On the continuity of the equilibrium price density and its uses

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3 March 2005
CDAM Research Report LSE-CDAM-2005-04

Abstract

With L^{∞} as the commodity space, the equilibrium price density is shown to be a continuous function of the commodity characteristics. The result is based on symmetry ideas from the Hardy-Littlewood-Pólya theory of rearrangements. It includes, but is not limited to, the case of symmetric (rearrangement-invariant) production costs and additively separable consumer utility. For example, in continuous-time peak-load pricing of electricity, it applies also when there is a storage technology and demands are cross-price dependent. In this context, a continuously varying price has two uses. First, it precludes demand jumps that would arise from discontinuous switches from one price rate to another. Second, in the problems of operating and valuing hydroelectric and pumped-storage plants (studied elsewhere), price continuity guarantees that their capacities (viz., the reservoir and the converter), the energy stocks, and in the case of hydro also the river flows, have well-defined marginal values.

Keywords: Price continuity, equilibrium existence, efficiency rents, energy storage.

2000 *AMS Mathematics Subject Classification*: Primary 91B50. Secondary 46E30.

1991 *Journal of Economic Literature Classification*: C62, D51, D58, L94.

1 Introduction

Discontinuous pricing, e.g., in time-of-use (TOU) tariffs, is likely to result in disequilibrium by creating demand discontinuities that are incompatible with pricing at marginal cost: e.g., with a sudden price drop, any consequential demand jump means that the marginal cost also rises and thus begins to differ from the ruling price. To avoid this, an equilibrium price must be *continuous* as a function of time. This property has other useful implications: for example, with marginal cost pricing of electricity, price continuity guarantees that the short-run profit function is differentiable in the fixed inputs, and hence that their efficiency rents are uniquely defined: see [15] or [18], and [16, Theorem 1] or [17].[1] And these values are fundamental to the short-run approach to long-run equilibrium that we develop in [20].

When both demand and supply are cross-price independent, the price-continuity result can be obtained by the elementary method of supply and demand curves: their intersection varies continuously with time if the curves do. This applies to, e.g., the case of cross-price independent demand for electricity supplied by thermal plants (Section 2). By exploiting the results of [15] or [18], the method of curves can be extended to include energy storage (Section 3). This is useful because, as we show in [15] or [18], price continuity guarantees that the capacities of a storage plant (viz., the reservoir and the converter) have definite marginal values.

An alternative and ultimately much more general method of proving price continuity in equilibrium begins by observing that, in examples like thermal electricity generation, the short-run supply curve remains unchanged over the cycle, which is represented by the interval $[0, T]$. So, on the assumption of no start-up or shutdown costs, the short-run cost is a symmetric (a.k.a. rearrangement-invariant) function of the output bundle $y = (y(t))_{t\in[0,T]}$—and hence so is the long-run cost. Symmetry means that the production cost, $C(y)$, depends on the values of y but *not* on their particular arrangement on $[0, T]$. In the language of electricity suppliers, the cost is a function of the load-duration curve, which mathematically is the decreasing rearrangement of y (Definition 5). Cost symmetry is useful because it implies that the trajectories of output y and of the supporting price p are similarly arranged (Lemma 9 with Remark 10), which means that it cannot be that $p(t') < p(t'')$ and $y(t') > y(t'')$. With symmetry, this holds globally on $[0, T]$, i.e., for any instants t' and t''. But, on the production side, it suffices to impose a weaker, local condition which formalises the notion that a price jump cannot entail a drop in supply (Definition 11). On the demand side, a similar, though slightly stronger, condition means that a price jump must entail a drop in demand (Definition 22). Together, these conditions rule out a price jump in equilibrium (Theorem 26). The assumption on consumer demand captures more than just the case of additively separable utility, and

[1] For the profit to be differentiable, the price does not have to be a pure density function: it suffices that the density part be continuous [15]. This is useful when the price contains also a "singular" part.

thus goes beyond the case of independent demands (Example 24). The assumption about supply generalises both the case of symmetric costs and the case of additively separable costs, but it has to be weakened further to include technologies such as energy storage (Definition 14 and Lemma 35).[2]

Although our analysis is motivated by a continuous-time problem, it applies to any good differentiated by a commodity characteristic that ranges over a topological space, denoted also by T, which carries an "underlying" measure σ and thus generalises the interval $[0, T]$ with the Lebesgue measure (for example, σ could be a probability on a "continuum" of events). In such a context, quantities of goods and their values are integrals with respect to (w.r.t.) σ—and so the commodity space consists of functions, from T into the real line $\mathbb{R}$, that are integrable, square-integrable or bounded (depending on the problem). It must be paired with a suitable price space. A pair of Lebesgue spaces, L^{ϱ} and $L^{\varrho'}$ with $(1/\varrho) + (1/\varrho') = 1$, is an example; and the price space $L^{\varrho'}$ is the norm-dual of L^{ϱ} when $\varrho < +\infty$.

But the case relevant for peak-load pricing is that of $\varrho = +\infty$: the functions representing commodity bundles must be bounded because the problem involves capacity costs or constraints. The norm-dual of $L^{\infty}[0, T]$ is larger than $L^{1}[0, T]$, and the elements of $L^{\infty *} \setminus L^{1}$, called "price singularities", have an essential role as capacity charges when the output $y \in L^{\infty}[0, T]$ has a pointed peak: if the set $\{t : y(t) = \mathrm{Sup}(y)\}$ has zero Lebesgue measure then the subdifferential $\partial \, \mathrm{Sup}(y)$ lies wholly in $L^{\infty *} \setminus L^{1}$. When the equilibrium allocation actually lies in the smaller commodity space of continuous $\mathbb{R}$-valued functions, $\mathcal{C}[0, T]$, such a price functional can be restricted to $\mathcal{C}$ and represented by a singular measure. Thus it acquires a tractable mathematical form and can be used as part of a TOU tariff. For example, when the demand for electricity has a firm pointed peak, a point measure represents the capacity charge in *\$* per kW demanded at the peak instant, whilst the fuel charge is a price density in *\$*/kWh: see [14]. Thus the price system lies in the space of measures $\mathcal{M}[0, T]$.

The type of equilibrium that the price space $L^{1}[0, T]$ does accommodate is one in which the capacity charge is spread as a density over a peak plateau in the output. Such an equilibrium arises if the users' utility and production functions are Mackey continuous (which means that consumption is interruptible, i.e., that a brief interruption causes only a small loss of utility or output): see [19]. For this case, we identify a set of conditions on which the equilibrium price function, $p^{\star}$, is not only integrable but also continuous. These conditions—viz., symmetry of production costs, additive separability of consumer utility, and their generalisations—are specific to commodity spaces of measurable functions (such as L^{∞}). Our price-continuity result is therefore quite different from those of Hindy et al. [7], Horsley [8] and [9], Jones [23], and Ostroy and Zame [28], which apply to the commodity space of measures $\mathcal{M}(T)$, and therefore to a different class of problems

[2]Even in the context of thermal generation with storage, our general framework improves on the method of curves because it applies also with interdependent demands.

(Section 10). They employ the standard approach to price representation, which relies mainly on topological assumptions on consumer preferences and production sets in a commodity space paired with the "target" price space. When the price space is $\mathcal{C}(T)$, that method necessitates the use of $\mathcal{M}(T)$ as the commodity space; so it could not serve our purposes because production costs such as the capacity cost are undefined outside the space L^{∞}. However, like Ostroy and Zame, here as in [13] we exploit the "automatic" continuity of the essential limit, which reduces the task to showing that the limit of $p^{\star}$ exists everywhere on T.

The method of curves, which does not require a fully formalised vector-space framework, is presented first (Sections 2 and 3). The commodity and price spaces L^{∞} and L^1 are introduced in Section 4. This is followed by a discussion of symmetry and its generalisations, in Sections 5 and 6. Section 7 gives the general price-continuity theorem. The case of additively separable utility (without cost separability) is spelt out in Section 8 (with additional results showing that both price and quantity trajectories are continuous and bounded). The application to electricity pricing with storage and with a general, cross-price dependent demand is presented in Section 9, which extends and supersedes Section 3. Appendix A gives the proofs for Sections 5 to 9. Appendix B reviews the concept of essential value (or limit). Appendix C reviews some properties of continuous functions.

2 Peak-load pricing with cross-price independent demands

The simplest model of equilibrium consists of supply and demand curves, S and D, in the price-quantity plane; and if one or both curves vary continuously with a parameter such as time, then so does their intersection point. This observation is useful in continuous-time peak-load pricing, i.e., pricing a cyclically demanded good which is produced by one or more techniques with capacity costs (in addition to variable costs). In this context, a continuous price can serve as an equilibrium solution to the problem of demand jumps caused by discontinuous switches from one price rate to another in a TOU tariff. A local demand maximum arises on the wrong side of such an instant, viz., just after a price drop (or just before a price jump). For example, the introduction of a two-rate tariff for electricity usually results in a surge of demand just after the switch from the daytime rate to the night-time rate: see, e.g., [27, pp. 65–66 with Figure 2.2]. Since this is a typical example, the cyclically priced flow in question is henceforth referred to as electricity, although the model applies to other goods as well.

With a one-station technology, the long-run marginal cost (LRMC) tariff has the form $p^{\star}_{\mathrm{LR}}(t) = w + r\gamma^{\star}(t)$, where r is the unit capacity cost, w is the unit running cost, and $\gamma^{\star}$ (with $\int \gamma^{\star}(t)\,\mathrm{d}t = 1$) is the distribution of the capacity charge, which is concentrated on the (global) maxima of the long-run equilibrium output $y^{\star}_{\mathrm{LR}}$. To keep demand constant during the peak, the price varies continuously with time. With a multi-station technology, the tariff structure is more complex: the offpeak price varies between the lowest and the highest of the unit variable costs of the various station types. Marginal cost pricing means that the offpeak price is the generating system's marginal variable cost, i.e., the unit fuel cost of the marginal station on line. (In the long run the system must also be optimal, i.e., it must minimise the total cost of meeting the demand.) Thus the marginalist principle might appear to imply discontinuous price changes: with, say, a two-station technology with variable costs $w_1 < w_2$, it seems that the price must drop from w_2 to w_1 as soon as the demand (at price w_2) has fallen to k_1, the capacity of the first, base type. But the users' response to such a sudden price drop is likely to reverse, albeit temporarily, the downward trend of demand—in which case, to meet the demand at the price w_1, the second station must immediately be switched back on, and the marginal fuel cost increases back to w_2. This undermines the tariff because the ruling price, w_1, differs from the marginal cost. And if the tariff is revised to take account of the new demand trajectory, new price discontinuities are created, so the difficulty arises afresh. As we show, there is nevertheless an equilibrium solution: it consists in lowering the price gradually, from w_2 to w_1, to keep the demand constant and equal to k_1 for a time after the peak station has been switched off. The price keeps falling just enough to maintain the demand (which would fall below k_1 if the price were kept constant at w_2). After such a transition period, the price "freezes" at w_1, and demand starts falling again. Price and quantity move alternately (along the vertical and horizontal segments of the supply curve in Figure 1a), i.e., the price and output trajectories have alternating plateaux: see Figures 1b and 1d.[3] Thus price continuity implies that, in addition to the peak plateau, the output has offpeak plateaux during which the price changes from w_2 to w_1 and *vice versa*.

In practice, a continuous price change could be approximated by a number of small price jumps. A cruder but effective device is to stagger a price drop by timing it differently for different consumers. For example, since 1977 Electricité de France has spread the onset of its night-time rate over one and a half hours;[4] each consumer is notified of his particular night period but is given no choice in the matter. Since the effect on market demand is akin to facing the "average consumer" with a price varying between the two rates, this can be viewed as a rough implementation, workable even with two-rate or three-rate meters, of the exact pricing solution.

With a cross-price independent demand for electricity and a purely thermal generating

[3] The graphs in Figures 1b and 1d are not periodic. They can be thought of in two ways: either as representing only a part of the cycle, or as representing the whole cycle, but after a rearrangement of time which produces nonincreasing price- and load-duration curves (and which exists by Lemma 9 and Remark 7).

[4] With a uniformly timed night-time rate, the EdF's experience in 1976 was that demand would surge, just after the start of the low rate, by over 3GW (ca. 7% of maximum demand).

technology, the method of supply and demand curves applies directly to the short-run equilibrium, and it extends to the long-run equilibrium by the short-run approach. The perfectly competitive short-run supply curve depends on the generating capacities (k_θ) and their unit running costs (w_θ), where $\theta = 1, \ldots, \Theta$ are the various station types. If the current electricity price is p, then the supply from station type θ is: $S_\theta(\mathsf{p}) = 0$ for $\mathsf{p} < w_\theta$, $S_\theta(\mathsf{p}) = k_\theta$ for $\mathsf{p} > w_\theta$, and $S_\theta(\mathsf{p}) = [0, k_\theta]$ for $\mathsf{p} = w_\theta$ (in which case $S_\theta(\mathsf{p})$ is multi-valued). The total supply is $S_{\mathrm{Th}}(\mathsf{p}) = \sum_{\theta=1}^{\Theta} S_\theta(\mathsf{p})$. For, say, $\Theta = 2$ with $w_1 < w_2$, the total supply from a two-station thermal system $k = (k_1, k_2)$ is

$$S_{\mathrm{Th}}(\mathsf{p}) = \begin{cases} 0 & \text{for } \mathsf{p} < w_1 \\ [0, k_1] & \text{for } \mathsf{p} = w_1 \\ k_1 & \text{if } w_1 < \mathsf{p} < w_2 \\ [k_1, k_1 + k_2] & \text{for } \mathsf{p} = w_2 \\ k_1 + k_2 & \text{for } \mathsf{p} > w_2 \end{cases} \tag{1}$$

(Figure 1a). This is, of course, the supply schedule of a producer whose short-run cost is additively separable over the cycle $[0, T]$. Here, for the thermal technology,

$$C_{\mathrm{SR}}(y(\cdot)) = \int_0^T c_{\mathrm{SR}}(y(t))\,\mathrm{d}t \tag{2}$$

with (see also Figure 1c)

$$\begin{aligned} c_{\mathrm{SR}}(\mathsf{y}) &= \int_0^{\mathsf{y}} S_{\mathrm{Th}}^{-1}(\mathsf{q})\,\mathrm{d}\mathsf{q} = \int_0^{\mathsf{y}} \left(w_1 1_{[0,k_1]}(\mathsf{q}) + w_2 1_{[k_1,k_1+k_2]}(\mathsf{q})\right)\mathrm{d}\mathsf{q} \\ &= w_1\mathsf{y} + (w_2 - w_1)(\mathsf{y} - k_1)^+ \end{aligned} \tag{3}$$

if $0 \le \mathsf{y} \le k_1 + k_2$ (otherwise $c_{\mathrm{SR}} = +\infty$).

The demand $D_t(\mathsf{p})$ is, at any time t, a function of the current price alone. It can be interpreted as the demand of a household maximising the utility function

$$U(x(\cdot), m) = m + \int_0^T u(t, x(t))\,\mathrm{d}t$$

over $x(\cdot) \ge 0$ and $m \ge 0$ subject to the budget constraint $m + \int_0^T p(t)\,x(t)\,\mathrm{d}t \le M$, where M is the income and $p(\cdot)$ is a TOU price in terms of the numeraire (which represents all the other goods and thus closes the model). With this behaviour, the equilibrium price can be expressed in terms of marginal utility and thus shown to be continuous in t if $\partial u/\partial \mathsf{x}$ is continuous. For each t, the instantaneous utility $u(t, \mathsf{x})$ is taken to be a strictly concave, increasing and differentiable function of the consumption rate $\mathsf{x} \in \mathbb{R}_+$, with $(\partial u/\partial \mathsf{x})(t, 0) > w_1$ (to ensure that the short-run equilibrium demand is positive at every t, if $k_1 > 0$). For simplicity, all demand is assumed to come from a single household. Its income M is the sum of an endowment of the numeraire (m^{En}) and the pure profit from electricity sales, i.e.,

$$M = m^{\mathrm{En}} + \sum_{\theta=1}^{2} \left(\int_0^T (p(t) - w_\theta)^+ \, \mathrm{d}t - r_\theta \right) \cdot k_\theta$$

where r_1 and r_2 are the unit capacity costs (per cycle), and $\pi^+ = \max\{\pi, 0\}$ is the nonnegative part of π. To guarantee a positive demand for the numeraire, assume that $m^{\mathrm{En}} > \sum_{\theta=1}^{2} (T w_\theta + r_\theta) k_\theta$. Then the demand at any time t depends only on the current price $p(t)$, and it is determined from the equation

$$\frac{\partial u}{\partial \mathsf{x}} (t, x(t)) = p(t) .$$

In other words, $D_t(\mathsf{p}) = ((\partial u / \partial \mathsf{x})(t, \cdot))^{-1} (\mathsf{p})$. When $w_2 < (\partial u / \partial \mathsf{x})(t, k_1 + k_2)$, this value of $\partial u / \partial \mathsf{x}$ is the price needed to equate demand to $k_1 + k_2$. Similarly, when $w_1 \leq (\partial u / \partial \mathsf{x})(t, k_1) \leq w_2$, the middle term is the price needed to bring the demand down to k_1. So the short-run equilibrium price can be given as

$$p^\star_{\mathrm{SR}}(t) = w_1 + \min \left\{ \left(\frac{\partial u}{\partial \mathsf{x}} (t, k_1) - w_1 \right)^+, w_2 - w_1 \right\} + \left(\frac{\partial u}{\partial \mathsf{x}} (t, k_1 + k_2) - w_2 \right)^+ \quad (4)$$

which is continuous in t if $\partial u / \partial \mathsf{x}$ is (for any fixed $\mathsf{x} > 0$). If additionally $w_1 > \min_t (\partial u / \partial \mathsf{x})(t, k_1)$ and $w_2 < \max_t (\partial u / \partial \mathsf{x})(t, k_1)$, then those times t with $p^\star_{\mathrm{SR}}(t)$ between w_1 and w_2 (and with the equilibrium output equal to k_1) form a set of positive measure.[5] With $k_2 > 0$, this is an offpeak plateau in the output (Figure 1d).

The long-run equilibrium is obtained from the short-run equilibrium by solving the simultaneous equations $r_\theta = \int_0^T (p^\star_{\mathrm{SR}}(t, k_1, k_2) - w_\theta)^+ \, \mathrm{d}t$ for k and putting the solution $k^\star$ into $p^\star_{\mathrm{SR}}(t, k)$.[6]

The short-run price formula (4) extends to the case of any number, Θ, of stations. Also, no inequalities between the w_θ's need be assumed. This is useful when w_θ depends on time, in which case there may be no fixed merit order among the stations. Denote by $w^\uparrow \in \mathbb{R}^\Theta$ the nondecreasing rearrangement of the vector $w = (w_\theta)_{\theta=1}^{\Theta} \in \mathbb{R}^\Theta$ (i.e., $w_1^\uparrow$ is the smallest entry in w, $w_2^\uparrow$ is the second smallest, and so on). In these terms,

$$p^\star_{\mathrm{SR}}(t) = w_1^\uparrow(t) + \sum_{\theta=1}^{\Theta-1} \min \left\{ \left(\frac{\partial u}{\partial \mathsf{x}} \left(t, \sum_{\alpha:\, w_\alpha(t) \leq w_\theta^\uparrow(t)} k_\alpha \right) - w_\theta^\uparrow(t) \right)^+, \left(w_{\theta+1}^\uparrow(t) - w_\theta^\uparrow(t) \right) \right\} \quad (5)$$

[5] Note that $0 < \operatorname{meas}\{t : w_1 < (\partial u / \partial \mathsf{x})(t, k_1) < w_2\}$ because this set is nonempty and open.

[6] In the case of a corner solution with $k_2 = 0$, only the inequality $r_2 \geq \int_0^T (p^\star_{\mathrm{SR}} - w_2)^+ \, \mathrm{d}t$ holds.

$$+\left(\frac{\partial u}{\partial \mathsf{x}}\left(t, \sum_{\theta=1}^{\Theta} k_\theta\right) - w_\Theta^\uparrow (t)\right)^+ .$$

3 Peak-load pricing with storage and independent demands

In addition to eliminating demand jumps, price continuity is useful in the problem of operating and valuing storage facilities for cyclically priced goods. In the context of electricity this applies to hydroelectric and pumped-storage plants. Here we deal with pumped storage (PS); the case of hydro is similar. Unlike a thermal plant, a storage plant has two capital inputs, viz., the reservoir capacity k_{St} (in kWh) and the conversion capacity k_{Co} (in kW) which transforms the stored energy into electricity and *vice versa*. (For a more detailed description of the technology, see [18] and Section 9 here.) Given a TOU electricity price p and the plant's capacities $k_{\mathrm{PS}} = (k_{\mathrm{St}}, k_{\mathrm{Co}})$, the stock of energy can be assigned a TOU shadow price $\psi(t)$, which is its marginal value in maximisation of the operating profit. As we show in [18, Lemma 8], $\psi(t)$ is unique for every t if $p(t)$ is continuous in t. In general, there is a set of such stock price functions $\hat{\Psi}(p, k_{\mathrm{PS}})$, but it has just one element, $\hat{\psi}(p, k_{\mathrm{PS}})$, if p is continuous. It then follows that the capacities have definite and separate marginal values, and so does the river flow in the case of hydro: see [16, Theorem 1] and [17], in addition to [18, Theorem 9]. This brings out the importance of price continuity in the general equilibrium.

In terms of any $\psi \in \hat{\Psi}(p, k_{\mathrm{PS}})$, the storage plant's optimal output rate $y(t)$ can be given as in (6) below: $y(t) = \pm k_{\mathrm{Co}}$ if $p(t) \neq \psi(t)$, with $y(t) \in [-k_{\mathrm{Co}}, k_{\mathrm{Co}}]$ if $p(t) = \psi(t)$. For each t, this defines the plant's supply curve $S_{\mathrm{PS},t}$ in the price-quantity plane, but the curve is *not* cross-price independent because it depends on $\psi(t)$, which depends on the whole function p. This means that, with a combined generation and storage system, the short-run equilibrium price cannot be found by intersecting curves as in the purely thermal case. Nevertheless, if $p^\star_{\mathrm{SR}}$ is an equilibrium tariff, and S_t is the system's supply curve constructed from k_θ, w_θ, k_{Co} and a $\psi \in \hat{\Psi}(p^\star_{\mathrm{SR}}, k_{\mathrm{PS}})$, then—for a certain choice, $\psi^\star_{\mathrm{SR}}$, of ψ—the curve S_t intersects the demand curve D_t at $p^\star_{\mathrm{SR}}(t)$. This fact can still be used to show that $p^\star_{\mathrm{SR}}$ is continuous, but the argument requires an extra step, which is to show that $\psi^\star_{\mathrm{SR}}(t)$ is continuous in t (and hence that S_t varies continuously with t). Once $\psi^\star_{\mathrm{SR}}$ is known to be continuous, continuity of $p^\star_{\mathrm{SR}}$ follows, as in the purely thermal case, from (5), which is now applied with $\Theta + 1$ instead of Θ and with $w_{\Theta+1}(t) := \psi^\star_{\mathrm{SR}}(t)$. (It also follows that the set $\hat{\Psi}(p^\star_{\mathrm{SR}})$ is actually a singleton $\hat{\psi}(p^\star_{\mathrm{SR}})$, and that this is $\psi^\star_{\mathrm{SR}}$.)

It remains to show that $\psi^\star_{\mathrm{SR}}$ is indeed continuous. This can be deduced from equilibrium conditions and two properties of every $\psi \in \hat{\Psi}$, viz.: (i) that ψ is of bounded variation, so it has the two one-sided limits $\psi(t\pm)$,[7] and (ii) that ψ rises or falls (possi-

[7] Since the set of t's with $\psi(t-) \neq \psi(t+)$ is at most countable, it does not matter which of the two

bly with a jump or a drop) only when the reservoir is full or empty, respectively. Consider a system with, say, two thermal stations as in (1) and one storage station (with conversion capacity k_{Co}). With $p^{\star}_{\mathrm{SR}}$ and $\psi^{\star}_{\mathrm{SR}}$ abbreviated to $p^{\star}$ and $\psi^{\star}$, introduce the curve

$$S_{\mathrm{PS},t}(\mathsf{p}) = \begin{cases} -k_{\mathrm{Co}} & \text{for } \mathsf{p} < \psi^{\star}(t) \\ [-k_{\mathrm{Co}}, k_{\mathrm{Co}}] & \text{for } \mathsf{p} = \psi^{\star}(t) \\ k_{\mathrm{Co}} & \text{for } \mathsf{p} > \psi^{\star}(t) \end{cases} \tag{6}$$

and add it to the S_{Th} of (1) to form

$$S_t(\mathsf{p}) = S_{\mathrm{Th}}(\mathsf{p}) + S_{\mathrm{PS},t}(\mathsf{p}).$$

For every t, this curve intersects D_t at $p^{\star}(t)$. When $p^{\star}(t) = \psi^{\star}(t)$, the rate of equilibrium output from storage can be read off as the horizontal distance from the intersection point to the centre of the horizontal segment of length $2k_{\mathrm{Co}}$ which S_t has at the price $\psi^{\star}(t)$: see Figure 2. As the point is left or right of centre, so the output is negative or positive, i.e., the reservoir is being charged or discharged, respectively.

Suppose that $\psi^{\star}$ has a jump at some t, i.e., $\psi^{\star}(t-) < \psi^{\star}(t+)$. Say there is no w_{θ} between $\psi^{\star}(t-)$ and $\psi^{\star}(t+)$. (If there is, it only helps the argument.) Figure 2 shows the case of $\Theta = 2$ with $w_1 < \psi^{\star}(t-) < \psi^{\star}(t+) < w_2$; the curve given by (1) plus (6) with $\psi^{\star}(t-)$ or $\psi^{\star}(t+)$ in place of $\psi^{\star}(t)$ is denoted by S_{t-} or S_{t+}. Now note that D_t cannot intersect S_{t+} below $\psi^{\star}(t+)$ or to the left of centre of the horizontal segment at the level $\psi^{\star}(t+)$, since this would mean that the reservoir is being charged for a time just after t, which is infeasible because the reservoir is full at t.[8] Similarly, D_t cannot intersect S_{t-} above $\psi^{\star}(t-)$ or to the right of centre of the horizontal segment at the level $\psi^{\star}(t-)$, since this would mean that the reservoir is being discharged for a time just before t, which is again infeasible. (In Figure 2, the lines that D_t cannot intersect are the heavy lines.) So, being monotone, D_t must have a vertical segment, from the centre at level $\psi^{\star}(t+)$ to the centre at level $\psi^{\star}(t-)$. But such a vertical segment contradicts the *strict* monotonicity of D_t in p, i.e., the differentiability of $u(t,\cdot)$. This shows that $\psi^{\star}$ is continuous, and hence so is $p^{\star}$.

In Section 9, continuity of $p^{\star}(\cdot)$ and $\hat{\psi}(p^{\star})(\cdot)$ is re-derived in a different way, and in the *other* order: first it is proved for $p^{\star}$ (by applying Theorem 26). And if $p(\cdot)$ is continuous then so is $\hat{\psi}(p)(\cdot)$: see [15] or [18, Lemma 8].[9]

values is chosen for $\psi(t)$ itself.

[8] Continuity of D_t in t is used here.

[9] Obviously that result cannot be used to derive the continuity of $p^{\star}$ from that of $\psi^{\star}$ as in this section: such an argument would be circular.

4 Commodity and price spaces for a general framework

The method of curves is limited to the case of independent demands (Sections 2 and 3). However, a more general price-continuity result can be based on the same idea that a price jump would cause a drop in the demand trajectory but a jump in the supply trajectory. The demand drop must be nonzero to include the case when supply does not actually jump but does not drop either—as can be the case in, e.g., thermal electricity generation (see Figure 1a, where y stays at k_1 if p jumps from w_1 to w_2). For a general result (Theorem 26), such responses of demand and supply to price jumps are simply assumed. We name them *sub-symmetry* and *quasi-symmetry* of preferences and technologies, since these properties follow from the stronger condition of symmetry, i.e., from invariance under rearrangement. A function C of $y = (y(t))_{t\in[0,T]}$ is called *symmetric* (a.k.a. rearrangement-invariant) if $C(y)$ depends only on the distribution of y w.r.t. the Lebesgue measure on $[0,T]$.[10] For example, the short-run cost of thermal electricity generation, the C_{SR} of (2), is symmetric, and so is its long-run cost (which is not additively separable like C_{SR}). When C is convex, its symmetry guarantees that y and $p = \nabla C(y)$ are *similarly arranged*, i.e., that for (almost) every t' and t'' if $p(t') < p(t'')$ then $y(t') \leq y(t'')$: see [12, Theorem 1]. Applied to a joint cost as a function of the output trajectory y, this means that outputs are always higher (or at least not lower) at higher-priced times. In other words, price and output increments do not have opposite signs anywhere on $[0,T]$.

Similarity of arrangement of prices and outputs is thus a global consequence of cost symmetry. Its full strength is not necessary for proving price continuity, which is a local property of the equilibrium price function $p^\star\colon [0,T] \to \mathbb{R}$. It suffices to assume a local and approximate version of the arrangement similarity between quantities and the supporting prices—and this is sub-symmetry (Definition 11). On the production side, the assumption is further weakened to quasi-symmetry (Definition 14), to make it hold for technologies such as energy storage (Lemma 35). On the consumption side, the assumption is slightly strengthened (Definition 22). It is verified for differentiable additively separable utility (Example 23); extensions to other forms of utility are sketched (Example 24). These results (Lemma 35 and Example 23) make Theorem 26 apply to peak-load pricing of electricity with cross-price independent demands and with (or without) energy storage, thus re-establishing the results of Section 3 (and Section 2).

Such an analysis requires the duality framework of a pair of function spaces to represent commodities and prices. In peak-load pricing, an output bundle is always bounded by the productive capacity, so the commodity space is $L^\infty[0,T]$. It is paired with $L^1[0,T]$ as the price space—but our task is to show that the equilibrium price density $p^\star$ is con-

[10] Equivalently, C is symmetric if $C(y) = C(y \circ \rho)$ for every Lebesgue measure-preserving transformation $\rho\colon [0,T] \to [0,T]$.

tinuous (and not just integrable) on $[0,T]$. Before this application, T standing alone denotes an abstract set of commodities which carries a topology with a countable base of open sets. Additionally, T carries a finite nonatomic (and nonnegative) measure σ on a sigma-algebra $\mathcal{A}$ that contains all the Borel subsets of T. Every nonempty open subset of T is assumed to be σ-nonnull, i.e., to have a positive measure. The vector space of all σ-equivalence classes of $\mathcal{A}$-measurable real-valued functions on T is denoted by $L^0(T,\sigma)$. The commodity space of all σ-essentially bounded functions, $L^\infty(T)$, is paired with $L^1(T)$, the price space of all σ-integrable functions.[11]

Apart from T (which may represent a single differentiated good), there is a finite number of homogeneous goods numbered by 1, 2, ..., $G \geq 0$. So a complete commodity bundle is a $(y,q) \in L^\infty(T) \times \mathbb{R}^G$, and its value at a price system $(p,r) \in L^1(T) \times \mathbb{R}^G$ is $\int_T p(t)\, y(t)\, \sigma(\mathrm{d}t) + r \cdot q$, abbreviated to $\langle p \,|\, y \rangle + r \cdot q$.

5 Symmetry and weaker conditions on production sets

We next formalise the idea that a jump in the price trajectory cannot coincide with a drop in the supply trajectory. First, this is shown to follow from symmetry of the cost function (or of the input correspondence when the inputs are not aggregated into a scalar cost). Symmetry implies an even stronger "similarity" of the price and output trajectories, viz., that they rise and fall simultaneously (Lemma 9 and Remark 10). This is more than is actually needed for price continuity, and the assumption is too strong for some applications: in electricity pricing, the cost of energy storage is not symmetric (although the cost of thermal generation is). We therefore weaken the similarity condition (Definitions 11, 14 and 17). Like symmetry, the weaker properties are preserved in summation of production sets. Some "general" examples meeting the weak conditions are given at this stage, but the motivating example of energy storage is dealt with in Section 9 (Lemmas 35 and 37).

Definition 1 *A function C on $L^0(T)$ is σ-symmetric (a.k.a. rearrangement-invariant) if, for every y and z in $L^0(T)$, the condition $\sigma(y^{-1}(B)) = \sigma(z^{-1}(B))$ for every Borel set $B \subset \mathbb{R}$ implies that $C(y) = C(z)$. (In other words, C is symmetric if its value depends only on the distribution of its argument w.r.t. σ.)*

Definition 2 *A set $S \subset L^0(T)$ is σ-symmetric if its indicator function is symmetric, i.e., if the conditions: $y \in S$, $z \in L^0$ and $\sigma(y^{-1}(B)) = \sigma(z^{-1}(B))$ for every Borel set $B \subset \mathbb{R}$ imply that $z \in S$ also. (In other words, S is symmetric if $z \in S$ whenever z has the same distribution, w.r.t. σ, as some $y \in S$.)*

[11] The analysis can be adapted for use with other L^ϱ-spaces in problems involving unbounded commodity bundles.

When this concept is used here, S is the section of a production set $\mathbb{Y} \subset L^{\infty}(T) \times \mathbb{R}^{G}$ through a $q \in \mathbb{R}^{G}$, i.e., S is the q-restricted production set

$$\mathbb{Y}(q) := \{y \in L^{\infty}(T) : (y, q) \in \mathbb{Y}\}. \tag{7}$$

The set of "output" bundles $\mathbb{Y}(q)$ is symmetric for each "input" $q \in \mathbb{R}^{G}$ if and only if the "input requirement" set $\mathbb{Y}_{y} := \{q : (y, q) \in \mathbb{Y}\}$ depends only on the distribution of y. In such a case, the production cost

$$C(y) := \inf_{q} \{-\langle r \mid q \rangle : (y, q) \in \mathbb{Y}\}$$

is a symmetric function of y (for each input price system $r \in \mathbb{R}^{G}$).

For the purpose of proving price continuity on T, the relevant implication of symmetry is similarity of arrangement for the functions p and y (on T) which represent a price system and an output bundle that maximises the (q-restricted) profit on $S = \mathbb{Y}(q)$. This result (Lemma 9) is preceded by a discussion of similarity of arrangement, a concept introduced by Day [2, p. 932].

Definition 3 *Two elements, p and y, of $L^{0}(T)$ are similarly arranged if, for any measurable sets A' and A'',*[12]

$$\operatorname{ess\,sup}_{A'} p < \operatorname{ess\,inf}_{A''} p \Rightarrow \operatorname{ess\,sup}_{A'} y \leq \operatorname{ess\,inf}_{A''} y. \tag{8}$$

After replacing p and y by any of their variants $\check{p}$ and $\check{y}$—which are literally functions rather than equivalence classes of almost everywhere (a.e.) equal functions—similarity of arrangement can be usefully reformulated in terms of values at any *points* t' and t'' (instead of values on sets A' and A'').

Remark 4 *Two elements, p and y, of $L^{0}(T)$ are similarly arranged if and only if*

$$p(t') < p(t'') \Rightarrow y(t') \leq y(t'')$$

for σ-almost every (a.e.) t' and t'' in T—i.e., if and only if for any variants $\check{p}$ and $\check{y}$ (of p and y) there is a σ-null set Z such that for every t' and t'' in $T \setminus Z$

$$\check{p}(t') < \check{p}(t'') \Rightarrow \check{y}(t') \leq \check{y}(t''). \tag{9}$$

As is shown by Day [2, p. 939, 5.6], similarity of arrangement is equivalent to the existence of a common ranking pattern. To state this, we first introduce the concepts.

[12] It obviously suffices to verify this for any σ-almost disjoint pair of σ-nonnull sets, A' and A''.

Definition 5 *The nonincreasing rearrangement $y_{\downarrow}$ of a $y \in L^0(T, \sigma)$ is the nonincreasing function on $[0, \sigma(T)]$ with the same distribution, relative to the Lebesgue measure (*meas*), as the distribution of y w.r.t. σ. That is, $y_{\downarrow}$ is nonincreasing and, for every Borel set $B \subset \mathbb{R}$,*[13]

$$\text{meas}\left\{\tau \in [0, \sigma(T)] : y_{\downarrow}(\tau) \in B\right\} = \sigma\left\{t \in T : y(t) \in B\right\}.$$

Definition 6 *A ranking pattern of a $y \in L^0(T, \sigma)$ is any measure-preserving map ρ: $T \to [0, \sigma(T)]$ such that $y = y_{\downarrow} \circ \rho$. The set of all such maps is denoted by $\mathcal{R}(y)$.*[14]

Comments:

1. $\mathcal{R}(y) \neq \emptyset$ (if σ is nonatomic). This is the Lorentz-Ryff Lemma [31, Lemma 1], stated also in, e.g., [3, 3.3].

2. If y has no plateau (i.e., $\sigma\{t : y(t) = \mathsf{y}\} = 0$ for each $\mathsf{y} \in \mathbb{R}$ or, equivalently, $y_{\downarrow}$ is *strictly* decreasing), then the pattern of y is unique, and it is

$$\rho_y = (y_{\downarrow})^{-1} \circ y.$$

   Note that $\rho_y(t) = \sigma\{\tau \in T : y(\tau) \geq y(t)\}$, i.e., $\rho_y(t)/\sigma(T)$ is t's "percentage above"—the fraction of T on which y is above its "current" value $y(t)$. Thus ρ_y ranks the elements of T by the value of y (hence its name, "the ranking pattern").

Remark 7 (Day) *Assume that σ is nonatomic (on $\mathcal{A}$). Two functions p and y, in $L^0(T, \mathcal{A}, \sigma)$, are similarly arranged if and only if*

$$\mathcal{R}(p) \cap \mathcal{R}(y) \neq \emptyset \tag{10}$$

i.e., if and only if both $p = p_{\downarrow} \circ \rho$ and $y = y_{\downarrow} \circ \rho$ for some measure-preserving map ρ: $T \to [0, \sigma(T)]$.

Comments:

1. It is obvious that (10) implies (9) or, equivalently, (8). To prove the converse—that (8) implies (10)—Day [2, p. 939, 5.6] shows that (8) implies that the pair (p, y) is *jointly* equidistributed to $(p_{\downarrow}, y_{\downarrow})$, i.e., that (p, y) has the same joint distribution as $(p_{\downarrow}, y_{\downarrow})$, and that this in turn implies (10). Thus he extends the Lorentz-Ryff Lemma to pairs (and also n-tuples) of functions, and adds the joint equidistribution as another equivalent condition.

[13] When y is the output and p is a TOU tariff, the graphs of $y_{\downarrow}$ and $p_{\downarrow}$ are known in electricity pricing as the load-duration and price-duration curves. On L^{∞}, the operation $x \mapsto x_{\downarrow}$ is $\mathrm{m}\left(L^{\infty}, L^1\right)$-continuous: see [10].

[14] "Measure-preserving" means that $\sigma\left(\rho^{-1}(B)\right) = \text{meas}\, B$ for every Borel set $B \subset [0, \sigma(T)]$.

2. In (8) and (9), the inequalities in the antecedent and the consequent must be strict and nonstrict, respectively.

3. As is obvious from Remark 4 (or Remark 7), similarity of arrangement is a symmetric binary relation in L^0—i.e., p and y can be interchanged in (8) or (9). It is *not* a transitive relation (since every function is arranged similarly to a constant).

As we note next, similarity of arrangement is preserved in summation.

Remark 8 *If each of two functions, y and z, is arranged similarly to p, then so is $y+z$.*

As has been mentioned, if p represents a linear functional supporting a symmetric set S at a point y, then p and y are similarly arranged (or, equivalently, have a common pattern). This is next spelt out for the case of $p \in L^1$ and $y \in S \subset L^\infty$. (The same holds for L^ϱ and $L^{\varrho'}$ instead of L^∞ and L^1.)

Lemma 9 *Assume that the measure σ is nonatomic (on $\mathcal{A}$) and that S is a symmetric subset of $L^\infty(T, \mathcal{A}, \sigma)$. If $p \in L^1(T)$ and y maximises $\langle p \,|\, \cdot \rangle$ on S—i.e., $y \in S$ and $\langle p \,|\, y \rangle = \sup\{\langle p \,|\, z \rangle : z \in S\}$—then p and y are similarly arranged.*

The corresponding result for functions follows [12, Theorem 1].

Remark 10 *If σ is nonatomic, $C\colon L^\infty(T, \sigma) \to \mathbb{R}$ is a symmetric convex function and $p \in \partial C(y) \cap L^1(T, \sigma)$, i.e., a $p \in L^1$ is a subgradient of C at y, then p and y are similarly arranged.*

Applied to a (restricted) production set $S = \mathbb{Y}(q)$, Lemma 9 shows that, in an output bundle $y \in S \subset L^\infty(T)$ and a supporting price system $p \in L^1(T)$, the quantity and price move up and down together over "time". But a weaker property, introduced next, suffices for the purpose of proving price continuity.

Notation The set of all neighbourhoods of t is denoted by $\mathcal{N}(t)$.

Definition 11 *A set $S \subset L^\infty(T)$ is sub-symmetric if: for every $p \in L^1(T)$ and every y that maximises $\langle p \,|\, \cdot \rangle$ on S, and for every $t \in T$ and $\epsilon > 0$, there exists an $H \in \mathcal{N}(t)$ such that for any two measurable sets $A' \subset H$ and $A'' \subset H$*

$$\epsilon + \operatorname*{ess\,sup}_{A'} p < \operatorname*{ess\,inf}_{A''} p \;\Rightarrow\; \operatorname*{ess\,sup}_{A'} y \le \operatorname*{ess\,inf}_{A''} y. \tag{11}$$

Equivalently, S is sub-symmetric if, for every p, y, t and ϵ such as above, there exists an $H \in \mathcal{N}(t)$ such that, for σ-almost every t' and t'' in H,[15]

$$\epsilon + p(t') < p(t'') \Rightarrow y(t') \le y(t''). \tag{12}$$

[15] As in Remark 4, the p and y in (12) must be interpreted as any variants $\check{p}$ and $\check{y}$, and the phrase "for a.e. t' and t'' in H" is to be interpreted as meaning "for every t' and t'' in H but outside of some σ-null set Z". The excepted set Z depends on the choice of variants $\check{p}$ and $\check{y}$, but it can be chosen independently of ϵ and, also, of t (since the topology of T has a countable open base).

Comments:

1. Unlike the case of a symmetric S, in which p and y are similarly arranged by Lemma 9, if S is only sub-symmetric then the relationship between p and y is *not* symmetric, i.e., p and y *cannot* be interchanged in (11) or (12).

2. Because of the ϵ, the strict inequality between the values of p in the antecedent of (11) or (12) can be made nonstrict without changing the concept. But the inequality between the values of y in the consequent of (11) or (12) must be *non*-strict, as in (8) or (9).

Every symmetric set is obviously sub-symmetric.[16] A proper example of sub-symmetry in production is the additively separable convex cost $\int_T c\,(t, y\,(t))\,\mathrm{d}t$: it is *not* a symmetric function of y unless the "instantaneous" cost is independent of t directly (i.e., unless the integrand $c\,(t, \mathsf{y})$ is actually independent of t, as in (2)). But if the cost curve $c\,(t, \cdot)$, together with its y-derivative, varies continuously with t, then it can be approximated in a neighbourhood of any $t_0 \in T$ by the fixed (time-independent) curve $c\,(t_0, \cdot)$. This is why $\int c\,(t, y\,(t))\,\mathrm{d}t$ is "locally and approximately" symmetric: in precise terms, its sublevel sets are sub-symmetric, as is shown next.

Example 12 *Assume that* c: $T \times (-\infty, \mathsf{k}] \to \mathbb{R}$, *where* $\mathsf{k} \in \mathbb{R}$ *is a constant, is a differentiable convex integrand, i.e., the function* $t \mapsto c\,(t, \mathsf{y})$ *is* σ*-integrable on* T *(for every* $\mathsf{y} \in \mathbb{R}$*), whilst the function* $\mathsf{y} \mapsto c\,(t, \mathsf{y})$ *is convex and differentiable on* $(-\infty, \mathsf{k}]$*, for every* $t \in T$*. Then*

$$C\,(y) := \int_T c\,(t, y\,(t))\,\sigma\,(\mathrm{d}t) \tag{13}$$

is a convex integral functional on $L^\infty\,(T)$*, defined effectively for* $y \leq \mathsf{k}$*: see, e.g., [29].*

If additionally $\partial c/\partial \mathsf{y}$ *is (jointly) continuous on* $T \times (-\infty, \mathsf{k}]$ *then, for every* $a \in \mathbb{R}$*, the set*

$$S = \{y \in L^\infty\,(T) : C\,(y) \leq a\} \tag{14}$$

is sub-symmetric, provided that $C\,(y) < a$ *for some* y *(Slater's Condition).*

Comments:

1. $(\partial c/\partial \mathsf{y})\,(t, \mathsf{k})$ means the left (one-sided) derivative (w.r.t. y, at $\mathsf{y} = \mathsf{k}$); it is assumed to be finite.

[16] To satisfy (11) when S is symmetric, it suffices to set $H = T$ regardless of ϵ (by Lemma 9 and Definition 3).

2. With c differentiable in y on $(-\infty, \mathsf{k})$, even a *strict* inequality holds between the values of y in (12), except when $y(t') = y(t'') = \mathsf{k}$. The exception is caused by the kink which $c(t, \cdot)$ has at $\mathsf{y} = \mathsf{k}$ (where the curve is "cut off" by setting $c(t, \mathsf{y}) = +\infty$ for $\mathsf{y} > \mathsf{k}$).

3. Typically, a negative $y(t)$ can arise *only* from free disposal, and so the "instantaneous" production cost $c(t, \mathsf{y})$ is nondecreasing in y, with $c(t, \mathsf{y}) = 0$ for $\mathsf{y} \leq 0$. In such a case, $c(t, \cdot)$ usually has a kink at $\mathsf{y} = 0$ but, like its kink at $\mathsf{y} = \mathsf{k}$, this does not spoil the sub-symmetry result: Example 12 extends to the case of $(\partial c / \partial \mathsf{y})(t, 0+) > 0 = (\partial c / \partial \mathsf{y})(t, 0-)$.

Example 12 is next reoriented for application to an industrial customer, who uses a differentiated input $z \in L^{\infty}_{+}(T)$ to produce a quantity $\int_T f(t, z(t))\, \mathrm{d}t$ of a homogeneous output good.

Example 13 *Assume that $f \colon T \times \mathbb{R}_+ \to \mathbb{R}$ is a differentiable concave integrand, i.e., the function $t \mapsto f(t, \mathsf{z})$ is σ-integrable on T (for every $\mathsf{z} \in \mathbb{R}$), and that the function $\mathsf{z} \mapsto f(t, \mathsf{z})$ is concave and differentiable on $\mathbb{R}_+$. Then*

$$F(z) := \int_T f(t, z(t))\, \sigma(\mathrm{d}t) \tag{15}$$

is a concave integral functional on $L^{\infty}_{+}(T)$: see, e.g., [29].

If additionally $\partial f / \partial \mathsf{z}$ is (jointly) continuous on $T \times \mathbb{R}_+$ then, for every $\zeta \in \mathbb{R}$, the set

$$S = \left\{ -z \in L^{\infty}_{-}(T) : F(z) \geq \zeta \right\} \tag{16}$$

is sub-symmetric, provided that $F(z) > \zeta$ for some z (Slater's Condition).

There is a significantly weaker condition on the technologies that, together with sub-symmetry of preferences, ensures price continuity in equilibrium. To formulate it, we use the concept of the *essential value* of p at t. Denoted by $\operatorname{ess} p(t)$, it exists if and only if the *lower* and *upper essential values*, $\underline{p}(t)$ and $\overline{p}(t)$, are equal *and* finite. In other words, $t \notin \operatorname{dom} \operatorname{ess} p$ if and only if either $-\infty < \underline{p}(t) < \overline{p}(t) < +\infty$ or $\underline{p}(t) = -\infty$ or $\overline{p}(t) = +\infty$. These concepts are reviewed in Appendix B.

Definition 14 *A set $S \subset L^{\infty}(T, \sigma)$ is* quasi-symmetric *if: for every $p \in L^1(T)$ and every y that maximises $\langle p \,|\, \cdot \rangle$ on S, and for any $t_{\ddagger} \in T \setminus \operatorname{dom} \operatorname{ess} p$, there is a number $\alpha > 0$ such that every neighbourhood $N \in \mathcal{N}(t_{\ddagger})$ has a pair of σ-nonnull subsets, $A' \subset N$ and $A'' \subset N$, such that*

$$\alpha + \operatorname*{ess\,sup}_{A'} p \leq \operatorname*{ess\,inf}_{A''} p \tag{17}$$

$$\operatorname{ess\,sup}_{A'} y \leq \operatorname{ess\,inf}_{A''} y \tag{18}$$

i.e., for σ-almost every $t' \in A'$ and $t'' \in A''$,[17]

$$\alpha + p(t') \leq p(t'') \tag{19}$$

$$y(t') \leq y(t''). \tag{20}$$

The "price part" (17) of the quasi-symmetry condition is always met because, as is noted next, it follows purely from the hypothesis of a price discontinuity (i.e., from the nonexistence of $\operatorname{ess} p$ at $t_\ddagger$).

Remark 15 *For any $t_\ddagger \in T \setminus \operatorname{dom} \operatorname{ess} p$, there is an $\alpha > 0$ such that every $N \in \mathcal{N}(t_\ddagger)$ has σ-nonnull subsets, A' and A'', with $\alpha + \operatorname{ess\,sup}_{A'} p \leq \operatorname{ess\,inf}_{A''} p$. More specifically:*

1. *If $\underline{p}(t_\ddagger) = -\infty$ or $\overline{p}(t_\ddagger) = +\infty$, then every $\alpha > 0$ has this property.*
2. *If $-\infty < \underline{p}(t_\ddagger) < \overline{p}(t_\ddagger) < +\infty$, then any positive $\alpha < \overline{p}(t_\ddagger) - \underline{p}(t_\ddagger)$ has this property.*

Corollary 16 *Every sub-symmetric set (and hence every symmetric set) is quasi-symmetric.*

The next condition on the technologies does not come into the general price-continuity result itself (Theorem 26). But it is needed to verify that, in equilibrium, the theorem's strong sub-symmetry assumption on consumer preferences holds in specific cases, such as that of differentiable additively separable utility (Section 8). It serves to establish first that the equilibrium price function is bounded and that consumption is therefore bounded away from zero (which is necessary for strong sub-symmetry to hold). This condition, formulated next, requires merely that the profit-maximising output must be arbitrarily close to its peak at the "times" of sufficiently high prices (if the price function is unbounded).[18]

Definition 17 *A set $S \subset L^\infty(T)$ is pseudo-symmetric if, for every $p \in L^1(T)$ and every y that maximises $\langle p \,|\, \cdot \rangle$ on S,*

$$\lim_{\mathsf{p} \nearrow +\infty} \operatorname*{ess\,inf}_{t:\, p(t) > \mathsf{p}} y(t) \geq \operatorname{EssSup}(y) := \operatorname*{ess\,sup}_{t \in T} y(t) \tag{21}$$

i.e., if for every $\delta > 0$ there is a $\mathsf{p} \in \mathbb{R}$ such that, for σ-almost every $t \in T$,

$$p(t) > \mathsf{p} \Rightarrow y(t) \geq \operatorname{EssSup}(y) - \delta. \tag{22}$$

[17] This means "for every t' in A' and t'' in A'' but outside of some σ-null set Z". The excepted set Z depends on the choice of variants of p and y, but it can be chosen independently of N and $t_\ddagger$.

[18] In the case of an industrial user, the condition means that his profit-maximising input must be arbitrarily close to its minimum when the price is high, since his net output of the differentiated good is, of course, the negative of his input.

Comment: The question arises only when p is unbounded: when $p \in L^{\infty}$, Condition (22) is met vacuously by $\mathsf{p} = \mathrm{EssSup}\,(p)$ or larger (since this means that $p\,(t) \leq \mathsf{p}$ for a.e. t). In other words, if p is bounded from above, then (21) holds for every $y \in L^{\infty}$ because the inequality in (21) has $+\infty$ on its left-hand side; this is the only case in which the inequality in (21) is strict.

Lemma 18 *Every symmetric set is pseudo-symmetric.*

An industrial user of a differentiated good meets a pseudo-symmetry condition if his production function is additively separable.[19]

Example 19 *Under the assumptions of Example 13 on the concave functional* $F\,(z) := \int_T f\,(t, z\,(t))\,\sigma\,(\mathrm{d}t)$ *for* $z \in L_+^{\infty}\,(T)$*, if additionally* $\sup_{t \in T}\,(\partial f / \partial \mathsf{z})\,(t, 0) < +\infty$ *(as is the case when* T *is compact and* $\partial f / \partial \mathsf{z}$ *is continuous in* t*), then, for every* $\zeta \in \mathbb{R}$*, the set* $S = -\{z \geq 0 : F\,(z) \geq \zeta\}$ *is pseudo-symmetric (provided that* $F\,(z) > \zeta$ *for some* z*).*

The "symmetry-like" concepts are applied to the sections of a production set $\mathbb{Y} \subset L^{\infty}\,(T) \times \mathbb{R}^G$ through a $q \in \mathbb{R}^G$. To say that such a set has *symmetric T-sections* (or sub-symmetric T-sections, etc.) means that, for every $q \in \mathbb{R}^G$, the set $\mathbb{Y}\,(q)$ defined by (7) is symmetric (or sub-symmetric). These properties are mostly preserved in the summation of sets (although quasi-symmetry for the sum requires sub-symmetry for all but one of the sets). To establish this, note first that

$$(\mathbb{Y}' + \mathbb{Y}'')\,(q) = \bigcup_{(q', q'') :\, q' + q'' = q} (\mathbb{Y}'\,(q') + \mathbb{Y}''\,(q'')) \,. \tag{23}$$

Furthermore, the components of a profit-maximising output bundle in the sum's section are also profit maxima: in precise terms, if

$$y = y' + y'' \text{ maximises } \langle p \,|\, \cdot \rangle \text{ on } \mathbb{Y}\,(q) := (\mathbb{Y}' + \mathbb{Y}'')\,(q) \tag{24}$$

$$q = q' + q'' \text{ and } (y', q') \in \mathbb{Y}' \text{ and } (y'', q'') \in \mathbb{Y}'' \tag{25}$$

then

$$y' \text{ maximises } \langle p \,|\, \cdot \rangle \text{ on } \mathbb{Y}'\,(q') \text{ and } y'' \text{ maximises } \langle p \,|\, \cdot \rangle \text{ on } \mathbb{Y}''\,(q'') \,. \tag{26}$$

Lemma 20 *If two subsets,* $\mathbb{Y}'$ *and* $\mathbb{Y}''$*, of* $L^{\infty}\,(T) \times \mathbb{R}^G$ *have symmetric* T*-sections that are additionally convex and* $\mathrm{w}\,(L^{\infty}, L^1)$*-closed (weakly* closed) in* $L^{\infty}\,(T)$*, then also their sum* $\mathbb{Y} := \mathbb{Y}' + \mathbb{Y}''$ *has symmetric sections.*

Lemma 21 *For any two subsets,* $\mathbb{Y}'$ *and* $\mathbb{Y}''$*, of* $L^0\,(T) \times \mathbb{R}^G$*:*

[19] Similarly, a supplier of the good meets the pseudo-symmetry condition if his cost is additively separable, as in Example 12.

1. *If both $\mathbb{Y}'$ and $\mathbb{Y}''$ have sub-symmetric T-sections, then so has their sum $\mathbb{Y} := \mathbb{Y}' + \mathbb{Y}''$.*

2. *If $\mathbb{Y}'$ has quasi-symmetric sections, and $\mathbb{Y}''$ has sub-symmetric sections, then their sum $\mathbb{Y} := \mathbb{Y}' + \mathbb{Y}''$ has quasi-symmetric sections.*

3. *If both $\mathbb{Y}'$ and $\mathbb{Y}''$ have pseudo-symmetric sections, then so has their sum $\mathbb{Y} := \mathbb{Y}' + \mathbb{Y}''$.*

6 Sub-symmetry of preferences

A variant of the sub-symmetry concept is needed to formulate a condition on consumer preferences which, together with quasi-symmetry of the production set, ensures price continuity in equilibrium. For use in this context, the condition is reoriented to minimisation of "expenditure" instead of maximisation of "profit" as in Definition 11. Also, it is formulated "pointwise" because its verification requires a condition on the particular consumption bundle x: Example 23 requires that $\operatorname{Inf}(x) > 0$ (which is needed because the demand $x(t)$ cannot drop when it is already zero). Recall that $\mathcal{N}(t)$ means the set of all neighbourhoods of t.

Definition 22 *A set $S \subset L^{\infty}(T)$ is strongly sub-symmetric at a point $x \in S$ if: for every $p \in L^1(T)$ such that x minimises $\langle p \mid \cdot \rangle$ on S, and for every $t \in T$ and $\epsilon > 0$, there exist an $H \in \mathcal{N}(t)$ and a $\delta > 0$ such that, for any two measurable sets $A' \subset H$ and $A'' \subset H$,*

$$\epsilon + \operatorname*{ess\,sup}_{A'} p < \operatorname*{ess\,inf}_{A''} p \ \Rightarrow \ \operatorname*{ess\,inf}_{A'} x \geq \delta + \operatorname*{ess\,sup}_{A''} x. \tag{27}$$

Equivalently, S is strongly sub-symmetric at x if, for any t and ϵ such as above, there exists an $H \in \mathcal{N}(t)$ and a $\delta > 0$ such that, for σ-almost every t' and t'' in H,

$$\epsilon + p(t') < p(t'') \Rightarrow x(t') \geq x(t'') + \delta. \tag{28}$$

Comment: A symmetric set need *not* be *strongly* sub-symmetric at every point. For example, if $c(t, y(t))$ in (13) is actually $c(y(t))$, a convex function of $y(t)$ alone, then the sublevel set (14) is symmetric and hence sub-symmetric, but not strongly so (at a point $x = -y$), unless c is differentiable and $\operatorname{EssSup}(y) < k$.

This concept is used with S equal to a superlevel set for the ordering of $L_{+}^{\infty}(T)$ obtained by fixing an $\widetilde{m} \in \mathbb{R}^G$ in an ordering $\preccurlyeq$ of $L_{+}^{\infty}(T) \times \mathbb{R}_{+}^{G}$, i.e., with S equal to

$$S(\widetilde{x}, \widetilde{m}, \preccurlyeq) := \left\{ x \in L_{+}^{\infty}(T) : (\widetilde{x}, \widetilde{m}) \preccurlyeq (x, \widetilde{m}) \right\} \tag{29}$$

which is a "preferred set" for the section of $\preccurlyeq$ through $\widetilde{m}$. When $\preccurlyeq$ is represented by a function $\mathbb{U}$ on $L_{+}^{\infty}(T) \times \mathbb{R}_{+}^{G}$, this set is $\{x : \mathbb{U}(\widetilde{x}, \widetilde{m}) \leq \mathbb{U}(x, \widetilde{m})\}$, a superlevel set of $U = \mathbb{U}(\cdot, \widetilde{m})$.

In the following example of strong sub-symmetry in consumption, the utility function has the additively separable form, $U(x) = \int_T u(t, x(t))\,\mathrm{d}t$. This is mathematically similar to the case of additively separable cost and production functions (Examples 12 and 13).

Example 23 *Assume that* $u\colon T \times \mathbb{R}_+ \to \mathbb{R}$ *is a differentiable concave integrand, i.e., the function* $t \mapsto u(t, \mathsf{x})$ *is* σ*-integrable on* T *(for every* $\mathsf{x} \in \mathbb{R}_+$*), whilst the function* $\mathsf{x} \mapsto u(t, \mathsf{x})$ *is concave, (strictly) increasing and differentiable on* $\mathbb{R}_{++}$ *(and continuous also at* $\mathsf{x} = 0$*), for every* $t \in T$*. Then*

$$U(x) := \int_T u(t, x(t))\,\sigma(\mathrm{d}t) \tag{30}$$

is a concave integral functional on $L_+^\infty(T)$*: see, e.g., [29].*

If additionally $\partial u/\partial \mathsf{x}$ *is (jointly) continuous on* $T \times \mathbb{R}_{++}$ *then, for every* $\widetilde{x} \in L^\infty(T)$ *with* $\operatorname{EssInf}(\widetilde{x}) > 0$*, the set*

$$S = \left\{x \in L_+^\infty(T) : U(x) \geq U(\widetilde{x})\right\} \tag{31}$$

is strongly sub-symmetric at $\widetilde{x}$*.*

The strong sub-symmetry condition can be verified for other functional forms of utility. For example, the additively separable form can be generalised by adding further terms.

Example 24 *Assume that* $v\colon T \times \mathbb{R}_+ \times T \times \mathbb{R}_+ \to \mathbb{R}$ *has the properties:*

1. *The function* $(\mathsf{x}', \mathsf{x}'') \mapsto v(t', \mathsf{x}', t'', \mathsf{x}'')$ *is jointly concave, increasing and continuously differentiable on* $\mathbb{R}_+^2$*, for every* $(t', t'') \in T \times T$*.*

2. *The function* $(t', t'') \mapsto v(t', \mathsf{x}', t'', \mathsf{x}'')$ *is* $\sigma \times \sigma$*-integrable on* $T \times T$*, for every* $(\mathsf{x}', \mathsf{x}'') \in \mathbb{R}_+^2$*.*

With $u\colon T \times \mathbb{R}_+ \to \mathbb{R}$ *as in Example 23 (and* $\partial u/\partial \mathsf{x}$ *continuous), define*

$$U(x) := \int_T u(t, x(t))\,\sigma(\mathrm{d}t) + \frac{1}{2}\int_T \int_T v(t', x(t'), t'', x(t''))\,\sigma(\mathrm{d}t')\,\sigma(\mathrm{d}t'') \tag{32}$$

for every $x \in L_+^\infty(T)$*. Assume also, without loss of generality, that* $v(t', \mathsf{x}', t'', \mathsf{x}'') = v(t'', \mathsf{x}'', t', \mathsf{x}')$*. If additionally* $\partial^2 v/\partial \mathsf{x}' \partial \mathsf{x}'' \leq 0$ *everywhere and* $\operatorname{EssInf}(\widetilde{x}) > 0$*, then the set*

$$S = \left\{x \in L_+^\infty(T) : U(x) \geq U(\widetilde{x})\right\}$$

is strongly sub-symmetric at $\widetilde{x}$*.*[20]

[20] A weaker sufficient condition on the derivatives is that $\sigma(T) \sup_{\mathsf{x}', \mathsf{x}''} \partial^2 v/\partial \mathsf{x}' \partial \mathsf{x}''$ does not exceed $\inf_{\mathsf{x}', \mathsf{x}''} \left(-\partial^2 u/\partial \mathsf{x}^2\right)$, where x' and x'' range over $[\operatorname{Inf}(\widetilde{x}), \operatorname{Sup}(\widetilde{x})]$.

7 Continuity of the equilibrium price density function

As is shown next, the weak symmetry-like conditions are sufficient for price continuity in equilibrium. The sets of producers and households (or consumers) are denoted by Pr and Ho. The production set of producer $i \in \mathrm{Pr}$ is $\mathbb{Y}_i \subset L^\infty(T) \times \mathbb{R}^G$, and $\mathbb{Y} := \sum_{i \in \mathrm{Pr}} \mathbb{Y}_i$ is the total production set. The consumption set of each household $h \in \mathrm{Ho}$ is the nonnegative orthant $L^\infty_+(T) \times \mathbb{R}^G_+$. Consumer preferences, taken to be complete and transitive, are given by a total (a.k.a. complete) weak preorder $\preccurlyeq_h$. The corresponding strict preference is denoted by $\prec_h$. The household's initial endowment is denoted by $\left(x_h^{\mathrm{En}}, m_h^{\mathrm{En}}\right) \in L^\infty_+ \times \mathbb{R}^G_+$; the household's share in the profits of producer i is $\varsigma_{hi} \geq 0$, with $\sum_h \varsigma_{hi} = 1$ for each i.[21]

Definition 25 *A price system $(p^\star, r^\star) \in L^1(T) \times \mathbb{R}^G$ supports an allocation, $(x_h^\star, m_h^\star) \geq 0$ and $(y_i^\star, q_i^\star) \in \mathbb{Y}_i$ for each $h \in \mathrm{Ho}$ and $i \in \mathrm{Pr}$, as a competitive equilibrium if:*

1. $\sum_h \left(x_h^\star - x_h^{\mathrm{En}}, m_h^\star - m_h^{\mathrm{En}}\right) = (y^\star, q^\star) := \sum_i (y_i^\star, q_i^\star)$.
2. $\langle p^\star \mid y_i^\star \rangle + \langle r^\star \mid q_i^\star \rangle = \sup_{y,q} \left\{ \langle p^\star \mid y \rangle + \langle r^\star \mid q \rangle : (y, q) \in \mathbb{Y}_i \right\}$.
3. $\langle p^\star \mid x_h^\star \rangle + \langle r^\star \mid m_h^\star \rangle = \left\langle p^\star \mid x_h^{\mathrm{En}} + \sum_i \varsigma_{hi} y_i^\star \right\rangle + \left\langle r^\star \mid m_h^{\mathrm{En}} + \sum_i \varsigma_{hi} q_i^\star \right\rangle$.
4. *For every $(x, m) \geq 0$, if $\langle p^\star \mid x \rangle + \langle r^\star \mid m \rangle \leq \langle p^\star \mid x_h^\star \rangle + \langle r^\star \mid m_h^\star \rangle$, then $(x, m) \preccurlyeq_h (x_h^\star, m_h^\star)$.*

Theorem 26 *Assume that:*

1. *A price system $(p^\star, r^\star) \in L^1(T, \sigma) \times \mathbb{R}^G$ supports a competitive equilibrium with a consumption allocation $(x_h^\star, m_h^\star) \in L^\infty_+(T) \times \mathbb{R}^G_+$ (for $h \in \mathrm{Ho}$) and with a total input-output bundle $(y^\star, q^\star) \in \mathbb{Y}$.*
2. *The section $\mathbb{Y}(q^\star)$ of the total production set is a quasi-symmetric subset of $L^\infty(T)$.*
3. *The set $S(x_h^\star, m_h^\star, \preccurlyeq_h)$, defined by (29), is strongly sub-symmetric at $x_h^\star$, for each household h.*
4. *For each h, the initial endowment is nonnegative and has a continuous variant, i.e., $\operatorname{ess} x_h^{\mathrm{En}} \in \mathcal{C}_+(T)$.*

Then the equilibrium price has a continuous variant, i.e., $\operatorname{ess} p^\star \in \mathcal{C}(T)$.

[21] The ranges of running indices in summations, etc., are always taken to be the largest possible with any specified restrictions.

8 Continuity and boundedness of price and quantity with additively separable utility

As is shown below, the price-continuity result (Theorem 26) applies when the consumer's utility is, up to a monotone transformation, a *differentiable* additively separable function of a consumption bundle $x \in L_{+}^{\infty}(T)$, for a fixed consumption of the other goods, m. With such preferences, verification of the *strong* sub-symmetry condition rests on Example 23 and on the following first-order condition (FOC) for utility maximisation or expenditure minimisation.

Lemma 27 *Assume that $u\colon T \times \mathbb{R}_{+} \times \mathbb{R}_{+}^{G} \to \mathbb{R}$ is a concave integrand parameterised by $\mathbb{R}_{+}^{G}$, i.e.:*

1. *For every $t \in T$, the function $(\mathsf{x}, m) \mapsto u(t, \mathsf{x}, m)$ is concave.*
2. *The function $t \mapsto u(t, \mathsf{x}, m)$ is σ-integrable on T, for every $\mathsf{x} \in \mathbb{R}_{+}$ and $m \in \mathbb{R}_{+}^{G}$.*
3. *The function $\mathsf{x} \mapsto u(t, \mathsf{x}, m)$ is concave, (strictly) increasing and continuous on $\mathbb{R}_{+}$, for every $t \in T$ and $m \in \mathbb{R}_{+}^{G}$.*

Assume also that $W\colon \mathbb{R}^{1+G} \to \mathbb{R}$ is differentiable, concave and increasing in each variable, and define

$$\mathbb{U}(x, m) := W(U(x, m), m) \tag{33}$$

where, for $x \in L_{+}^{\infty}(T)$ and $m \in \mathbb{R}_{+}^{G}$,

$$U(x, m) := \int_{T} u(t, x(t), m)\, \sigma(\mathrm{d}t) \tag{34}$$

(with $u(t, \mathsf{x}, m) := -\infty$ for $\mathsf{x} < 0$ and $\mathbb{U}(x, m) := -\infty$ for $(x, m) \ngeq 0$). If additionally $p \in L^{1}(T)$, $M > 0$, and $(\widetilde{x}, \widetilde{m})$ maximises $\mathbb{U}(x, m)$ over x and m subject to: $\langle p \,|\, x \rangle + r \cdot m \le M$, $x \ge 0$ and $m \ge 0$, then there exists a scalar $\widetilde{\lambda} > 0$ such that

$$\widetilde{\lambda} p(t) \in \partial_{\mathsf{x}} u(t, \widetilde{x}(t), \widetilde{m}) \quad \text{for } \sigma\text{-almost every } t \in T. \tag{35}$$

When $(\partial u / \partial \mathsf{x})(0) = +\infty$, it follows that $\widetilde{x} \gg 0$ (i.e., $\widetilde{x}(t) > 0$ for a.e. $t \in T$). If additionally $u(t, \cdot, m)$ is differentiable on $\mathbb{R}_{++}$ (for every $t \in T$ and $m \in \mathbb{R}_{+}^{G}$), then there is a unique scalar $\widetilde{\lambda} > 0$ such that

$$\widetilde{\lambda} p(t) = \frac{\partial u}{\partial \mathsf{x}}(t, \widetilde{x}(t), \widetilde{m}) \quad \text{for } \sigma\text{-almost every } t \in T. \tag{36}$$

Comment: That $p \in L^1$ is part of (36), but that part is *assumed* and *not* proved in Lemma 27: unless $\mathrm{EssInf}(\widetilde{x}) > 0$, $\widetilde{x}$ can be optimal also when $p \in L^{\infty *} \setminus L^1$.

Utility maximisation is usually equivalent to expenditure minimisation, as is noted next.

Remark 28 *Assume that $\preccurlyeq$ is a total preorder on a closed convex subset X of a real topological vector space $(L, \mathcal{T})$, and that $\preccurlyeq$ is $\mathcal{T}$-locally nonsatiated and lower semicontinuous along each linear segment of X.*[22] *Then the following three conditions on a point $\widetilde{x} \in X$ and a continuous linear functional $p \in (L, \mathcal{T})^*$ are equivalent to one another, provided that there exists an $x^{\mathrm{S}} \in X$ with $\langle p \,|\, x^{\mathrm{S}} \rangle < \langle p \,|\, \widetilde{x} \rangle$:*

1. *For every $x \in X$, if $\langle p \,|\, x \rangle \leq \langle p \,|\, \widetilde{x} \rangle$ then $x \preccurlyeq \widetilde{x}$.*

2. *For every $x \in X$, if $\langle p \,|\, x \rangle < \langle p \,|\, \widetilde{x} \rangle$ then $x \preccurlyeq \widetilde{x}$.*

3. *For every $x \in X$, if $\langle p \,|\, x \rangle < \langle p \,|\, \widetilde{x} \rangle$ then $x \prec \widetilde{x}$ (i.e., if $x \succcurlyeq \widetilde{x}$ then $\langle p \,|\, x \rangle \geq \langle p \,|\, \widetilde{x} \rangle$).*

For each t, the FOC (36) establishes a monotone correspondence between the "current" price $p(t)$ and the "instantaneous" consumption rate $x_h(t)$. If additionally the net output rate is near its peak whenever the current price is sufficiently high (which is the pseudo-symmetry condition), it follows that the equilibrium price function $p^\star$ is bounded, and that the equilibrium consumption $x_h^\star$ is therefore bounded away from zero. These two preliminary results are established next (Proposition 30 and Corollary 31). For the rest of this section, the utility of each household h, is taken to have the form (33)–(34). The "instantaneous" utility a.k.a. felicity function, u_h, is assumed to meet the following conditions, in addition to those of Lemma 27.

Continuity of Marginal Utility. For every h and $m \in \mathbb{R}_+^G$

$$\frac{\partial u_h}{\partial \mathsf{x}} (\cdot, \cdot, m) \in \mathcal{C}(T \times \mathbb{R}_{++}) \tag{37}$$

i.e., the function $(t, \mathsf{x}) \mapsto (\partial u_h / \partial \mathsf{x})(t, \mathsf{x}, m)$ is (jointly) continuous on $T \times \mathbb{R}_{++}$.[23]

Boundedness of Marginal Utility (in t). For every h and $(\mathsf{x}, m) \in \mathbb{R}_{++} \times \mathbb{R}_+^G$

$$\sup_{t \in T} \frac{\partial u_h}{\partial \mathsf{x}} (t, \mathsf{x}, m) < +\infty. \tag{38}$$

When T is compact, this follows from the continuity of $\partial u_h / \partial \mathsf{x}$ in t.

Unboundedness of Marginal Utility (in x). For every h and $m \in \mathbb{R}_+^G$

$$\frac{\partial u_h}{\partial \mathsf{x}} (t, \mathsf{x}, m) \nearrow +\infty \quad \text{uniformly in } t \in T \text{ as } \mathsf{x} \searrow 0. \tag{39}$$

[22] By definition, $\preccurlyeq$ is $\mathcal{T}$-locally nonsatiated if $x' \in \mathrm{cl}_{\mathcal{T}} \{x \in X : x' \prec x\}$ for every $x' \in X$ (where $x' \prec x$ means that $x' \preccurlyeq x$ and $x' \not\succcurlyeq x$). Even for the strongest choice of $\mathcal{T}$, this follows from nonsatiation if $\preccurlyeq$ can be represented by a concave function U.

[23] Partial continuity of $\partial u_h / \partial \mathsf{x}$ in x follows from its monotonicity, i.e., from the concavity of u_h in x.

Remark 29 *Assume (39). If* $x_h^{\mathrm{En}} = 0$ *for each* h*, then* $y^\star \gg 0$.

Proposition 30 *In addition to (38) and (39), assume that:*

1. *The section* $\mathbb{Y}(q^\star)$ *of the total production set is a pseudo-symmetric subset of* $L^\infty(T)$.

2. $x_h^{\mathrm{En}} \geq 0$ *for each* h.

3. $\langle p^\star \mid x_h^\star \rangle + r \cdot m_h^\star > 0$ *for each* h*, i.e., the equilibrium expenditures are positive.*

4. $0 < \mathrm{EssSup}(y^\star) := \operatorname{ess\,sup}_{t \in T} y^\star(t)$*, i.e., the equilibrium net output of the differentiated good is nonzero (as is the case by Remark 29 if* $x_h^{\mathrm{En}} = 0$ *for each* h*).*

Then $p^\star \in L^\infty(T)$*, i.e., the equilibrium price is essentially bounded.*

Remark 29 can now be strengthened so that Example 23 can be applied to verify the strong sub-symmetry condition of Theorem 26.

Corollary 31 *On the assumptions of Proposition 30,* $\mathrm{EssInf}(x_h^\star) > 0$ *for each* h.

Corollary 32 *In addition to the assumptions of Proposition 30, assume (37). Then the set* $S(x_h^\star, m_h^\star, \preccurlyeq_h)$*, defined by (29), is strongly sub-symmetric at* $x_h^\star$*, for each* h.

Therefore, as is spelt out next, Theorem 26 applies to the case of additively separable utility (if $\partial u_h / \partial \mathsf{x}$ is continuous in (t, x) and infinite at $\mathsf{x} = 0$).

Theorem 33 *In addition to (37), (38) and (39), assume that:*

1. *A price system* $(p^\star, r^\star) \in L^1(T) \times \mathbb{R}^G$ *supports a competitive equilibrium with a consumption allocation* $(x_h^\star, m_h^\star)_{h \in \mathrm{Ho}}$ *and a total input-output bundle* $(y^\star, q^\star) \in \mathbb{Y}$.

2. *The section* $\mathbb{Y}(q^\star)$ *of the total production set is quasi-symmetric, and also pseudo-symmetric.*

3. $x_h^{\mathrm{En}} \in \mathcal{C}_+(T)$ *for each* h.

4. $\langle p^\star \mid x_h^\star \rangle + r \cdot m_h^\star > 0$ *for each* h.

5. $\mathrm{EssSup}(y^\star) > 0$ *(as is the case by Remark 29 when* $x_h^{\mathrm{En}} = 0$ *for each* h*).*

Then the equilibrium price has a continuous variant (which is also bounded and strictly positive), i.e., $\operatorname{ess} p^\star \in \mathcal{C}_{++}^{\mathrm{B}}(T)$*.*[24]

[24] Because of (38), $\operatorname{ess} p^\star$ is bounded even if T is not compact.

Finally, when the equation $\lambda \mathsf{p} = (\partial u_h / \partial \mathsf{x})(t, \mathsf{x})$ has a unique solution for x, price continuity implies that the equilibrium quantities, $x_h^\star(t)$ and $y^\star(t)$, are also continuous in t.

Corollary 34 *On the assumptions of Theorem 33, if additionally T is compact and $(\partial u_h / \partial \mathsf{x})(t, \cdot)$ is (strictly) decreasing, for each t—i.e., the "instantaneous" utility function $\mathsf{x} \mapsto u_h(t, \mathsf{x})$ is strictly concave (as well as differentiable on $\mathbb{R}_{++}$)—then the equilibrium consumption of the differentiated good has a continuous variant (which is also strictly positive), i.e.,* $\operatorname{ess} x_h^\star \in \mathcal{C}_{++}(T)$.

9 Application to peak-load pricing with storage

The price-continuity theorem is next applied to electricity pricing with (or without) pumped storage. This extends the results of Sections 2 and 3 to the case of cross-price dependent demand, provided that the preferences and technologies of electricity users meet the weak symmetry-like conditions. To present this application rigorously yet briefly, we assume that:

1. As a result of aggregating commodities on the basis of some fixed relative prices, there are just two consumption goods apart from electricity—viz., a numeraire (measured in \$) and a homogeneous final good produced with an input of electricity.

2. The various kinds of thermal generating capacity and fuel have fixed prices, $r_{\mathrm{Th}} = (r_1, \ldots, r_\Theta)$ and $w = (w_1, \ldots, w_\Theta)$, in terms of the numeraire (i.e., in \$/kW and \$/kWh, respectively). The prices of storage and conversion capacities, $r_{\mathrm{PS}} = (r_{\mathrm{St}}, r_{\mathrm{Co}})$ in \$/kWh and \$/kW, are also fixed.

A complete commodity bundle consists therefore of electricity (a differentiated good) and of a number of homogeneous goods, viz., the thermal capacities, the fuels, the storage and conversion capacities, the produced final good and the numeraire. The quantities are always written in this order; but those which are irrelevant in a particular context (and can be set equal to zero) may be omitted. For example, a consumption bundle consists of electricity, the produced final good and the numeraire; so it may be written as $(x; \varphi, m) \in L^\infty[0,T] \times \mathbb{R}^2$. A matching consumer price system is $(p; \varrho, 1) \in L^{\infty*}[0,T] \times \mathbb{R}^2$ (whilst a complete price system is $(p; r_{\mathrm{Th}}, w; r_{\mathrm{PS}}; \varrho, 1)$). There is a finite set, Ho, of households; and for each $h \in \mathrm{Ho}$ the utility function U_h is $\mathrm{m}(L^\infty \times \mathbb{R}^2, L^1 \times \mathbb{R}^2)$-continuous (Mackey continuous) on the consumption set $L_+^\infty[0,T] \times \mathbb{R}_+^2$. Each household's initial endowment is a quantity $m_h^{\mathrm{En}} > 0$ of the numeraire only; and nonsatiation in this commodity is assumed.

There is also an industrial user, producing the final good from inputs of electricity and the numeraire (z, n). The user's production function $F\colon L_+^\infty[0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be m($L^\infty \times \mathbb{R}, L^1 \times \mathbb{R}$)-continuous, concave, nondecreasing and nonzero, with $F(0,0) = 0$.[25] In the case of decreasing returns to scale, each household's share ς_h in the user industry's profits must also be specified.

The electricity supplier uses a multi-station thermal technology and pumped storage. A thermal technique, θ, generates an output flow y from the inputs of fuel v_θ (in kWh), and of generating capacity k_θ (measured in kW, like the output rate $y(t)$). The long-run production set of technique $\theta = 1, \ldots, \Theta$ is the cone

$$\mathbb{Y}_\theta := \left\{ (y; -k_\theta, -v_\theta) \in L^\infty [0,T] \times \mathbb{R}^2_- : \operatorname{EssSup}\left(y^+\right) \le k_\theta, \int_0^T y^+ \, \mathrm{d}t \le v_\theta \right\} \tag{40}$$

and the total thermal production set is

$$\mathbb{Y}_{\mathrm{Th}} := \left\{ \left(\sum_{\theta=1}^{\Theta} y_\theta; -(k_\theta, v_\theta)_{\theta\in\Theta} \right) : (y_\theta; -k_\theta, -v_\theta) \in \mathbb{Y}_\theta \text{ for } \theta = 1, \ldots, \Theta \right\}.$$

This is the sum of the $\mathbb{Y}_\theta$'s, each of which is embedded in $L^\infty \times \left(\mathbb{R}^2_-\right)^\Theta$; for simplicity, different types of station are assumed to use different fuels. To justify formally the fixed prices of inputs for electricity supply (r_{Th}, w and r_{PS}), there is also the production set equal to the hyperplane perpendicular to the vector $(r, w, 1)$ and passing through the origin in the space of the electricity supplier's inputs and the numeraire.

Thermal generation is supplemented by pumped storage. Energy is moved in and out of storage with a converter, which is taken to be perfectly efficient and symmetrically reversible: this means that in unit time a unit converter can either turn a unit of the marketed good (electricity) into a unit of the stocked intermediate good (a storable form of energy), or *vice versa*. On this simplifying assumption, the signed outflow of energy from the reservoir, $-\dot{s}(t)$, is equal to the storage plant's net output rate, $y(t) = y^+(t) - y^-(t)$. The converter's capacity is denoted by k_{Co} (measured in kW). The reservoir's capacity is k_{St} (in kWh); stock can be held in storage at no running cost (or loss of stock). The long-run production set is, therefore,

$$\mathbb{Y}_{\mathrm{PS}} := \left\{ (y; -k_{\mathrm{St}}, -k_{\mathrm{Co}}) \in L^\infty [0,T] \times \mathbb{R}^2_- : |y| \le k_{\mathrm{Co}}, \text{ and } \exists s \; \dot{s} = -y, \; s(0) = s(T) \text{ and } 0 \le s \le k_{\mathrm{St}} \right\}.$$

The minimum requirements for storage capacity and conversion capacity, when the (signed) output from storage is y with $\int_0^T y \, \mathrm{d}t = 0$, are:[26]

$$\check{k}_{\mathrm{St}}(y) = \max_{t\in[0,T]} \int_0^t y(t) \, \mathrm{d}t + \max_{t\in[0,T]} \int_t^T y(t) \, \mathrm{d}t \tag{41}$$

$$\check{k}_{\mathrm{Co}}(y) = \|y\|_\infty = \operatorname{ess} \sup_{t\in[0,T]} |y(t)|. \tag{42}$$

[25] It follows that $F(z,n) > 0$ for every $n > 0$ and every z with $\operatorname{EssInf}(z) > 0$.

[26] Formula (41) is derived in [15].

In these terms, $(y, -k_{\mathrm{St}}, -k_{\mathrm{Co}}) \in \mathbb{Y}_{\mathrm{PS}}$ if and only if

$$\int_0^T y(t)\,\mathrm{d}t = 0,\ \check{k}_{\mathrm{St}}(y) \le k_{\mathrm{St}} \text{ and } \check{k}_{\mathrm{Co}}(y) \le k_{\mathrm{Co}}. \tag{43}$$

Unlike the thermal capacity and fuel requirements in (40), and unlike $\check{k}_{\mathrm{Co}}(y)$, the storage capacity requirement $\check{k}_{\mathrm{St}}(y)$ is *not* a symmetric function of y. But the storage technology does meet the quasi-symmetry condition. We verify this by formalising the following argument, in which ψ is the shadow price of stock: take an electricity price p which jumps at some t, i.e., $p(t-) < p(t+)$. Can the supply (from the storage plant) drop? If it does, i.e., $y(t-) > y(t+)$, then obviously $y(t-) > -k_{\mathrm{Co}}$ and $y(t+) < k_{\mathrm{Co}}$, so $p(t-) \ge \psi(t-)$ and $p(t+) \le \psi(t+)$. Hence $0 < p(t+) - p(t-) \le \psi(t+) - \psi(t-)$, so $s(t) = k_{\mathrm{St}}$, i.e., the reservoir must be full at t. So it cannot be being discharged just before t or charged just after t, i.e., $y(t-) \le 0 \le y(t+)$. This contradicts the drop in y. But the argument is not rigorous because p and y may fail to have the one-sided limits. The need to make it rigorous is what has led us to the concept of quasi-symmetry.

Lemma 35 *For every $k_{\mathrm{PS}} = (k_{\mathrm{St}}, k_{\mathrm{Co}}) \in \mathbb{R}^2_+$, the set of feasible flows from storage, $\mathbb{Y}_{\mathrm{PS}}(-k_{\mathrm{PS}}) \subset L^\infty[0,T]$, is quasi-symmetric (with the usual topology on $[0,T]$).*

As we show in [19], the assumed Mackey continuity of the users' utility and production functions, U_h and F, means that electricity consumption is interruptible (i.e., a brief interruption causes only a small loss of utility or output), and this guarantees that the equilibrium TOU price is a density, i.e., a time-dependent rate in \$/kWh. As we state next, sub-symmetry conditions on U_h and F guarantee that the price density is continuous.

Theorem 36 *The electricity pricing model has a long-run competitive equilibrium. Furthermore:*

1. *If an equilibrium tariff $p^\star \in L^{\infty *}_+[0,T]$ supports (together with a price $\varrho^\star \in \mathbb{R}_+$ for the other produced good) an equilibrium allocation with a nonzero electricity output $y^\star_{\mathrm{Th}} + y^\star_{\mathrm{PS}}$ from thermal generation and pumped storage, then $p^\star \in L^1_+[0,T]$.*

2. *Assume additionally that:*

   (a) *For each household h, the set*

   $$\left\{x \in L^\infty_+ : U_h(x, \varphi^\star_h, m^\star_h) \ge U_h(x^\star_h, \varphi^\star_h, m^\star_h)\right\}$$

   is strongly sub-symmetric at $x^\star_h$.

(b) The section of the industrial user's production set through any $(\zeta, -n) \in \mathbb{R} \times \mathbb{R}_-$, i.e., the set

$$\mathbb{Y}_{\mathrm{IU}}(\zeta, -n) = \left\{-z \in L^{\infty}_{-}[0,T] : F(z,n) \geq \zeta\right\}$$

is sub-symmetric.

Then $p^\star$ has a continuous variant, i.e., $\operatorname{ess} p^\star \in \mathcal{C}[0,T]$.

The conditions on the users are met when U_h and F are additively separable utility and production functions with a continuous marginal utility or productivity: see Sections 5, 6 and 8. Since this case uses Proposition 30 and Corollary 31 (to apply Example 23), it requires also the following result on the storage technology.

Lemma 37 *For each $k_{\mathrm{PS}} = (k_{\mathrm{St}}, k_{\mathrm{Co}}) \in \mathbb{R}^2_+$, the set $\mathbb{Y}_{\mathrm{PS}}(-k_{\mathrm{PS}})$ is pseudo-symmetric.*

Corollary 38 *Assume that:*

1. *As in Section 8, each household's utility U_h has the (concave) integral form (34), and the instantaneous utility from electricity consumption $u_h(\cdot,\cdot,\varphi,m)$ satisfies (37), (38) and (39) for any $(\varphi, m) \in \mathbb{R}^2_+$ (in place of $m \in \mathbb{R}^G_+$).*

2. *The industrial user's production function F has the integral form (15).*

Then any equilibrium tariff has a continuous variant (which is also strictly positive), i.e., $\operatorname{ess} p^\star \in \mathcal{C}_{++}[0,T]$. *Furthermore, for each h, if additionally u_h is strictly concave (in its second variable, x) then the equilibrium consumption of electricity has a continuous variant (which is also strictly positive), i.e.,* $\operatorname{ess} x^\star_h \in \mathcal{C}_{++}[0,T]$.

10 Comparisons with other frameworks for price continuity

Our analysis applies to the commodity space $L^\infty(T)$, and potentially to other Lebesgue function spaces. A different kind of price-continuity result applies to the commodity space of all (Borel) measures, $\mathcal{M}(T)$. This choice of space describes a commodity bundle which can either be concentrated on a single characteristic (the point measure at a $t \in T$) or be spread out as a density (w.r.t. an underlying measure σ on T). It has been used to model physical commodity differentiation by Horsley [8] and [9], Jones [22] and [23], Mas-Colell [25], and Ostroy and Zame [28]. It applies also to continuous-time intertemporal problems, but only if the good in question can be consumed instantly as well as over time, like aggregate wealth in the consumption-savings problem of Hindy et al. [7]. By contrast, for a good which can only be consumed over time rather than instantly, a consumption or production bundle can be represented by a density function but not by a point measure. With capacity constraints, the feasible quantity densities are also bounded, and the commodity space must be $L^{\infty}[0,1]$ or a subspace thereof.

It might seem possible to embed the commodity space $L^{\infty}[0,1]$ in $\mathcal{M}[0,1]$ and then apply a price-continuity result formulated for the larger space. This strategy may to some extent succeed with pure exchange, but not with production because, in problems such as peak-load pricing, the production sets in the original commodity space, $L^{\infty}[0,1]$, do not have a usable extension to $\mathcal{M}[0,1]$. An example of central interest is the capacity cost function

$$C(y) = \operatorname*{ess\,sup}_{t \in [0,1]} y^{+}(t) \tag{44}$$

which obviously does not have a finite, nondecreasing extension beyond $L^{\infty}[0,1]$. Extension of consumer preferences is less problematic. For example, additively separable concave utility, originally defined by $U(x) = \int_0^1 u(t, x(t))\,\mathrm{d}t$ for $x \in L_{+}^{\infty}$, does have a $\mathrm{w}(\mathcal{M}, \mathcal{C})$-upper semicontinuous extension to $\mathcal{M}_{+}[0,1]$, viz.,

$$U(\mu) = \int_0^1 u\left(t, \frac{\mathrm{d}\mu_{\mathrm{AC}}}{\mathrm{d}\,\mathrm{meas}}(t)\right)\mathrm{d}t + \int_0^1 \frac{\partial u}{\partial \mathsf{x}}(t, +\infty)\,\mu_{\mathrm{S}}(\mathrm{d}t)$$

where $\mathrm{d}\mu_{\mathrm{AC}}/\mathrm{d}\,\mathrm{meas}$ is the density (w.r.t. the Lebesgue measure) of the absolutely continuous part of μ, and μ_{S} is the singular part of μ: see, e.g., [29, (4.4)] with $n = 1$, or [28, p. 599] for the case of u independent of t directly with $(\mathrm{d}u/\mathrm{d}\mathsf{x})(+\infty) = 0$. This U fails Jones' condition of continuity for the weak topology $\mathrm{w}(\mathcal{M}, \mathcal{C})$,[27] but Ostroy and Zame [28, Theorem 1] have succeeded in weakening that assumption to $\mathrm{w}(\mathcal{M}, \mathcal{C})$-upper semicontinuity (and continuity for the variation norm of $\mathcal{M}$). Although they study only the case of pure exchange, it is possible to include production, as in Jones' work. This cannot, of course, remove the basic obstacle to deriving results for L^{∞} from those for $\mathcal{M}$—which is that production costs such as (44) cannot be extended to $\mathcal{M}$.

A useful (though equally inapplicable to L^{∞}) variant of the analysis for the commodity space $\mathcal{M}$ is given by Hindy et al. [7], who develop Jones' idea [23, p. 525] of replacing the instantaneous consumption rate $x(t)$ by its average over a small ϵ-neighbourhood of each t, in which case the utility function extends to $U_{\epsilon}(\mu) = (1/2\epsilon) \int_0^1 u(t, \mu[t-\epsilon, t+\epsilon])\,\mathrm{d}t$, defined for $\mu \in \mathcal{M}_{+}[0,1]$. Such preferences result in an absolutely continuous, and actually Lipschitz, equilibrium price function; roughly speaking, this is because the "moving average" $(1/2\epsilon) \int_{t-\epsilon}^{t+\epsilon} p(\tau)\,\mathrm{d}\tau$ is continuous in t whenever $p \in L^1$, and it is a Lipschitz function of t if $p \in L^{\infty}$. The same applies to various weighted averages. Hindy et al. [7, Proposition 7 and Theorem 2] show that such a utility function (but not the additively

[27] A *strictly* concave, additively separable utility on L_{+}^{∞} is $\mathrm{w}(L^{\infty}, L^1)$-discontinuous—see, e.g., [1, p. 539]—and *a fortiori* it can have no $\mathrm{w}(\mathcal{M}, \mathcal{C})$-continuous extension to $\mathcal{M}_{+}$ (since $\mathcal{C} \subset L^1$).

separable utility) is continuous[28] and uniformly proper for a norm $\|\cdot\|$ on $\mathcal{M}[0,1]$ for which

$$(\mathcal{M}[0,1], \|\cdot\|)^* = \mathrm{Lip}[0,1] \tag{45}$$

i.e., the norm-dual of $\mathcal{M}$ is identified as the space of Lipschitz functions by means of the bilinear form

$$\langle p \mid \mu \rangle = \int_{[0,1]} p(t)\, \mu(\mathrm{d}t)$$

for $p \in \mathrm{Lip}$ and $\mu \in \mathcal{M}$. Given this, Lipschitz continuity of the equilibrium price follows from Mas-Colell and Richard's general framework [26]. The norm in question is

$$\|\mu\| := \int_0^1 |\mu[0,t]|\, \mathrm{d}t + |\mu[0,1]| \tag{46}$$

i.e., it is the L^1-norm of the cumulative distribution function (c.d.f.) of μ plus the total mass of μ.

This formula extends the Kantorovich-Rubinshtein-Vassershtein (KRV) norm $\|\cdot\|_{\mathrm{KRV}}$, and the dual's representation (45) can be derived from the corresponding result for $\|\cdot\|_{\mathrm{KRV}}$. Given a compact metric space (K,d), the KRV norm of a null measure μ (a signed measure of zero total mass on K) is defined as the optimal value of the Monge-Kantorovich mass transfer problem in which μ^+ and μ^- (the nonnegative and nonpositive parts of μ) represent the initial and final distributions of the mass to be transferred: see, e.g., [5, p. 329, line 2 f.b.] or [24, VIII.4.4: (25)]. The KRV norm turns out to be dual to the Lipschitz norm

$$\|p\|_{\mathrm{L}} := \sup_{t',t'' \in K} \frac{|p(t') - p(t'')|}{d(t',t'')}.$$

This, the best Lipschitz constant for p, is a seminorm on $\mathrm{Lip}(K)$; it is a norm on the subspace $\mathrm{Lip}_0(K)$ that consists of those Lipschitz functions vanishing at a fixed point $t_0 \in K$. This space is, in other words, isometric to the KRV norm-dual of the space of null measures $\mathcal{M}^{\mathrm{N}}(K)$, i.e.,

$$\left(\mathcal{M}^{\mathrm{N}}, \|\cdot\|_{\mathrm{KRV}}\right)^* = (\mathrm{Lip}_0, \|\cdot\|_{\mathrm{L}}) \tag{47}$$

by the Kantorovich-Rubinshtein Theorem: see, e.g., [5, 11.8.2] or [24, VIII.4.5: Theorem 1].

For the case of $K = [0,1]$ with the metric $d(t',t'') = |t' - t''|$, an explicit formula for the KRV norm is

$$\|\mu\|_{\mathrm{KRV}} = \int_0^1 |\mu[0,t]|\, \mathrm{d}t \tag{48}$$

for $\mu \in \mathcal{M}^{\mathrm{N}}[0,1]$: see, e.g., [5, Problem 11.8.1].[29] Also, $\|p\|_{\mathrm{L}} = \operatorname{ess\,sup}_{[0,1]} |\dot{p}|$.

[28] On $\mathcal{M}_+[0,1]$, the $\|\cdot\|$-topology, $\mathrm{w}(\mathcal{M}, \mathcal{C})$ and $\mathrm{w}(\mathcal{M}, \mathrm{Lip})$ are equivalent to one another.

[29] This defines a distance between two measures, μ' and μ'', of equal mass as $\|\mu'' - \mu'\|_{\mathrm{KRV}}$.

To derive (45) from (47), represent $\mathcal{M}[0,1]$ as the direct sum of $\mathcal{M}^{\mathrm{N}}[0,1]$ and $\mathrm{span}(\varepsilon_1)$, the one-dimensional space of point measures at $t_0 = 1$. Then the norm (46) on $\mathcal{M}[0,1]$ is the direct sum of $\|\cdot\|_{\mathrm{KRV}}$ and the usual norm $|\cdot|$ of the real line: from (48), and from the fact that the c.d.f. of ε_1 is zero on $[0,1)$,[30]

$$\begin{aligned}\|\mu - \mu[0,1]\,\varepsilon_1\|_{\mathrm{KRV}} + |\mu[0,1]| &= \|(\mu - \mu[0,1]\,\varepsilon_1)[0,\cdot]\|_{L^1} + |\mu[0,1]| \\ &= \|\mu[0,\cdot]\|_{L^1[0,1]} + |\mu[0,1]| =: \|\mu\|.\end{aligned}$$

It follows that

$$(\mathcal{M}[0,1], \|\cdot\|)^* = \mathrm{Lip}_0[0,1] \oplus \mathbb{R} \simeq \mathrm{Lip}[0,1]$$

with Lip mapped to $\mathrm{Lip}_0 \oplus \mathbb{R}$ by $p \mapsto (p - p(1), p(1))$. It also follows that the dual norm on $\mathrm{Lip}[0,1]$ is the maximum of $\|\cdot\|^*_{\mathrm{KRV}}$ and $|\cdot|$, i.e.,

$$\|p\|^* = \max\{\|\dot{p}\|_\infty, |p(1)|\}.$$

11 Conclusions

Equilibrium pricing of a continuous-time flow such as electricity is likely to require a continuously varying price to eliminate the demand jumps caused by sudden switches between different price rates. Price continuity is also useful for other reasons, notably in rental valuation of storage plants. With cross-price independent demand and supply curves, an equilibrium price varies continuously if the curves do. A more general result is based on ideas from the Hardy-Littlewood-Pólya theory of rearrangements. In particular, symmetry of the production cost implies "similarity of arrangement" of price and output trajectories, which can be used to prove price continuity. But it is important that the assumption can be weakened for use with non-symmetric costs (such as the reservoir cost in energy storage), and that it can be adapted for use with preferences also. This yields what we believe to be the first applicable price-continuity result for competitive equilibrium in the Lebesgue commodity and price spaces $L^\infty(T)$ and $L^1(T)$, i.e., the spaces of bounded and of integrable functions on a (topological) measure space of commodity characteristics.

A Proofs

Proof of Lemma 8. This follows from, e.g., Definition 3 and the inequalities

$$\operatorname*{ess\,sup}_{A'}(y+z) \le \operatorname*{ess\,sup}_{A'} y + \operatorname*{ess\,sup}_{A'} z \tag{49}$$

$$\operatorname*{ess\,inf}_{A''} y + \operatorname*{ess\,inf}_{A''} z \le \operatorname*{ess\,inf}_{A''}(y+z) \tag{50}$$

[30] This is why the choice of $t_0 = 1$ is convenient.

for every y and z. ■

Proof of Lemma 9. It suffices to show that $\langle p \,|\, y\rangle \geq \langle p_{\downarrow} \,|\, y_{\downarrow}\rangle$, which is the reverse of the Hardy-Littlewood-Pólya Inequality, and then to apply Day's characterisation of the case of equality [2, 5.2, pp. 937–938].[31] To this end, since σ is nonatomic, take any $\rho \in \mathcal{R}_p$ (i.e., $p = p_{\downarrow} \circ \rho$). Since S is symmetric, $y_{\downarrow} \circ \rho \in S$; so

$$\langle p \,|\, y\rangle \geq \langle p \,|\, y_{\downarrow} \circ \rho\rangle = \langle p_{\downarrow} \circ \rho \,|\, y_{\downarrow} \circ \rho\rangle = \langle p_{\downarrow} \,|\, y_{\downarrow}\rangle$$

(since ρ is measure-preserving). So $\langle p \,|\, y\rangle = \langle p_{\downarrow} \,|\, y_{\downarrow}\rangle$, and Day's result [2, 5.2] shows that $\mathcal{R}(p) \cap \mathcal{R}(y) \neq \emptyset$. ■

Comment: In the Proof of Lemma 9, ρ is a pattern of p but not of y, in general. If, however, p has no plateau, then it has a single pattern, and so ρ must be the common pattern of p and y.

Proof of Example 12. This is proved in the same way as Example 13, with Corollary 43 applied this time to M equal to a negative scalar multiple of $\partial c/\partial \mathsf{y}$. ■

Proof of Example 13. Take any $t \in T$, $\epsilon > 0$, $p \in L^1(T)$ and y that maximises $\langle p \,|\, \cdot\rangle$ on S, i.e., $z := -y$ minimises $\langle p \,|\, \cdot\rangle$ on $-S = \{x : F(x) \geq \zeta\}$. There exists a scalar $\mu \geq 0$ such that[32]

$$p(\tau) = \mu \frac{\partial f}{\partial \mathsf{z}}(\tau, z(\tau)) \quad \text{for } \sigma\text{-almost every } \tau \in T \text{ with } z(\tau) > 0 \tag{51}$$

$$p(\tau) \geq \mu \frac{\partial f}{\partial \mathsf{z}}(\tau, z(\tau)) \quad \text{for } \sigma\text{-almost every } \tau \in T \text{ with } z(\tau) = 0. \tag{52}$$

If $\mu > 0$ then, by Corollary 43 applied to $M = \mu \partial f/\partial \mathsf{z}$,[33] there exists an $H \in \mathcal{N}(t)$ such that: for every t' and t'' in H and every z' and z'' in $[\operatorname{EssInf}(z), \operatorname{EssSup}(z)] \subset \mathbb{R}_+$

$$\mathsf{z}' < \mathsf{z}'' \Rightarrow \frac{\partial f}{\partial \mathsf{z}}(t', \mathsf{z}') \geq \frac{\partial f}{\partial \mathsf{z}}(t', \mathsf{z}'') \geq \frac{\partial f}{\partial \mathsf{z}}(t'', \mathsf{z}'') - \frac{\epsilon}{\mu}.$$

From this and (51)–(52), for a.e. t' and t'' in H,

$$z(t') < z(t'') \Rightarrow p(t') \geq \mu \frac{\partial f}{\partial \mathsf{z}}(t', z(t')) \geq \mu \frac{\partial f}{\partial \mathsf{z}}(t'', z(t'')) - \epsilon = p(t'') - \epsilon.$$

In other words, for a.e. t' and t'' in H,

$$\epsilon + p(t') < p(t'') \Rightarrow z(t') \geq z(t'')$$

which proves (12), since $y = -z$.

[31] The inequality is: $\langle p \,|\, y\rangle \leq \langle p_{\downarrow} \,|\, y_{\downarrow}\rangle$, with equality if and *only* if p and y are similarly arranged. This is a special case of [2, 5.2], for $\varphi(x_1, x_2) = x_1 x_2$.

[32] This is where Slater's Condition is used.

[33] Here it suffices to use the partial continuity of $\partial c/\partial \mathsf{y}$ in t, which holds uniformly in y.

Finally, if $\mu = 0$ then $p(t'') > p(t')$ implies that $z(t'') = 0 \le z(t')$, as required. ■

Proof of Remark 15. For every $N \in \mathcal{N}(t_\ddagger)$

$$\operatorname*{ess\,inf}_N p \le \underline{p}(t_\ddagger) \quad \text{and} \quad \overline{p}(t_\ddagger) \le \operatorname*{ess\,sup}_N p.$$

Case 1: If $\overline{p}(t_\ddagger) = +\infty$ then $\operatorname{ess\,sup}_N p = +\infty$; so for every α there is an $A'' \subset N$ with $p \ge \operatorname{ess\,inf}_N p + 1 + \alpha$ almost everywhere on A''. There is also an $A' \subset N$ with $p \le \operatorname{ess\,inf}_N p + 1$ a.e. on A'; hence (17). When $\underline{p}(t_\ddagger) = -\infty$, the argument is similar.

Case 2: If $-\infty < \underline{p}(t_\ddagger) < \overline{p}(t_\ddagger) < +\infty$, then $\operatorname{ess\,inf}_{N_0} p$ and $\operatorname{ess\,sup}_{N_0} p$ are finite for some $N_0 \in \mathcal{N}(t_\ddagger)$. For every $N \in \mathcal{N}(t_\ddagger)$ and $\epsilon > 0$ there exist A', $A'' \subset N \cap N_0$ such that

$$p \le \operatorname*{ess\,inf}_{N \cap N_0} p + \frac{\epsilon}{2} \quad \text{a.e. on } A' \quad \text{and} \quad p \ge \operatorname*{ess\,sup}_{N \cap N_0} p - \frac{\epsilon}{2} \quad \text{a.e. on } A''.$$

Used with an $\epsilon \le \overline{p}(t_\ddagger) - \underline{p}(t_\ddagger) - \alpha$, this gives

$$\alpha \le \overline{p}(t_\ddagger) - \underline{p}(t_\ddagger) - \epsilon \le \operatorname*{ess\,sup}_{N \cap N_0} p - \operatorname*{ess\,inf}_{N \cap N_0} p - \epsilon \le \operatorname*{ess\,inf}_{A''} p - \operatorname*{ess\,sup}_{A'} p$$

as required. ■

Proof of Corollary 16. Given a sub-symmetric $S \subset L^\infty(T)$ and any p, $t_\ddagger$ and y as in Definition 14, fix any $\alpha > 0$ that is less than $\overline{p}(t_\ddagger) - \underline{p}(t_\ddagger)$ if $\overline{p}(t_\ddagger)$ and $\underline{p}(t_\ddagger)$ are finite. Fix any $\epsilon < \alpha$, and choose an $H \in \mathcal{N}(t_\ddagger)$ as in Definition 11. For every $N \in \mathcal{N}(t_\ddagger)$, apply Remark 15 (to $N \cap H$) to choose nonnull subsets, A' and A'', of $N \cap H$ with

$$\epsilon + \operatorname*{ess\,sup}_{A'} p < \alpha + \operatorname*{ess\,sup}_{A'} p \le \operatorname*{ess\,inf}_{A''} p.$$

Since $A' \cup A'' \subset H$, it follows that $\operatorname{ess\,sup}_{A'} y \le \operatorname{ess\,inf}_{A''} y$, by (11). ■

Proof of Lemma 18. Given a symmetric $S \subset L^\infty(T)$ and any p and y maximising $\langle p \,|\, \cdot \rangle$ on S, fix any $\delta > 0$ and denote the set of δ-near-peaks of y by

$$P_\delta(y) := \{t \in T : y(t) \ge \operatorname{EssSup}(y) - \delta\}.$$

Take any two numbers p'' and p' with $\mathsf{p}'' > \mathsf{p}' > \operatorname{ess\,inf}_{P_\delta(y)} p$, and define the sets

$$A' := \{t \in P_\delta : p(t) \le \mathsf{p}'\} \text{ and } A'' := \{t \in T : p(t) \ge \mathsf{p}''\}.$$

Then $\operatorname{ess\,inf}_{A''} p \ge \mathsf{p}'' > \mathsf{p}' \ge \operatorname{ess\,sup}_{A'} p$, which implies (by Lemma 9 and Definition 3) that

$$\operatorname*{ess\,inf}_{A''} y \ge \operatorname*{ess\,sup}_{A'} y \ge \operatorname*{ess\,inf}_{A'} y \ge \operatorname{EssSup}(y) - \delta$$

(the penultimate inequality holds because A' is nonnull). This means that (22) holds for $\mathsf{p} = \mathsf{p}''$ (or larger). ■

Proof of Example 19. As in the Proof of Example 13, take a $z := -y$ minimises $\langle p \,|\, \cdot \rangle$ on $-S = \{z : F(z) \geq \zeta\}$ and a $\mu \in \mathbb{R}_+$ satisfying (51)–(52). Set $\mathsf{p} := \mu \sup_{t \in T} (\partial f / \partial \mathsf{z})(t, 0)$. Then $\mathsf{p} \geq \mu (\partial f / \partial \mathsf{z})(t, \mathsf{z})$ for every $t \in T$ and $\mathsf{z} \in \mathbb{R}_+$, and so

$$p(t) > \mathsf{p} \Rightarrow z(t) = 0 \leq \operatorname{EssInf}(z)$$

which means that (22) holds even for $\delta = 0$. (To prove it only for $\delta > 0$, which is all that is required, it would suffice to assume $\sup_{t \in T} (\partial f / \partial \mathsf{z})(t, 0) < +\infty$ for every $\mathsf{z} > 0$.) ■

Proof of Lemma 20. This uses parts of the Hardy-Littlewood-Pólya Theorem, which characterises the majorisation order $\prec_{\mathrm{HLP}}$, abbreviated to $\prec$ (defined below).

Take any $q \in \mathbb{R}^G$, $y \in \mathbb{Y}(q)$ and any $z \in L^0$ with the same distribution as y, i.e., $\sigma \circ z^{-1} = \sigma \circ y^{-1}$ (so $z \in L^\infty$). Since y and z have the same distribution (w.r.t. σ), there is a σ-doubly stochastic (d.s.) linear operator[34] D: $L^\infty \to L^\infty$ with $z = Dy$: see [31, Theorem 1]. Use (23) to choose $(y', q') \in \mathbb{Y}'$ and $(y'', q'') \in \mathbb{Y}''$ with $y = y' + y''$ and $q = q' + q''$. Since D is d.s., $Dy' \prec y' \in \mathbb{Y}'(q')$: see, e.g., [3, 4.9] or [31, Theorem 3].

It follows that $Dy' \in \mathbb{Y}'(q')$. This step uses a characterisation of symmetry in terms of $\prec$, which is the partial nonstrict preorder (a reflexive and transitive binary relation) defined on $L^1(T)$ by: $w \prec x$ if and only if $\int_0^\tau w_\downarrow(t)\,\mathrm{d}t \leq \int_0^\tau x_\downarrow(t)\,\mathrm{d}t$ for every $\tau \in [0, \sigma(T)]$, with equality when $\tau = \sigma(T)$.[35] (For $x_\downarrow$, see Definition 5.) Denote the set of all functions on T majorised by x by

$$\operatorname{maj}(x) := \{w : w \prec x\}$$

and denote the set of those functions equidistributed (a.k.a. equimeasurable) to x by

$$\operatorname{eqd}(x) := \left\{w : \sigma \circ w^{-1} = \sigma \circ x^{-1}\right\}.$$

If $x \in L^\infty(T)$ and σ is nonatomic, then $\operatorname{maj}(x)$ is convex and $\mathrm{w}(L^\infty, L^1)$-compact, and it is the $\mathrm{w}(L^\infty, L^1)$-closed convex hull of $\operatorname{eqd}(x)$: see, e.g., [3, 5.2].[36] So a closed convex S is symmetric if and *only* if the conditions $w \prec x \in S$ imply that $w \in S$. Applied to Dy', y' and $\mathbb{Y}'(q')$ in place of w, x and S, this shows that $Dy' \in \mathbb{Y}'(q')$.

Similarly, $Dy'' \in \mathbb{Y}''(q'')$. Hence $z = Dy' + Dy'' \in \mathbb{Y}(q)$. ■

Comment: For our purposes, the useful implication of symmetry is the similarity of arrangement for p and $y \in \operatorname{argmax}_{\mathbb{Y}(q)} \langle p \,|\, \cdot \rangle$, by Lemma 9. Preservation of *this* property, in summation, is simpler to prove than preservation of symmetry: if y maximises $\langle p \,|\, \cdot \rangle$ on $\mathbb{Y}(q)$, use (23) to decompose it as in (24)–(25). Then, by (26) and the assumed property of $\mathbb{Y}'(q')$ and $\mathbb{Y}''(q'')$, both y' and y'' are arranged similarly to p; and it follows that so

[34] A σ-d.s. operator is also known as a Markov operator with σ as an invariant measure.

[35] Roughly speaking, $w \prec x$ means that the distribution of w (w.r.t. σ) is "more concentrated about the average" than the distribution of x.

[36] A stronger result is that $\operatorname{eqd}(x)$ is the set of all the extreme points of $\operatorname{maj}(x)$: see [32, p. 1026]. A similar result holds for weak majorisation [11].

is $y = y' + y''$ (Remark 8). A similar argument, spelt out next, applies to the weaker conditions.

Proof of Lemma 21. For all three parts of the lemma, given any $q \in \mathbb{R}^G$, any $p \in L^1(T)$ and y that maximises $\langle p \,|\, \cdot \rangle$ on $\mathbb{Y}(q)$, use (23) to choose $(y', q') \in \mathbb{Y}'$ and $(y'', q'') \in \mathbb{Y}''$ with $y = y' + y''$ and $q = q' + q''$, as in (24)–(25). This ensures (26). From here on the proof depends on the part.

For Part 1, given also any $t \in T$ and $\epsilon > 0$, use (26) and the sub-symmetry of both $\mathbb{Y}'(q')$ and $\mathbb{Y}''(q'')$ to take $H' \in \mathcal{N}(t)$ and $H'' \in \mathcal{N}(t)$ as in Definition 11 with y' or y'' in place of y. Set $H := H' \cap H''$; then for σ-a.e. t_1 and t_2 in H, if $\epsilon + p(t_1) < p(t_2)$ then both $y'(t_1) \leq y'(t_2)$ and $y''(t_1) \leq y''(t_2)$, and so $y(t_1) \leq y(t_2)$ by adding up. This proves Part 1.

For Part 2, one extends the Proof of Corollary 16 by combining it with the decomposition of y. Given also any $t_\ddagger \in T \setminus \operatorname{dom}\operatorname{ess} p$, use (26) and the quasi-symmetry of $\mathbb{Y}'(q')$ to take an $\alpha > 0$ as in Definition 14 (with y' in place of y). Fix any positive $\epsilon < \alpha$, and use (26) and the sub-symmetry of $\mathbb{Y}''(q'')$, to choose an $H \in \mathcal{N}(t_\ddagger)$ as in Definition 11 (with y'' in place of y). For every $N \in \mathcal{N}(t_\ddagger)$, by the quasi-symmetry of $\mathbb{Y}'(q')$, there exist nonnull subsets, A_1 and A_2, of $N \cap H$ with

$$\operatorname*{ess\,sup}_{A_1} y' \leq \operatorname*{ess\,inf}_{A_2} y' \tag{53}$$

$$\epsilon + \operatorname*{ess\,sup}_{A_1} p < \alpha + \operatorname*{ess\,sup}_{A_1} p \leq \operatorname*{ess\,inf}_{A_2} p. \tag{54}$$

Since $A_1 \cup A_2 \subset H$, it follows—from (54) and (11) applied to $y'' \in \mathbb{Y}''(q'')$ in place of $y \in S$—that

$$\operatorname*{ess\,sup}_{A_1} y'' \leq \operatorname*{ess\,inf}_{A_2} y''.$$

Adding this to (53) and applying (49)–(50) completes the argument.

For Part 3, given also any $\delta > 0$, use (26) and the pseudo-symmetry of both $\mathbb{Y}'(q')$ and $\mathbb{Y}''(q'')$ to take p'' and p' such that, for a.e. t,

$$p(t) > \mathsf{p}' \Rightarrow y'(t) \geq \operatorname{EssSup}(y') - \frac{\delta}{2} \quad \text{and} \quad p(t) > \mathsf{p}'' \Rightarrow y''(t) \geq \operatorname{EssSup}(y'') - \frac{\delta}{2}$$

and set $\mathsf{p} := \max\{\mathsf{p}', \mathsf{p}''\}$. Then, for a.e. t,

$$p(t) > \mathsf{p} \Rightarrow y(t) = y'(t) + y''(t) \geq \operatorname{EssSup}(y') + \operatorname{EssSup}(y'') - \delta \geq \operatorname{EssSup}(y) - \delta$$

as required. ■

Proof of Example 23. Take any $t \in T$, $\epsilon > 0$ and $p \in L^1(T)$ such that x minimises $\langle p \,|\, \cdot \rangle$ on S (so $p \geq 0$). If $p = 0$, there is nothing to prove. If $p \neq 0$, then (since

EssInf $(\widetilde{x}) > 0$) there is a (unique) scalar $\widetilde{\lambda} > 0$ such that, for a.e. $t \in T$,[37]

$$\widetilde{\lambda} p(t) = \frac{\partial u}{\partial \mathsf{x}}(t, \widetilde{x}(t)). \tag{55}$$

Since EssInf $(\widetilde{x}) > 0$, Corollary 43 applies to $M = \partial u_h / \partial \mathsf{x}$ with $\underline{\mathsf{x}} :=$ EssInf $(\widetilde{x})$ and $\overline{\mathsf{x}}$:= EssSup $(\widetilde{x})$; so for every $t \in T$ and $\epsilon > 0$ there exist a number $\delta > 0$ and an $H \in \mathcal{N}(t)$ such that: for each h, every t' and t'' in H and every x' and x'' in $[\underline{\mathsf{x}}, \overline{\mathsf{x}}] \subset \mathbb{R}_{++}$

$$\mathsf{x}' < \mathsf{x}'' + \delta \Rightarrow \frac{\partial u}{\partial \mathsf{x}}(t', \mathsf{x}') > \frac{\partial u}{\partial \mathsf{x}}(t'', \mathsf{x}'') - \widetilde{\lambda}\epsilon.$$

From this and (55), for a.e. t' and t'' in H,

$$\widetilde{x}(t') < \widetilde{x}(t'') + \delta \Rightarrow p(t') = \frac{1}{\widetilde{\lambda}} \frac{\partial u_h}{\partial \mathsf{x}}(t', \widetilde{x}(t')) > \frac{1}{\widetilde{\lambda}} \frac{\partial u}{\partial \mathsf{x}}(t'', \widetilde{x}(t'')) - \epsilon = p(t'') - \epsilon.$$

In other words, for a.e. t' and t'' in H,

$$\epsilon + p(t') \le p(t'') \Rightarrow \widetilde{x}(t') \ge \widetilde{x}(t'') + \delta$$

which proves (28). ■

Proof of Theorem 26. Recall that T has a countable base of open sets which are σ-nonnull. So, by Corollary 41, it suffices to show that $\operatorname{ess} p^\star$ exists *everywhere* on T. Suppose contrarily that there is a $t \in T \setminus \operatorname{dom} \operatorname{ess} p^\star$. Since $y^\star$ maximises $\langle p^\star \mid \cdot \rangle$ on $Y(q^\star)$, use the quasi-symmetry of $Y(q^\star)$ to choose a number $\alpha > 0$ as in Definition 14. Fix also any positive $\epsilon \le \alpha$.[38] Since $x_h^\star$ minimises $\langle p^\star \mid \cdot \rangle$ on $S(x_h^\star, m_h^\star, \preccurlyeq_h)$,[39] use the strong sub-symmetry of this set to choose a number $\delta > 0$ and an $H \in \mathcal{N}(t)$ such that, for each h and for σ-almost every t' and t'' in H,

$$\epsilon + p^\star(t') < p^\star(t'') \Rightarrow x_h^\star(t') \ge x_h^\star(t'') + \delta. \tag{56}$$

Fix any positive number $\beta \le \delta$. By the continuity of x_h^{En}, there is an $I \in \mathcal{N}(t)$ such that for a.e. t' and t'' in I, and for each h,

$$\left| x_h^{\mathrm{En}}(t') - x_h^{\mathrm{En}}(t'') \right| < \beta \le \delta. \tag{57}$$

[37] To prove (55), use the formula for the normal cone to a superlevel set of a continuous concave function: see, e.g., [21, 4.3: Proposition 2]. (Slater's Condition holds here because u is increasing in x; and U is norm-continuous at $\widetilde{x}$ because EssInf $(\widetilde{x}) > 0$.) This shows that $p \in \lambda^{-1} \partial U(\widetilde{x})$ for some $\lambda > 0$; and the formula for ∂U—given in [29, Corollary 2C] and [21, 8.3: Theorem 3]—completes the argument. (Here ∂U reduces to ∇U.) It also follows that $p \gg 0$. Alternatively, the FOC for an expenditure minimum can be deduced from the FOC for a utility maximum, i.e., from Lemma 27 with $G = 0$ and Remark 28 (which applies here because $\langle p \mid \widetilde{x} \rangle > 0$).

[38] The value $\epsilon = \alpha$ will do, but a smaller ϵ dispels any impression that the argument "hangs by a thread", in the sense of relying on the contradiction between a strict inequality $a < b$ and the reverse inequality $a \ge b$. The same goes for the choice of a $\beta \le \delta$.

[39] With $\preccurlyeq_h$ locally nonsatiated, utility maximisation implies expenditure minimisation: see Remark 28.

By the quasi-symmetry of $Y(q^\star)$—i.e., by (17)–(18) applied to $N = H \cap I$—there exist σ-nonnull sets $A' \subset H \cap I$ and $A'' \subset H \cap I$ with

$$\operatorname*{ess\,sup}_{A'} y^\star \leq \operatorname*{ess\,inf}_{A''} y^\star \tag{58}$$

$$\alpha + \operatorname*{ess\,sup}_{A'} p^\star \leq \operatorname*{ess\,inf}_{A''} p^\star. \tag{59}$$

From this and (56) it follows that, for each h,

$$x_h^\star(t') \geq x_h^\star(t'') + \delta \quad \text{for a.e. } t' \in A' \text{ and } t'' \in A'' \tag{60}$$

(i.e., for every $t' \in A' \setminus Z$ and every $t'' \in A'' \setminus Z$, where Z is some σ-null set). Since $y^\star = \sum_h \left(x_h^\star - x_h^{\mathrm{En}}\right)$, it follows from (57) and (60) that $y^\star(t') > y^\star(t'')$ for a.e. $t' \in A'$ and $t'' \in A''$; and *a fortiori* (since $\sigma(A') > 0$ and $\sigma(A'') > 0$)

$$\operatorname*{ess\,sup}_{A'} y^\star > \operatorname*{ess\,inf}_{A''} y^\star.$$

This contradicts (58) and thus proves that $\operatorname{dom} \operatorname{ess} p^\star = T$. ■

Proof of Lemma 27. One way to derive (36) is to use the FOC of smooth optimisation, after extending $u(t, \cdot, m)$, if possible, to a differentiable function on the whole of $\mathbb{R}$. A more general method, which applies also when u is nondifferentiable in x, is the Kuhn-Tucker Condition of convex optimisation. (This does not require a concave extension of $u(t, \cdot, m)$ beyond $\mathbb{R}_+$, so it applies also when $\partial u / \partial \mathsf{x} \nearrow +\infty$ as $\mathsf{x} \searrow 0$, as is assumed from (39) on.) For an $(\widetilde{x}, \widetilde{m}) \geq 0$ to be a maximum point it is necessary (and sufficient) that there exists a $(\lambda, \mu, \nu) \in \mathbb{R}_{++} \times L_+^{\infty *} \times \mathbb{R}_+^G$ such that, with $\widehat{\partial}$ denoting the superdifferential,[40]

$$(\lambda p - \mu, \lambda r - \nu) \in \widehat{\partial}_{x,m} \mathbb{U}(\widetilde{x}, \widetilde{m}) \tag{61}$$

$$\langle p \,|\, \widetilde{x} \rangle + r \cdot \widetilde{m} = M \tag{62}$$

$$\mu \text{ is concentrated on } \{t \in T : \widetilde{x}(t) \leq \epsilon\} \text{ for every } \epsilon > 0 \tag{63}$$

$$\nu \cdot \widetilde{m} = 0. \tag{64}$$

(See, e.g., [30, Examples 4', 4"], where [30, (8.12)] implies, through [30, Theorems 18 (a) and 17 (a)], "the strong form of the Kuhn-Tucker Theorem", i.e., the equivalence in [30, Corollary 15A], so that every primal optimum is "supported" by a dual optimum. And, with $M > 0$, the generalised Slater's Condition of [30, (8.12)] is verified at, e.g., the point $(x, m) = (\epsilon 1_T; \epsilon, \ldots, \epsilon)$: for a sufficiently small $\epsilon > 0$, it meets the budget constraint strictly, and it lies in the interior of $L_+^\infty(T) \times \mathbb{R}_+^G$ for the supremum norm. Finally, λ cannot be 0 because of nonsatiation, i.e., because u is increasing in x, and W is increasing in U.) Obviously (61) implies that

$$\lambda p - \mu \in \widehat{\partial}_x \mathbb{U}(\widetilde{x}, \widetilde{m}) = \frac{\partial W}{\partial U} \widehat{\partial}_x U(\widetilde{x}, \widetilde{m}). \tag{65}$$

[40] Since U is concave and W is concave and increasing, their composition $\mathbb{U}$ is also concave.

From this and the formula for $\widehat{\partial}_x U$ given in, e.g., [29, Corollary 2C] and [21, 8.3: Theorem 3] it follows that (with $\widetilde{m}$ suppressed from the notation)

$$(\lambda p - \mu)_{\mathrm{CA}}(t) = \lambda p - \mu_{\mathrm{CA}}(t) \in \delta \widehat{\partial}_{\mathsf{x}} u(t, \widetilde{x}(t)) \quad \text{for a.e. } t \in T \tag{66}$$

where $\delta := \partial W / \partial U > 0$, and μ_{CA} is the countably additive part of $\mu \in L^{\infty *}$ (in the Hewitt-Yosida decomposition, for which see, e.g., [1, Appendix I: (26)–(27)] or [6, III.7.8]). Since, for σ-a.e. t, the term $\mu_{\mathrm{CA}}(t) / \delta$ is nonnegative and actually vanishes unless $\widetilde{x}(t) = 0$, it can be absorbed into $\widehat{\partial}_{\mathsf{x}} u(t, 0) = [(\partial u / \partial \mathsf{x})(t, 0), +\infty)$, i.e., $\mu_{\mathrm{CA}}/\delta + \widehat{\partial}_{\mathsf{x}} u \subseteq \widehat{\partial}_{\mathsf{x}} u$. Thus (66) implies (35) with $\widetilde{\lambda} := \lambda/\delta$. When $\partial u / \partial \mathsf{x} = +\infty$ at $\mathsf{x} = 0$, it follows that $\widetilde{x} \gg 0$ because $\widehat{\partial}_{\mathsf{x}} u = \emptyset$ at $\mathsf{x} = 0$. When u is additionally differentiable w.r.t. x on $\mathbb{R}_{++}$, Condition (35) simplifies to (36), since $\widehat{\partial}_{\mathsf{x}} u(t, \widetilde{x}(t)) = \{(\partial u / \partial \mathsf{x})(t, \widetilde{x}(t))\}$. ■

Proof of Remark 29. By Lemma 27, $0 \ll \sum_h x_h^\star = y^\star$. ■

Proof of Proposition 30. The idea is that a sufficiently high price would depress the demand so much that it could not equal the supply, given that the output is then close to its peak by pseudo-symmetry.

Since $p^\star \geq 0$, it suffices to show that $p^\star$ is bounded from above. By (36), for each h there exists a $\lambda_h^* > 0$ such that, for a.e. $t \in T$,

$$\lambda_h^* p(t) = \frac{\partial u_h}{\partial \mathsf{x}}(t, x_h^*(t), m_h^*). \tag{67}$$

Fix any positive numbers y and (x_h) with

$$\sum_{h \in \mathrm{Ho}} \mathsf{x}_h < \mathsf{y} < \mathrm{EssSup}(y^\star)$$

and use (38) to define

$$\mathsf{p}' := \max_{h \in \mathrm{Ho}} \sup_{t \in T} \frac{1}{\lambda_h^*} \frac{\partial u_h}{\partial \mathsf{x}}(t, \mathsf{x}_h, m_h^\star) < +\infty.$$

By (67), with $m_h^\star$ suppressed from the notation henceforth,

$$x_h^\star(t) \geq \mathsf{x}_h \Rightarrow p^\star(t) = \frac{1}{\lambda_h^*} \frac{\partial u_h}{\partial \mathsf{x}}(t, x_h^\star(t)) \leq \frac{1}{\lambda_h^*} \frac{\partial u_h}{\partial \mathsf{x}}(t, \mathsf{x}_h) \leq \mathsf{p}'.$$

Since $x_h^{\mathrm{En}} \geq 0$, it follows that for a.e. t

$$p^\star(t) > \mathsf{p}' \Rightarrow y^\star(t) \leq \sum_h x_h^\star(t) < \sum_{h \in \mathrm{Ho}} \mathsf{x}_h < \mathsf{y} < \mathrm{EssSup}(y^\star). \tag{68}$$

Since $\mathsf{y} < \mathrm{EssSup}(y^\star)$ and $\mathbb{Y}(q^\star)$ is pseudo-symmetric, there exists a $\mathsf{p}'' \in \mathbb{R}$ such that, for a.e. t,

$$p(t) > \mathsf{p}'' \Rightarrow y(t) \geq \mathsf{y}. \tag{69}$$

Set $\mathsf{p} = \max\{\mathsf{p}', \mathsf{p}''\}$. Then

$$p^\star(t) \le \mathsf{p} \quad \text{for almost every } t \in T \tag{70}$$

because (68) and (69) contradict each other unless $\{t : p^\star(t) > \mathsf{p}\}$ is a null set. ■

Comments: When $\mathbb{Y}(q^\star)$ is symmetric, the Proof of Proposition 30 simplifies and strengthens:

1. The number p' itself is a bound on $p^\star$. This is because, in terms of the sets

   $$A' := \{t \in T : p^\star(t) > \mathsf{p}'\} \quad \text{and} \quad A'' := \{t \in T : y^\star(t) \ge \mathsf{y}\}$$

   the implication (68) means that $\operatorname{ess\,sup}_{A'} y^\star < \mathsf{y} \le \operatorname{ess\,inf}_{A''} y^\star$; and, since $y^\star$ and $p^\star$ are similarly arranged (Lemma 9), it follows that

   $$\operatorname*{ess\,sup}_{A'} p^\star \le \operatorname*{ess\,inf}_{A''} p^\star \le \operatorname*{ess\,sup}_{A''} p^\star \le \mathsf{p}'.$$

   (The penultimate inequality holds because A'' is σ-nonnull, whilst the last inequality holds by (68) again.) This shows that

   $$\operatorname*{ess\,sup}_{t:\, p^\star(t) > \mathsf{p}'} p^\star(t) \le \mathsf{p}'$$

   which cannot be—unless $p^\star \le \mathsf{p}'$ a.e. (in which case A' is null and the supremum on A' is $-\infty$).

2. In the one-consumer case with $x^{\mathrm{En}} = 0$, the number p' (which depends on x) becomes the exact upper bound on $p^\star$ as $\mathsf{x} \nearrow \operatorname{EssSup}(y^\star)$, since $\lambda^\star \mathsf{p}'$ decreases then to

   $$\inf_{\mathsf{x}:\, \mathsf{x} < \operatorname{Sup}(y^\star)} \sup_{t \in T} \frac{\partial u}{\partial \mathsf{x}}(t, \mathsf{x}) = \sup_{t \in T} \inf_{\mathsf{x}:\, \mathsf{x} < \operatorname{Sup}(y^\star)} \frac{\partial u}{\partial \mathsf{x}}(t, \mathsf{x}) = \sup_{t \in T} \frac{\partial u}{\partial \mathsf{x}}(t, \operatorname{EssSup}(y^\star)) = \lambda^\star \operatorname{EssSup}(p^\star).$$

Proof of Corollary 31. The idea is that a bounded price must mean a positive minimum consumption rate (since the marginal utility becomes infinite at zero).

By Proposition 30 and uniformity of the divergence in (39), there is a constant $\underline{\mathsf{x}}_h > 0$ with

$$\frac{\partial u_h}{\partial \mathsf{x}}(t, \underline{\mathsf{x}}_h) > \lambda_h^\star \operatorname{EssSup}(p^\star) \quad \text{for every } t \in T$$

(with $m_h^\star$ suppressed from the notation). It follows that

$$x_h^\star(t) > \underline{\mathsf{x}}_h \quad \text{for almost every } t \in T$$

since $x_h^\star(t) \le \underline{\mathsf{x}}_h$ would imply that $(\partial u_h / \partial \mathsf{x})(t, x_h^\star(t)) \ge (\partial u_h / \partial \mathsf{x})(t, \underline{\mathsf{x}}_h) > \lambda_h^\star p^\star(t)$, which contradicts (67). ■

Proof of Corollary 32. This follows from Corollary 31 and Example 23. ■

Proof of Theorem 33. This follows from Theorem 26, since its strong sub-symmetry assumption on preferences holds by Corollary 32. ■

Proof of Corollary 34. As is shown below, $x_h^\star$ has a variant

$$\breve{x}_h^\star \colon T \to [\underline{\mathsf{x}}_h, \overline{\mathsf{x}}_h] = [\mathrm{EssInf}\,(x_h^\star)\,, \mathrm{EssSup}\,(x_h^\star)] \subset \mathbb{R}_{++}$$

for which (67) holds *everywhere* on T, i.e.,

$$\lambda_h^* \operatorname{ess} p^\star (t) = \frac{\partial u_h}{\partial \mathsf{x}} (t, \breve{x}_h^\star (t)) \quad \text{for every } t \in T. \tag{71}$$

(To see this, start from any variant $\breve{x}_h^\star \colon T \to [\underline{\mathsf{x}}_h, \overline{\mathsf{x}}_h]$, for which (71) holds for t outside of some σ-null set Z. For any $t \in Z$ choose a sequence $t_n \in T \setminus Z$ with $t_n \to t$ as $n \to +\infty$, then choose a limit point x of the sequence $(\breve{x}_h^\star (t_n))$, and redefine $\breve{x}_h^\star (t)$ as x. Since (71) holds along the sequence t_n, it also holds at $t = \lim_n t_n$ because $\partial u_h / \partial \mathsf{x}$ and $\operatorname{ess} p^\star$ are continuous by (37) and Theorem 33.)

Since $u_h (t, \cdot)$ is *strictly* concave, (71) can be inverted to give $\breve{x}_h^\star$ as the composition

$$\breve{x}_h^\star (t) = \left(\frac{\partial u_h}{\partial \mathsf{x}} (t, \cdot) \right)^{-1} (\lambda_h^* \operatorname{ess} p^\star (t)) \quad \text{for every } t \in T. \tag{72}$$

By (37), Lemma 44 applies to $M = \partial u_h / \partial \mathsf{x}$ with $K = [\underline{\mathsf{x}}_h, \overline{\mathsf{x}}_h]$; this shows that the function

$$(t, \mathsf{r}) \mapsto \left(\frac{\partial u_h}{\partial \mathsf{x}} (t, \cdot) \right)^{-1} (\mathsf{r})$$

is continuous (in t and r jointly). Since $\operatorname{ess} p^\star$ is continuous on T by Theorem 33, it follows from (72) that so is $\breve{x}_h^\star$ (and also that $\breve{x}_h^\star = \operatorname{ess} x_h^\star$). ■

Proof of Lemma 35. Given a $p \in L^1 [0, T]$ and any $t_\ddagger \in [0, T] \setminus \operatorname{dom} \operatorname{ess} p$, take a storage policy y that maximises the operating profit $\langle p \,|\, \cdot \rangle$ on $\mathbb{Y}_{\mathrm{PS}} (-k_{\mathrm{PS}})$. If $k_{\mathrm{St}} = 0$ or $k_{\mathrm{Co}} = 0$, then $y = 0$ and there is nothing to prove (since (17) holds by Remark 15). So assume that $k_{\mathrm{St}} > 0$ and $k_{\mathrm{Co}} > 0$. The stock trajectory associated with y is

$$s (t) = - \int_0^t y (\tau) \, \mathrm{d}\tau + \max_{t \in [0, T]} \int_0^t y (\tau) \, \mathrm{d}\tau \tag{73}$$

(since the second summand is the initial stock required for $s (t)$ never to fall below 0). The sets of those times when the reservoir is empty or full or neither are

$$E := \{t \in [0, T] : s (t) = 0\} \tag{74}$$

$$F := \{t \in [0, T] : s (t) = k_{\mathrm{St}}\} \,. \tag{75}$$

As we show in [15] and [18], there exists a function ψ on $[0, T]$—which is the TOU marginal value of stock, dependent also on p and k of course—with the following properties:

1. ψ is of bounded variation on the interval $(0,T)$, with $\psi(0) := \psi(0+)$ and $\psi(T) := \psi(T-)$ by convention.

2. ψ rises only on F and falls only on E. In formal terms, the sets F and E support, respectively, the nonnegative and nonpositive parts of the signed Borel measure defined by
$$\mathrm{d}\psi\,[t',t''] := \psi(t''+) - \psi(t'-) \tag{76}$$
for $t' \le t''$, with $\psi(0-) = \psi(T+)$ set equal to any number between $\psi(0+)$ and $\psi(T-)$.[41] In symbols, $\operatorname{supp}(\mathrm{d}\psi)^+ \subseteq F$ and $\operatorname{supp}(\mathrm{d}\psi)^- \subseteq E$.

3. The optimum output is of the "bang-bang" type on the set $\{t : p(t) \ne \psi(t)\}$, i.e., for a.e. $t \in [0,T]$
$$y(t) = \begin{cases} k_{\mathrm{Co}} & \text{if } p(t) > \psi(t) \\ -k_{\mathrm{Co}} & \text{if } p(t) < \psi(t) \end{cases} . \tag{77}$$

For simplicity choose variants of p and y which satisfy (77) for every t. There are two cases, and each requires a different argument: "from prices to quantities" if ψ is continuous, and "from quantities to prices" if ψ is discontinuous.[42]

Case 1: If ψ is continuous at $t_\ddagger$, take any $\alpha > 0$ that is less than $\overline{p}(t_\ddagger) - \underline{p}(t_\ddagger)$ if $\overline{p}(t_\ddagger)$ and $\underline{p}(t_\ddagger)$ are finite. Fix any positive number $\beta \le \alpha$ and an $I \in \mathcal{N}(t_\ddagger)$ such that $|\psi(t) - \psi(t_\ddagger)| < \beta/2$ for every $t \in I$. By Remark 15, for every $N \in \mathcal{N}(t_\ddagger)$ there exist sets $A' \subset N \cap I$ and $A'' \subset N \cap I$, both of positive measure, with $\alpha + p(t') \le p(t'')$ for every $t' \in A'$ and $t'' \in A''$. Suppose that $y(t') > y(t'')$ for some $t' \in A'$ and $t'' \in A''$; then of course $y(t') > -k_{\mathrm{Co}}$ and $y(t'') < k_{\mathrm{Co}}$, so $p(t') \ge \psi(t')$ and $p(t'') \le \psi(t'')$. Hence the contradiction
$$\alpha \le p(t'') - p(t') \le \psi(t'') - \psi(t') < \beta \le \alpha$$
which shows that actually $y(t') \le y(t'')$ for every $t' \in A'$ and $t'' \in A''$. (This case includes the cases of $t_\ddagger = 0$ and $t_\ddagger = T$, since at the endpoints ψ is defined by continuity, which means one-sided continuity.)

Case 2: If ψ is discontinuous, at a $t_\ddagger \in (0,T)$, then $\psi(t_\ddagger-) \ne \psi(t_\ddagger+)$, so $t_\ddagger \in E \cup F$. Consider, e.g., the case of $\psi(t_\ddagger-) < \psi(t_\ddagger+)$, in which $t_\ddagger \in F$. Take any positive $\alpha < \psi(t_\ddagger+) - \psi(t_\ddagger-)$ and any positive $\epsilon \le \psi(t_\ddagger+) - \psi(t_\ddagger-) - \alpha$; fix an $I \in \mathcal{N}(t_\ddagger)$ such that $|\psi(t) - \psi(t_\ddagger-)| \le \epsilon/2$ for every $t \in I \cap (0,t_\ddagger)$ and $|\psi(t) - \psi(t_\ddagger+)| \le \epsilon/2$ for every $t \in I \cap (t_\ddagger, T)$. Being full at $t_\ddagger$, the reservoir cannot be being discharged just before $t_\ddagger$ or charged just after $t_\ddagger$. In formal terms, for every $N \in \mathcal{N}(t_\ddagger)$ it cannot be that $y > 0$ a.e. on $N \cap I \cap (0,t_\ddagger)$, i.e., $y \le 0$ on some nonnull set $A' \subseteq N \cap I \cap (0,t_\ddagger)$. *A fortiori* $p \le \psi$ on

[41] So $\mathrm{d}\psi$ has zero total mass on $[0,T]$, and $\mathrm{d}\psi\{0\}$ and $\mathrm{d}\psi\{T\}$ do not have opposite signs (in this context, 0 and T can also be viewed as a single point of a circle).

[42] Case 2 cannot arise if p is continuous [15], but here the continuity of p must not be assumed (since it is, of course, to be proved by using the present lemma).

A'. Similarly, on some nonnull set $A'' \subseteq N \cap I \cap (t_{\ddagger}, T)$ one has $y \geq 0$ and, hence, $p \geq \psi$ on A''. So, for every $t' \in A' \subseteq N$ and $t'' \in A'' \subseteq N$,

$$p(t') \leq \psi(t') \leq \psi(t_{\ddagger}-) + \frac{\epsilon}{2} \quad \text{and} \quad p(t'') \geq \psi(t'') \geq \psi(t_{\ddagger}+) - \frac{\epsilon}{2}$$

and therefore

$$p(t'') - p(t') \geq \psi(t_{\ddagger}+) - \psi(t_{\ddagger}-) - \epsilon \geq \alpha.$$

This completes the proof, since $y(t'') \geq 0 \geq y(t')$ by the very choice of A' and A''. (The case of $\psi(t_{\ddagger}-) > \psi(t_{\ddagger}+)$, in which $t_{\ddagger} \in E$, is handled in a similar way.) ■

Comment: In Case 2, the Proof of Lemma 35 shows also that, for $t_{\ddagger} \in F$,

$$0 \leq \psi(t_{\ddagger}+) - \psi(t_{\ddagger}-) \leq \overline{p}(t_{\ddagger}) - \underline{p}(t_{\ddagger}).$$

When the last inequality is strict, the choice of α for Case 2 can be improved: as in Case 1, any $\alpha < \overline{p}(t_{\ddagger}) - \underline{p}(t_{\ddagger})$ will do. For $\alpha > \psi(t_{\ddagger}+) - \psi(t_{\ddagger}-)$, this is shown as in Case 1.

Proof of Theorem 36. By [1, Theorem 1] there is an equilibrium price system with $p^{\star} \in L_{+}^{\infty *}$. And actually $p^{\star} \in L^{1}$, as we show in [19]. This proves Part 1.

For Part 2, we verify the assumptions of Theorem 26. Since each of the $\mathbb{Y}_{\theta}$'s has symmetric sections, so does their sum $\mathbb{Y}_{\mathrm{Th}}$ (Lemma 20). Since $\mathbb{Y}_{\mathrm{PS}}$ has quasi-symmetric sections (Lemma 35), and $\mathbb{Y}_{\mathrm{IU}}$ has sub-symmetric sections (by assumption), the total production set has quasi-symmetric sections (Lemma 21). For households, strong sub-symmetry at $x_h^{\star}$ is assumed. And $x_h^{\mathrm{En}} = 0 \in \mathcal{C}_{+}[0,T]$ trivially. ■

Proof of Lemma 37. The function ψ on $[0,T]$, introduced in the Proof of Lemma 35, is bounded (since it is of bounded variation). By (77), if $p(t) > \mathrm{Sup}(\psi)$ then $y(t) = k_{\mathrm{Co}} \geq \mathrm{EssSup}(y)$. So (22) holds even with $\delta = 0$ (and with $\mathsf{p} = \mathrm{Sup}(\psi)$). ■

Proof of Corollary 38. To show that $p^{\star}$ is continuous, we verify the assumptions of Theorem 33. As in the Proof of Theorem 36, the set $\mathbb{Y}_{\mathrm{Th}}$ has symmetric sections and $\mathbb{Y}_{\mathrm{PS}}$ has quasi-symmetric sections. Additionally $\mathbb{Y}_{\mathrm{IU}}$ has sub-symmetric sections (Example 13), so the total production set has quasi-symmetric sections (by Parts 1 and 2 of Lemma 21). Also, both $\mathbb{Y}_{\mathrm{PS}}$ and $\mathbb{Y}_{\mathrm{IU}}$ have pseudo-symmetric sections (Lemma 37 and Example 19); so the total production set has pseudo-symmetric sections (by Part 3 of Lemma 21). Finally, $x_h^{\mathrm{En}} = 0$ for each h by assumption (so $y_{\mathrm{Th}}^{\star} + y_{\mathrm{PS}}^{\star} - z^{\star}$, the electricity output net of industrial consumption, is strictly positive by Remark 29).

By assumption, u_h is increasing (in x), so $p^{\star} \gg 0$ by (36). Also, with u_h strictly concave (in x), $x_h^{\star}$ is continuous by Corollary 34. And $\mathrm{Min}(x_h^{\star}) > 0$ by Corollary 31. ■

B Lower and upper essential values

Assume that T is a (Hausdorff) topological space, and that σ is a measure on a sigma-algebra, $\mathcal{A}$, of subsets of T that contains the Borel sigma-algebra. Recall that $L^{0}(T, \sigma)$ is the vector space of all (σ-equivalence classes of) measurable $\mathbb{R}$-valued functions on T.

Definition 39 *For every $p \in L^0(T)$, the lower essential value of p at t is*

$$\underline{\mathrm{ess}}\, p(t) := \sup_{N \in \mathcal{N}(t)} \operatorname{ess\,inf}_N p \tag{78}$$

i.e., it is the supremum, on the neighbourhood system $\mathcal{N}(t)$, of the essential infimum of p in any neighbourhood, N, of t. The upper essential value of p at t is

$$\overline{\mathrm{ess}}\, p(t) := \inf_{N \in \mathcal{N}(t)} \operatorname{ess\,sup}_N p. \tag{79}$$

The notations are abbreviated to $\underline{p}$ and $\overline{p}$. Where $\underline{p}(t)$ and $\overline{p}(t)$ are equal and finite, their common value is the essential value of p at t, denoted by $\operatorname{ess} p(t)$. Its domain is

$$\operatorname{dom} \operatorname{ess} p = \left\{ t \in T : -\infty < \underline{p}(t) = \overline{p}(t) < +\infty \right\}.$$

Comments:

1. The essential values are literally functions (rather than equivalence classes).

2. For $p \in L^1$, equivalent definitions of $\underline{p}$, $\overline{p}$ and $\operatorname{ess} p$ are given in, e.g., [34, II.9: pp. 89–90].

3. $\operatorname{ess\,inf}_T p \leq \underline{p}(t)$ and $\overline{p}(t) \leq \operatorname{ess\,sup}_T p$, with equality at some $t \in T$ in each case. So unless $p \in L^\infty$, both $\underline{p}$ and $\overline{p}$ are extended real-valued functions, from T into $\mathbb{R} \cup \{\pm\infty\}$. But $\operatorname{ess} p$ is finite (on $\operatorname{dom} \operatorname{ess} p$).

4. The lower and upper essential *limits* are usually defined by means of *pierced* neighbourhoods, i.e.,

   $$\operatorname{ess} \liminf_{\tau \to t} p(\tau) := \sup_{N \in \mathcal{N}(t)} \operatorname{ess} \inf_{N \setminus \{t\}} p \quad \text{and} \quad \operatorname{ess} \limsup_{\tau \to t} p(\tau) := \inf_{N \in \mathcal{N}(t)} \operatorname{ess} \sup_{N \setminus \{t\}} p.$$

   The limits are identical to the values (78)–(79) if $\sigma\{t\} = 0$. This is the case in [4, IV.36–IV.37]—where $\sigma =$ meas, and T is an interval of $\mathbb{R}$. The one-sided, left or right, essential limits are then defined as well. These are used in the theory of stochastic processes to establish the continuity of sample paths.

5. If $\sigma\{t\} > 0$ (i.e., t is an atom for σ) then

   $$\underline{\mathrm{ess}}\, p(t) = \min\left\{ p(t), \operatorname{ess} \liminf_{\tau \to t} p(\tau) \right\} \text{ and } \overline{\mathrm{ess}}\, p(t) = \max\left\{ p(t), \operatorname{ess} \limsup_{\tau \to t} p(\tau) \right\}.$$

6. The essential limit is the limit for the essential topology [4, p. 105].

The following account of the key results on essential values combines those of [4, IV.37] and [34, II.9: pp. 90, 91, 94].

Lemma 40 *For every* $p \in L^0(T, \sigma)$*:*

1. *The lower value* $\underline{p}$ *is lower semicontinuous, and the upper value* $\overline{p}$ *is upper semi-continuous (each as a function from* T *into* $\mathbb{R} \cup \{\pm\infty\}$*).*
2. *If the topology of* T *has a countable open base, then* $\underline{p} \leq p \leq \overline{p}$ *almost everywhere on* T *(w.r.t.* σ*).*
3. *If every nonempty open subset of* T *is* σ*-nonnull, then* $\underline{p} \leq \overline{p}$ *everywhere on* T*.*

So when the essential value of p exists *everywhere* on T, it is "automatically" continuous, and it is a variant of p. (This is also stated in [28, Appendix: Lemma 5].)

Corollary 41 *Assume that* T *has a countable base of open sets that are* σ*-nonnull. If* $\operatorname{dom} \operatorname{ess} p = T$*, then* $\operatorname{ess} p \in \mathcal{C}(T)$ *and* $\operatorname{ess} p = p$ *almost everywhere on* T *(w.r.t.* σ*).*

C Uniform continuity and inverse continuity

For a jointly continuous map M on a product of two topological spaces, the implications of compactness of one or both spaces are spelt out (for use in the Proofs of Examples 12, 13 and 23).

Lemma 42 *Assume that* T *is a (Hausdorff) topological space,* K *is a metrisable compact with a metric* d_K*, and* d_R *is a metric on a set* R*. Then any continuous map* $M\colon T \times K \to R$ *is (jointly) continuous uniformly in the second variable; i.e., for every* $t \in T$ *and every number* $\epsilon > 0$ *there exist a neighbourhood* H *of* t *and a number* $\delta > 0$ *such that for every* t' *and* t'' *in* H *and for every* x' *and* x'' *in* K *with* $d_K(\mathsf{x}', \mathsf{x}'') < \delta$ *one has* $d_R(M(t', \mathsf{x}'), M(t'', \mathsf{x}'')) < \epsilon$*.*

Comment: If T is, like K, a compact metric space, then so is $T \times K$; in which case every continuous map M is uniformly continuous on $T \times K$ (i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that, for every t' and t'' in T and for every x' and x'' in K, if $d_T(t', t'') < \delta$ and $d_K(\mathsf{x}', \mathsf{x}'') < \delta$ then $d_R(M(t', \mathsf{x}'), M(t'', \mathsf{x}'')) < \epsilon$). This—joint continuity that is uniform in *both* variables—is obviously stronger than the property established in Lemma 42.

Corollary 43 *If a continuous function* $M\colon T \times [\underline{\mathsf{x}}, \overline{\mathsf{x}}] \to \mathbb{R}$ *is nonincreasing in the second variable (which ranges over a bounded closed interval of* $\mathbb{R}$*), then for every* $t \in T$ *and* $\epsilon > 0$ *there exist a neighbourhood* H *of* t *and a* $\delta > 0$ *such that for every* t' *and* t'' *in* H *and for every* x' *and* x'' *in* $[\underline{\mathsf{x}}, \overline{\mathsf{x}}]$*, if* $\mathsf{x}' < \mathsf{x}'' + \delta$ *then* $\epsilon + M(t', \mathsf{x}') > M(t'', \mathsf{x}'')$*.*

When the inverse of $M(t,\cdot)$ exists for each t, it is jointly continuous if the domain of M is compact. This is a special case of the result stated in, e.g., [33, 5.9.1].

Lemma 44 *Assume that both T and K are compact spaces, and $M\colon T\times K\rightarrow R$ is a (jointly) continuous map into another (Hausdorff) topological space. If for each $t\in T$ the map $M_t=M(t,\cdot)$ is invertible, then so is the map*

$$T\times K\ni(t,\mathsf{x})\mapsto(t,M(t,\mathsf{x}))\in T\times R. \tag{80}$$

The inverse is defined, on the compact range of the map (80), by

$$(t,\mathsf{r})\mapsto\left(t,M_t^{-1}(\mathsf{r})\right)\in T\times K \tag{81}$$

and it is also continuous. Hence the map $(t,\mathsf{r})\mapsto M_t^{-1}(\mathsf{r})$ is continuous.

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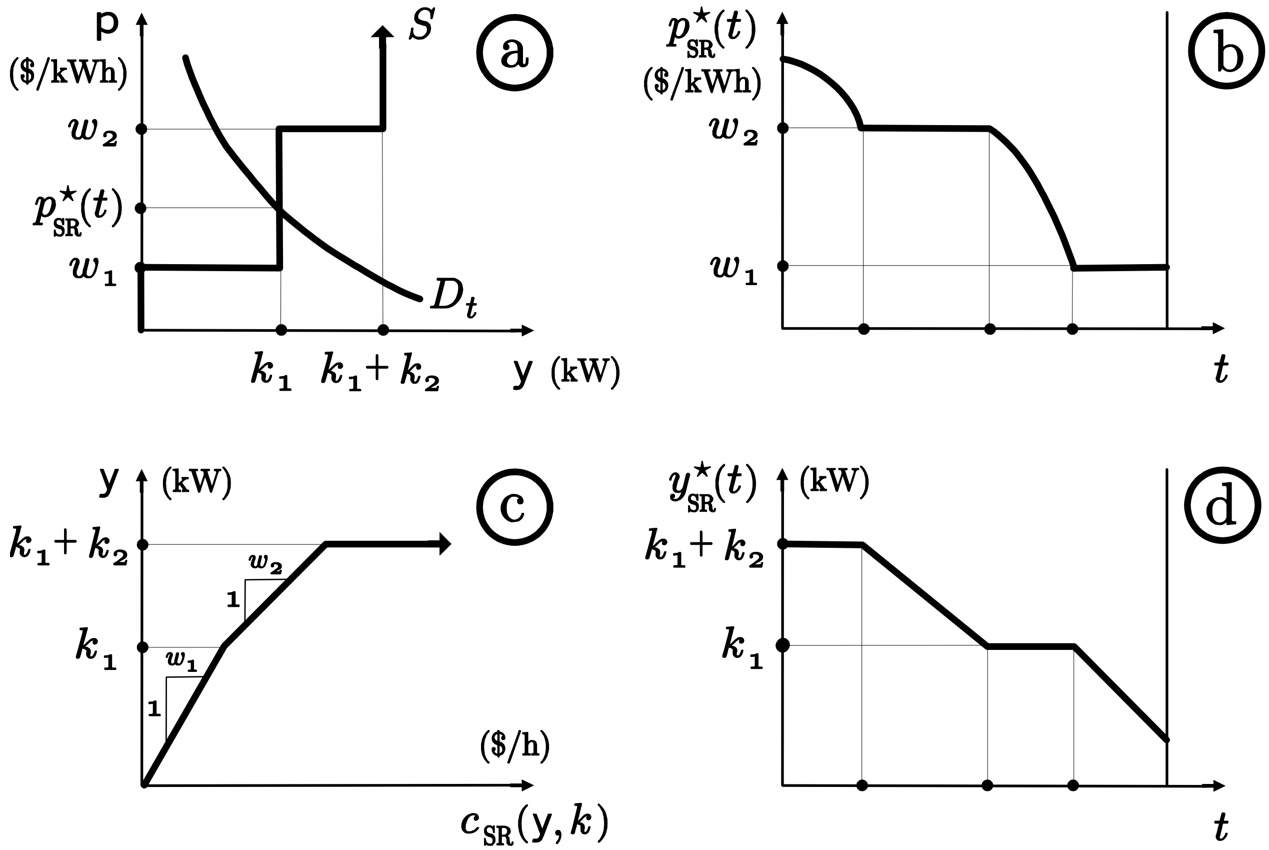

Figure 1: Short-run equilibrium of supply and (cross-price independent) demand for thermally generated electricity: (a) determination of the price and output for each instant t; (b) and (d) the trajectories of price and output; (c) the short-run cost curve (the integral of S w.r.t. y).

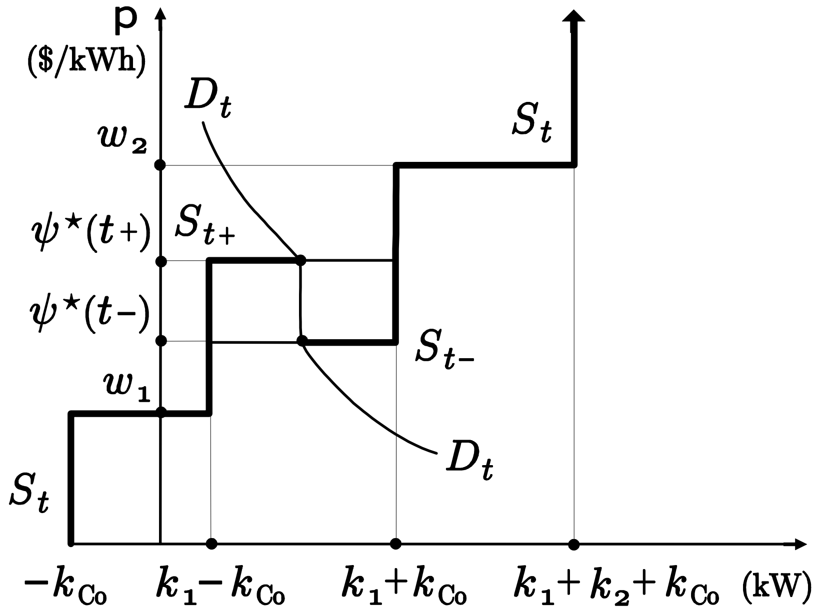

Figure 2: Proof of continuity of short-run equilibrium price for electricity supplied by a combined generation-and-storage system: the energy stock price $\psi^\star$ cannot be discontinuous if the (cross-price independent) demand curve D_t is strictly decreasing; and if $\psi^\star(t)$ is continuous in t, then so is the supply curve S_t and hence also $p^\star(t)$.