the vertices in S are assigned colours in a special way, related to the partition π.

We shall define a *π-standard element* of $\mathcal{V}_{k,r}$ as follows:

$$[[S \mid \sigma_i^S]] \ = \sum_{g\in C\ h\in R} sign(g)\ [S \mid gh\sigma_i^S].$$

(This is the analogue of what is often called a *polytabloid.*) Clearly, the number of π-standard elements is $\binom{r}{|\pi|}d(\pi)$. By analogy with the general theory [**11**, p.70] it can be shown that they are linearly independent in $\mathcal{V}_{k,r}$. It follows that they span a subspace $\mathcal{Y}$ of dimension $\binom{r}{|\pi|}d(\pi)$. In fact there are $m_\pi(k)$ subspaces of $\mathcal{V}_{k,r}$ isomorphic to $\mathcal{Y}$, where m_π is the polynomial defined in Section 2. These subspaces are obtained by choosing different correspondences between the cells of π^k and the colours. For our purposes it suffices to study just one of these subspaces, defined by the canonical correspondence described above.

Example 3.1: the π-standard elements when $\pi = [21]$

In this case $[\pi^k]$ has $k-3$ cells in row 0, two cells in row 1, and one cell in row 2. Denoting the colour c_i by i, and identifying colours with the cells of $[\pi^k]$ in the canonical way, we have the diagram

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k-3 \\ k-2 & k-1 & & & \\ k & & & & \end{array} .$$

The stabilizers are:

$$R \ = \ Sym\{k-2, k-1\}, \quad C \ = \ Sym\{1, k-2, k\} \times Sym\{2, k-1\}.$$

There are two π-standard tableaux σ_1, σ_2, given by $\sigma_1(1,2,3) \ = \ (k-2, k-1, k)$ and $\sigma_2(1,2,3) = (k-2, k, k-1)$. Thus for each 3-subset S of V, there are two π-standard injections. For example, if $S = \{s,t,u\}$ with $s < t < u$, σ_1^S and σ_2^S are given by

$$\sigma_1^S(s) = k-2,\ \sigma_1^S(t) = k-1,\ \sigma_1^S(u) = k;$$
$$\sigma_2^S(s) = k-2,\ \sigma_2^S(t) = k,\ \sigma_2^S(u) = k-1.$$

The subspace $\mathcal{Y}$ is spanned by the two π-standard elements $[[S \mid \sigma_1^S]]$ and $[[S \mid \sigma_2^S]]$, where

$$\begin{aligned} [[S \mid \sigma_1^S]] &= [[s,t,u \mid k-2, k-1, k]] \\ &= [s,t,u \mid k-2, k-1, k] - [s,t,u \mid 1, k-1, k] - [s,t,u \mid k-2, k-1, 1] \\ &\quad - [s,t,u \mid k, k-1, k-2] + [s,t,u \mid k, k-1, 1] + [s,t,u \mid 1, k-1, k-2] \\ &\quad - [s,t,u \mid k-2, 2, k] + [s,t,u \mid 1, 2, k] + [s,t,u \mid k-2, 2, 1] \\ &\quad + [s,t,u \mid k, 2, k-2] - [s,t,u \mid k, 2, 1] - [s,t,u \mid 1, 2, k-2] \\ &\quad + 12 \text{ other terms.} \end{aligned}$$

Each of the 12 terms displayed corresponds to taking $h = id$, with g being one of the 12 permutations in C; the remaining 12 terms are obtained by taking $h = (k-2\ k-1)$. $\diamond$

4 Construction of the matrices N^π

We say that two proper k-colourings α, β of K_r are *compatible* if $\alpha(v) \neq \beta(v)$ for all $v \in V$. The *compatibility matrix* N is the matrix whose rows and columns correspond to the colourings, with entries

$$N_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are compatible;} \\ 0 & \text{otherwise.} \end{cases}$$

It is a standard result, easily proved, that the number of k-colourings of $B(r, n)$ is the trace of N^n.

In order to calculate this number we shall consider, for each partition π with $|\pi| \leq r$, the action of N on the space $\mathcal{Y}$ spanned by the π-standard elements $[[S \mid \sigma_i^S]]$. These elements are linear combinations of terms $[S \mid gh\sigma_i^S]$, and N is a linear operator, so we shall require the general formula for the effect of N on $[S \mid \xi]$, proved in [**4**, Theorem 2]:

$$N[S \mid \xi] = \sum_{T,\tau} (-1)^{|S \cap T|} f_{r-|S \cup T|} \, [T \mid \tau].$$

The sum is taken over all pairs (T, τ) such that

$$\text{(i) } \tau(T) \subseteq \xi(S) \quad \text{ and } \quad \text{(ii) } \tau \text{ agrees with } \xi \text{ on } S \cap T,$$

and the coefficients $f_{r-|S \cup T|}$ correspond to the numbers denoted by $c_{|S \cup T|}$ in [**4**]. These numbers are defined, for given positive integers k, r and $0 \leq i \leq r$ by

$$f_i = F(i, k - r + i),$$

where, for non-negative integers a and all $z \in \mathbb{C}$,

$$F(a, z) = \sum_{j=0}^{a} (-1)^j \binom{a}{j} (z - j)(z - j - 1) \cdots (z - a + 1).$$

(More details will be given in Section 10.)

Example 4.1: the action of N on a π-standard element, $\pi = [21]$

Consider the partition $\pi = [21]$, and a subset $S = \{s, t, u\}$ of vertices, with $s < t < u$ in the fixed ordering of V. In Example 3.1 we described the two standard elements associated with π, the first one being

$$[[s, t, u \mid k - 2, k - 1, k]] \; = \; [[S \mid \sigma]] \qquad (\sigma = \sigma_1^S)$$

This is the signed sum of 24 elements $[S \mid gh\sigma]$, corresponding to the pairs (g, h) with $g \in C$, $h \in R$. Since N is a linear operator, the effect of N on $[[S \mid \sigma]]$ can be obtained by considering its effect on each of these 24 elements, using the formula given above.

From conditions (i) and (ii) it follows that a term $[T \mid gh\tau]$ occurs in $N[S \mid gh\sigma]$ if and only a term $[T \mid \tau]$ occurs in $N[S \mid \sigma]$. For example, $N(sign(g)[S \mid gh\sigma])$ contains a term obtained by taking T to be $\{s\}$ and τ to be the restriction of σ to $\{s\}$. Since $\sigma(s) = k - 2$, this term is

$$sign(g)\ [s \mid gh(k-2)].$$

By definition,taking the sum of these terms over the 24 pairs (g,h) gives $[[T \mid \tau]]$. In this case we can work out the 24 terms individually, and it turns out that they cancel in pairs, so that $[[T \mid \tau]] = 0$. The following theorem asserts that this will always happen when $|T| < |S|$. $\diamond$

Theorem 4.2 Let π be a partition, S a subset of the r-set V with $|S| = |\pi|$, and $\sigma : S \to [\pi^k]$ a π-standard injection. Then $N[[S \mid \sigma]]$ can be expressed as linear combination of terms $[[T \mid \tau]]$ with $|T| = |S|$.

Proof It follows from the definition of $[[S \mid \sigma]]$ that

$$N[[S \mid \sigma]] \;=\; \sum_{g\in C, h\in R} sign(g)\ N[S \mid gh\sigma].$$

The formula for the action of N tells us that a term $[T \mid \tau]$ occurs in $N[S \mid \sigma]$ if and only (i) $\tau(T) \subseteq \sigma(S)$; and (ii) τ and σ agree on $S \cap T$. It follows that a term $[T \mid gh\tau]$ occurs in $N[S \mid gh\sigma]$ under the same conditions. Hence

$$\begin{aligned} N[[S \mid \sigma]] &= \sum_{g\in C, h\in R} sign(g) \sum_{T,\tau} (-1)^{|S\cap T|} f_{r-|S\cup T|}\ [T \mid gh\tau] \\ &= \sum_{T,\tau} (-1)^{|S\cap T|} f_{r-|S\cup T|} \sum_{g\in C, h\in R} sign(g)\ [T \mid gh\tau] \\ &= \sum_{T,\tau} (-1)^{|S\cap T|} f_{r-|S\cup T|}\ [[T \mid \tau]]. \end{aligned}$$

Now suppose that $|T| < |\pi|$. Let $\tau : T \to [\pi^k]$ be an injection satisfying conditions (i) and (ii). Then $\tau(T)$ is a proper subset of $[\pi]$, and for each $h \in R$, $h\tau(T)$ is also a proper subset of $[\pi]$.

Let Φ_h be the pointwise stabilizer in C of $h\tau(T)$, and let $C_i = g_i\Phi_h$ $(1 \le i \le m)$ be the left cosets of Φ_h in C. If g is in C_i, then $[T \mid gh\tau] = [T \mid g_i h\tau]$. Thus

$$\begin{aligned} [[T \mid \tau]] \;&=\; \sum_{h\in R}\sum_{g\in C} sign(g)\ [T \mid gh\tau] \\ &= \sum_{h\in R}\sum_{i=1}^{m}\sum_{g\in C_i} sign(g)\ [T \mid gh\tau] \\ &= \sum_{h\in R}\sum_{i=1}^{m} [T \mid g_i h\tau] \sum_{g\in C_i} sign(g). \end{aligned}$$

Now $h\tau(T)$ is a proper subset of $[\pi]$, so there is a cell in $[\pi]$ that is not in $h\tau(T)$. Suppose $(x,y) = c'$ is such a cell, and let $(0,y) = c''$. Then the transposition $(c'c'')$ is in Φ_h. Thus Φ_h contains equal numbers of odd and even permutations, as does each coset C_i, and the sum of $sign(g)$ over each coset is zero. We have proved that $[[T \mid \tau]] = 0$ whenever $|T| < |S|$, and the result follows. $\diamond$

We now establish a significant strengthening of Theorem 4.2. Not only can the terms $[[T \mid \tau]]$ be constrained so that $|T| = |S|$, but τ can be taken to be a π-standard injection. Let σ_j be a given π-standard tableau. In the proof of Theorem 4.2 we obtained the formula

$$N[[S \mid \sigma_j^S]] \;=\; \sum_{T,\tau} (-1)^{|S\cap T|} f_{r-|S\cup T|}\,[[T \mid \tau]],$$

where the sum is taken over injections τ for which (i) $\tau(T) = \sigma_j^S(S) = [\pi]$, and (ii) τ agrees with σ_j^S on $S \cap T$. We shall now specify these injections more clearly. Let the members of S and T be $s_1 < s_2 < \cdots < s_\ell$ and $t_1 < t_2 < \cdots < t_\ell$, and define a subset $\Omega(ST)$ of S_ℓ as follows:

$$\Omega(ST) = \{\alpha \in S_\ell \mid \alpha(q) = p \text{ whenever } s_p = t_q\}.$$

Note that when $|S \cap T| = b$, the set $\Omega(ST)$ is a coset of the stabilizer of b elements in S_ℓ, and so it contains $(\ell - b)!$ permutations.

Lemma 4.3 An injection $\tau : T \to [\pi^k]$ satisfies conditions (i) and (ii) with respect to σ_j^S if and only if $\tau = (\sigma_j\alpha)^T$ for some $\alpha \in \Omega(ST)$.

Proof Let $\tau = (\sigma_j\alpha)^T$; then clearly, $\tau(T) \subseteq [\pi] = \sigma_j^S(S)$. Suppose $x \in S \cap T$, say $x = s_p = t_q$. Since α is in $\Omega(ST)$, $\alpha(q) = p$ and we have

$$\tau(x) = (\sigma_j\alpha)^T(x) = (\sigma_j\alpha)^T(t_q) = \sigma_j\alpha(q) = \sigma_j(p) = \sigma_j^S(s_p) = \sigma_j^S(x).$$

Thus τ satisfies conditions (i) and (ii). Conversely, suppose τ satisfies the conditions. Then we can define

$$\alpha(q) = \begin{cases} p & \text{if } t_q = s_p \in S \cap T; \\ \sigma_j^{-1}\tau(t_q) & \text{if } t_q \notin S \cap T. \end{cases}$$

and $\alpha \in \Omega(ST)$, $\tau = (\sigma_j\omega)^T$, as required. $\diamond$

As a result of Theorem 4.2 and Lemma 4.3 we know that $N[[S \mid \sigma_j^S]]$ can be expressed as a linear combination of terms $[[T \mid \tau]]$ with $|T| = |S|$ and $\tau = (\sigma_j\alpha)^T$, for some $\alpha \in \Omega(ST)$. That is,

$$N[[S \mid \sigma_j^S]] \;=\; \sum_{|T|=|S|} (-1)^{|S\cap T|} f_{r-|S\cup T|} \sum_{\alpha\in\Omega(ST)} [[T \mid (\sigma_j\alpha)^T]],$$

where the inner sum contains $(\ell - |S \cap T|)!$ terms. It remains to show that this expression can be written as a linear combination of terms $[[T \mid \sigma_i^T]]$, where σ_i $(1 \le i \le d)$ are the π-standard tableaux.

Young's natural representation R^π is a representation of the symmetric group S_ℓ by $d \times d$ matrices, defined as follows. The theory associates with any bijection $\gamma : \{1, 2, \ldots, \ell\} :\longrightarrow$

$[\pi]$ an object $\{\gamma\}$, so that $\alpha \in S_\ell$ acts on these objects by taking $\{\gamma\}$ to $\{\gamma\alpha\}$. When γ is one of the d π-standard tableaux σ_j, the associated object $\{\sigma_j\}$ is known as a *polytabloid* (see Section 3). The key step in the theory [**8** pp.114-124, **11** pp.70-74] is to establish that $\{\sigma_j\alpha\}$ is a linear combination of $\{\sigma_i\}$:

$$\{\sigma_j\alpha\} \;=\; \sum_i (R^\pi(\alpha))_{ji}\,\{\sigma_i\}.$$

Since our π-standard elements are constructed by analogy with the polytabloids, it follows that

$$[[T \mid (\sigma_j\alpha)^T]] \;=\; \sum_i (R^\pi(\alpha))_{ji}\,[[T \mid (\sigma_i)^T]].$$

These results can be summarized as follows.

Theorem 4.4 Let k and r be given integers, with $k \geq 2r$. Let π be a partition with $|\pi| \leq r$ and $d(\pi) = d$, and let $\mathcal{Y}$ be the subspace of $\mathcal{V}_{k,r}$ spanned by the elements $[[S \mid \sigma]]$, where $|S| = |\pi|$ and $\sigma : S \to [\pi^k]$ is a π-standard injection. Then $\mathcal{Y}$ is invariant under N. Furthermore, the action of N on $\mathcal{Y}$ is represented by a matrix N^π that can be partitioned into $d \times d$ submatrices N^π_{ST}, one for each pair (S,T) of $|\pi|$-subsets of the underlying r-set V. These submatrices are given by the formula

$$N^\pi_{ST} \;=\; (-1)^{|S\cap T|} f_{r-|S\cup T|} \sum_{\alpha\in\Omega(ST)} R^\pi(\alpha).$$

$\diamond$

Techniques for calculating Young's natural representation are well-known, and from now on we shall assume that the submatrices N^π_{ST} can be worked out using the formula given above, as in the following example. However, it must be stressed that further work is required before we can calculate the eigenvalues of N^π.

Example 4.5: a submatrix of $N^{[21]}$ when $r = 7$

Let $\pi = [21]$ and $V = \{v_1, v_2, \ldots, v_7\}$, with the natural ordering. Since $\binom{7}{3} = 35$ and $d(\pi) = 2$, the matrix N^π has size 70×70 and is partitioned into submatrices N^π_{ST} of size 2×2.

For example, take $S = \{v_2, v_4, v_7\}$ and $T = \{v_1, v_2, v_5\}$. The intersection of S and T contains only one element v_2, which comes first in S and second in T. It follows that $\Omega(ST)$ consists of the two permutations in S_3 which take 2 to 1, that is

$$\alpha' = (21), \quad \alpha'' = (321).$$

In [**11** pp.39,75] we find

$$R^{[21]}(\alpha') = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad R^{[21]}(\alpha'') = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Thus

$$N_{ST}^{[21]} \;=\; (-1)^1 f_{7-5}\left(\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right) \;=\; \begin{pmatrix} k^2-13k+43 & 0 \\ k^2-13k+43 & 0 \end{pmatrix}.$$

$\diamond$

5 Generalized distance matrices

In this section we denote the ring of $n \times n$ matrices over a ring $\mathcal{R}$ by $Mat(n, \mathcal{R})$. The isomorphism

$$Mat(n_1, Mat(n_2, \mathcal{R})) \approx Mat(n_1 n_2, \mathcal{R})$$

obtained by taking the entries of an $n_1 \times n_1$ matrix to be matrices size $n_2 \times n_2$ will be assumed.

We shall consider the ℓ-subsets of the r-set V as vertices of the *Johnson graph* $J(r, \ell)$, in which S, T are *adjacent* when $|S \cap T| = \ell - 1$. In this graph the *distance* between two ℓ-subsets S and T is given by $\partial(S, T) = \ell - |S \cap T|$. It is well-known that, since the graph $J(r, \ell)$ is distance-regular, its spectral properties can be studied by using the relevant distance matrices (see Example 5.1).

The main result obtained so far (Theorem 4.4) can be expressed in terms of a generalization of the distance matrices of $J(r, \ell)$, using coefficients in $\mathbb{Q}S_\ell$, the rational group-ring of S_ℓ. For $0 \le m \le \ell$ define

$$A^{(m)} \;\in\; Mat\left(\binom{r}{\ell}, \mathbb{Q}S_\ell\right)$$

by setting $A^{(0)} = (id)I$ and, for $m \ge 1$,

$$A_{ST}^{(m)} = \begin{cases} (1/m!) \sum_{\alpha \in \Omega(ST)} \alpha & \text{if } \partial(S,T) = m\ ; \\ 0 & \text{otherwise.} \end{cases}$$

For any partition π with $|\pi| = \ell$, we can extend Young's natural representation to a homomorphism of rings

$$R^\pi : \mathbb{Q}S_\ell \to Mat(d(\pi), \mathbb{Q}),$$

which induces a homomorphism

$$R_*^\pi \;:\; Mat\left(\binom{r}{\ell}, \mathbb{Q}S_\ell\right) \to Mat\left(\binom{r}{\ell}, Mat(d(\pi), \mathbb{Q})\right) \;\approx\; Mat\left(\binom{r}{\ell} d(\pi), \mathbb{Q}\right)$$

defined by

$$(R_*^\pi(M))_{ST} = R^\pi(M_{ST}).$$

In Theorem 4.4 we obtained a formula for N_{ST}^π involving the sum

$$\sum_{\alpha\in\Omega(ST)} R^\pi(\alpha) = R^\pi\Big(\sum_{\alpha\in\Omega(ST)} \alpha\Big) = R^\pi(m!\,A^{(m)}_{ST}) = m!\,(R^\pi_*(A^{(m)}))_{ST},$$

where $m = \partial(S,T)$. Thus, in the current framework, we can think of Theorem 4.4 as a formula for the submatrix N^π_{ST} in terms of $R^\pi_*(A^{(m)})_{ST}$:

$$N^\pi_{ST} \;=\; (-1)^{|S\cap T|}\, f_{r-|S\cup T|}\, m!\,(R^\pi_*(A^{(m)}))_{ST} \qquad (m = \partial(S,T)).$$

Since $A^{(m)}_{ST} \neq 0$ only when $\partial(S,T) = m$ we can present this result as a formula for the matrix N^π, arranged as a sum over m. Recalling that S and T are ℓ-subsets of an r-set, we have

$$\partial(S,T) = \ell - |S\cap T| = |S\cup T| - \ell,$$

and the range of values of $|S\cup T|$ runs from ℓ to $\min(2\ell, r)$. Thus the range of values of m runs from 0 to $\min(\ell, r-\ell)$, and we obtain

$$N^\pi \;=\; \sum_{m=0}^{\min(\ell,r-\ell)} (-1)^{\ell+m}\, f_{r-\ell-m}\, m!\, R^\pi_*(A^{(m)}).$$

Example 5.1: the principal representation By taking π to be the partition $[\ell]$ we obtain the *principal representation* $R^{pri} = R^{[\ell]}$ of S_ℓ. In this case $d = 1$ and $R^{pri}(\omega) = 1$ for all $\omega \in S_\ell$. Since $|\Omega(ST)| = m!$ when $\partial(S,T) = m$, the matrices $R^{pri}_*(A^{(m)})$ are simply the ordinary distance matrices of the Johnson graph.

As a consequence of distance-regularity, each of these matrices with $m > 1$ can be expressed as a polynomial $v_m(A)$, where $A = A^{(1)}$ is the *adjacency matrix* [**3**]. This fact, together with the general expression for N^π given above, means that N^{pri} itself is a polynomial function of A. Thus its eigenvalues can be calculated, given the eigenvalues λ_i of A and the polynomials v_m. In fact, the eigenvalues are:

$$\lambda_i = (\ell-i)(r-\ell-i) - i \quad \text{with multiplicity } \binom{r}{i} - \binom{r}{i-1} \quad (0 \le i \le \ell),$$

and the polynomials v_m are related to the Eberlein polynomials E_m. Details may be found in [**4**]. ◇

In the Sections 6-9 we shall approach the problem of calculating the eigenvalues of N^π in the following way. According to the formula displayed above, $N^\pi = R^\pi_*(X)$, where X is the matrix

$$\sum_{m=0}^{\min(\ell,r-\ell)} (-1)^{\ell+m}\, f_{r-\ell-m}\, m!\, A^{(m)}.$$

This is an integral linear combination of the $\binom{r}{\ell} \times \binom{r}{\ell}$ matrices $A^{(m)}$ with entries in $\mathbb{Q}S_\ell$. We shall discover that the eigenvalues of these matrices are elements of $\mathbb{Q}S_\ell$ in a number of cases, and that the same is true for the eigenvalues of X. For each π with $|\pi| = \ell$ the eigenvalues of N^π can then be obtained by applying R^π.

6 Geodesic walks in $J(r, \ell)$

In this section we obtain a relationship between the generalized distance matrix $A^{(m)}$ and the mth power of the generalized adjacency matrix $A = A^{(1)}$. We begin by obtaining an explicit form for the entries of A.

When $\partial(S,T) = 1$, $\Omega(ST)$ consists of a single permutation α_{ST}, and $A_{ST} = \alpha_{ST}$. Define $od(S,T)$ to be the position in S of the unique element of S that is not in T. That is,

$$od(S,T) = a \quad \text{when} \quad S = (S \cap T) \cup \{s_a\}.$$

With this notation, $od(T,S) = b$, where $T = (S \cap T) \cup \{t_b\}$.

Lemma 6.1 If $\partial(S,T) = 1$, let $a = od(S,T)$, $b = od(T,S)$. The non-zero entries of the generalized adjacency matrix A are given by

$$\alpha_{ST} = \begin{cases} (a\; a+1\; a+2\; \cdots\; b) & \text{if } a < b; \\ id & \text{if } a = b; \\ (a\; a-1\; a-2\; \cdots\; b) & \text{if } a > b. \end{cases}$$

Proof If $a < b$ the members of S and T can be listed as follows, where each row is in increasing order from left to right, and equal elements are in the same column.

$$\begin{array}{ccccccccccc} s_1 & s_2 & \ldots & s_a & s_{a+1} & s_{a+2} & \ldots & s_b & & s_{b+1} & \ldots & s_\ell \\ t_1 & t_2 & \ldots & & t_a & t_{a+1} & \ldots & t_{b-1} & t_b & t_{b+1} & \ldots & t_\ell \end{array}$$

It follows that $\Omega(ST)$ contains only the cyclic permutation

$$\alpha_{ST} = (a\; a+1\; a+2\; \cdots\; b).$$

If $a = b$ or $a > b$ the obvious modifications apply. $\diamond$

We note that this result is a significant restriction on the permutations in S_ℓ that can occur as α_{ST}. In fact is it easy to see that only $\ell^2 - 2\ell + 2$ permutations can occur in this way.

It is a standard result that the entries of the mth power of the ordinary adjacency matrix count the walks of length m in the graph. For the generalized adjacency matrix we have

$$(A^m)_{ST} = \sum \alpha_{SU} \ldots \alpha_{YZ}\alpha_{ZT},$$

where the sum is taken over walks $S, U, \ldots, Y, Z, T$ of length m. When $\partial(S,T) = m$, such a walk is a *geodesic*, and $E = S \cap T$ is a set of size $\ell - m$, with the property that E is also a subset of the intermediate sets $U, \ldots, Y, Z$.

Given a geodesic walk $\mathcal{G} = S, U, \ldots, Y, Z, T$, let $\alpha(\mathcal{G}) = \alpha_{SU} \ldots \alpha_{YZ}\alpha_{ZT}$. Write

$$S = E \cup S', \quad T = E \cup T', \qquad \text{where} \quad |S'| = |T'| = m.$$

At each step of $\mathcal{G}$, one element of S' is removed and one element of T' is inserted. So we can define a bijection $\beta_{\mathcal{G}} : S' \to T'$ by the rule

$$\beta_{\mathcal{G}}(s') = t', \text{ when } s' \text{ is removed and } t' \text{ inserted at the same step.}$$

Lemma 6.2 Let $\mathcal{G}$ be a geodesic walk from S to T. Then

(i) $\alpha(\mathcal{G})$ is in $\Omega(ST)$;

(ii) $\alpha(\mathcal{G})$ is determined by $\beta_{\mathcal{G}}$.

Proof (i) Suppose s_p is in $S \cap T$, say $s_p = t_q = e \in E$. Then for each of the intermediate vertices on $\mathcal{G}$ we can write

$$e = u_{i(U)} = \cdots = y_{i(Y)} = z_{i(Z)},$$

where $i(U)$ is the position of e in the order on U, and so on . By definition

$$\alpha_{ZT}(q) = i(Z), \quad \alpha_{YZ}(i(Z)) = i(Y), \quad \ldots, \quad \alpha_{SU}(i(U)) = p.$$

Thus

$$\alpha(\mathcal{G})(q) = \alpha_{SU} \ldots \alpha_{YZ}\alpha_{ZT}(q) = p,$$

which means that $\alpha(\mathcal{G})$ is in $\Omega(ST)$, as claimed.

(ii) Suppose s_p is in S'. Then $\beta_{\mathcal{G}}(s_p) = t_q$, for some t_q in T'. We shall prove that in this case $\alpha(\mathcal{G})(q) = p$.

Suppose that s_p is removed and t_q inserted at the step from G to H on $\mathcal{G}$. Then $s_p = g_i$ and $t_q = h_j$, say. Write $\alpha(\mathcal{G})$ as $\alpha_1\alpha_{GH}\alpha_2$, where α_1 and α_2 are the appropriate composite permutations. Then

$$s_p = g_i \implies \alpha_1(i) = p, \qquad h_j = t_q \implies \alpha_2(q) = j,$$

and $\alpha_{GH}(j) = i$. Hence $\alpha(\mathcal{G})(q) = p$, as claimed. ⋄

The significance of the lemma is that $\alpha(\mathcal{G})$ does not depend on the order in which the members of S' are removed as we proceed along $\mathcal{G}$. It depends only on $\beta_{\mathcal{G}}$, which tells us which pairs of elements (s', t') are removed/inserted at the same step. There are $(m!)^2$ geodesic walks from S to T: one class of size $m!$ (corresponding to the the $m!$ possible orders) for each of the $m!$ pairings. All the walks $\mathcal{G}$ in a given class have the same $\alpha(\mathcal{G})$. So when $\partial(S,T) = m$,

$$(A^m)_{ST} = \sum_{\mathcal{G}} \alpha(\mathcal{G}) = m! \sum_{\alpha \in \Omega(ST)} \alpha = (m!)^2 A^{(m)}_{ST}.$$

This result not only describes the relationship between $A^{(m)}$ and A^m, it also provides an alternative definition of $A^{(m)}$ in terms of the permutations α_{ST} and geodesic walks.

Theorem 6.3 Let S, T be ℓ-subsets of an r-set, regarded as vertices of the Johnson graph $J(r, \ell)$. Then, for $m = 1, 2, \ldots, \ell$,

$$A^{(m)}_{ST} \;=\; \begin{cases} 1/(m!)^2 \sum_{\mathcal{G}} \alpha(\mathcal{G}) & \text{if } \partial(S,T)=m, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum $\sum_{\mathcal{G}}$ is taken over geodesic walks from S to T. Equivalently

$$A^{(m)}_{ST} \;=\; \begin{cases} 1/(m!)^2 (A^m)_{ST} & \text{if } \partial(S,T)=m, \\ 0 & \text{otherwise.} \end{cases}$$

$\diamond$

Example 6.4: some geodesic walks in $J(9,5)$ Let $r=9$, $\ell=5$, and consider the geodesic walk $\mathcal{G}$ of length 4:

$$S = 13568 \longrightarrow 35678 \longrightarrow 36789 \longrightarrow 23679 \longrightarrow 24679 = T.$$

Here the elements paired by $\beta_{\mathcal{G}}$ are $(1,7),(5,9),(8,2),(3,4)$. In terms of the order on S and T, they are $(s_1,t_4),(s_3,t_5),(s_5,t_1),(s_2,t_2)$; we also have $s_4=t_3$. According to the lemma, we can deduce immediately that $\alpha(\mathcal{G}) = (1534)$. The answer can be checked by using the definition of $\alpha(\mathcal{G})$:

$$(1234)(2345)(4321)\,id \;=\; (1534).$$

There are 4! geodesic walks from S to T with the same pairing, but in a different order. Each determines the same element of $\Omega(ST)$, for example, for the walk

$$13568 \longrightarrow 13689 \longrightarrow 14689 \longrightarrow 12469 \longrightarrow 24679,$$

we have

$$(345)\,id\,(432)(1234) = (1534).$$

$\diamond$

In the following section we shall outline a strategy for calculating the eigenvalues of A, based on evaluation of products like $A^{(i)}A$. Theorem 6.3 provides simple answer to part of this problem. We have

$$(A^{(i)}A)_{ST} = \sum_U A^{(i)}_{SU} A_{UT} = \frac{1}{(i!)^2} \sum_U \sum_{\mathcal{G}} \alpha(\mathcal{G})\alpha_{UT}.$$

Here the first sum is over vertices U such that $\partial(S,U)=i$ and $\partial(U,T)=1$, and the second sum is over geodesic walks $\mathcal{G}$ from S to U. The conditions on U imply that the answer is non-zero only when $\partial(S,T)=i-1,i,i+1$. Thus we need to consider three sums

$$\Sigma_j \;=\; \frac{1}{(i!)^2} \sum_U \sum_{\mathcal{G}} \alpha(\mathcal{G})\alpha_{UT} \qquad (j=i-1,i,i+1),$$

where Σ_j denotes the sum over the relevant pairs $(U, \mathcal{G})$ when $\partial(S,T) = j$. In particular, Theorem 6.3 implies that Σ_{i+1} depends only on the distance $\partial(S,T) = i+1$, not the pair (S,T).

Theorem 6.5 Let S, T be vertices of $J(r,\ell)$ such that $\partial(S,T) = i+1$ $(1 \leq i \leq \ell - 1)$. Then

$$(A^{(i)}A)_{ST} \;=\; (i+1)^2\, A^{(i+1)}_{ST}.$$

Proof Given S, T such that $\partial(S,T) = i+1$, consider

$$\Sigma_{i+1} \;=\; \frac{1}{(i!)^2} \sum_{U} \sum_{\mathcal{G}} \; \alpha(\mathcal{G})\alpha_{UT},$$

where the sum is taken over the set of pairs $(U, \mathcal{G})$, consisting of a vertex U such that $\partial(S,U) = i$, $\partial(U,T) = 1$, and a geodesic walk $\mathcal{G}$ from S to U. There is a bijective correspondence between this set and the set of geodesic walks $\mathcal{G}^+$ from S to T, where $\mathcal{G}^+$ is obtained by adding the edge UT to $\mathcal{G}$. It follows that $\alpha(\mathcal{G}^+) = \alpha(\mathcal{G})\alpha_{UT}$, so that

$$\Sigma_{i+1} \;=\; \frac{1}{(i!)^2} \sum_{\mathcal{G}^+} \alpha(\mathcal{G}^+) \;=\; (i+1)^2\, \frac{1}{((i+1)!)^2} \sum_{\mathcal{G}^+} \alpha(\mathcal{G}^+) \;=\; (i+1)^2\, A^{(i+1)}_{ST}.$$

$\diamond$

7 Introduction to the algebra of distance matrices

The elementary theory of distance-regular graphs [**3**] is based on the relationship between the *ordinary* distance matrices $A^{(0)}, A^{(1)}, A^{(2)}, \ldots$ and the powers of the adjacency matrix $A = A^{(1)}$. This relationship arises from consideration of the matrix product $A^{(i)}A$. Given vertices s, t we have

$$(A^{(i)}A)_{st} \;=\; \sum_{u} A^{(i)}_{su} A_{ut},$$

where the sum can be taken over vertices u such that $\partial(s,u) = i$ and $\partial(u,t) = 1$, since all other terms are zero by definition. These conditions imply that $\partial(s,t)$ must be one of $i-1, i$, or $i+1$. Furthermore, if the numbers of vertices u satisfying the conditions is b_{st}, a_{st}, c_{st} in these three cases, we have the equation

$$(A^{(i)}A)_{st} = b_{st}A^{(i-1)}_{st} + a_{st}A^{(i)}_{st} + c_{st}A^{(i+1)}_{st}.$$

When the graph is distance-regular the numbers b_{st}, a_{st}, c_{st} do not depend on the vertices s, t, but only on $\partial(s,t)$, and they can therefore be denoted by b_{i-1}, a_i, c_{i+1}. These numbers are known as the *intersection numbers*. For the Johnson graph $J(r,\ell)$ they are

$$b_{i-1} = (\ell - i + 1)(r - \ell - i + 1), \qquad a_i = i(r - 2i), \qquad c_{i+1} = (i+1)^2.$$

The ordinary distance matrices form a basis for an algebra $\mathcal{A}$, and the equations describing the action of A by post-multiplication on the members of this basis can be regarded as defining a representation of A. In other words, the intersection numbers are the entries of an *intersection matrix* B, which represents A. It follows that the minimal equations of A and B are the same, and so the eigenvalues of A can be deduced from those of B. (The case $\ell = 2$ is discussed in Example 7.6 below.)

At the end of Section 6 we explained how part of the ordinary theory can be extended to the generalized distance matrices for $J(r, \ell)$. Roughly speaking, we should like to find elements ξ, η, χ of $\mathbb{Q}S_\ell$ such that

$$(A^{(i)}A)_{ST} = \xi A^{(i-1)}_{ST} + \eta A^{(i)}_{ST} + \chi A^{(i+1)}_{ST}.$$

Theorem 6.5 asserts that $\chi = (i+1)^2$, just as in the ordinary theory, where the intersection number c_{i+1} is equal to $(i+1)^2$. Unfortunately, this is only part of the story: the other coefficients are not so straightforward. In this section we shall deal with the cases $i = 1$ and $i = \ell$, where some progress can be made.

When $i = 1$ the product $A^{(1)}A$ is just A^2, and $(A^2)_{ST}$ is the sum of terms $\alpha_{SU}\alpha_{UT}$ taken over vertices U such that $\partial(S, U) = \partial(U, T) = 1$. There are three cases, depending on whether $\partial(S, T) = 0, 1$ or 2, so we have to consider three sums Σ_0, Σ_1, Σ_2, where

$$\Sigma_j = \sum_U \alpha_{SU}\alpha_{UT} \qquad (j = 0, 1, 2),$$

the sum being taken over those U for which $\partial(S, U) = \partial(U, T) = 1$ when $\partial(S, T) = j$. The immediate objective is to determine ξ, η, χ in $\mathbb{Q}S_\ell$ such that

$$\Sigma_0 = \xi A^{(0)}_{ST}, \qquad \Sigma_1 = \eta A^{(1)}_{ST}, \qquad \Sigma_2 = \chi A^{(2)}_{ST}.$$

The first and third cases are straightforward. When $\partial(S, T) = 0$, we have $T = S$, and a walk of length 2 is simply S, U, S, where U is one of the $b_0 = \ell(r - \ell)$ vertices adjacent to S. Since $\alpha_{SU}\alpha_{US} = id$, Σ_0 is equal to $\ell(r - \ell)id$. Furthermore, $A^{(0)} = (id)I$, so we have

$$\xi = \ell(r - \ell)id.$$

When $\partial(S, T) = 2$, we have the situation of Theorem 6.5 with $i = 1$, and so

$$\chi = 4id.$$

When $\partial(S, T) = 1$ the situation is more complicated, because there are two types of vertex U that satisfy the conditions $\partial(S, U) = \partial(U, T) = 1$. First, for each of the $r - \ell - 1$ elements $x \notin S \cup T$ there is a vertex U of the form

$$U = (S \cap T) \cup \{x\}; \quad \text{we refer to this as } \textit{Type I}.$$

Secondly, for each of the $\ell - 1$ elements $y \in S \cap T$, there is a vertex U of the form

$$U = ((S \cap T) \setminus y) \cup \{s_a, t_b\}; \quad \text{we refer to this as } \textit{Type II}.$$

Recall that $od(S, T) = a$ and $od(T, S) = b$, where $S \setminus (S \cap T) = \{s_a\}$ and $T \setminus (S \cap T) = \{t_b\}$.

Lemma 7.1 Let S, U, T be a walk of length 2 such that $\partial(S,T) = 1$. Then

(i) if U is of Type I, $od(S,U) = od(S,T)$;
(ii) if U is of Type II, then $od(S,U) = c$, where $y = s_c$.

Proof Suppose $od(S,T) = a$, so that $S = (S \cap T) \cup \{s_a\}$. If U is of Type I we have $S \cap U = S \cap T$, and so $S = (S \cap U) \cup \{s_a\}$ and $od(S,U) = a$.
If U is of Type II, we have $S = (S \cap U) \cup \{s_c\}$ and so $od(S,U) = c$. ◇

Lemma 7.2 Let S, T be vertices of $J(r,\ell)$ such that $\partial(S,T) = 1$ and $od(S,T) = i$. Then

$$(A^2)_{ST} = \eta\alpha_{ST}, \qquad \text{where} \qquad \eta = (r-\ell-1)id + \kappa_i,$$

and $\kappa_i \in \mathbb{Q}S_\ell$ is defined by

$$\kappa_i = \sum_{j \neq i} (ij) \quad (1 \leq i \leq \ell).$$

Proof There are $r - \ell - 1$ vertices U of Type I, and for these vertices $S \cap T \subseteq U$. Thus, for any $e \in S \cap T$, say $e = s_p = t_q$, we can also write $e = u_r$. It follows that

$$\alpha_{SU}\alpha_{UT}(q) = \alpha_{SU}(r) = p,$$

and so $\alpha_{SU}\alpha_{UT} = \alpha_{ST}$.
On the other hand, suppose U is of Type II. Then according to Lemma 7.1, $od(S,U) = c$, where $y = s_c \neq s_a$. We shall show that in this case

$$\alpha_{SU}\alpha_{UT} = \theta\alpha_{ST},$$

where θ is the transposition (ac). There are several cases, and we give the full proof only in a couple of them.
Suppose $a = b$, $t_a < s_a$, and $c < a$. The sets S, U, T can tabulated as follows, using the same conventions as in the proof of Lemma 6.1.

s_1	$\dots$	s_c	s_{c+1}	$\dots$	s_{a-1}		s_a	s_{a+1}	$\dots$	s_ℓ
u_1	$\dots$		u_c	$\dots$	u_{a-2}	u_{a-1}	u_a	u_{a+1}	$\dots$	u_ℓ
t_1	$\dots$	t_c	t_{c+1}	$\dots$	t_{a-1}	t_a		t_{a+1}	$\dots$	t_ℓ.

It follows that

$$\alpha_{SU} = (c\; c+1\; \cdots\; a-1), \quad \alpha_{UT} = (a\; a-1\; \cdots\; c), \quad \alpha_{ST} = id,$$

and hence $\alpha_{SU}\alpha_{UT} = (ac)\alpha_{ST}$ in this case.
If $a < c < b$ we have

s_1	$\dots$	s_a	s_{a+1}	$\dots$		s_c	s_{c+1}	$\dots$	s_b		$\dots$	s_ℓ
u_1	$\dots$	u_a	u_{a+1}	$\dots$			u_c	$\dots$	u_{b-1}	u_b	$\dots$	t_ℓ
t_1	$\dots$		t_a	$\dots$		t_{c-1}	t_c	$\dots$	t_{b-1}	t_b	$\dots$	t_ℓ.

It follows that

$$\alpha_{SU} = (c\, c+1\ \cdots\ b), \quad \alpha_{UT} = (a\, a+1\ \cdots\ c-1), \quad \alpha_{ST} = (a\, a+1\ \cdots b),$$

and $\alpha_{SU}\alpha_{UT} = (ac)\alpha_{ST}$ in this case also.

The other cases can be verified similarly. Since the $\ell-1$ vertices U of Type II are obtained by taking s_c to be any one of the $\ell-1$ members of S, except s_a, the result follows. $\diamond$

It is convenient to express the foregoing results as a single matrix equation. Define the matrix $K = (\kappa_{ST})$ in $Mat\left(\binom{r}{\ell}, \mathbb{Q}S_\ell\right)$ by

$$\kappa_{ST} \;=\; \begin{cases} \kappa_i & \text{if } \partial(S,T) = 1 \text{ and } od(S,T) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.3 The generalized distance matrices $A^{(0)} = (id)I$, $A^{(1)} = A$, and $A^{(2)}$ for $J(r,\ell)$ satisfy the equation

$$A^{(1)}A = A^2 = \ell(r-\ell)id\, I + (r-\ell-1)id\, A + K \circ A + 4id\, A^{(2)},$$

where $\circ$ denotes the pointwise product of matrices. $\diamond$

We can use this equation to determine the eigenvalues arising from the alternating representation of S_ℓ, for all ℓ and r.

Example 7.4: the alternating representation

For $\ell \geq 2$ the *alternating representation* R^{alt} of S_ℓ is defined by $R^{alt}(\omega) = sign(\omega)$. It corresponds to the partition $[1^\ell]$, and clearly its dimension is $d = 1$.

If $\partial(S,T) = \ell \geq 2$, $\Omega(ST)$ is a coset of the stabilizer of $|S \cap T|$ elements in S_ℓ, and contains equal numbers of odd and even permutations. It follows from the definition that $R^{alt}_*(A^{(\ell)})$ is the zero matrix for all $\ell \geq 2$. This observation can be used to determine the eigenvalues of $A_* = R^{alt}_*(A)$, as follows.

Let $K_* = R^{alt}_*(K)$. Applying R^{alt}_* to the formula in Theorem 7.3 gives

$$(A_*)^2 = \ell(r-\ell)I + (r-\ell-1)A_* + K_* \circ A_*.$$

Each non-zero entry of K is one of the κ_i, the sum of $\ell-1$ transpositions, so the corresponding entry of K_* is $-(\ell-1)$, and $K_* \circ A_* = -(\ell-1)A_*$. Hence

$$\begin{aligned} (A_*)^2 &= \ell(r-\ell)I + (r-\ell-1)A_* - (\ell-1)A_* \\ &= \ell(r-\ell)I + (r-2\ell)A_* \end{aligned}.$$

Therefore the eigenvalues of A_* are roots of the equation

$$\lambda^2 - (r-2\ell)\lambda - \ell(r-\ell) = 0,$$

that is $r-\ell$ and $-\ell$. Since the trace of A_* is 0, their multiplicities μ', μ'' satisfy

$$\mu'(r-\ell) + \mu''(-\ell) = 0, \qquad \mu' + \mu'' = \binom{r}{\ell},$$

that is,

$$\mu' = \binom{r-1}{\ell-1}, \qquad \mu'' = \binom{r-1}{\ell}.$$

$\diamond$

In addition to the case $i = 1$ covered above, the case $i = \ell$ can also be treated quite simply. The key here is the fact that, when $\partial(S,T) = \ell$, the sets S and T are disjoint. Hence, by definition, $\Omega(ST)$ is the whole group S_ℓ, and

$$A^{(\ell)}_{ST} = \frac{\sigma}{\ell!}, \qquad \text{where } \sigma = \sum_{\omega \in S_\ell} \omega.$$

Noting that $\sigma\tau = \sigma$ for all $\tau \in S_\ell$, we can prove the following result.

Theorem 7.5 The generalized distance matrices for $J(r,\ell)$ $(r \geq 2\ell)$ satisfy

$$A^{(\ell)}A \;=\; \frac{(r-2\ell+1)\sigma}{\ell!}\,A^{(\ell-1)} + \ell(r-2\ell)id\,A^{(\ell)}.$$

Proof Given S, T we have

$$(A^{(\ell)}A)_{ST} = \sum_U A^{(\ell)}_{SU}A_{UT} = \sum_U \frac{\sigma}{\ell!}\alpha_{UT} = \sum_U \frac{\sigma}{\ell!}.$$

The sum is taken over the relevant set of vertices U, and since the diameter of $J(r,\ell)$ is ℓ, there are only two possible values of $\partial(S,T)$ in this case, $\ell-1$ and ℓ.

Suppose $\partial(S,T) = \ell-1$. Then there are $b_{\ell-1} = r-2\ell+1$ vertices U that contribute to the sum, and the sum is

$$(r-2\ell+1)\,\frac{\sigma}{\ell!} \;=\; \left(\frac{(r-2\ell+1)\sigma}{\ell!}\right)\,A^{(\ell-1)}_{ST},$$

since $\sigma A^{(\ell-1)}_{ST} = \sigma$ always. On the other hand, suppose $\partial(S,T) = \ell$. Then there are $a_\ell = \ell(r-2\ell)$ vertices U that contribute to the sum, and it is

$$\ell(r-2\ell)\,\frac{\sigma}{\ell!} \;=\; \ell(r-2\ell)\,A^{(\ell)}_{ST}.$$

$\diamond$

Example 7.6: algebra over $\mathbb{Q}S_2$ for $J(r,2)$

We shall show that the equations obtained in Theorems 7.3 and 7.5 completely cover the case $\ell = 2$. This is because S_2 contains just two permutations id and $\theta = (12)$, and every non-zero entry of the matrix K is equal to θ. Hence $K \circ A = \theta A$.

When $r = 3$, $J(3,2)$ is the complete graph on three vertices, with diameter 1. Hence $A^{(2)}$ vanishes and the equation (Theorem 7.3) for A^2 takes the simple form

$$A^2 = 2id\,I + K \circ A = 2id\,I + \theta A.$$

So the equation for the eigenvalues over $\mathbb{Q}S_2$ is

$$\lambda^2 - \theta\lambda - 2id = 0,$$

with roots $\lambda = 2\theta$ and $\lambda = -\theta$. Applying the principal and alternating representations gives the eigenvalues $2, -1$ and $1, -2$ respectively.

When $r \geq 4$, the diameter of $J(r, 2)$ is 2 and the ordinary intersection matrix is [**3**, p.168]

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2r-4 & r-2 & 4 \\ 0 & r-3 & 2r-8 \end{pmatrix}.$$

The rows of B give the coefficients in the ordinary equations for $A^{(i)}A$. That is, the first row corresponds to the trivial equation $A^{(0)}A = A^{(1)}$, and the second and third rows to the equations

$$\begin{aligned} A^{(1)}A &= (2r-4)I + (r-2)A + 4A^{(2)}, \\ A^{(2)}A &= (r-3)A + (2r-8)A^{(2)}. \end{aligned}$$

On eliminating $A^{(2)}$ we get a cubic equation for the (ordinary) adjacency matrix A, and in fact this equation is just the characteristic equation of B.

For the generalized $A^2 = A^{(1)}A$ with $r \geq 4$ and $\ell = 2$, Theorem 7.3 gives the equation

$$A^{(1)}A \;=\; (2r-4)id\,I + \big((r-3)id + \theta\big)\,A + 4id\,A^{(2)}.$$

This clearly corresponds to the second-row equation displayed above. The counterpart of the third-row equation is given by Theorem 7.5:

$$A^{(2)}A \;=\; \frac{1}{2}(r-3)(id+\theta)\,A + (2r-8)id\,A^{(2)}.$$

From these equations we can eliminate $A^{(2)}$ and obtain a cubic equation for A. It is the characteristic equation of the *generalized intersection matrix*

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2r-4 & (r-3)id+\theta & 4 \\ 0 & (r-3)/2\,(id+\theta) & 2r-8 \end{pmatrix},$$

which is

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0,$$

where

$$\alpha_2 = -(3r-11)id - \theta, \quad \alpha_1 = (2r^2 - 18r + 34)id - 2\theta, \quad \alpha_0 = 2(2r-4)(r-4).$$

Rather surprisingly this equation factorizes over $\mathbb{Q}S_2$ as $(\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda - \lambda_2)$, where

$$\lambda_0 = (2r-6)id + 2\theta, \quad \lambda_1 = (r-3)id - \theta, \quad \lambda_2 = -2id.$$

Applying R^{pri} we obtain the ordinary eigenvalues of $J(r,2)$:

$$\lambda_0^{pri} = 2r-4, \quad \lambda_1^{pri} = r-4, \quad \lambda_2^{pri} = -2.$$

Applying R^{alt} we have

$$\lambda_0^{alt} = 2r-8, \quad \lambda_1^{alt} = r-2, \quad \lambda_2^{alt} = -2.$$

Comparison with Example 7.4 shows that the 'eigenvalue' λ_0^{alt} is spurious. This is not a problem, because our analysis only claims that the eigenvalues satisfy the cubic equation, not that all roots of the equation are eigenvalues. In fact it is clear that the anomaly arises in this case because $R_*^{alt}(A^{(2)})$ is the zero matrix, and only the second-row equation is relevant. $\diamond$

8 The extended algebra of generalized distance matrices

The foregoing investigations show that the algebra $\mathcal{A}$ spanned by the ordinary distance matrices for $J(r,\ell)$ does not generalize immediately. For example, the formula for A^2 obtained in Theorem 7.3 contains an 'extra' term $K \circ A$, and only when $\ell = 2$ can $K \circ A$ be written as a multiple of A, and the standard algebraic manipulations carried out (Example 7.6).

In this section we consider the possibility of constructing an algebra $\mathcal{A}^+$ that contains the generalized distance matrices. By analogy with the ordinary case, we hope to obtain general formulae for the action of A by multiplication on a basis of $\mathcal{A}^+$. This would afford a representation of A, analogous to the intersection matrix B in the ordinary case, enabling the eigenvalues of A to be calculated.

Denote the matrix $K \circ A$ by Z, so that when $\partial(S,T) = 1$

$$Z_{ST} = \kappa_i \alpha_{ST} \qquad \text{where} \qquad S = (S \cap T) \cup \{s_i\}.$$

The analysis of ZA, given in Theorem 8.1 below, suggests the following generalization of Z. For $m \geq 2$ we define a matrix $Z^{(m)}$ as follows. The entries $Z_{ST}^{(m)}$ are 0 unless $\partial(S,T) = m$ when

$$Z_{ST}^{(m)} = \frac{1}{m!}(\kappa_{i_1} + \kappa_{i_2} + \cdots \kappa_{i_m}) A_{ST}^{(m)} \qquad \text{where} \qquad S = (S \cap T) \cup \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\}.$$

Invoking the the explicit definition of $A^{(m)}$ we can write this more concisely as

$$Z_{ST}^{(m)} = \frac{1}{(m!)^2}\Big(\sum_i \kappa_i\Big)\Big(\sum_\alpha \alpha\Big),$$

where the first sum is taken over those i for which $s_i \notin S \cap T$, and the second sum is taken over $\alpha \in \Omega(ST)$.

In particular, when $m=\ell$, we have $Z^{(\ell)} = (2c_1/m!)A^{(\ell)}$ where

$$c_1 = \frac{1}{2}(\kappa_1 + \kappa_2 + \cdots + \kappa_\ell)$$

is the formal sum of all 2-cycles in S_ℓ. In other words, $Z^{(\ell)}$ is just a multiple of $A^{(\ell)}$. However $Z^{(1)} = Z$ and $Z^{(2)}, \ldots, Z^{(\ell-1)}$ are 'new' objects. The following theorem shows that they enter naturally into the algebra of generalized distance matrices.

Theorem 8.1 With the notation as above

$$ZA \;=\; 2(r-\ell)c_1\, I + (r-\ell-1)id\, Z + K \circ Z + 4id\, Z^{(2)}.$$

Proof We have

$$(ZA)_{ST} \;=\; \sum_U \kappa_{SU}\alpha_{SU}\alpha_{UT},$$

where the sum is over vertices U such that S, U, T is a walk of length 2. We shall consider the cases $\partial(S,T) = 0, 1, 2$ in turn.

Suppose that $\partial(S,T) = 0$, that is, $T = S$. Then $\alpha_{SU}\alpha_{US} = id$ for each of the $\ell(r-\ell)$ vertices U adjacent to S, so $(ZA)_{ST} = \sum_U \kappa_{SU}$. The number of U for which $od(S,U) = i$ is $r-\ell$ for each $i \in \{1, 2, \ldots, \ell\}$, so κ_{SU} takes the value κ_i exactly $r-\ell$ times. Hence

$$(ZA)_{SS} \;=\; (r-\ell)(\kappa_1 + \cdots + \kappa_\ell) = 2(r-\ell)c_1.$$

Suppose that $\partial(S,T) = 1$. Then $(ZA)_{ST}$ can be split into two sums

$$\sum_{(I)} \kappa_{SU}\alpha_{SU}\alpha_{UT} \;+\; \sum_{(II)} \kappa_{SU}\alpha_{SU}\alpha_{UT}.$$

Here the sums are taken over vertices U of Type I and II, respectively, as defined in Section 7. For Type I vertices $od(S,U) = od(S,T)$, and according to Lemmas 7.1 and 7.2 we have $\kappa_{SU} = \kappa_{ST}$, $\alpha_{SU}\alpha_{UT} = \alpha_{ST}$. There are $r-\ell-1$ such vertices, so

$$\sum_{(I)} \;=\; (r-\ell-1)\kappa_{ST}\alpha_{ST} = (r-\ell-1)Z_{ST}.$$

For Type II vertices, we have $\kappa_{SU} = \kappa_c$, $\alpha_{SU}\alpha_{UT} = (ac)\alpha_{ST}$, where $c = od(S,U)$ and $a = od(S,T)$. There is one such U for each of the $\ell-1$ values of $c \neq a$, hence

$$\sum_{(II)} \;=\; \left(\sum_{c\neq a} \kappa_c(ac)\right)\alpha_{ST} \;=\; \left(\sum_{c\neq a} (ac)\kappa_a\right)\alpha_{ST} \;=\; \kappa_a^2\alpha_{ST} = (K \circ Z)_{ST}.$$

Finally, suppose $\partial(S,T) = 2$, say $S = (S \cap T) \cup \{s_i, s_j\}$. There are four vertices U such that $\partial(S,U) = \partial(U,T) = 1$. For two of them $od(S,U) = i$, and for the other two $od(S,U) = j$. Let $\Omega(ST) = \{\alpha', \alpha''\}$. According to the proof of Lemma 6.1, the four terms $\alpha_{SU}\alpha_{UT}$ are equal α' and α'' for the two vertices U with $od(S,U)=i$ and similarly for the two vertices U with $od(S,U)=j$. Hence

$$\sum_U \kappa_{SU}\alpha_{SU}\alpha_{UT} \;=\; \kappa_i\alpha' + \kappa_i\alpha'' + \kappa_j\alpha' + \kappa_j\alpha'' = (\kappa_i+\kappa_j)(\alpha'+\alpha'') \;=\; 4id\, Z^{(2)}_{ST}.$$

Hence the formula is verified in all cases. ◇

There is a useful alternative form for the term $K \circ Z$. Define $\phi_i \in \mathbb{Q}S_\ell$ to be the formal sum of all 3-cycles in S_ℓ that contain i, that is,

$$\phi_i = \sum_{x,y} (ixy),$$

and define $F \in Mat\left(\binom{r}{\ell}, \mathbb{Q}S_\ell\right)$ by

$$F_{ST} \;=\; \begin{cases} \phi_i & \text{if } \partial(S,T)=1 \text{ and } od(S,T)=i; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.2 With the notation defined above,

$$K \circ Z = (\ell-1)idA + F \circ A.$$

Proof If $od(S,T)=i$, we have $(K\circ Z)_{ST} = \kappa_i^2\alpha_{ST}$. Now κ_i is the sum of the $\ell-1$ transpositions containing i, so expanding κ_i^2 we obtain $\ell-1$ identity terms and $(\ell-1)(\ell-2)$ terms of the form $(ix)(iy)=(iyx)$, that is,

$$\kappa_i^2 = (\ell-1)id + \phi_i.$$

The result follows.

◇

9. Algebra over $\mathbb{Q}S_3$ for $J(r,3)$

Denote the generators (12), (23) of S_3 by g, h respectively, so that

$$gh=(123), \quad hg=(321), \quad ghg=hgh=(13).$$

It will turn out that all the coefficients in our equations are linear combinations of the *class sums*

$$c_0 = id; \qquad c_1 = g+h+ghg; \qquad c_2 = gh+hg.$$

The elements c_0, c_1, c_2 form a basis for the centre of the group algebra, $Z(\mathbb{Q}S_3)$, and they satisfy the equations

$$c_1^2 = 3c_0+3c_2; \quad c_2^2 = 2c_0+c_2; \quad c_1c_2=c_2c_1=2c_1.$$

(Generally, for any ℓ the class sums form a basis for $Z(\mathbb{Q}S_\ell)$, and they satisfy relations of the form

$$c_i c_j = c_j c_i = \sum_k a_{ijk} c_k,$$

where the a_{ijk} are non-negative integers that can be obtained from the character table of S_ℓ [**9** pp.318-320].)

From now on we shall often write c_0 instead of id, and abbreviate qc_0 to q ($q \in \mathbb{Q}$). With this convention the equations obtained in Theorems 7.3 and 8.1 for $\ell = 3$ are as follows:

$$A^2 = (3r-9)I + (r-4)A + K \circ A + A^{(2)},$$

$$ZA = (2r-6)c_1\, I + (r-4)Z + K \circ Z + 4Z^{(2)}.$$

Recall that $Z = K \circ A$ and, according to Lemma 8.2, $K \circ Z = 2A + F \circ A$, where $F_{ST} = \phi_i$ when $\partial(S,T) = 1$ and $od(S,T) = i$. Furthermore, when $\ell = 3$ we have $\phi_1 = \phi_2 = \phi_3 = c_2$, so $F \circ A = c_2 A$. Thus the equations reduce to

$$A^2 = (3r-9)I + (r-4)A + Z + 4A^{(2)},$$

$$ZA = (2r-6)c_1\, I + (2c_0 + c_2)\, A + (r-4)Z + 4Z^{(2)}.$$

Example 9.1: calculations for $J(4,3)$

When $r = 4$ the graph $J(4,3)$ is the complete graph K_4 with diameter 1, so $A^{(2)}$ and $Z^{(2)}$ are both zero. The two basic equations reduce to

$$A^2 = 3c_0 I + Z, \qquad ZA = 2c_1 I + (2c_0 + c_2)A.$$

Eliminating Z we obtain

$$A^3 - (5c_0 + c_2)A - 2c_1 I = O.$$

(In fact, this equation can be proved directly by considering walks of length 3 in $J(4,3) = K_4$.) Thus the eigenvalues of A over $\mathbb{Q}S_3$ satisfy the equation

$$\lambda^3 - (5c_0 + c_2)\lambda - 2c_1 = 0.$$

We can use the relationships between the class sums to factorize this equation over $\mathbb{Q}S_3$. First we note that there is a factor $\lambda - c_1$, and write the equation as

$$(\lambda - c_1)(\lambda^2 + c_1\lambda - (2c_0 - 2c_2)) = 0.$$

The discriminant of the quadratic term is

$$c_1^2 + (8c_0 - 8c_2) = 11c_0 - 5c_2 = (3c_0 - c_2)^2.$$

Hence the three roots of the equation are

$$c_1, \qquad (1/2)(3c_0 - c_1 - c_2), \qquad (1/2)(-3c_0 - c_1 + c_2).$$

There are three irreducible representations of S_3. The principal and alternating representations are 1-dimensional, and correspond to the partitions [3] and [111]. The third representation is 2-dimensional and corresponds to the partition [21]. It is given by [**11**, p.75]

$$R^{[21]}(g) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad R^{[21]}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and it is easy to check that under $R^{[21]}$

$$c_0 \mapsto I, \quad c_1 \mapsto O, \quad c_2 \mapsto -I.$$

The images of the $\mathbb{Q}S_3$-eigenvalues in the three cases are thus

$$\begin{array}{rccc} [3]: & 3 & -1 & -2 \\ [111]: & -3 & 2 & 1 \\ [21]: & O & 2I & -2I. \end{array}$$

Comparing these results with the general ones for the principal and alternating representations (Examples 5.1 and 7.4) we see that the algebraic manipulations have thrown up one extraneous possibility in each case: in fact the eigenvalues are respectively $3, -1$ and $1, -3$, with multiplicities 1 and 3 in both cases.

In the case of $R^{[21]}$ all three possible eigenvalues $0, 2, -2$ occur, with multiplicities $2, 3, 3$. Note that the total multiplicity is 8 because the representation is 2-dimensional and A_* is an 8×8 matrix. $\diamond$

In order to describe what happens in the case $\ell = 3$ when $r > 4$, we require equations for the action of A (by right-multiplication) on the matrices $A^{(2)}, Z^{(2)}, A^{(3)}$. These equations are given in the following result.

Theorem 9.2 When $r \geq 6$, the matrices $A^{(2)}, Z^{(2)}, A^{(3)}$ for $J(r, 3)$ satisfy the equations

$$A^{(2)}A = (r-4)A + \frac{1}{2}(r-4)Z + (r-4)A^{(2)} + 9A^{(3)};$$

$$Z^{(2)}A = (r-4)(2c_0 + c_1 + c_2)A + \frac{1}{2}(r-4)Z + (r-4)Z^{(2)} + 3c_1A^{(3)};$$

$$A^{(3)}A = \frac{1}{6}(r-5)(c_0 + c_1 + c_2)\,A^{(2)} + (3r-18)A^{(3)}.$$

Proof For the first two equations we use Theorem 6.3, and the fact that $Z^{(2)}_{SU}$ is a multiple of $A^{(2)}_{SU}$, say $K_{SU}A^{(2)}_{SU}$. Thus

$$(A^{(2)}A)_{ST} = \sum_U A^{(2)}_{SU}\alpha_{UT} = \frac{1}{4}\sum_U \Big(\sum_{\mathcal{G}} \alpha(\mathcal{G})\Big)\alpha_{UT},$$

$$(Z^{(2)}A)_{ST} = \sum_U Z^{(2)}_{SU}\alpha_{UT} = \sum_U K_{SU}A^{(2)}_{SU}\alpha_{UT}.$$

In these equations, $\sum_U$ denotes a sum over vertices U such that $\partial(S,U) = 2$, $\partial(U,T) = 1$, and $\sum_{\mathcal{G}}$ denotes a sum ove geodesic walks $\mathcal{G}$ from S to U.

Throughout the following calculation we take $S = \{x, y, z\}$, and suppose that $x = s_h, y = s_i, z = s_j$, where h, i, j is the permutation of $1, 2, 3$ that makes $s_1 < s_2 < s_3$.

When $\partial(S,T) = 1$, we can take $T = \{x, y, n\}$, which implies that $od(S,T) = j$. Then there are $b_1 = 2(r-4)$ vertices U that contribute to the sum displayed above, $r-4$ of each of the forms $\{n, x, *\}, \{n, y, *\}$, where $*$ is any symbol except x, y, z, n. For each such U there are four geodesic walks $\mathcal{G}$ from S to U. Let $\Omega(SU) = \{\omega_1, \omega_2\}$, so that $\alpha(\mathcal{G})$ takes the values ω_1, ω_2 twice each. One of the geodesic walks is S, T, U, so we may take $\omega_1 = \alpha_{ST}\alpha_{TU}$. Furthermore, $\omega_2(\omega_1)^{-1}$ fixes f, where s_f is the common element of S and U. So we can write $\omega_2 = \tau\omega_1$, where τ transposes the two members of $\{1, 2, 3\} \setminus \{f\}$.

When $U = \{n, x, *\}$ the common member of S and U is $x = s_h$. Hence $\omega_2 = (ij)\omega_1$. Similarly, when $U = \{n, y, *\}$, $\omega_2 = (hj)\omega_1$.

Since $od(S,T) = j$ and $\omega_1\alpha_{UT} = \alpha_{ST}\alpha_{TU}\alpha_{UT} = \alpha_{ST}$, we have

$$\begin{aligned}(A^{(2)}A)_{ST} &= \frac{1}{4}\Big((r-4)(2id + 2(ij))\omega_1\alpha_{UT} + (r-4)(2id + 2(hj))\omega_1\alpha_{UT}\Big) \\ &= \frac{1}{2}(r-4)(2id + \kappa_j)\alpha_{ST}.\end{aligned}$$

For $Z^{(2)}A$, we note that, for all U, $y = s_i$ and $z = s_j$ are the elements of S that are not in U, and so according to the definition of $Z^{(2)}$, $K_{SU} = \frac{1}{2}(\kappa_i + \kappa_j)$, independent of U. Thus

$$\begin{aligned}(Z^{(2)}A)_{ST} &= \frac{1}{2}(\kappa_i + \kappa_j)(A^{(2)}A)_{ST} \\ &= \frac{1}{2}(\kappa_i + \kappa_j)\frac{1}{2}(r-4)(2id + \kappa_j)\alpha_{ST} \\ &= \frac{1}{2}(r-4)(4c_0 + 2c_1 + 2c_2 + \kappa_j)\alpha_{ST}.\end{aligned}$$

When $\partial(S,T) = 2$, we can take $T = \{x, v, w\}$. The four geodesic walks from S to T are defined by the four vertices

$$P_1 = \{x, v, z\}, \quad P_2 = \{x, y, v\}, \quad Q_1 = \{x, y, w\}, \quad Q_2 = \{x, v, z\}.$$

The elements θ_1, θ_2 of $\Omega(ST)$ are thus

$$\theta_i = \alpha_{SP_i}\alpha_{P_iT} = \alpha_{SQ_i}\alpha_{Q_iT} \quad (i = 1, 2).$$

There are $a_2 = 2(r-4)$ vertices U that contribute to the sum

$$\sum_U \Big(\sum_{\mathcal{G}} \alpha(\mathcal{G})\Big),$$

specifically $r-4$ of each of the forms $\{x,v,*\},\{x,w,*\}$, where $*$ is any symbol except y,z,v,w. Writing $\Omega(SU)=\{\omega_1(U),\omega_2(U)\}$ the sum is

$$2\Big(\sum_{U\ni v}+\sum_{U\ni w}\Big)\,(\omega_1(U)+\omega_2(U)).$$

When $U=\{x,v,*\}$, two of the four geodesic walks from S to U are S,P_1,U and S,P_2,U, so $\omega_i(U)=\alpha_{SP_i}\alpha_{P_iU}$, and

$$\omega_i(U)\alpha_{UT}=\alpha_{SP_i}\alpha_{P_iU}\alpha_{UT}=\alpha_{SP_i}\alpha_{P_iT}=\theta_i.$$

When $U=\{x,w,*\}$, two of the four geodesic walks from S to U are S,Q_1,U and S,Q_2,U, so $\omega_i(U)=\alpha_{SQ_i}\alpha_{Q_iU}$, and the same result holds. Hence

$$\begin{aligned}(A^{(2)}A)_{ST}&=\frac{1}{4}\sum_U\Big(\sum_{\mathcal{G}}\alpha(\mathcal{G})\Big)\alpha_{UT}\\&=\frac{1}{2}\sum_U(\omega_1(U)+\omega_2(U))\alpha_{UT}\\&=\frac{1}{2}(r-4)(\theta_1+\theta_2)\\&=(r-4)A^{(2)}_{ST}.\end{aligned}$$

As in the previous case, for all U, $y=s_i$ and $z=s_j$ are the elements of S that are not in U, and so $K_{SU}=\frac{1}{2}(\kappa_i+\kappa_j)$. For the same reason, $K_{ST}=\frac{1}{2}(\kappa_i+\kappa_j)$. Hence

$$\begin{aligned}(Z^{(2)}A)_{ST}&=\frac{1}{2}(\kappa_i+\kappa_j)(A^{(2)}A)_{ST}\\&=\frac{1}{2}(\kappa_i+\kappa_j)(r-4)A^{(2)}_{ST}\\&=(r-4)Z^{(2)}_{ST}.\end{aligned}$$

When $\partial(S,T)=3$, we can take $T=\{u,v,w\}$.

For $A^{(2)}A$ we already have Theorem 6.4, which states that $(A^{(2)}A)_{ST}=9A^{(3)}_{ST}$. For $Z^{(2)}A$ we have the equation

$$(Z^{(2)}A)_{ST}=\sum_U K_{SU}A^{(2)}_{SU}\alpha_{UT},$$

where the sum is taken over the $c_3=9$ vertices U such that $\partial(S,U)=2$ and $\partial(U,T)=1$. There are three such vertices containing any one of the symbols x,y,z. For example, when $x\in U$, these vertices are $\{x,u,v\},\{x,u,w\},\{x,v,w\}$, and for each of them $K_{SU}=\frac{1}{2}(\kappa_i+\kappa_j)$. Thus

$$\sum_{U\ni x}K_{SU}A^{(2)}_{SU}\alpha_{UT}=\frac{1}{2}(\kappa_i+\kappa_j)\sum_{U\ni x}A^{(2)}_{SU}\alpha_{UT}.$$

For each of these three vertices there are four geodesic paths $\mathcal{G}$ from S to U, each of which can be extended to a geodesic path $\mathcal{G}^+$ from S to T by the addition of the edge UT. Hence

$$\sum_{U \ni x} A^{(2)}_{SU} \alpha_{UT} = \sum_{U \ni x} \Big(\frac{1}{4} \sum_{\mathcal{G}} \alpha(\mathcal{G}) \Big) \alpha_{UT} = \frac{1}{4} \sum_{U \ni x} \sum_{\mathcal{G}^+} \alpha(\mathcal{G}^+).$$

For each of the three U, the four geodesic paths $\mathcal{G}$ from S to U give two distinct values of $\alpha(\mathcal{G})$ and $\alpha(\mathcal{G}^+)$. The six resulting values of $\alpha(\mathcal{G}^+)$ are distinct, and comprise all elements of S_3. Hence

$$\sum_{U \ni x} K_{SU} A^{(2)}_{SU} \alpha_{UT} = \frac{1}{2}(\kappa_i + \kappa_j) \frac{1}{4}(2\sigma).$$

Thus the sum over all nine vertices U is

$$\frac{1}{2}\big((\kappa_i + \kappa_j) + (\kappa_h + \kappa_j) + (\kappa_h + \kappa_i)\big)\frac{1}{2}\sigma = \frac{1}{2} c_1 \sigma = 3 c_1 A^{(3)}_{ST},$$

since $A^{(3)}_{ST} = \sigma/6$ for all S, T.

Finally, the equation for $A^{(3)}A$ is a special case of Theorem 7.5. ◇

For the case $\ell = 3$ we now have equations describing the action of A by right-multiplication on the matrices $I, A, Z, A^{(2)}, Z^{(2)}, A^{(3)}$. These equations provide a 6-dimensional representation of A. Putting $s = r - 4$ and

$$t = (r-4)(2c_0 + c_1 + c_2), \qquad u = \frac{1}{6}(r-5)(c_0 + c_1 + c_2),$$

the matrix is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3s+3 & s & 1 & 4 & 0 & 0 \\ (2s+2)c_1 & 2c_0 + c_2 & s & 0 & 4 & 0 \\ 0 & s & s/2 & s & 0 & 9 \\ 0 & t & s/2 & 0 & s & 3c_1 \\ 0 & 0 & 0 & u & 0 & 3s-6 \end{pmatrix}.$$

10 The eigenvalues of N^π as polynomials

We now study the family of graphs $\mathcal{B}_r = \{B(r,n)\}$, for a given value of r. In Section 4 the starting point was the basic result that, for a given number of colours k, the value of the chromatic polynomial $P(B(n,r);k)$ can be expressed as the trace of N^n, where the matrix N has $k(k-1)\cdots(k-r+1)$ rows and columns. This result essentially determines the dependence of $P(B(n,r);k)$ on n, by showing that it can be expressed as the sum of the nth powers of the eigenvalues of N.

In order to deal with the dependence on k we showed that N can be decomposed into matrices N^π, representing its action on invariant subspaces, where there are $m_\pi(k)$ equivalent representations corresponding to each partition π with $|\pi| \le r$. This decomposition has the practical advantage that the size of N^π is independent of k. However, the entries of N^π do depend on k, and the foregoing analysis has described this dependence in detail. As pointed out in Section 5, when $|\pi| = \ell$, N^π is the image under R^π_* of the matrix in $Mat(\binom{r}{\ell}, \mathbb{Q}S_\ell)$ defined by

$$X \;=\; \sum_{m=0}^{\min(\ell,r-\ell)} \;(-1)^{\ell+m} f_{r-\ell-m}(k)\, m!\, A^{(m)},$$

where $A^{(m)}$ is a generalized distance matrix for $J(r,\ell)$. We aim to use this expression to determine the eigenvalues of N^π.

The matrix X is a function of r, k and ℓ. Since we are regarding r as fixed, we shall write $X = X_\ell(k)$, noting that $X_\ell(k)$ depends on k only through the terms $f_{r-\ell-m}(k)$. As in Section 4, given k and $0 \le i \le r$, these terms are defined by

$$f_i(k) = F(i, k - r + i),$$

where

$$F(i,z) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \langle z - j\rangle_{i-j}.$$

Here $\langle\ \rangle$ is the extended 'falling factorial' defined for $w \in \mathbb{C}$ and non-negative integers ℓ by

$$\langle w\rangle_0 = 1, \qquad \langle w\rangle_\ell = (w - \ell + 1)\langle w\rangle_{\ell-1} \quad (\ell \ge 1).$$

It is easy to verify that F satisfies the recurrence

$$F(0,z) = 1, \qquad F(1,z) = z - 1,$$

$$F(i,z) = (z-1)F(i-1, z-1) + (i-1)F(i-2, z-2).$$

Elementary algebra leads directly to the following result.

Lemma 10.1 Suppose r is given. Then the sequence $f_i(k) = F(i, k - r + i)$ is defined for $(0 \le i \le r$ by the recurrence

$$f_0(k) = 1, \quad f_1(k) = k - r,$$
$$f_i(k) = (k - r + i - 1)f_{i-1}(k) + (i-1)f_{i-2}(k) \qquad (2 \le i \le r).$$

$\diamond$

It follows that $f_i(k)$ is a monic polynomial of degree i in k, with integer coefficients that alternate in sign. The leading terms are

$$f_i(k) = k^i - \frac{1}{2}i(2r - i + 1)k^{i-1} + \ \cdots\ .$$

In order to calculate the eigenvalues of N^π for each π with $|\pi| = \ell$, we use the formula for $X_\ell(k)$ given above, followed by an application of R_*^π. It turns out that (in some cases) the only additional information required is the spectrum of $A_* = R_*^\pi(A)$, where $A = A^{(1)}$ is the generalized adjacency matrix.

Example 10.2: the eigenvalues for $\mathcal{B}_4$

The f-polynomials for $r = 4$ are

$$f_0(k) = 1, \quad f_1(k) = k - 4, \quad f_2(k) = k^2 - 7k + 13,$$
$$f_3(k) = k^3 - 9k^2 + 29k - 34, \quad f_4(k) = k^4 - 10k^3 + 41k^2 - 84k + 73,$$

and so in this case the matrices $X_\ell(k)$ are

$$\begin{aligned}
X_0(k) &= (k^4 - 10k^3 + 41k^2 - 84k + 73)A^{(0)}; \\
X_1(k) &= -(k^3 - 9k^2 + 29k - 34)A^{(0)} + (k^2 - 7k + 13)A^{(1)}; \\
X_2(k) &= (k^2 - 7k + 13)A^{(0)} - (k - 4)A^{(1)} + 2A^{(2)}; \\
X_3(k) &= -(k - 4)A^{(0)} + A^{(1)}. \\
X_4(k) &= A^{(0)}.
\end{aligned}$$

In this set of equations $A^{(0)}, A^{(1)}, A^{(2)}$ denote matrices associated with $J(r, \ell)$, with ℓ taking the values $0, 1, 2, 3, 4$ in the respective equations. In order to obtain N^π we must replace them with their images under R_*^π, where $\ell = |\pi|$. It is significant that when $\ell = 2$ the matrix $A^{(2)}$ is a polynomial function of $A^{(1)}$ (see Example 7.6). This means that every $X_\ell(k)$ in the above list yields a polynomial function of k and A_* when R_*^π is applied to it, and the eigenvalues of $X_\ell(k)$ are the same polynomial functions of the eigenvalues of A_*. In the Appendix this observation is used to calculate all the eigenvalues for $\mathcal{B}_4$. The eigenvalues for $\mathcal{B}_2$ and $\mathcal{B}_3$ are also given. ◇

In the cases $r = 2, 3, 4$ it turns out that the eigenvalues inherit the alternating-coefficients property of the f-polynomials. We conjecture that this is true for all $r \geq 2$. In other words, each eigenvalue $\lambda(k)$ has the form

$$\pm(k^{r-\ell} - bk^{r-\ell-1} + ck^{r-\ell-2} - \cdots),$$

where the coefficients $b, c, \ldots$ are positive integers.

References

1 S. Beraha, J. Kahane, N.J. Weiss. Limits of zeros of recursively defined families of polynomials. *Stud. Found. Combin. Adv. Math. Suppl. Studies* 1 (1978) 213-232.

2 N.L. Biggs, R.M. Damerell, D.A. Sands. Recursive families of graphs. *J. Comb. Theory Ser. B* 12 (1972) 52-65.

3 N.L. Biggs. *Algebraic Graph Theory - Second Edition*, Cambridge University Press, 1993

4 N.L. Biggs. Chromatic polynomials and representations of the symmetric group. *Lin. Alg. Applications* 356 (2002) 3-26.

5 N.L. Biggs, M.H.Klin, P. Reinfeld. Algebraic methods for chromatic polynomials. *Europ. J. Comb.* 25 (2004) 147-160.
6 N.L. Biggs, Specht modules and chromatic polynomials. *J. Comb. Theory Ser. B* 92 (2004) 359-377.
7 S.C. Chang. Exact chromatic polynomials for toroidal chains of complete graphs. *Physica A* 313 (2002) 397-410.
8 G.D. James The Representation Theory of the Symmetric Group. *Lecture Notes in Mathematics 682* Springer, Berlin 1978.
9 G.D.James and M. Liebeck, *Representations and Characters of Groups.* Cambridge U.P. 1993.
10 R.C.Read and G.F. Royle. Chromatic roots of families of graphs. In: *Graph Theory, Combinatorics, and Applications* ed. Y. Alavi, G. Chartrand, O.R. Ollerman, A.J. Schwenk, Wiley, 1991, pp.1009-1029.
11 B.E. Sagan. *The Symmetric Group.* Springer 2001.
12 A.D. Sokal. Bounds on the complex zeros of (di-)chromatic polynomials and the Potts-model partition function. *Combinatorics, Probability and Computing*, 10 (2001) 41-77.
13 W.T. Tutte. On chromatic polynomials and the golden ratio, *J. Comb. Theory*, 9 (1970) 289-296.
14 D.R. Woodall, A zero-free interval for chromatic polynomials, *Discrete Math.* 101 (1992) 333-341.

Appendix: Eigenvalues for $\mathcal{B}_r$, $r = 2, 3, 4$.

We shall calculate the eigenvalues for $\mathcal{B}_4$, using the methods developed in the paper. Equations for the matrices $X_\ell(k)$, $\ell = 0, 1, 2, 3, 4$ are given in Section 10. In each equation $A^{(0)}, A^{(1)}, A^{(2)}$ denote matrices associated with $J(4, \ell)$ for the relevant value of ℓ, and we shall consider each value in turn. For each partition π with $|\pi| = \ell$ the polynomial obtained by replacing A by $A_* = R_*^\pi(A)$ will be denoted by $X_\pi(k, A_*)$.

- $\ell = 0$. Here $A^{(0)}$ is the 1×1 matrix (id). There is only one representation, corresponding to the empty partition o, and $X_o(k, A_*)$ is the 1×1 matrix $(k^4 - 10k^3 + 41k^2 - 84k + 73)I$, with the eigenvalue,

$$\lambda_1 = k^4 - 10k^3 + 41k^2 - 84k + 73.$$

- $\ell = 1$. Here $A^{(0)}$ and $A^{(1)}$ are the 4×4 matrices $(id)I$ and $(id)(J - I)$. There is only one representation with $\ell = 1$, the principal representation $R^{[1]}$, and the images under $R_*^{[1]}$ of $A^{(0)}$ and $A^{(1)}$ are I and $A_* = J - I$. The eigenvalues of A_* are 3 and -1, with multiplicities 1 and 3 respectively. We have

$$X_{[1]}(k, A_*) = -(k^3 - 9k^2 + 29k - 34)I + (k^2 - 7k + 13)A_*,$$

so the eigenvalues are

$$\lambda_2 = -(k^3 - 9k^2 + 29k - 34) + (3)(k^2 - 7k + 13)$$
$$= -k^3 + 12k^2 - 50k + 73;$$
$$\lambda_3 = -(k^3 - 9k^2 + 29k - 34) + (-1)(k^2 - 7k + 13)$$
$$= -k^3 + 8k^2 - 22k + 21;$$

with multiplicities 1 and 3 respectively.

• $\ell = 2$. Here $A^{(0)} = (id)I, A^{(1)} = A$, and $A^{(2)}$ are 6×6 matrices. In Example 7.6 we showed that when $\ell = 2$ $A^{(2)}$ is a polynomial in A: putting $r = 4$ and $\theta = (12) \in S_2$, the result is

$$A^{(2)} = \frac{1}{4}\big(A^2 - (id + \theta)A - (4id)I\big).$$

(Of course, in this case it is easy to write down $A^{(1)}$ and $A^{(2)}$ and check the equation directly.)

Here we have to consider the principal and alternating representations, corresponding to the partitions $[2], [11]$. We have

$$\begin{aligned} X_{[2]}(k, A_*) &= (k^2 - 7k + 13)I - (k-4)A_* + \frac{1}{2}\big(A_*^2 - 2A_* - 4I\big) \\ &= (k^2 - 7k + 11)I - (k-3)A_* + \frac{1}{2}A_*^2; \\ X_{[11]}(k, A_*) &= (k^2 - 7k + 13)I - (k-4)A_* + \frac{1}{2}\big(A_*^2 - 4I\big) \\ &= (k^2 - 7k + 11) - (k-4)A_* + \frac{1}{2}A_*^2. \end{aligned}$$

The eigenvalues and multiplicities of A_* were obtained for $\pi = [2], [11]$ and all r in Example 7.6. Putting $r = 4$ we get

$$\begin{array}{r} \text{eigenvalue :} \\ \text{multiplicity :} \end{array} \quad [2] : \begin{array}{ccc} 4 & 0 & -2 \\ 1 & 3 & 2 \end{array} \qquad [11] : \begin{array}{cc} 2 & -2 \\ 3 & 3 \end{array}.$$

It follows that the eigenvalues of $X_{[2]}(k, A_*)$ are

$$\lambda_4 = (k^2 - 7k + 11) - (4)(k-3) + (4^2)(1/2)$$
$$= k^2 - 11k + 31;$$
$$\lambda_5 = (k^2 - 7k + 11);$$
$$\lambda_6 = (k^2 - 7k + 11) - (-2)(k-3) + (-2)^2(1/2)$$
$$= k^2 - 5k + 7;$$

with multiplicities $1, 3, 2$ respectively.

The eigenvalues of $X_{[11]}(k, A_*)$ are

$$\lambda_7 = (k^2 - 7k + 11) - (2)(k-4) + 2^2(1/2) \\ = k^2 - 9k + 21;$$

$$\lambda_8 = (k^2 - 7k + 11) - (-2)(k-4) + (-2)^2(1/2) \\ = k^2 - 5k + 5;$$

with multiplicities $3, 3$ respectively.

- $\ell = 3$. There are three irreducible representations of S_3, corresponding to the partitions $\pi = [3], [111], [21]$. In each case

$$X_\pi(k, A_*) = -(k-4)I + A_*.$$

The spectrum of A_* in each of the three cases was obtained in Example 9.1:

$$\begin{matrix} \text{eigenvalue :} \\ \text{multiplicity :} \end{matrix} \qquad [3] : \begin{matrix} 3 & -1 \\ 1 & 3 \end{matrix} \qquad [111] : \begin{matrix} 1 & -3 \\ 1 & 3 \end{matrix} \qquad [21] : \begin{matrix} 2 & 0 & -2 \\ 3 & 2 & 3 \end{matrix}.$$

Thus we get the following eigenvalues

$$\begin{aligned} [3] :&\quad \lambda_9 = -(k-7), &\quad \lambda_{10} = -(k-3), \\ [111] :&\quad \lambda_{11} = -(k-1), &\quad \lambda_{12} = -(k-5), \\ [21] :&\quad \lambda_{13} = -(k-6), &\quad \lambda_{14} = -(k-4), &\quad \lambda_{15} = -(k-2), \end{aligned}$$

with multiplicities $1, 3, 1, 3, 3, 2, 3$ respectively.

- $\ell = 4$. Here $X_4(k) = A^{(0)}$ is simply the 1×1 matrix (id), and $X_\pi(k, A_*) = (1)$ for all partitions π of 4. Hence for each such partition we have an eigenvalue 1, with multiplicity 1. Note that the coefficient of 1^n in the formula for $P(B(4,n);k)$ is the sum of the $m_\pi(k)$ for all partitions with $|\pi| = 4$.

The complete formula for $P(B(4,n);k)$ can now be checked against the one obtained by less systematic arguments in [**4**].

For reference, the f-polynomials and eigenvalues for $r = 2, 3$ are also listed here. They can be obtained easily by the methods used above. The numbering is chosen to conform with a system that would apply to all $r \geq 2$, which is why there is no λ_6 when $r = 3$, for example.

$\mathbf{r = 2}$

$$f_0 = 1, \qquad f_1 = z - 2, \qquad f_2 = z^2 - 3z + 3.$$

$$\lambda_1 = z^2 - 3z + 3, \qquad -\lambda_2 = z - 3, \qquad -\lambda_3 = z - 1, \qquad \lambda_4 = 1.$$

$\mathbf{r = 3}$

$$f_0 = 1, \qquad f_1 = z - 3, \qquad f_2 = z^2 - 5z + 7, \qquad f_3 = z^3 - 6z^2 + 14z - 13.$$

$$\begin{aligned} \lambda_1 &= z^3 - 6z^2 + 14z - 13, \\ -\lambda_2 &= z^2 - 7z + 13, \qquad -\lambda_3 = z^2 - 4z + 4, \\ \lambda_4 &= z - 5, \qquad \lambda_5 = z - 1, \\ \lambda_7 &= z - 4, \qquad \lambda_8 = z - 2. \end{aligned}$$

◇

On the basis of these results, it is not unreasonable to conjecture that the eigenvalues of N^π are polynomials in k with coefficients that are integers and alternate in sign, for all partitions π.