Dividend Policy Irrelevance and Eigenvalue Location

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Abstract

This note gives a qualified affirmative answer to a natural question, asked by Ohlson and motivated by some earlier work, concerning the irrelevance of dividend policy to the calculation of equity in the context of an Ohlson style general linear accounting dynamic. Does Dividend Irrelevance occur when discounting at a rate of interest R if and only if R is set equal uniquely to the dominant eigenvalue of the principal submatrix? The latter submatrix relates the accounting variables to each other in the absence of any dividend payout. The question reduces to the assertion that the maximum eigenvalue $\kappa_{\max}$ of the following 'bordered diagonal matrix'

$$\begin{bmatrix} \lambda_1 & 0 & & 0 & 1 \\ 0 & \lambda_2 & & & 1 \\ & & \cdots & & \\ 0 & 0 & & \lambda_n & 1 \\ \omega_1 & \omega_2 & & & \omega_{n+1} \end{bmatrix}.$$

lies between the first largest and second largest among $|\lambda_1|, ..., |\lambda_n|$. An affirmative answer necessarily restricts the dividend policy vector $(\omega_1, ..., \omega_{n+1})$. The results show that an algebraic condition equivalent to dividend irrelevance derived previously is not vacuous.

Key words: Dividend irrelevance, dominant eigenvalue, bordered diagonal matrix.

1. Introduction

Consider the following first-order recurrence, which we term the Ohlson accounting dynamics.

$$\left.\begin{array}{rcllll} z_{t+1} & = & Az_t & + bd_t & + av_t, \\ d_{t+1} & = & wz_t & + \beta d_t & + \alpha v_t, \\ v_{t+1} & = & & & + \gamma v_t, \end{array}\right\} \qquad (\Omega)$$

with

$$0 \leq \gamma < 1.$$

Here $z_t = (z_t^1, ..., z_t^n) \in R^n$, is called the **accounting state vector**, while d_t and v_t are reals defined for $t = 0, 1, 2, ..$, representing respectively the dividend and the information variable (a proxy for un-accounted value); the value of the latter is assumed to dwindle into insignificance over time. The vector

$$\omega_{\text{div}} = (w, \beta, \alpha)$$

is the dividend policy vector. A is a real matrix of size $n \times n$, called the **reduced matrix** of the system (Ω). Its eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ are listed in order of decreasing modulus, i.e.

$$|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|.$$

The **augmented matrix** is given by

$$\bar{A} = \bar{A}(w, \beta) = \begin{bmatrix} A & b \\ w & \beta \end{bmatrix},$$

regarded as a function of (w, β). Its eigenvalues will likewise be regarded as functions of (w, β) and referred to as

$$\kappa_1, \kappa_2, ..., \kappa_{n+1},$$

identified through the condition

$$\kappa_i(0, ..., 0, \beta) = \lambda_i \text{ for } i = 1, ..., n, \text{ and } \kappa_{n+1}(0, ..., 0, \beta) = \beta. \tag{1.1}$$

We will be working for the most part in an equivalent canonical setting where $\bar{A}$ is replaced by

$$\begin{bmatrix} \lambda_1 & 0 & & 0 & \delta_1 \\ 0 & \lambda_2 & & & \delta_2 \\ & & \cdots & & \\ 0 & 0 & & \lambda_n & \delta_n \\ \omega_1 & \omega_2 & & & \omega_{n+1} \end{bmatrix}, \tag{1.2}$$

where $\delta_i = \pm 1$ for each i.

Equity, defined as a discounted series of dividends, is regarded here as a function of the discount rate R; thus

$$P_0(R; d) = \sum_{t=1}^{\infty} R^{-t} d_t. \tag{1.3}$$

We say that equity is **dividend irrelevant** at R if the value of $P_0(R; d)$ is unchanged as the dividend policy vector ω_{div} changes. This amounts to requiring that $P_0(R, d)$ depends only on the initial data: A, b, z_0, d_0. The dividend irrelevance question (in particular whether or not dividends should be irrelevant to stockholders) has been a live issue since Modigliani and Miller (1961). See, for example, Dybvig and Zender (1991). The current quest for dividend irrelevance comes from the general possibility of restating equity in terms of an identically discounted alternative series based on accounting numbers, as first pointed out by Preinreich (1936). If (Ω) models the evolution of the firm and z_t models its observable accounting numbers, interest focuses on whether valuations are possible at time $t = 0$, based on the accounting numbers alone, that is to say in the absence of access to the currently unobservable information ω_{div}. See Proposition 2 below.

In earlier work, Ostaszewski (2003), generalizing from a one-dimensional result of Ohlson (2003) to the n-dimensional setting given above, has shown that if dividend irrelevance occurs at R, then R is an eigenvalue of A, and has also derived a necessary and sufficient condition that dividend irrelevance occur at R in terms which make the given vector b, the vector of **dividend significance coefficients**, depend on A in a specified way. That result was derived under what is a **convergence assumption**, namely that R exceeds in modulus the eigenvalues of the augmented matrix $\bar{A}$ (which control the growth rates of the accounting state variables), a requirement for securing convergence of the equity series (1.3). Further so-called 'implicit assumptions' required in particular that the augmented matrix and the reduced matrix have eigenvalues distinct. This was read as an acceptable 'genericity' assumption, since accounting systems must be stable under the small perturbations which would cause eigenvalues to become distinct.

This paper inter alia identifies circumstances under which dividend irrelevance does indeed occur at an eigenvalue of A and so shows that the earlier derived equivalence is non vacuous.

2. Conjecture and Results

Professor Jim Ohlson asks the natural questions: are there circumstances such that dividend irrelevance occurs iff R takes the value of the dominant eigenvalue (the largest in modulus) of the reduced matrix A. Asymptotic considerations warrant this conjecture, since for generic initial conditions, and for large t the dominant eigenvalue growth of $\bar{A}$ dwarfs into insignificance the other state components, both in the accounting state vector and in the dividend, (provided of course that the dividend policy vector gives the dominant growth component a non-zero coefficient). Asymptotic considerations thus turn the multi-dimensional system apparently into an essentially one-dimensional one to which Ohlson's Principle (in dimension one) might apply; that is to say, assuming dividend and dominant state variable are inter-linked, dividend irrelevance occurs if and only if R takes a unique value, that value being equal to the eigenvalue of A corresponding to the dominant state. (Of course in the long run observation of the dividend sequence permits the inference of the dividend policy vector.)

In this note we offer an answer for small ω_{div} based on algebraic considerations, some complex analysis (including an inessential reference to Marden's 'Mean-Value Theorem for polynomials'), and graphical analysis. These complement a standard text-book analysis based on Gerschgorin's circle theorem $-$ see for instance Noble(1969).

Unsurprisingly, *the eigenvalues of $\bar{A}$ may be located arbitrarily,* but only if no restrictions are placed on the dividend policy (w, β). However, Ohlson's question implicitly assumes the convergence assumption as a boundedness assumption on the eigenvalues of $\bar{A}$. It transpires (see Proposition 3) that the dividend policy vector is restricted by this assumption to the interior of an appropriate polytope.

Conditions may be placed on the vector b such that when $\omega = (w, \beta)$ lies in an open region of parameter space, it is the case that the dominant eigenvalue of the augmented matrix lies between the first largest and the second largest eigenvalue of the reduced matrix. This is the substance of Theorem 1 stated below and deduced at the end of this section.

Theorem 1 (An Eigenvalue Dominance Theorem). *Suppose that A has real eigenvalues. In the canonical setting (1.2) we have as follows.*

(i) *If* $\text{sign}[\delta_1] = -1$ *and* $\text{sign}[\delta_i] = +1$ *for* $i = 2, ..., n$, *then the open set*

$$\{\omega : \bar{A} \text{ has real distinct roots and } \lambda_2 < \kappa_1(\omega) < \lambda_1\},$$

is non-empty and intersects the set

$$\{\omega : \omega_1 > 0, ..., \omega_{n+1} > 0\}.$$

Moreover the eigenvalue κ_2 is, for small ω, increasing in ω. Under these circumstances dividend irrelevance holds uniquely at $R = \lambda_1$.

(ii) *More generally the open set*

$$\{\omega : \bar{A} \text{ has real distinct roots and } \lambda_2 < \kappa_1(\omega) < \lambda_1\},$$

is non-empty and intersects the set

$$\{\omega : \delta_1\omega_1 < 0, \delta_2\omega_2 > 0, ..., \delta_n\omega_n > 0\},$$

and again under these circumstances dividend irrelevance holds uniquely at $R = \lambda_1$.

Remark. We see therefore that for an appropriate vector b there is a region of parameter space for which the eigenvalues of the augmented matrix $\bar{A}$ remain in modulus strictly bounded by the dominant eigenvalue of A (the one of maximum modulus). Note the re-emergence of the side conditions $\delta_1\omega_1 < 0$ analogous to the condition $\omega_{12}\omega_{21} < 0$ in Ohlson's Theorem.

We are able to provide some information about the extent of the subspace (see the formula (2.7) of section 2.2) where we obtain when $\delta_1 < 0$ the upper bound on positive $\omega_1$of

$$\frac{1}{4}(\lambda_1 - \beta)^2 + \{\omega_2\delta_2 + ...\},$$

for the case $\delta_2\omega_2 > 0$. Moreover, calculations of section 2.2 appear to imply that even if ω_1 rises above this bound the two roots of the characetristic polynomial of $\bar{A}$ which are forced into coincidence move towards the boundary of the disc $|\zeta| < |\lambda_1|$ in the complex ζ-plane. By contrast we find for $\omega_1\delta_1 < 0$ and $\omega_2\delta_2 < 0$ the top two roots of the augmented matrix $\bar{A}$ both approach λ_2 from opposite sides; this again is in keeping with the expectation that dividend irrelevance occurs only at the dominant root λ_1.

Our results link to recent work concerned with the real spectral radius of a matrix, see Hinrichsen, D., Kelb,B, (1994), which investigates by how much a matrix may be perturbed without moving its spectrum out of a given open set in the complex plane. In the cited work the open set of concern is usually either the unit disc or the open left half-plane in connection with stability issues. Our interest however focuses additionally on the open set described by the annulus defined by the first and second largest eigenvalues of A, as defined above, to which weand wewhereas We note that there is a well-established Sturmian algorithm for counting the number of zeros of a polynomial in the unit disc in the complex plane (see Marden (1949), section 42, p. 148), and therefore in principle the Ohlson question is resolvable for a given policy vector ω_{div} by reference to the number of zeros in the unit circle of the two polynomials

$$\chi_{\bar{A}}(\kappa/|\lambda_1|), \qquad \chi_{\bar{A}}(\kappa/|\lambda_2|).$$

Specifically the first should have $n+1$ zeros and the second no more than n. The Schur-Cohn criterion might perhaps also be invoked to count the number of roots in the unit disc of

$$a_0 + a_1 z + ... + a_m z^m$$

and we recall that this requires counting the number of sign changes in the determinantal sequence $\{\Delta_0, \Delta_1, ..., \Delta_m\}$ defined by

$$\Delta_0 = 1, \Delta_1 = \begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_0 & 0 & a_3 & a_2 \\ a_1 & a_0 & 0 & a_3 \\ a_3 & 0 & a_0 & a_1 \\ a_2 & a_3 & 0 & a_0 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_0 & 0 & 0 & a_3 & a_2 & a_1 \\ a_1 & a_0 & 0 & 0 & a_3 & a_2 \\ a_2 & a_1 & a_0 & 0 & 0 & a_3 \\ a_3 & 0 & 0 & a_0 & a_1 & a_2 \\ a_2 & a_3 & 0 & 0 & a_0 & a_1 \\ a_1 & a_2 & a_3 & 0 & 0 & a_0 \end{vmatrix},$$

etc. provided that no roots are located on the unit disc and that these determinants are non-zero.

2.1. Preliminaries

Our analysis is based on two results embodied in Proposition 1 and in the equivalences given in Proposition 2. The arbitrary placement of the zeros, the substance of Proposition 3, is also a consequence of Proposition 2.

Proposition 1 (Under the assumption of distinct eigenvalues). *For the canonical form of the dynamics, after diagonalization of the reduced matrix, the dividend significance coefficients, denoted δ_i, are all non-zero and hence we may assume for each $i \leq n$ that*

$$\delta_i = +1 \text{ or } -1, \text{ if preferred.}$$

For any $i \leq n$ dividend irrelevance occurs at $R = \lambda_i$ provided

$$R > \max\{\kappa_i : i = 1, ..., n+1\}, \tag{2.1}$$

in which case we have

$$d_0 + P_0(R; d) = -\frac{RZ_0^i}{\delta_i}.$$

The proof is given in section 3.

Remark 1. Apparently, if the eigenvalues of $\bar{A}$ are bounded below the radius of any other eigenvalue of A the Propositions permits dividend irrelevance to occur at several rates of return. We will show below that subject to (2.1) such an anomalous behaviour is definitely excluded when $\omega_1 \neq 0$ and also $\omega_j \neq 0$ for some $1 < j \leq n$.

Remark 2. In principle we might want to allow $\delta_i = -1$, to pick up a restriction in the directional sense of a re-scaling of accounting variables (if appropriate); it transpires from the next Proposition that the sign of δ_i can be absorbed by ω_i and the choice of sign is only a matter of convenience. The right-hand side perforce does not refer to the eigenvalues κ_i despite the fact that these control the growth rates of the canonical accounting variables.

The following algebraic equivalences lie at the heart of all our arguments.

Proposition 2 (Inverse relations). *Put $\lambda_{n+1} = \beta = \omega_{n+1}$. The following equations are all equivalent.*

$$\chi_{\bar{A}}(\kappa, \omega_1, ..., \omega_{n+1}) = 0, \tag{2.2}$$

$$\prod_{i=1}^{n+1}(\kappa - \lambda_i) = \omega_1\delta_1 \prod_{i=2}^{n}(\kappa - \lambda_i) + \omega_2\delta_2 \prod_{i\neq 2}^{n}(\kappa - \lambda_i) + ... + \omega_n\delta_n \prod_{i\neq n}^{n}(\kappa - \lambda_i), \tag{2.3}$$

$$\omega_1\delta_1 + \omega_2\delta_2 + ... + \omega_n\delta_n = (\kappa - \lambda_j)(\kappa - \beta) + \sum_{i\neq j}^{n} \frac{\omega_i\delta_i(\lambda_j - \lambda_i)}{\kappa - \lambda_i}, \text{ for } j \leq n \tag{2.4}$$

$$\beta = \kappa - \frac{\omega_1\delta_1}{\kappa - \lambda_1} - \frac{\omega_2\delta_2}{\kappa - \lambda_2} - ... - \frac{\omega_n\delta_n}{\kappa - \lambda_n}, \text{ i.e. for } j = n+1. \tag{2.5}$$

In particular with $j = 1$ we obtain the equivalent equation

$$-\omega_1\delta_1 = -(\kappa - \lambda_1)(\kappa - \beta) + \{\omega_2\delta_2 + ..\} - \frac{\omega_2\delta_2(\lambda_1 - \lambda_2)}{\kappa - \lambda_2} - ...$$

Each of the above identities enables a different analytic approach. Proof is offered in section 4.

Our first conclusion regards the potentially arbitrary placement of the zeros of (2.2).

Proposition 3 (Zero placement). *In the canonical setting of Proposition 1, for an appropriate choice of* ω *the characteristic polynomial*

$$\chi_{\bar{A}}(\kappa,\omega) = |\kappa I - \bar{A}(\omega)|$$

may take the form

$$\kappa^{n+1} - c_0\kappa^n + c_2\kappa^{n-1} + ... + (-1)^{n+1}c_n,$$

for arbitrary choice of real coefficients $c_0, ..., c_n$. *The transformation* $(c_0, ..., c_n) \to (\omega_1, ..., \omega_{n+1})$ *is affine invertible. The roots* $\kappa_1, ..., \kappa_{n+1}$ *of the characteristic polynomial may therefore be located at will, subject only to the inclusion, for each selected complex root, of its conjugate.*

This result is proved in section 5 and indicates that in principle the region of parameter space in which the boundedness assumption holds may be obtained as the transform of the set of vectors $(c_0, ..., c_n)$ satisfying a criterion derived from Cauchy's Theorem, namely

$$|c_0| + |c_1||\lambda_1| + ... + |c_n||\lambda_1|^n < |\lambda_1|^n.$$

Since the set of vectors $(c_0, ..., c_n)$ so described is the interior of a polytope, the corresponding region in parameter space is therefore likewise seen to be the interior of a polytope. Let us term this the **Cauchy polytope**.

Evidently $(0, ..., 0, \beta)$ is on the boundary of the Cauchy polytope, since then

$$\chi_{\bar{A}}(\kappa,\omega) = (\kappa - \lambda_1)..(\kappa - \lambda_n)(\kappa - \beta).$$

The situation with general placement of eigenvalues alters if β is a positive real, and lies below the eigenvalues of A. The formula (2.5) confines the non-real eigenvalues κ_i to an infinite strip, while the formula (2.3) allows us to confine all the eigenvalues still further when ω is itself bounded. The formula (2.4) offers a graphical approach to the analysis of the real root location.

Proposition 4 (Strip and two circles theorem). *Suppose that* $\beta \leq \lambda_n < ... < \lambda_1$ *that* $\omega_1 \neq 0$ *and that*

$$\delta_1\omega_1, ..., \delta_n\omega_n \geq 0.$$

(i) *All the non-real roots of the characteristic equation (2.2) lie in the infinite strip of the complex* ζ*-plane given by*

$$\beta \leq \mathrm{Re}(\zeta) \leq \lambda_1.$$

(ii) *Let* ε *be arbitrary but positive. Let* $K(\varepsilon)$ *be the real interval* $[\beta - \eta, \lambda_1 + \eta]$ *expanded so that*

$$\beta - \eta = \frac{(\beta + \lambda_1) - \sqrt{(\lambda_1 - \beta)^2 + 4\varepsilon}}{2}, \quad \lambda_1 + \eta = \frac{(\beta + \lambda_1) + \sqrt{(\lambda_1 - \beta)^2 + 4\varepsilon}}{2}$$

If

$$\omega_1 + ... + \omega_n \leq \varepsilon, \delta_1\omega_1, ..., \delta_n\omega_n \geq 0,$$

then all the roots of (2.2) lie in the star-shaped region $S(K, \pi/(n+1))$ *comprising two circles subtending angles of* $\pi/(n+1)$ *on* K.

Remark 1. Taken together parts (i) and (ii) may operate simultaneously. These results should however be taken together with Gerschgorin's Circle Theorem which implies immediately that the eigenvalues lie in the union of the discs in the complex ζ-plane given by $|\zeta - \lambda_i| \leq |\omega_i|$ and by $|\zeta - \beta| \leq |\omega_1| + ..|\omega_n|$. Thus the eigenvalues are bounded not only to the above mentioned vertical strip, but also to a horizontal strip of width $2\max\{|\omega_j| : j \leq n\}$ around the real axis.

Remark 2. It is obvious that for $\omega_2 = .. = \omega_n = 0$ and with $|\omega_1| \le \varepsilon$ that the real roots of (2.2) lie in $K(\varepsilon)$ by continuity. Gerschgorin's Circle Theorem limits the real roots to the slightly larger interval $[\beta - \varepsilon, \lambda_1 + \varepsilon]$. Thus the two circle result is merely a sharpening of the bounds.

Remark 3. If $\lambda_n < \beta$, less elegant improvements can be made so that K extends only as far as λ_1 on the left.

We can state, ahead of the proof of the proposition, our theorem on eigenvalue location.

Theorem 2 (Eigenvalue bounds). *Suppose that $\beta \le \lambda_n < ... < \lambda_1$ and that*

$$|\omega_1|, ..., |\omega_n| \le \varepsilon.$$

Non-real eigenvalues lie in the rectangle bounded by $\gamma = \beta, \zeta = \lambda_1, \zeta = \pm\varepsilon$. Real eigenvalues lie in the interval $K(\varepsilon)$.

The theorem follows from the Proposition - see Figure 1. The two circle result gives useful bounds only for the real roots.

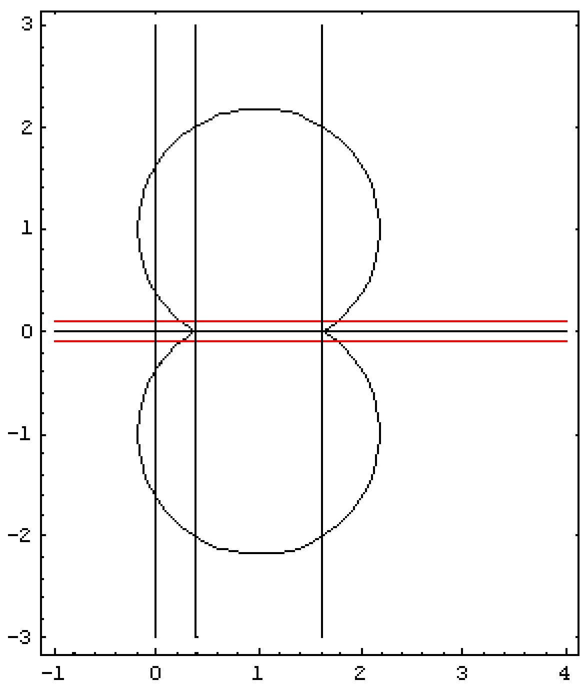

Figure 1. Vertical strip and two-circle bounds. Horizontal bound implied by Gerschgorin's Theorem.

Proof of Proposition 4. For part (i) we argue as follows. Suppose that $\omega_i \delta_i \ge 0$ and $\beta \le \lambda_n$. Suppose z satisfies

$$z - \beta = \frac{\omega_1 \delta_1}{z - \lambda_1} + \frac{\omega_2 \delta_2}{z - \lambda_2} + ... + \frac{\omega_n \delta_n}{z - \lambda_n},$$

If z is to the right strictly of λ_1 then we may also assume that z has positive imaginary part (otherwise switch to the conjugate root $\bar{z}$). The argument of $z - \lambda_j$ is for each j therefore positive and that of $1/(z - \lambda_j)$ therefore negative, i.e. has negative imaginary part. The right-hand side therefore sums to a complex number with negative imaginary part. However, $z - \beta$ has positive imaginary part.

If z is to the left of β we may suppose it has negative imaginary part. The argument of $\lambda_j - z$ is for each j therefore positive, as also for $\beta - z$. Now apply the previous reasoning to the identity

$$\beta - z = \frac{\omega_1 \delta_1}{\lambda_1 - z} + \frac{\omega_2 \delta_2}{\lambda_2 - z} + ... + \frac{\omega_n \delta_n}{\lambda_n - z}.$$

For part (ii), let ε_j be arbitrary real for $j = 1, ...n$. We will apply Marden's 'Mean-Value Theorem for polynomials' (Marden, p.23) to the polynomials h_j for $j = 1, ..., n$ and the polynomial $f(z)$ as defined by

$$f(z) = \prod_{i=1}^{n+1} (z - \lambda_i), \qquad h_j(z) = \varepsilon_j \prod_{j \neq 2}^{n} (z - \lambda_i).$$

We must, however, first find for each j the location of the roots of the equation $f(z) = h_j(z)$. The roots are of course $z = \lambda_k$ for $k \neq j$ taken together with the two real roots of

$$(z - \beta)(z - \lambda_j) = \varepsilon_j,$$

which are to the left of β and the right of λ_j. The exact and approximative formulas are

$$u_j^{\pm} = \frac{(\beta + \lambda_j) \pm \sqrt{(\lambda_j - \beta)^2 + 4\varepsilon_j}}{2} = \beta - \frac{\varepsilon_j}{4(\lambda_j - \beta)}, \lambda_j + \frac{\varepsilon_j}{4(\lambda_j - \beta)},$$

and require that

$$-\frac{1}{4}(\lambda_j - \beta)^2 \leq \varepsilon_j.$$

Thus the roots of all the equations lie in the interval $K = (u_1^-, u_1^+)$. By Marden's Theorem in the special case of real positive scalars m_j summing to unity, the roots of

$$f(z) = \sum m_j h_j(z),$$

lie in the star-shaped region $S(K, \pi/(n+1))$.Thus if we take $\varepsilon_j = \varepsilon$ small and $m_j \varepsilon = \delta_j \omega_j$ so that

$$\delta_1 \omega_1 + ... + \delta_n \omega_n = \varepsilon, \text{ with } \delta_1 \omega_1, ..., \delta_n \omega_n \geq 0.$$

then indeed $\sum m_j = 1$ and all the roots of (2.2) lie in the said star-shaped region.

In fact one may take $\varepsilon_1 = \varepsilon$ small and $m_1 \varepsilon_1 = \delta_1 \omega_1$ and $\varepsilon_j = \mu = \min\{\lambda_j - \lambda_{j+1} : \text{for } j > 1\}$ m $m_j \mu = \delta_j \omega_j > 0$ leading to the restriction

$$1 = m_1 + ... + m_n = \frac{\delta_1 \omega_1}{\varepsilon_1} + \frac{1}{\mu}(\delta_2 \omega_2 + ...)$$

i.e.

$$\omega_1 + \frac{\varepsilon}{\mu}(\delta_2 \omega_2 + ... + \delta_n \omega_n) = \varepsilon, \delta_1 \omega_1, ..., \delta_n \omega_n \geq 0.$$

Remark. The above analysis so far does not yet exclude the possibility of all eigenvalues being located to the left of λ_2.We next offer a graphical analysis of the real root locations in the following subsection which shows that at least one root has to be to the right of λ_1 when $\omega_1 \neq 0$ and $\omega_j \neq 0$ for some $j > 1$.

2.2. Graphical analysis

The purpose of this subsection is to show that under suitable restrictions one can guarantee the existence of a real eigenvalue in the range (λ_2, λ_1).Specifically we show that if $\delta_1 = -1$ and $\delta_2 = ... = +1$ there is a real eigenvalue in the range λ_2, λ_1 for all small enough positive ω_1.

Interest naturally focuses on $\omega_1$as the link coefficient with the dominant state vector. Treating ω_1 as a free variable, with the remaining dividend policy coefficient fixed we use (2.4) to study the map $\kappa \to \omega_1$ and its local inverses. We have

$$-\omega_1\delta_1 = -(\kappa - \lambda_1)(\kappa - \beta) + \{\omega_2\delta_2 + ..\} - \frac{\omega_2\delta_2(\lambda_1 - \lambda_2)}{\kappa - \lambda_2} - ..., \tag{2.6}$$

so that the graph of ω_1 against κ has $(n-1)$ vertical asymptotes from right to left at $\kappa = \lambda_2, \lambda_3, ...\lambda_n$ all of which are manifestly simple poles. The asymptotes break up the leading inverted-U-shaped quadratic (if $\delta_1 < 0$) into n connected components corresponding to the intervals $(-\infty, \lambda_n), (\lambda_n, \lambda_{n-1}), ..., (\lambda_n, +\infty)$. The equation

$$\frac{\partial\omega_1}{\partial\kappa} = 0,$$

is equivalent to an n degree polynomial equation with n solutions (taking into account multiplicity), some of which may be conjugate complex roots. Thus we may expect up to n stationary points in the graph.

In the interval $(\lambda_{i+1}, \lambda_i)$ the component has either an even, or an odd, number of stationary points depending on whether the sign of $\omega_{i+1}\delta_{i+1}\omega_i\delta_i$ is $+1$ or -1. In view of the behaviour of the leading quadratic term not all the components may be monotone (possess a zero number of stationary points!). Thus at least one component is non-monotonic.

The components may be interpreted as graphs/loci of the eigenvalues $\kappa_i(\omega_1)$. More precisely, the differentiable local inverses of the mapping $\kappa \to \omega_1$ are the graphs of $\kappa_i(\omega_1)$. That is to say, each non-monotonic component must be first partitioned into monotone parts on either side of its stationary points. The labelling of these inverses from right to left respects the cyclic order on the set $\{1, ..., n\}$ and also one at least of the identifications

$$\lim_{\kappa \nearrow \lambda_i} \omega_1(\kappa) = \kappa_i, \qquad \lim_{\kappa \searrow \lambda_i} \omega_1(\kappa) = \kappa_i.$$

The latter may require the point at infinity on the asymptote $\kappa = \lambda_i$ to be considered as the intersection of consecutive loci.

Note further for $\kappa = \lambda_1$ we obtain zero on the right-hand side of (2.6) and so $\kappa = \lambda_1$ is a root, of the expression on the right, i.e. $\kappa_1(0) = \lambda_1$. (This is consistent with the matrix $M - \kappa I$ having a first column with zeros in all but the last row.)

Since $\kappa_1(0) = \lambda_1$, the asymptotic features of the graph ensure that for all, small enough, positive ω_1 the eigenvalue $\kappa_1(\omega_1)$ is in the range (λ_2, λ_1) as we now demonstrate.

With our assumption that $\delta_1 = -1$ and $\delta_2 = \delta_3 = ... = +1$, we can arrange for $\kappa_1(\omega_1)$ to be large and positive in the vicinity to the right of λ_1 by taking $\omega_2 < 0$. With $\omega_2 < 0$ the domain of κ_1 is infinite so that

$$\lim_{\omega_1 \to \infty} \kappa_1(\omega_1) = \lambda_2.$$

Thus the largest real eigenvalue of κ_1 remains above λ_2. See Figure 2a. Of course for small enough $\omega_1$the remaining roots $\kappa_i(\omega_1)$, even if complex, remain in an open vertical complex strip including the closed real interval $[\beta, \lambda_2]$.

We can similarly arrange for $\kappa_1(\omega_1)$ to be large and negative in the vicinity to the right of λ_1 by taking $\omega_2 > 0$. In view of the behaviour of the graph for large $\kappa > \lambda_1$ this implies the existence of two roots in (λ_1, λ_2) under these circumstances. With $\omega_2 > 0$ the domain of κ_1 is bounded, say by $\omega_1 \leq \omega_1^* = \omega_1^*(\omega_2, ..., \omega_{n+1})$ it is the case that

$$\lim_{\omega_1 \searrow \omega_1^*} \kappa_1(\omega_1) = \lim_{\omega_1 \nearrow \omega_1^*} \kappa_2(\omega_1).$$

See Figure 2b. As the eigenvalue κ_1 remains above λ_2 dividend irrelevance can occur only at λ_1. An upper bound for ω_1^* is provided by the maximum value of the quadratic term

$$-(\kappa - \lambda_1)(\kappa - \beta) + \{\omega_2\delta_2 + ..\}$$

obtained by evaluation at $\kappa = \frac{1}{2}(\lambda_1 + \beta)$.namely:

$$\frac{1}{4}(\lambda_1 - \beta)^2 + \{\omega_2\delta_2 + ...\}. \tag{2.7}$$

This gives β, the coefficient at the previous date's dividend, a significant bounding role.

Note that in both scenarios the locus of κ_1 is decreasing, as ω_1 increases from zero.

Graphs of $\omega_1(\kappa)$ are illustrated in Figure 2 for the case $n = 2$; the graphs of $\omega_2(\kappa)$,not shown, inevitably corroborate this picture.

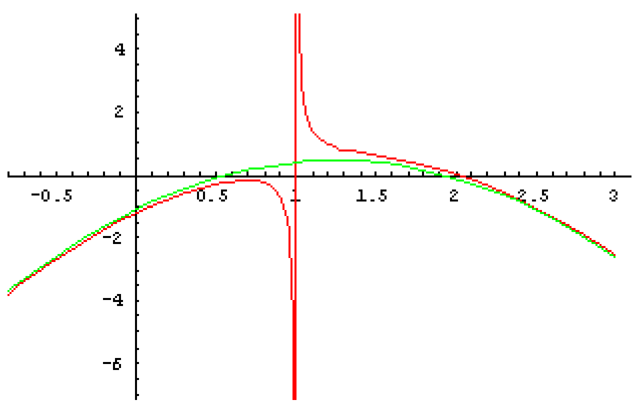

Figure 2a. Graph of $\omega_1(\kappa_1)$ with $\delta_1 = -1, \delta_2 = +1, \omega_2 < 0$.

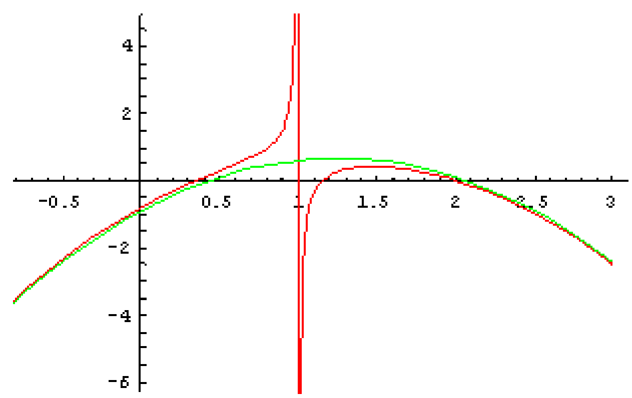

Figure 2b. Graph of $\omega_1(\kappa_1)$ with $\delta_1 = -1, \delta_2 = +1, \omega_2 > 0$.

We conclude this section by illustrating the effect of the coefficient β on the three eigenvalues in the case $\omega_1 = \omega_1 = .1$. In the range $\beta < \lambda_2$ we see that the root κ_1 decreasing and κ_2 increasing as β increases; κ_3 is increasing for all β, as might be expected, with λ_2. Intuitively speaking, the push away from the origin created by the two increasing roots κ_2 and κ_3 causes the location of the coincident root $\kappa_1 = \kappa_2$ to execute a jump up to a new coincidence location above λ_1, by way of a continuous root locus in the complex ζ-plane (see the Remark on bifurcation in the next section). The push can in fact be physically interpreted. The partial fraction expansion terms may be regarded as modelling an electric charge placed at the pole and acting according to an inverse distance law (see Marden (1949), page 7). Thus for β large enough for both κ_2 and κ_1 to have been re-located above λ_1, we see the locus of κ_1 resume its downward path towards the origin (but tending only as far as the barrier λ_1), while κ_2 resumes its upward path away from the origin. The locus dynamics are investigated more properly in the next section.

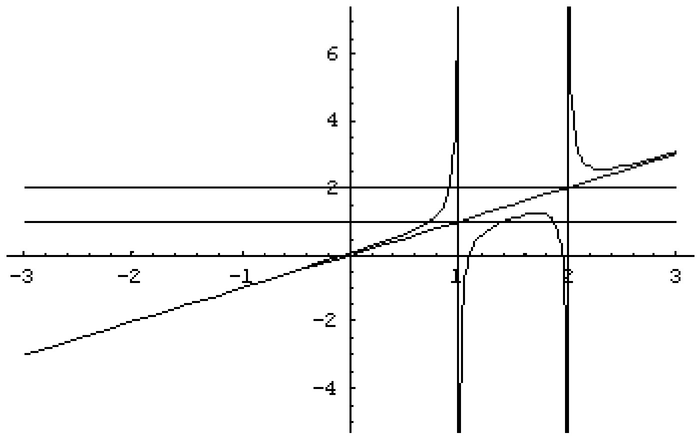

Figure 3. Graph of $\omega_3(\kappa_1)$ with
$\delta_1 = -1, \delta_2 = +1, \omega_1, \omega_2 > 0$.

2.3. Differential properties of eigenvalues: some bifurcation analysis

The purpose of this section is to show briefly how to analyse the root locus in the ζ-plane. We conduct a partial analysis mostly concentrated on the dynamics of the dominant eigenvalue as ω_1 changes (with the remaining policy parameters are fixed) with a view to completing the proof of the dominance theorem in the next section. Our starting point is the following proposition which follows from (2.3) by implicit differentiation.

Proposition 5. (Under the assumption of distinct eigenvalues). *In the canonical setting of Proposition 1, let the dividend policy vector be represented by* $\omega = (\omega_1, ..., \omega_{n+1})$ *with* $\omega_{n+1} = \beta$. *The eigenvalues* κ_i *of the augmented matrix* $\bar{A}$ *regarded here, together with* $\bar{A}$, *as functions of* $\omega = (\omega_1, ..., \omega_{n+1})$, *satisfy the following differential properties. For* $1 \leq i \leq n+1$ *and for* $1 \leq j \leq n$ *it is the case that*

$$\frac{\partial \kappa_i}{\partial \omega_j} = -\frac{\delta_j(\lambda_1 - \kappa_i)(\lambda_2 - \kappa_i)...(\lambda_{j-1} - \kappa_i)(\lambda_{j+1} - \kappa_i)...(\lambda_n - \kappa_i)}{(\kappa_1 - \kappa_i)(\kappa_2 - \kappa_i)...(\kappa_{i-1} - \kappa_i)(\kappa_{i+1} - \kappa_i)...(\kappa_{n+1} - \kappa_i)}, \tag{2.8}$$

and for $j = n+1$ *it is the case that*

$$\frac{\partial \kappa_i}{\partial \omega_{n+1}} = \frac{(\lambda_1 - \kappa_i)(\lambda_2 - \kappa_i)...(\lambda_n - \kappa_i)}{(\kappa_1 - \kappa_i)(\kappa_2 - \kappa_i)...(\kappa_{i-1} - \kappa_i)(\kappa_{i+1} - \kappa_i)...(\kappa_{n+1} - \kappa_i)}. \tag{2.9}$$

Proof. We begin by observing a simple consequence of the identity

$$\chi_M(\kappa) = (\kappa_1 - \kappa)(\kappa_2 - \kappa)...(\kappa_{n+1} - \kappa).$$

We have

$$\chi_M'(\kappa) = \frac{d}{d\kappa}\chi_M(\kappa) = -(\kappa_2 - \kappa)...(\kappa_{n+1} - \kappa) - (\kappa_1 - \kappa)(\kappa_3 - \kappa)...(\kappa_{n+1} - \kappa) - ...$$

so that

$$\chi_M'(\kappa_i) = -\prod_{j \neq i}(\kappa_j - \kappa_i),$$

and in particular

$$\chi'_M(\kappa_1) = -\prod_{j>1}(\kappa_j - \kappa_1).$$

Specifically, we have when

$$\omega = \omega_0 := (0, ..., 0, \beta),$$

that

$$\chi'_M(\kappa_1) = -(\lambda_2 - \lambda_1)...(\lambda_n - \lambda_1)(\omega_{n+1} - \lambda_1).$$

Now we apply implicit differentiation to the identity

$$\chi_M(\kappa_i(\omega), \omega) = 0.$$

That is to say we have

$$\frac{\partial \kappa_1}{\partial \omega_j} = -\frac{\partial \chi}{\partial \omega_j} \div \left(\frac{\partial \chi}{\partial \kappa}\right)_{\kappa=\kappa_1} = \frac{\partial \chi}{\partial \omega_j} \div \prod_{i>1}(\kappa_i - \kappa_1),$$

except at the critical points ω defined by

$$\prod_{i>1}(\kappa_i(\omega) - \kappa_1(\omega)) = 0,$$

e.g. where the locus of $\kappa_1(\omega)$ crosses $\kappa_2(\omega)$.The result of the Proposition now follows directly from (2.3).

Remark. For the assumption of distinct roots to hold we must manifestly disregard the non-generic critical points which are those points ω where any two of the functions κ_i agree in value; of particular importance to us are points ω where $\kappa_1(\omega)$ may cease to be the largest eigenvalue (ranking according to modulus), as for instance when it agrees in value with $\kappa_2(\omega)$. The first formula when $j = 1$ is to be read as

$$\frac{\partial \kappa_1}{\partial \omega_1} = -\frac{\delta_1(\lambda_2 - \kappa_1)...(\lambda_n - \kappa_1)}{(\kappa_2 - \kappa_1)...(\kappa_n - \kappa_1)(\kappa_{n+1} - \kappa_1)},$$

and note that, at $\omega = \omega_0 := (0, ..., 0, \beta)$, we have, by (1.1), $\kappa_{n+1} = \beta$ and for $i = 1, ..., n$

$$\kappa_i = \lambda_i,$$

as well as

$$\frac{\partial \kappa_1}{\partial \omega_1} = \frac{\delta_1}{(\lambda_1 - \beta)}, \tag{2.10}$$

and

$$\frac{\partial \kappa_1}{\partial \omega_j} = 0, \text{ for } j > 1.$$

The equation (2.10) implies that the choice of a β value close to λ_1 will accelerate the growth rate κ_1 of the leading canonical variable Z^1 relative to the first dividend policy coefficient.

Technical point. In the arguments that follow, it is important to realize that when the roots κ_i and κ_{i+1} are complex conjugates then for real κ_1 the following signature property is satisfied

$$\text{sign}[(\kappa_i - \kappa_1)(\kappa_{i+1} - \kappa_1)] = +1,$$

just as though κ_i and κ_{i+1} were real and both below κ_1. (Since the quadratic has no real roots, it is here positive definite.)

Corollary 1 (Bifurcation behaviour near $\kappa_1 = \kappa_2$) *Assume the eigenvalues of $A$are real and distinct and that for $j = 3, ..., n$ that κ_j is real and satisfies $\lambda_{j+1} < \kappa_j < \lambda_j$.For small enough positive increments in ω_1 it is the case that at any bifurcation point the conjugate complex roots κ_1 and κ_2 move closer to the origin if $\delta_1 > 0$, and away from the origin if $\delta_1 < 0$.*

Remark. The corollary thus confirms the intuition expressed in connection with Figure 3. where we alluded to the push away from the origin with $\delta_1 < 0$.

Proof. Suppose that $\kappa_1 = \kappa_2$ occurs at some point $\omega_1 = \omega_1^*$. If now $\omega_1 = \omega_1^* + \Delta\omega$ with $\Delta\omega > 0$ write, with κ real,

$$\begin{aligned} \kappa_1 &= \kappa + i\varepsilon, \kappa_2 = \kappa - i\varepsilon, \\ (\lambda_j - \kappa_1) &= \rho_j e^{i\theta_j}, (\kappa_{j+1} - \kappa_1) = \rho_j' e^{i\phi_j}, \end{aligned}$$

where i denotes $\sqrt{-1}$ for the purposes of this paragraph only. Thus, since $\rho_j < \rho_j'$, for $\Delta\omega$ small enough we shall have

$$\theta_j > \phi_j,$$

so that

$$\frac{\rho_j e^{i\theta_j}}{\rho_j' e^{i\phi_j}} = \frac{\rho_j}{\rho_j'} \exp[i(\theta_j - \phi_j)]$$

and hence that

$$\frac{\Delta\kappa_1}{\Delta\omega} = -\frac{1}{2\varepsilon i} \frac{\rho_2 e^{i\theta_2}}{\rho_2' e^{i\phi_2}} \cdot ... \cdot \frac{\rho_n e^{i\theta_n}}{\rho_n' e^{i\phi_n}} = \frac{1}{2\varepsilon} \frac{\rho_2}{\rho_2'} \cdot ... \cdot \frac{\rho_n}{\rho_n'} \exp[i(\frac{\pi}{2} + \psi)],$$

where ψ is small and positive. That is, the remaining ratios pull $\Delta\kappa_1$ in the same direction. In conclusion, the conjugate complex roots $\kappa_1$and κ_2 move closer to the origin.

Remark (Bifurcation behaviour elsewhere) Assuming that the first repeated root is elsewhere than at the dominant position, one may attempt to repeat the argument at the other locations to observe a tug of war between those ratios below the coincidence location pulling one way and those above it pulling the other way. (We have noted the electric force field interpretation.)Who wins this tug of war is determined by the geometric considerations and so we discover that there will be a critical point λ,a water-shed, such that to the right of λ the complex roots move towards the origin, whereas to the left they move away from the origin.

Corollary 2. *Suppose all the eigenvalues λ_i are real and positive and at some ω it is the case that κ_1 is the maximal eigenvector (in modulus) and is real, and further that*

$$\lambda_3 < \kappa_1 < \lambda_1.$$

Then we have as follows.

(i)

$$\begin{aligned} \text{sign}[\partial\kappa_1/\partial\omega_1] &= \text{sign}[\delta_1]\text{sign}[(\kappa_1 - \lambda_2)], \\ \text{sign}[\partial\kappa_1/\partial\omega_2] &= -\text{sign}[\delta_2], \end{aligned}$$

(iii) For $i = 3, ..., n$ we have

$$\text{sign}[\partial\kappa_1/\partial\omega_i] = -\text{sign}[\delta_i]\text{sign}[(\kappa_1 - \lambda_2)],$$

Finally

$$\text{sign}[\partial\kappa_1/\partial\omega_{n+1}] = -1.$$

Proof of Corollary 2. Let $i \geq 3$. We compute that

$$\begin{aligned}
\text{sign}[(\kappa_2-\kappa_1)(\kappa_3-\kappa_1)...(\kappa_{n+1}-\kappa_1)] &= (-1)^n, \\
\text{sign}[(\lambda_1-\kappa_1)(\lambda_3-\kappa_1)...(\lambda_n-\kappa_1)] &= (-1)^{n-2}, \\
\text{sign}[(\lambda_1-\kappa_1)(\lambda_2-\kappa_1)...(\lambda_{i-1}-\kappa_1)(\lambda_{i+1}-\kappa_1)...(\lambda_n-\kappa_1)] &= (-1)^{(n-3)}\text{sign}[(\lambda_2-\kappa_1)] \\
&= (-1)^{(n-4)}\text{sign}[(\kappa_1-\lambda_2)]
\end{aligned}$$

Corollary 3. *Suppose all the eigenvalues of A are real and positive. For all ω small enough so that*

$$|\kappa_{n+1}| < ... < |\kappa_2| < \kappa_1,$$

and so that κ_i is real with

$$\lambda_{i+1} < \kappa_i < \lambda_{i-1},$$

it is the case for each i with κ_i real that

$$\begin{aligned}
\text{sign}[\partial\kappa_i/\partial\omega_i] &= \text{sign}[\delta_i], \\
\text{sign}[\partial\kappa_i/\partial\omega_j] &= \text{sign}[\delta_i]\text{sign}[(\kappa_i-\lambda_i)]\text{sign}[(\kappa_i-\lambda_j)], \text{ for } i \neq j \leq n, \\
\text{sign}[\partial\kappa_i/\partial\omega_{n+1}] &= \text{sign}[(\kappa_i-\lambda_i)],
\end{aligned}$$

where $i > 1$. In particular, as long as $\kappa_2 < \kappa_1$,we have

$$\begin{aligned}
\text{sign}[\partial\kappa_2/\partial\omega_1] &= -\text{sign}[\delta_1]\text{sign}[(\kappa_2-\lambda_2)]\ , \qquad \text{sign}[\partial\kappa_2/\partial\omega_2] = \text{sign}[\delta_2], \\
\text{sign}[\partial\kappa_2/\partial\omega_j] &= \text{sign}[\delta_j]\text{sign}[(\kappa_2-\lambda_2)] \text{ for } 3 \leq j \leq n, \\
\text{sign}[\partial\kappa_2/\partial\omega_{n+1}] &= \text{sign}[(\kappa_2-\lambda_2)].
\end{aligned}$$

Proof of Corollary 3. Recalling that for $i \leq n$ we have

$$\frac{\partial\kappa_i}{\partial\omega_i} = -\frac{\delta_i(\lambda_1-\kappa_i)(\lambda_2-\kappa_i)...(\lambda_{i-1}-\kappa_i)(\lambda_{i+1}-\kappa_i)...(\lambda_n-\kappa_i)}{(\kappa_1-\kappa_i)(\kappa_2-\kappa_i)...(\kappa_{i-1}-\kappa_i)(\kappa_{i+1}-\kappa_i)...(\kappa_{n+1}-\kappa_i)},$$

we compute that

$$\begin{aligned}
\text{sign}[(\kappa_1-\kappa_i)(\kappa_2-\kappa_i)...(\kappa_{i-1}-\kappa_i)(\kappa_{i+1}-\kappa_i)...(\kappa_{n+1}-\kappa_i)] &= (-1)^{n-(i-1)} = (-1)^{(n+1-i)}, \\
\text{sign}[(\lambda_1-\kappa_i)(\lambda_2-\kappa_i)...(\lambda_{i-1}-\kappa_i)(\lambda_{i+1}-\kappa_i)...(\lambda_n-\kappa_i)] &= (-1)^{(n-1)-(i-1)} = (-1)^{(n-i)},
\end{aligned}$$

and so

$$\begin{aligned}
&= (-1)^{(n-i)}\text{sign}[(\lambda_i-\kappa_i)] \\
&= \text{sign}[(\lambda_1-\kappa_i)(\lambda_2-\kappa_i)...(\lambda_{i-1}-\kappa_i)(\lambda_i-\kappa_i)(\lambda_{i+1}-\kappa_i)...(\lambda_n-\kappa_i)] \\
&= \text{sign}[(\lambda_j-\kappa_i)]\text{sign}[(\lambda_1-\kappa_i)(\lambda_2-\kappa_i)...(\lambda_{j-1}-\kappa_i)(\lambda_{j+1}-\kappa_i)...(\lambda_n-\kappa_i)]
\end{aligned}$$

2.4. The dominance theorem

We may nor put together the analysis of the last sections to deduce our main result concerning the existence of a dominant eigenvalue.

Proof of Theorem. We consider the first part of the theorem only, as the more general result follows by a restatement of the same argument. By selecting the sign of δ_1 as (-1) and of δ_i for $i > 1$ as $(+1)$ we can arrange, given (2.10), for the eigenvalue function $\kappa_1(\omega)$ identified by the condition $\kappa_1(\omega_0) = \lambda_1$ to be decreasing as a function of ω in the region

$$\{\omega : \omega_1 > 0, ..., \omega_{n+1} > 0\},$$

and so to remain below λ_1. We have, however, to ensure that $\kappa_1(\omega)$ remains the maximal root. Recall that $\omega_0 = (0, ..., 0, \beta)$.Since the remaining eigenvalues $\lambda_i = \kappa_i(\omega_0)$ are below λ_1 we may, by appeal to continuity, ensure that the eigenvalues functions $\kappa_2(\omega), ..., \kappa_{n+1}(\omega)$ of $\bar{A}(\omega)$ also lie strictly below λ_1 and that moreover $\kappa_2(\omega) < \kappa_1(\omega)$.

Remark. By Corollary 3 it is possible that, following a path in parameter space, the locus of κ_2 intersects that of κ_1. Note, however, that if upon intersection at ω^* we were thereafter to have $\kappa_1(\omega') < \kappa_2(\omega')$ with ω' infinitesimally close to ω^*, then provided the remaining eigenvalues remain below $\kappa_1(\omega^*)$,the signs of all the derivatives $\partial\kappa_1/\partial\omega_j$ and of all the derivatives $\partial\kappa_2/\partial\omega_j$ would switch, i.e. both loci would turn around, a contradiction. Thus, subject to the assumption about the remaining eigenvalues, this implies that in fact ω^* is at the boundary of that region in policy parameter space where κ_1 and κ_2 are both real. Moreover according to (2.8) the graph has infinite slope at ω^*.We illustrate this point in the following simple example with $n = 1$, $\lambda_1 = 1$ and $|\beta| < 1$.

Example. For $\delta_1 = -1$,let

$$\bar{A} = \begin{bmatrix} 1 & -1 \\ \omega_1 & \beta \end{bmatrix}.$$

The characteristic polynomial is $\kappa^2 - (1+\beta)\kappa + (\beta + \omega_1)$. Here

$$\frac{d\kappa_1}{d\omega} = \frac{1}{(1+\beta-2\kappa_1)} = \frac{-b_1}{(\kappa_2 - \kappa_1)},$$

since $\kappa_1 + \kappa_2 = 1 + \beta$. The roots are real for $\omega_1 \leq \frac{1}{4}[(1+\beta)^2 - 4\beta] = \frac{1}{4}(1-\beta)^2 = \omega_1^*(\beta)$, and we have $\kappa_1 = \frac{1}{2}\left(1 + \beta + \sqrt{(1-\beta)^2 - 4\omega_1}\right)$

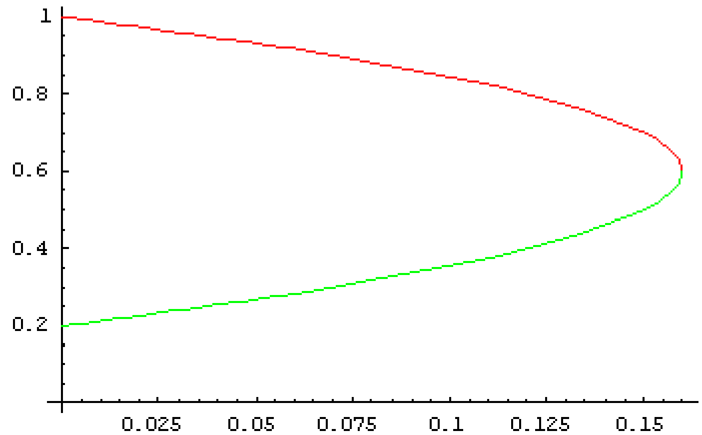

Figure 4. For fixed β the root locus of $\kappa_1(\omega_1)$ (in red) and $\kappa_2(\omega_1)$(in green).

decreasing down to $\kappa_1 = \frac{1}{2}(1+\beta)$ as $\omega_1$increases, and analogously $\kappa_2 = \frac{1}{2}\left(1+\beta-\sqrt{(1-\beta)^2-4\omega_1}\right)$ increasing up to $\kappa_2 = \frac{1}{2}(1+\beta)$.We note that as the roots become complex the real part stays constant at $(1+\beta)/2$, i.e. the root locus bifurcates and the conjugate roots move orthogonally to the real axis; there being no poles in this simple case there is no 'push' on the real part, neither away nor towards the origin.

3. Obtaining dividend irrelevance

In this section we prove the results in Proposition 1. Without loss of generality we may take $v_0 = 0$ and so ignore and hence suppress the variable v_t. (See earlier work).

After a change of accounting state variables from z_t to, say Z_t, the system becomes

$$\begin{bmatrix} Z_{t+1} \\ d_{t+1} \end{bmatrix} = M \begin{bmatrix} Z_t \\ d_t \end{bmatrix},$$

where

$$M = \begin{bmatrix} \lambda_1 & 0 & & 0 & \bar{b}_1 \\ 0 & \lambda_2 & & & \bar{b}_2 \\ & & \dots & & \\ 0 & 0 & & \lambda_n & \bar{b}_n \\ \omega_1 & \omega_2 & & & \omega_{n+1} \end{bmatrix}$$

is the new augmented matrix (with the same eigenvalues as the original augmented matrix $\bar{A}$) and where $\lambda_1, ..., \lambda_n$ are the eigenvectors of A assumed presented in decreasing modulus size (with $|\lambda_1|$ largest).

Evidently the characteristic polynomial

$$\chi_M(\kappa) = \chi_M(\kappa, \omega) = \chi_M(\kappa, \omega_1, ..., \omega_n, \omega_{n+1}) = |M - \kappa I|$$

is the same as $\chi_{\bar{A}}(\kappa)$. We assume its eigenvalues be $\kappa_1, ..., \kappa_{n+1}$ have distinct modulus.

The force of our implicit assumptions on the **dividend significance coefficients** is that they are all non-zero: $\bar{b}_i \neq 0$ for all i. For otherwise the eigenvalues of the augmented matrix would include all the eigenvalues of the reduced matrix. (To see this expand the characteristic determinant by the i-th row).

Henceforth we assume the canonical variables have been re-scaled by $\bar{b}_i$ and we may therefore take for the **canonical dividend significance coefficients** the symbol δ_i with the additional stipulation that

$$|\delta_1| = 1 \text{ for } i = 1, .., n.$$

As a first step we note the consequence for dividend irrelevance of the non-zero dividend significance coefficients. Writing $Z = (..., Z^i, ...)$ we have for each i that for some coefficient $l_1, .., l_{n+1}$ it is the case that

$$d_t = \sum l_j \kappa_j^t.$$

Now the equation

$$Z_{t+1}^i = \lambda_i Z_t^i + \delta_i \sum l_j \kappa_j^t$$

with the solution also given by the eigenvalues of the augmented matrix in the format

$$Z_t^i = \sum L_j \kappa_j^t,$$

must satisfy

$$\sum L_j \kappa_j^t (\kappa_j - \lambda_i) = \delta_i \sum l_j \kappa_j^t$$

for all t.Hence

$$L_j = \frac{\delta_i l_j}{\kappa_j - \lambda_i}.$$

We thus have, assuming $R > |\kappa_j|$ for all j, that the dividend series converges, and

$$\begin{aligned} P_0(R; d) &= \sum_{t=1}^{\infty} R^{-t} d_t = \sum_j \frac{1}{R - \kappa_j} l_j \kappa_j \\ &= \frac{1}{\delta_i} \sum_j \frac{\kappa_j - \lambda_i}{R - \kappa_j} L_j \kappa_j. \end{aligned}$$

Consequently, if $R = \lambda_i$ is permitted then

$$P_0(R; d) = -\frac{1}{\delta_i} \sum_j L_j \kappa_j = -\frac{Z_1^i}{\delta_1} = -\frac{\lambda_i Z_0^i + \delta_i d_0}{\delta_1} = -\frac{\lambda_i Z_0^i}{\delta_1} + d_0$$

indeed depends only on the initial data.

We recall the basis of this calculation is the identity

$$\begin{aligned} P_0(R; d) &= \sum_{t=1}^{\infty} R^{-t} d_t = \sum_{t=1}^{\infty} R^{-t} \sum_j l_j \kappa_j^t \\ &= \sum_j l_j \sum_{t=1}^{\infty} R^{-t} \kappa_j^t \\ &= \sum_j l_j \frac{\kappa_j / R}{1 - \kappa_j / R} = \sum_j \frac{1}{R - \kappa_j} l_j \kappa_j \end{aligned}$$

We will thus obtain dividend irrelevance at $R = \lambda_i$ provided all the eigenvalues κ_j are in modulus less than R. This is only likely to be the case when $R = \lambda_1$ is the eigenvalue of maximum modulus.

4. Derivation of equivalences

We begin by expanding by the bottom row

$$|M - \kappa I| = \begin{vmatrix} \lambda_1 - \kappa & 0 & & 0 & \delta_1 \\ 0 & \lambda_2 - \kappa & & & \delta_2 \\ & & \dots & & \\ 0 & 0 & & \lambda_n - \kappa & \delta_n \\ \omega_1 & \omega_2 & & & \lambda_{n+1} - \kappa \end{vmatrix}_{n+1} = 0$$

to obtain

$$(-1)^n \omega_1 D_1(\kappa) - \dots - \omega_n D_n(\kappa) + \prod_{i=1}^{n+1} (\lambda_i - \kappa) = 0,$$

or

$$(-1)^n \omega_1 D_1(\kappa) + (-1)^{n-1} \omega_2 D_2(\kappa) \dots - \omega_n D_n(\kappa) = (-1)(-1)^{n+1} \prod_{i=1}^{n+1} (\kappa - \lambda_i),$$

where

$$\begin{aligned} D_1(\kappa) &= \begin{vmatrix} 0 & 0 & 0 & \delta_1 \\ \lambda_2 - \kappa & & 0 & \delta_2 \\ \dots & & & \\ 0 & & \lambda_n - \kappa & \delta_n \end{vmatrix}_n \\ &= (-1)^{n-1} \delta_1 \prod_{i=2}^{n} (\lambda_i - \kappa) \\ &= \delta_1 \prod_{i=2}^{n} (\kappa - \lambda_i). \end{aligned}$$

Similarly

$$\begin{aligned} D_2(\kappa) &= \begin{vmatrix} \lambda_1 - \kappa & 0 & 0 & \delta_1 \\ 0 & 0 & 0 & \delta_2 \\ & \lambda_3 - \kappa & & \\ & & & \\ 0 & 0 & \lambda_n - \kappa & \delta_n \end{vmatrix}_n \\ &= (-1)^{n-2} \delta_2 \prod_{i \neq 2}^{n} (\lambda_i - \kappa) \\ &= -\delta_2 \prod_{i \neq 2}^{n} (\kappa - \lambda_i) \end{aligned}$$

This yields the equation

$$\omega_1\delta_1\prod_{i=2}^{n}(\kappa-\lambda_i)+\omega_2\delta_2\prod_{i\neq 2}^{n}(\kappa-\lambda_i)+...+\omega_n\delta_n\prod_{i\neq n}^{n}(\kappa-\lambda_i)=\prod_{i=1}^{n+1}(\kappa-\lambda_i),$$

Dividing by $\prod_{i\neq j}^{n}(\kappa-\lambda_i)$ we obtain

$$(\kappa-\lambda_j)(\kappa-\beta)=\omega_j\delta_j+\sum_{i\neq j}\omega_i\delta_i\frac{\kappa-\lambda_j}{\kappa-\lambda_i}=\omega_j\delta_j+\sum_{i\neq j}\omega_i\delta_i\left(1-\frac{\lambda_j-\lambda_i}{\kappa-\lambda_i}\right)$$

or

$$\omega_1\delta_1+\omega_2\delta_2+...=(\kappa-\lambda_j)(\kappa-\beta)+\sum_{i\neq j}^{n}\frac{\omega_i\delta_i(\lambda_j-\lambda_i)}{\kappa-\lambda_i},$$

as required.

Dividing by $\prod_{i=1}^{n}(\kappa-\lambda_i)$ we obtain

$$\kappa-\beta=\frac{\omega_1\delta_1}{\kappa-\lambda_1}+\frac{\omega_2\delta_2}{\kappa-\lambda_2}+...+\frac{\omega_n\delta_n}{\kappa-\lambda_n}.$$

5. Invertible parametrization and Zero placement

This section is devoted to a proof of Proposition 3. Let us write

$$\chi_A(\kappa)=|\kappa I-A|=\sum_{s=0}^{n}(-1)^s a_s\kappa^{n-s}=\prod_{i=1}^{n}(\kappa-\lambda_i)$$

so that

$$a_s=\sum_{j_1<...<j_s}\lambda_{j_1}...\lambda_{j_s}$$

and so

$$a_0=1,\qquad a_1=\lambda_1+...+\lambda_n,\qquad ...\qquad a_n=\lambda_1...\lambda_n.$$

Hence

$$\begin{aligned}\prod_{i=1}^{n+1}(\kappa-\lambda_i)&=(\kappa-\beta)\left[\sum_{s=0}^{n}(-1)^s a_s\kappa^{n-s}\right]\\&=\kappa^{n+1}-(a_1+\beta a_0)\kappa^n+...+(-1)^s[a_{s+1}+\beta a_s]\kappa^{n-s}+...+(-1)^{n+1}\beta a_n.\end{aligned}$$

As a first step we compute that

$$\lambda_2+...+\lambda_n=a_1-\lambda_1$$

and that

$$\begin{aligned}\sum_{1<u<v}\lambda_u\lambda_v&=\sum_{u<v}\lambda_u\lambda_v-\lambda_1\sum_{1<v}\lambda_v=\sum_{u<v}\lambda_u\lambda_v-\lambda_1(\sum_{v}\lambda_v-\lambda_1)\\&=a_1-\lambda_1(a_1-\lambda)=a_2-a_1\lambda_1+\lambda_1^2.\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{1<u<v<w} \lambda_u \lambda_v \lambda_w &= \sum_{u<v<w} \lambda_u \lambda_v \lambda_w - \lambda_1 \sum_{1<v<w} \lambda_v \lambda_w \\
&= a_3 - \lambda_1 [a_2 - a_1 \lambda_1 + \lambda_1^2] \\
&= a_3 - a_2 \lambda_1 + a_1 \lambda_1^2 - \lambda_1^3
\end{aligned}$$

The pattern is now clear, and we shall show by induction that

$$\sum_{1<j_1<...<j_s} \lambda_{j_1} ... \lambda_{j_s} = a_s - a_{s-1}\lambda_1 + \lambda_1^2 a_{s-2} + ... + (-1)^s a_0 \lambda_1^s.$$

Indeed

$$\begin{aligned}
\sum_{j_1<...<j_s} \lambda_{j_1} ... \lambda_{j_s} &= \sum_{j_1<...<j_s} \lambda_{j_1} ... \lambda_{j_s} - \lambda_1 \sum_{1<j_2<...<j_s} \lambda_{j_2} ... \lambda_{j_s} \\
&= a_s - \lambda_1 (a_{s-1} + ... + (-1)^{s-1} \lambda_1^{s-1}) \\
&= a_s - a_{s-1}\lambda_1 + \lambda_1^2 a_{s-2} + ... + (-1)^s a_0 \lambda_1^s.
\end{aligned}$$

Note that

$$a_n - a_{n-1}\lambda_1 + \lambda_1^2 a_{n-2} + ... + (-1)^n a_0 \lambda_1^n = 0,$$

so

$$\lambda_2 ... \lambda_n = \lambda_1^{-1} a_n = a_{n-1} - \lambda_1 a_{n-2} + ... + (-1)^{n-1} \lambda_1^{n-1}.$$

Our next step is to observe that the coefficients in the polynomial on the right-hand side of identity (2.3) may be expanded as follows

$$\begin{aligned}
D(\kappa) &= \omega_1 \delta_1 \prod_{i \neq 1}^{n} (\kappa - \lambda_i) + \omega_2 \delta_2 \prod_{i \neq 2}^{n} (\kappa - \lambda_i) + ... + \omega_n \delta_n \prod_{i \neq n}^{n} (\kappa - \lambda_i) \\
&= (\omega_1 \delta_1 + \omega_2 \delta_2 + ... + \omega_n \delta_n) \kappa^{n-1} - (\omega_1 \delta_1 [\lambda_2 + ...] + ...) \kappa^{n-2} \\
&\quad + (\omega_1 \delta_1 [\lambda_2 \lambda_3 + ...] + ...) \kappa^{n-3} + \\
&\quad ... + (-1)^s (\omega_1 \delta_1 \bar{a}_s(1) + ...) \kappa^{n-s} + .. + (-1)^{n-1} [\sum_{j=1}^{n} \omega_j \delta_j \prod_{i \neq j}^{n} \lambda_i],
\end{aligned}$$

where

$$\bar{a}_s(i) = \sum_{\substack{j_1<...<j_s \\ j_k \neq i}} \lambda_{j_1} .. \lambda_{j_s}$$

i.e. the summation refers to the omission of i from any of the components $j_1 ... j_s$.Note also that

$$\prod_{i \neq j}^{n} \lambda_i = \frac{a_n}{\lambda_j}.$$

We now consider for any constants c_s the identity

$$\begin{aligned}
&\prod_{i=1}^{n+1} (\kappa - \lambda_i) - \{\omega_1 \delta_1 \prod_{i \neq 1}^{n} (\kappa - \lambda_i) + \omega_2 \delta_2 \prod_{i \neq 2}^{n} (\kappa - \lambda_i) + ... + \omega_n \delta_n \prod_{i \neq n}^{n} (\kappa - \lambda_i)\} \\
= \ &\kappa^{n+1} - c_0 \kappa^n + c_2 \kappa^{n-1} + ... + c_n.
\end{aligned}$$

Comparing sides we obtain

$$\begin{aligned}
c_0 &= (a_1 + \beta a_0),\\
c_1 &= a_2 + \beta a_1 - (\omega_1\delta_1 + \omega_2\delta_2 + ... + \omega_n\delta_n),\\
c_2 &= a_3 + \beta a_2 - (\omega_1\delta_1\lambda_1 + ...)\\
&\quad ...\\
c_n &= a_{n+1} + \beta a_n - \left(\omega_1\delta_1\lambda_1^{-1} + ...\right).
\end{aligned}$$

Now given any c_0 we select β, so that

$$\beta = c_0 - a_1.$$

As for the remaining equations we have:

$$\begin{aligned}
c_1 - a_2 - \beta a_1 &= \omega_1\delta_1 + \omega_2\delta_2 + ... + \omega_n\delta_n,\\
c_2 - a_3 - \beta a_2 &= \omega_1\delta_1(a_1 - \lambda_1) + ...,\\
c_3 - a_4 - \beta a_3 &= \omega_1\delta_1(a_2 - s_1\lambda_1 + \lambda_1^2) +\\
&\quad ...\\
c_n - a_{n+1} - \beta a_n &= \omega_1\delta_1(a_{n-1} - a_{n-2}\lambda_1 + (-1)^{n-1}a_0\lambda_1^{n-1}) + ...,
\end{aligned}$$

where $a_{n+1} = 0$, i.e.

$$N\begin{bmatrix} \delta_1\omega_1 \\ \delta_2\omega_2 \\ ... \\ \delta_n\omega_n \end{bmatrix} = \begin{bmatrix} c_1 - a_2 - \beta a_1 \\ c_2 - a_3 - \beta a_2 \\ \\ c_n - a_{n+1} - \beta a_n \end{bmatrix}.$$

Here the coefficient matrix N is given as follows.

$$N = \begin{bmatrix} 1 & & 1 \\ a_1 - \lambda_1 & & a_1 - \lambda_n \\ a_2 - a_1\lambda_1 + \lambda_1^2 & ... & a_2 - a_1\lambda_n + \lambda_n^2 \\ ... & & ... \\ a_{n-1} - \lambda_1 a_{n-2} + ... + (-1)^{n-1}\lambda_1^{n-1} & & a_{n-1} - \lambda_1 a_{n-2} + ... + (-1)^{n-1}\lambda_1^{n-1} \end{bmatrix}.$$

Its determinant is equal, up to a possible sign change, to the van-der-Monde determinant $V(\lambda_1, ..., \lambda_n)$. Hence N is non-singular and the equation may be solved for any given vector $(c_1, ..., c_n)$.To see this note that N may be reduced to the alternant matrix $A(0, ..., n-1)$ in the variables $(-\lambda_1), ...(-\lambda_n)$

$$\begin{aligned}
&\begin{vmatrix} 1 & 1 & ... & 1 \\ a_1 - \lambda_1 & a_1 - \lambda_2 & ... & a_1 - \lambda_n \\ a_2 - a_1\lambda_1 + \lambda_1^2 & & & \\ ... & & & \\ a_{n-1} - \lambda_1 a_{n-2} + ... + (-1)^{n-1}\lambda_1^{n-1} & & & a_{n-1} - \lambda_n a_{n-2} + ... + (-1)^{n-1}\lambda_n^{n-1} \end{vmatrix}\\
= &\begin{vmatrix} 1 & 1 & ... & 1 \\ -\lambda_1 & -\lambda_2 & ... & -\lambda_n \\ \lambda_1^2 & & & \lambda_n^2 \\ ... & & & \\ (-\lambda_1)^{n-1} & & & (-\lambda_n)^{n-1} \end{vmatrix}\\
= &\ V(-\lambda_1, ..., -\lambda_n).
\end{aligned}$$

(taking $a_1$times the first row, $a_1$times the second row and so on).

It is now easy to find the inverse transformation by applying the elementary row operations just used to the original matrix equation. This leads to the following result. Putting $g_i = c_i - a_{i+1} - \beta a_i$the original equations:

$$N \begin{bmatrix} \delta_1\omega_1 \\ \delta_2\omega_2 \\ \ldots \\ \delta_n\omega_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \\ \\ g_n \end{bmatrix}$$

now transform to

$$V \begin{bmatrix} \delta_1\omega_1 \\ \delta_2\omega_2 \\ \ldots \\ \delta_n\omega_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \\ \\ h_n \end{bmatrix}$$

where

$$\begin{aligned} h_1 &= g_1, \\ h_2 &= g_2 - a_1 h_1, \\ h_3 &= g_3 - a_2 h_1 + a_1 h_2, \\ &\ldots \\ h_n &= g_n - a_{n-1} h_1 + a_{n-2} h_2 - \ldots \pm a_1 h_{n-1}. \end{aligned}$$

Note that with the sign adjustment h $h'_n = (-1)^n h_n$. Thus with $a_0 = 1$we have

$$\begin{bmatrix} \delta_1\omega_1 \\ \delta_2\omega_2 \\ \ldots \\ \delta_n\omega_n \end{bmatrix} = V^{-1} \begin{bmatrix} h_1 \\ h_2 \\ \\ \\ h_n \end{bmatrix},$$

where the inverse $V^{-1}$is given by (see Klinger,A.,(1967)) the matrix with ijentry

$$(-1)^{i+j} \frac{\bar{a}_{n-i}(j)}{\prod_{l=1}^{j-1}(\lambda_j - \lambda_l) \prod_{k=j+1}^{n}(\lambda_k - \lambda_j)}.$$

where the elementary symmetric function $\bar{a}_s(j)$ was defined above (sum over the s-fold product ommitting the variable λ_j).

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