# Embeddings and other mappings of rooted trees into complete trees

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**Abstract**

Let $T^n$ be the complete binary tree of height $n$, with root $1_n$ as the maximum element. For $T$ a tree, define $A(n;T) = |\{S \subseteq T^n : 1_n \in S, S \cong T\}|$ and $C(n;T) = |\{S \subseteq T^n : S \cong T\}|$. We disprove a conjecture of Kubicki, Lehel and Morayne, which claims that $\frac{A(n;T_1)}{C(n;T_1)} \leq \frac{A(n;T_2)}{C(n;T_2)}$ for any fixed $n$ and arbitrary rooted trees $T_1 \subseteq T_2$. We show that $A(n;T)$ is of the form $\sum_{j=0}^{l} q_j(n)2^{jn}$ where $l$ is the number of leaves of $T$, and each $q_j$ is a polynomial. We provide an algorithm for calculating the two leading terms of $q_l$ for any tree $T$. We investigate the asymptotic behaviour of the ratio $A(n;T)/C(n;T)$ and give examples of classes of pairs of trees $T_1, T_2$ where it is possible to compare $A(n;T_1)/C(n;T_1)$ and $A(n;T_2)/C(n;T_2)$. By calculating these ratios for a particular class of pairs of trees, we show that the conjecture fails for these trees, for all sufficiently large $n$. Kubicki, Lehel and Morayne have proved the conjecture when $T_1, T_2$ are restricted to being binary trees. We also look at embeddings into other complete trees, and we show how the result can be viewed as one of many possible correlation inequalities for embeddings of binary trees. We also show that if we consider strict order-preserving maps of $T_1, T_2$ into $T^n$ (rather than embeddings) then the corresponding correlation inequalities for these maps also generalise to arbitrary trees.

## 1 Introduction

We disprove a conjecture of Kubicki, Lehel and Morayne first stated in [2] concerning embeddings of rooted trees into a complete binary tree. Here, we assume all trees to be rooted, with the root being the unique maximum element. The *complete binary tree* $T^n$ of height $n$ is a ranked poset with $n$ levels, having $2^{n-1}$ leaves, where every element that is not a leaf has 2 distinct lower covers. So there are $2^{n-i}$ elements in level $i$, $i = 1, \ldots, n$. The root (in level $n$) is labelled $1_n$. For example, Figure 1 shows the Hasse diagram of $T^5$.

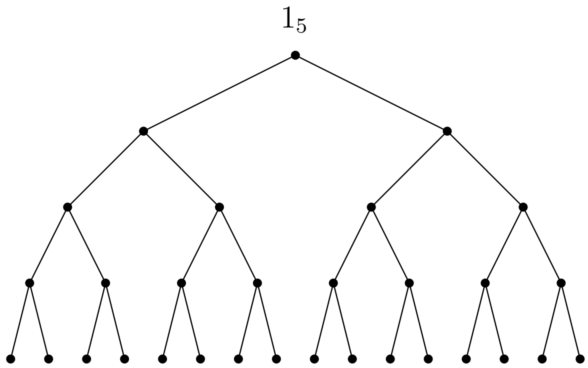



Figure 1: The complete binary tree, $T^5$

For $T$ a tree, define $A(n;T) = |\{S \subseteq T^n : 1_n \in S, S \cong T\}|$ and $B(n;T) = |\{S \subseteq T^n : 1_n \notin S, S \cong T\}|$. Define $C(n;T) = |\{S \subseteq T^n : S \cong T\}|$, so that $C(n;T) = A(n;T) + B(n;T)$. In [2], Kubicki, Lehel and Morayne proved that $\frac{A(n;T_1)}{B(n;T_1)} \leq \frac{A(n;T_2)}{B(n;T_2)}$ for any fixed $n$ and rooted binary trees $T_1, T_2$, such that $T_2$ contains a subposet isomorphic to $T_1$. They conjectured that the ratio $A/B$ also increases with $T$ for arbitrary trees. In this paper, we will use the ratio $\frac{A(n;T)}{C(n;T)}$ rather than $\frac{A(n;T)}{B(n;T)}$, but since $\frac{C(n;T)}{A(n;T)} = \frac{B(n;T)}{A(n;T)} + 1$ any statement about $\frac{A(n;T)}{C(n;T)}$ can be rewritten as an equivalent statement about $\frac{A(n;T)}{B(n;T)}$. So, the result above is equivalent to $\frac{A(n;T_1)}{C(n;T_1)} \leq \frac{A(n;T_2)}{C(n;T_2)}$ for any fixed $n$ and rooted binary trees $T_1, T_2$, such that $T_2$ contains a subposet isomorphic to $T_1$. The equivalent conjecture is that $A/C$ also increases with $T$ for arbitrary trees. This was proved for chains in [5] and for stars rooted at the centre in [4]. Informally, the conjecture claims that for larger trees there is a greater proportion of embeddings that map the root of the tree to $1_n$. This seems plausible; when constructing an embedding from a tree $T$ to $T^n$, the higher we choose to map the root of $T$, the more of $T^n$ there is to map the rest of the tree $T$ into. The intuition is that this extra space has more effect for a larger tree. Since a larger tree has more elements to embed, there should be relatively more embeddings that map the root to $1_n$. So, we expect the ratio $A/C$ to be larger for $T_2$ than for $T_1$. However, in Section 2 we show that the conjecture is false. For some trees $T_1 \subseteq T_2$ the extra elements in $T_2$ actually restrict the embeddings into $T^n$, so that this "extra space" gained by mapping the root to $1_n$ has less effect for $T_2$ than for $T_1$. It is the smaller tree that has relatively more embeddings mapping the root to $1_n$, and so the ratio $A/C$ is larger for $T_1$ than for $T_2$.

In fact, the conjecture fails even for ternary trees, as exhibited by the following example, where $T_1, T_2$ are as in Figure 2. This is a counterexample to the conjecture for small $n$ (less than 6) and also for all $n$ greater than 11. Intriguingly the inequality does hold for $n = 6, \ldots, 11$ for this pair of trees.

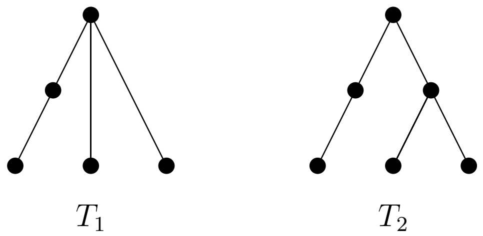


Figure 2: Counterexample to the conjecture of Kubicki, Lehel and Morayne

In [3], Kubicki, Lehel and Morayne proved an asymptotic version of the inequality, that $\lim_{n\to\infty} \frac{A(n;T_1)}{B(n;T_1)} \leq \lim_{n\to\infty} \frac{A(n;T_2)}{B(n;T_2)}$. They proved this by showing that $\lim_{n\to\infty} \frac{A(n;T)}{B(n;T)} = 2^{l(T)-1}-1$, where $l(T)$ is the number of leaves of $T$. Since $l(T_1) \leq l(T_2)$ the inequality follows. Working with $\frac{A(n;T)}{C(n;T)}$ the equivalent limit is $\lim_{n\to\infty} \frac{A(n;T)}{C(n;T)} = 1 - 1/2^{l(T)-1}$. In Section 4, we show that this limiting effect is misleading as regards the conjecture; we can find many examples $T_1 \subseteq T_2$ where $l(T_1) = l(T_2)$, so that the asymptotic inequality holds with equality, but for all sufficiently large $n$ we have $\frac{A(n;T_1)}{C(n;T_1)} > \frac{A(n;T_2)}{C(n;T_2)}$. Moreover, we show that for certain pairs of trees $T_1, T_2$ it is possible to show that $A(n;T_1)/C(n;T_1)$ is asymptotically larger than $A(n;T_2)/C(n;T_2)$, either by simply comparing the trees themselves, or by calculating the leading terms of $A(n;T)$ for each tree. Using this, we show that for a particular class of ternary trees $T_1 \subseteq T_2$, we have $\frac{A(n;T_1)}{C(n;T_1)} > \frac{A(n;T_2)}{C(n;T_2)}$ for arbitrarily large $n$. This seemingly destroys all hope of recovering a weaker-but-true statement from the conjecture; even when restricting $T_1, T_2$ to being ternary trees we can find counterexamples to the conjecture for arbitrarily large $n$.

In Section 5 we examine other generalisations to the binary case inequality. Let $T_p^n$ be the complete $p$-ary tree of height $n$, a ranked poset with $n$ levels, which has $p^{n-1}$ leaves and every element that is not a leaf has $p$ distinct lower covers. So, there are $p^{n-i}$ elements in level $i$, $i = 1, \ldots, n$. Define $A_p(n;T) = |\{S \subseteq T_p^n : 1_n \in S, S \cong T\}|$ and $C_p(n;T) = |\{S \subseteq T_p^n : 1_n \notin S, S \cong T\}|$ as the obvious analogues to the complete binary tree case. We prove that $\frac{A_p(n;T_1)}{C_p(n;T_1)} \leq \frac{A_p(n;T_2)}{C_p(n;T_2)}$ for any fixed $n$ and binary rooted trees $T_1 \subseteq T_2$. Moreover, we prove the result using the FKG-inequality, which places the result in the framework of correlation inequalities on distributive lattices. Using the FKG-inequality we can find many other correlation inequalities for embeddings of binary trees. We also show that if we look at strict order-preserving maps into $T_p^n$, rather than embeddings, the situation is simplified; here the corresponding correlation inequalities hold without any need for $T_1, T_2$ to be binary. An example of this is the result that if $T_1 \subseteq T_2$ are arbitrary trees, then the ratio of strict order-preserving maps that map the root of $T_1$ to $1_n$ to those that do not is smaller for $T_1$ than it is for $T_2$.

Let us introduce some notation. For a tree $T$, an embedding $\phi$ is a map from $T$ to $T^n$ such that $\phi(x) \geq \phi(y)$ in $T^n$ if and only if $x \geq y$ in $T$. That is, $T$ and $\phi(T)$ are isomorphic as labelled trees. For a tree $T$ with root $1_T$, define $A_T(n)$ to be the number of embeddings $\phi$ of $T$ into $T^n$ with $\phi(1_T) = 1_n$, and define $C_T(n)$ to be the total number of embeddings of $T$ into $T^n$.

As explained in [2], since $A(n;T)$ and $C(n;T)$ count the number of subposets of $T^n$ isomorphic to $T$ as *unlabelled* trees, whereas $A_T(n)$ and $C_T(n)$ count the number of subposets of $T^n$ isomorphic to $T$ as *labelled* trees, we have $A_T(n) = |G|A(n;T)$ and $C_T(n) = |G|C(n;T)$, where $G$ is the group of symmetries of the (unlabelled) tree $T$. Since the ratio $A/C$ is unaffected we can work with either labelled or unlabelled trees. We shall use labelled trees, and think of the labelled subposets of $T^n$ as embeddings.

# 2 Recurrence relations

We can use the regular structure of $T^n$ to find recurrence relations for $A_T(n)$ and $C_T(n)$. Let $t_1, t_2$ be the 2 lower covers of $1_n$ in $T^n$. Write $T_1^n$ for the set of all elements that are lower than or equal to $t_1$ in $T^n$, and similarly for $T_2^n$. So, $T_1^n$ and $T_2^n$ are both copies of $T^{n-1}$. For any embedding of a tree $T$ into $T^n$ the root $1_T$ of $T$ is either mapped to $1_n$, or mapped into $T_1^n$ or $T_2^n$. Counting these embeddings of $T$ into $T^n$ gives

$$C_T(n) - 2C_T(n-1) = A_T(n). \tag{1}$$

So, once we have calculated $A_T(n)$ we can solve a simple linear recurrence to find $C_T(n)$.

We now show that $A_T(n)$ also satisfies a linear recurrence relation. For any $x \in T$ we write $D[x]$ for the set of all elements in $T$ that are lower than or equal to $x$ in $T$. Let $T$ be a tree and suppose the root $1_T$ has $r$ lower covers $x_1, \ldots, x_r$. For any subset $L \subseteq [r]$ write $T_L$ for the tree formed by removing the subtrees $D[x_j]$ for all $j \in L^c$. (Here, $L^c = [r] \setminus L$.) Notice that $T_{\{j\}} \setminus \{1_T\} = D[x_j]$, $T_{[r]} = T$ and $T_\emptyset = \{1_T\}$.

We will count the embeddings of $T$ into $T^n$ by considering the possible places to map the elements $x_1, \ldots, x_r$. In particular we are interested in the partition of $\{x_1, \ldots, x_r\}$ defined by which of the two subtrees $T_1^n, T_2^n$ an element $x_i$ is mapped to.

Write $A^-_{T_L}(n)$ for the number of embeddings of $T_L$ into $T^n$ that map the root $1_T$ of $T_L$ to $1_n$ and map $x_j$ into $T_1^n$, for each $j \in L$. By the symmetry of $T^n$ this is the same as the number of embeddings of $T_L$ into $T^n$ that map $1_T$ to $1_n$ and map $x_j$ into $T_2^n$, for each $j \in L$.

For a fixed set $L \subseteq [r]$ we can count the number of embeddings $\phi$ of $T$ into $T^n$ with $\phi(x_i)$ in $T_1^n$ for all $i \in L$, and $\phi(x_i)$ in $T_2^n$ for all $i \in L^c$. Since the two trees $T_1^n$ and $T_2^n$ are below incomparable elements $t_1$ and $t_2$, we have that the number of such embeddings that also map $1_T$ to $1_n$ is exactly the product $A^-_{T_L}(n)A^-_{T_{L^c}}(n)$. So,

$$A_T(n) = \sum_{L \subseteq [r]} A^-_{T_L}(n)A^-_{T_{L^c}}(n). \tag{2}$$

For $L = \emptyset$, we have $T_\emptyset = \{1_T\}$ and $A^-_{T_L}(n)$ is equal to 1. For $L$ a singleton, $A^-_{T_L}(n)$ is the number of embeddings of $T_L \setminus \{1_T\} = D[x_j]$ into $T_1^n$, which itself is a copy of $T^{n-1}$. So $A^-_{T_L}(n) = C_{D[x_j]}(n-1)$. Finally, for $|L| \geq 2$, $A^-_{T_L}(n)$ is the number of embeddings that map $1_T$ to $1_n$ and map $x_j$ to an element of $T_1^n$ for all $j \in L$. Since $|L| \geq 2$ any such embedding

$\phi$ cannot map any of the $x_j$ to $t_1$. So, for each embedding $\phi$ we can construct a new embedding $\psi$ of $T$ into $T^n$ by defining $\psi(1_T) = t_1$ and $\psi(x) = \phi(x)$ for all $x \in T_L \setminus \{1_T\}$. Now, $\psi$ is an embedding into $T_1^n$ which maps $1_T$ to $t_1$, the root of $T_1^n$. Since $T_1^n$ is a copy of $T^{n-1}$ the number of these embeddings $\psi$ is $A_{T_L}(n-1)$. Since each $\phi$ corresponds uniquely to a $\psi$, and vice-versa, we must have $A^-_{T_L}(n) = A_{T_L}(n-1)$. To summarise,

$$A^-_{T_L}(n) = \begin{cases} 1 & L = \emptyset \\ C_{D[x_j]}(n-1) & L = \{j\} \\ A_{T_L}(n-1) & \text{otherwise} \end{cases} \tag{3}$$

for $i = 1, 2$.

It will also be useful to have another expression for $A^-_{T_L}(n)$ when $L = \{j\}$. We have that $A^-_{T_L}(n)$ is the number of embeddings of $T_L$ into $T^n$ that map $1_T$ to $1_n$ and map $x_j$ to an element in $T_1^n$. By symmetry of $T^n$ it is also the number of embeddings of $T_L$ into $T^n$ that map $1_T$ to $1_n$ and map $x_j$ to an element in $T_2^n$. Since, every embedding of $T$ into $T^n$ that maps $1_T$ to $1_n$ must map $x_j$ to an element in either $T_1^n$ or $T_2^n$ we have $2A^-_{T_L}(n) = A_{T_L}(n)$ or

$$A^-_{T_L}(n) = \frac{A_{T_L}(n)}{2} \tag{4}$$

for $L = \{j\}$.

We can use equations (1)–(4) to find $A_T(n)$ and $C_T(n)$ inductively. Recall that for a tree $T$, the number of leaves of $T$ is denoted by $l(T)$.

**Theorem 1.** *For any tree $T$, the number of embeddings of $T$ into $T^n$ is of the form*

$$C_T(n) = \sum_{j=0}^{l(T)} g_j(n) 2^{jn},$$

*where each $g_j$ is a polynomial.*

*For $T$ the 1-element tree, the number of these embeddings that map the root of $T$ to $1_n$, $A_T(n)$, is equal to 1. Otherwise, for $T$ with $|T| > 1$, the number is of the form*

$$A_T(n) = \sum_{j=0}^{l(T)} q_j(n) 2^{jn},$$

*where each $q_j$ is a polynomial.*

The following lemma on recurrence relations will be useful. The result is standard and the proof is omitted.

**Lemma 2.** *Suppose $l$ is some fixed positive integer. Then the solution to the equation*

$$y_n - 2y_{n-1} = \sum_{j=0}^{l} f_j(n) 2^{jn}, \quad y_1 = 0, \tag{5}$$

*where each $f_j$ is a polynomial, is*

$$y_n = \sum_{j=0}^{l} g_j(n)2^{jn}$$

*where each $g_j$ is a polynomial. Furthermore, for $j \neq 1$, the polynomial $g_j$ is the unique polynomial satisfying the identity*

$$g_j(n) - 2^{1-j}g_j(n-1) = f_j(n),$$

*and $g_1$ satisfies the identity*

$$g_1(n) - g_1(n-1) = f_1(n),$$

*where the constant term of $g_1$ is given by*

$$\sum_{j=0}^{l} g_j(1)2^j = 0$$

□

**Proof of Theorem 1.** We include the case of $T$ being a 1-element set for completeness. In this case, we see immediately that there are $2^n - 1$ embeddings of $T$ into $T^n$, which is exactly the number of elements in $T^n$. Also, only one of these embeddings maps the root of $T$ to $1_n$. So, $A_T(n) = 1$ as claimed, and $C_T(n) = 2^n - 1$ is of the required form.

For $|T| \geq 2$, we simultaneously prove that $A_T(n)$ and $C_T(n)$ are of the required form by induction on the size of $T$. We shall make use of Lemma 2 to solve recurrence relations for $A_T(n)$ and $C_T(n)$. We use induction to show that the recurrence is of the form of equation (5), and since we will only be considering trees with $|T| \geq 2$ we have the initial conditions $A_T(1) = 0, C_T(1) = 0$ as in (5).

For $|T| = 2$ the only tree is the 2-element chain, which has one leaf. Label the root $1_T$ and the leaf $x_1$. Since $1_T$ has only one lower cover, $r = 1$ in equation (2) and the subtrees of interest are $T_{\{1\}} = T$ and $T_\emptyset = \{1_T\}$. Using equations (2) and (3) we have

$$A_T(n) = A^-_{T_\emptyset}(n)A^-_{T_{\{1\}}}(n) + A^-_{T_{\{1\}}}(n)A^-_{T_\emptyset}(n) = 2C_{\{x_1\}}(n-1)$$

But we have shown earlier that $C_{\{x_1\}}(n) = 2^n - 1$. Therefore $A_T(n) = 2^n - 2$ which is of the required form (where $l(T) = 1$, $q_0(n) = -2$ and $q_1(n) = 1$).

In fact, we can see immediately that $A_T(n) = 2^n - 2$, since this is exactly the number of places to embed $x_1$ in $T^n$ (anywhere except at $1_n$, where $x$ is embedded). Using (1) and Lemma 2 we have that $C_T(n) = (n-2)2^n + 2$ which is of the required form ($g_1(n) = n - 2$ and $g_0(n) = 2$).

Suppose the result is true for all $T$ with $|T| < k$ and let $T$ be any tree with $|T| = k$. There are two cases to consider, depending on whether the root of $T$ has exactly one lower cover. If the root has exactly one lower cover, $x_1$, equation (2) reduces, in a similar way to the base case, to

$$A_T(n) = 2C_{D[x_1]}(n-1).$$

Applying the inductive hypothesis to $D[x_1]$, a tree with $l(D[x_1]) = l(T)$ leaves, we have that

$$C_{D[x_1]}(n) = \sum_{j=0}^{l(T)} g_j(n)2^{jn}$$

where $g_j$ are polynomials. Therefore, $A_T(n) = 2\sum_{j=0}^{l(T)} g_j(n-1)2^{j(n-1)} = \sum_{j=0}^{l(T)} q_j(n)2^{jn}$ where $q_j$ are polynomials.

If the root of $T$ has $r > 1$ lower covers $x_1, \ldots, x_r$ then we can write equation (2) as

$$A_T(n) = A^-_{T_\emptyset}(n)A^-_{T_{[r]}}(n) + A^-_{T_{[r]}}(n)A^-_{T_\emptyset}(n) + \sum_{\substack{L\subseteq[r] \\ L\neq\emptyset,[r]}} A^-_{T_L}(n)A^-_{T_{L^c}}(n)$$

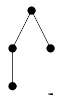
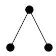
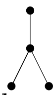
which can be rearranged to

$$A_T(n) - 2A_T(n-1) = \sum_{\substack{L\subseteq[r] \\ L\neq\emptyset,[r]}} A^-_{T_L}(n)A^-_{T_{L^c}}(n). \tag{6}$$

We use equations (3) and (4) in order to apply the inductive hypothesis. Terms in the sum where $L$ is not a singleton or complement of a singleton are of the form $A_{T_L}(n-1)A_{T_{L^c}}(n-1)$. Terms where $L$ is a singleton, but $L^c$ is not are of the form $A_{T_L}(n)A_{T_{L^c}}(n-1)/2$, terms where $L$ is not a singleton, but $L^c$ is are of the form $A_{T_L}(n-1)A_{T_{L^c}}(n)/2$ and terms where both $L$ and $L^c$ are singletons (this will only be for $r = 2$) are of the form $A_{T_L}(n)A_{T_{L^c}}(n)/4$.

By our inductive hypothesis we have $A_{T_L}(n) = \sum_{j=0}^{l(T_L)} q_j(n)2^{jn}$ for polynomials $q_j$. This means that the right hand side of equation (6) is of the form $\sum_{j=0}^{l(T)} h_j(n)2^{jn}$ for polynomials $h_j$. That is, $A_T(n)$ satisfies a recurrence relation and applying Lemma 2 gives the result for $A_T(n)$. Finally, we use (1) and Lemma 2 which gives the result for $C_T(n)$. □

Note that the proof of Theorem 1 actually shows how to find the polynomials $q_j$ and $g_j$ in the expressions for $A_T(n)$ and $C_T(n)$. However, for a particular tree $T$, in order to calculate $A_T(n)$ and $C_T(n)$ we need to calculate $A_{T_L}(n)$ for all subtrees $T_L$. For small trees the calculations are still relatively simple. We use the algorithm given in the proof of Theorem 1 to find explicit expressions for the two trees $T_1, T_2$ in Figure 2.

To find these expressions we need to also calculate $A_S$ and $C_S$ for subtrees $S$ of $T_1$ and $T_2$. Define the subtrees $S_1 =$ , $S_2 =$ , $S_3 =$ , $S_4 =$ , $S_5 =$ . In order to find $A_{T_1}$ we need to calculate $A_{S_1}, A_{S_2}, A_{S_3}, A_{S_4}$, and to find $A_{S_1}$ we need to calculate $C_{S_2}$. For $A_{T_2}$ we also need to calculate $A_{S_5}$ and to find this we need to calculate $C_{S_4}$. These calculations are left as an exercise for the reader. We have

$$A_{T_1}(n) = (n - 14/3)8^n + (-3n^2 + 24n - 34)4^n + (n^3/3 - 8n^2 + 65n/3 + 44/3)2^n + 24$$

$$A_{T_2}(n) = (2n/3 - 20/9)8^n + (-n^3 + 8n^2 - 30n + 58)4^n + (-2n^3/3 + 2n^2 - 40n/3 - 430/9)2^n - 8$$

and, using (1), we have

$$C_{T_1}(n) = (4n/3 - 20/3)8^n + (-6n^2 + 60n - 134)4^n \\ + (n^4/12 - 5n^3/2 + 83n^2/12 + 145n/6 + 494/3)2^n - 24$$

$$C_{T_2}(n) = (8n/9 - 88/27)8^n + (-2n^3 + 22n^2 - 110n + 250)4^n \\ + (-n^4/6 + n^3/3 - 35n^2/6 - 487n/9 - 6878/27)2^n + 8$$

So, $A_{T_1}(4)/C_{T_1}(4) = 99/101 > 67/69 = A_{T_2}(4)/C_{T_2}(4)$, a counterexample to the conjecture of Kubicki, Lehel and Morayne. We also have

$$\frac{A_{T_1}(5)}{C_{T_1}(5)} = \frac{2635}{2837} > \frac{1783}{1921} = \frac{A_{T_2}(5)}{C_{T_2}(5)}$$

but

$$\frac{A_{T_1}(6)}{C_{T_1}(6)} = \frac{44147}{49821} < \frac{31055}{34897} = \frac{A_{T_2}(6)}{C_{T_2}(6)}.$$

So, for $n = 4, 5$ these trees give a counterexample, but not for $n = 6$. In fact, for $n = 6, \ldots, 11$ the conjectured inequality holds, but for larger $n$ it does not. Asymptotically, we have

$$\frac{A_{T_1}(n)}{C_{T_1}(n)} = \frac{(n - 14/3)8^n + O(4^n)}{(4n/3 - 20/3)8^n + O(4^n)} = \frac{3}{4} + \frac{1}{4}n^{-1} + \frac{5}{4}n^{-2} + o(n^{-2})$$

and

$$\frac{A_{T_2}(n)}{C_{T_2}(n)} = \frac{(2n/3 - 20/9)8^n + O(4^n)}{(8n/9 - 88/27)8^n + O(4^n)} = \frac{3}{4} + \frac{1}{4}n^{-1} + \frac{11}{12}n^{-2} + o(n^{-2}),$$

so $A_{T_1}/C_{T_1}$ is asymptotically larger than $A_{T_2}/C_{T_2}$.

This asymptotic difference is very subtle. Here, the ratios $A_{T_1}/C_{T_1}$, $A_{T_2}/C_{T_2}$ differ only in the $n^{-2}$ terms and terms of lower order. We will show, in Section 4, that for any $T_1 \subseteq T_2$ which have $A_{T_1}/C_{T_1}$ asymptotically larger than $A_{T_2}/C_{T_2}$ the ratios differ only in the $n^{-2}$ terms and terms of lower order.

For small values of $n$ there are two competing factors which determine whether the conjectured inequality holds. Since $A_T$ and $C_T$ are related by (1), we have $A_T(n)/C_T(n) = 1 - 2C_T(n-1)/C_T(n)$. So, the conjectured inequality is equivalent to

$$\frac{C_{T_2}(n-1)}{C_{T_1}(n-1)} \leq \frac{C_{T_2}(n)}{C_{T_1}(n)}.$$

We can think of the ratio $C_{T_2}(n)/C_{T_1}(n)$ as the expected number of embeddings of $T_2$ into $T^n$ that are an extension of a randomly chosen embedding of $T_1$ into $T^n$. So, for $n = 3$, each embedding of $T_1$ into $T^3$ can only be extended one way (there is only one place in $T^3$ to which we can map the extra element of $T_2$), therefore $C_{T_2}(3)/C_{T_1}(3) = 1$. For larger values of $n$, some embeddings of $T_1$ into $T^n$ have no extensions to an embedding of $T_2$ into $T^n$, others will have many extensions to an embedding of $T_2$ into $T^n$. In this example, as $n$ increases there will tend to be a larger fraction of embeddings of $T_1$ into $T^n$ with no

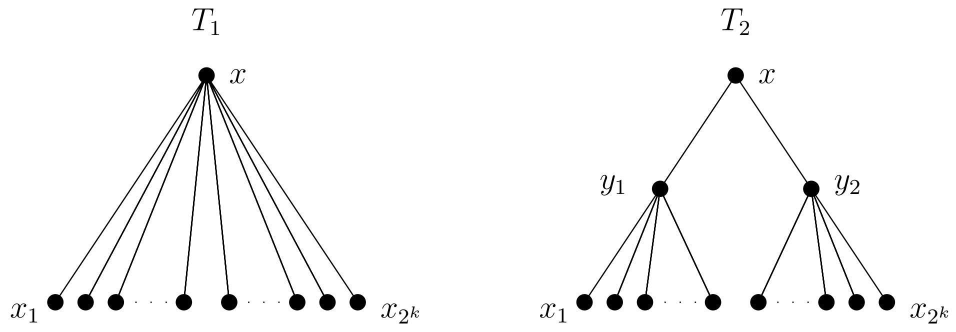



Figure 3: Counterexample to the conjecture of Kubicki, Lehel and Morayne

extension to an embedding of $T_2$ into $T^n$. However, those embeddings of $T_1$ into $T^n$ that do have extensions to embeddings of $T_2$ into $T^n$ will tend to have more of them, as $n$ increases. These two competing effects determine whether the ratio $C_{T_2}(n)/C_{T_1}(n)$ will increase or decrease for an increase in $n$. In this example the two effects are quite equally balanced, making it difficult to see intuitively why the inequality holds for some values of $n$ and fails for others.

The following example better illustrates the failure of the conjectured inequality, as in this example one effect dominates the other. Let $T_1$ and $T_2$ be as shown in Figure 3, where $k$ is some fixed integer. As we have explained, the conjecture claims that $C_{T_2}(n)/C_{T_1}(n)$ is increasing in $n$. However, we show that for these trees, the ratio is considerably larger for small $n$ than it is for large $n$, since for small $n$ there is a higher proportion of embeddings of $T_1$ that can be extended to an embedding of $T_2$.

For any $n$ with $n \geq k+1$, an embedding of $T_2$ into $T^n$ must map all the leaves $x_1, \ldots, x_{2^{k-1}}$ into the same half of $T^n$, and it must map all the leaves $x_{2^{k-1}+1}, \ldots, x_{2^k}$ into the same half of $T^n$. This is a restriction imposed by the elements $y_1$ and $y_2$. Embeddings of $T_1$ into $T^n$ do not have this restriction, and any embedding of $T_1$ into $T^n$, which does not partition the leaves in the same way cannot be extended to an embedding of $T_2$.

Now, for $n = k+1$, the tree $T^{k+1}$ has $2^k$ leaves, so all embeddings of $T_1$ into $T^{k+1}$ map the leaves of $T_1$ to the leaves of $T^{k+1}$. Therefore, we know that half the leaves of $T_1$ are mapped into one half of $T^{k+1}$ and the other half into the other half of $T^{k+1}$. So whether the embedding extends to an embedding of $T_2$ depends only on which particular set of $2^{k-1}$ leaves are mapped into one of the halves of $T^{k+1}$. Since there are $\binom{2^k}{2^{k-1}}$ subsets of size $2^{k-1}$, and two of these yield an extendible embedding (when we choose $\{x_1, \ldots, x_{2^{k-1}}\}$ or $\{x_{2^{k-1}+1}, \ldots, x_{2^k}\}$), each with one possible extension, the ratio $C_{T_2}(k+1)/C_{T_1}(k+1)$ is equal to $2/\binom{2^k}{2^{k-1}}$.

For $n \gg k+1$, most mappings from $T_1$ into $T^n$ are embeddings, but only those which partition the leaves as described above can be extended. Moreover, most of the embeddings that can be extended map the leaves $x_1, \ldots, x_{2^{k-1}}$ into one half of $T^n$, and the leaves $x_{2^{k-1}+1}, \ldots, x_{2^k}$ into the other half of $T^n$ (rather than the same half) and most of these extendible embeddings have only one possible extension. So of the total number of embeddings of $T_1$ into $T^n$ the fraction that are extendible is roughly $2^{-2^k}$ and most extendible embeddings have just one possible extension. Therefore, $C_{T_2}(n)/C_{T_1}(n)$ is roughly $1/2^{2^k}$, which is considerably smaller than $C_{T_2}(k+1)/C_{T_1}(k+1) = 2/\binom{2^k}{2^{k-1}}$.

## 3 Asymptotic behaviour of $A_T$ and $C_T$

We have shown that $A_T(n) = \sum_{j=0}^{l} q_j(n)2^{jn}$, where each $q_j$ is a polynomial. We wish to examine the asymptotic behaviour of $A_T(n)$ and so we need to calculate the leading terms of the dominant polynomial $q_l(n)$. Throughout this section we use the symbol $\sim$ to mean "asymptotically equivalent to"; we write $f(n) \sim g(n)$ if $f(n)/g(n)$ tends to 1 as $n$ tends to infinity. We also use $\sim$ in a shorthand for recurrence relations, writing for example $y_n - 2y_{n-1} \sim f(n)$ if $y_n - 2y_{n-1} = g(n)$ and $g(n) \sim f(n)$. We shall make use of the following lemma which gives the solutions to some particular recurrence relations.

**Lemma 3.** *The recurrence relation*

$$y_n - 2y_{n-1} \sim (\alpha n^d + \beta n^{d-1})2^{ln}$$

*where $d > 0$ has solution*

$$y_n \sim \begin{cases} \left(\frac{\alpha}{d+1}n^{d+1} + \left(\frac{\beta}{d} + \frac{\alpha}{2}\right) n^d\right) 2^n & \text{if } l = 1 \\ \frac{2^{l-1}}{2^{l-1}-1}\left(\alpha n^d + \left(\beta - \frac{\alpha d}{2^{l-1}-1}\right) n^{d-1}\right) 2^{ln} & \text{if } l \geq 2. \end{cases} \tag{7}$$

*The recurrence relation*

$$y_n - 2y_{n-1} \sim \alpha 2^{ln}$$

*has solution*

$$y_n \sim \begin{cases} \alpha n 2^n & \text{if } l = 1 \\ \frac{2^{l-1}}{2^{l-1}-1}\alpha 2^{ln} & \text{if } l \geq 2. \end{cases} \tag{8}$$

**Proof.** It is a simple exercise to check that equations (7) and (8) do give a particular solution to the exact recurrence relations $y_n - 2y_{n-1} = (\alpha n^d + \beta n^{d-1})2^{ln}$ and $y_n - 2y_{n-1} = \alpha 2^{ln}$. Since the complementary solution to both recurrences is $y_n = K2^n$, this is dominated by the particular solutions given, and so the asymptotic solution is as claimed. □

**Theorem 4.** *The leading polynomial $q_l(n)$ in the expression $A_T(n) = \sum_{j=0}^{l(T)} q_j(n)2^{jn}$ has degree $d(T)$, where $d(T) = |\{x \in T : x$ not the root or a leaf, $D[x]$ is a chain$\}|$. The coefficient $\alpha_T$ of $n^{d(T)}$ satisfies the following equations.*

*If $T$ is the 2-element chain, then $\alpha_T = 1$. Otherwise, if the root of $T$ has $r$ lower covers, then*

$$\alpha_T = \begin{cases} \dfrac{\alpha_{D[x_1]}}{d(T)} & r = 1 \text{ and } T \text{ a chain} \\[2ex] \dfrac{\alpha_{D[x_1]}}{2^{l(T)-1} - 1} & r = 1 \text{ and } T \text{ not a chain} \\[2ex] \dfrac{\alpha_{T_{\{1\}}}\alpha_{T_{\{2\}}} 2^{l(T)-2}}{2^{l(T)-1} - 1} & r = 2 \\[2ex] \dfrac{\sum_{j=1}^{r} \alpha_{T_{\{j\}}}\alpha_{T_{\{j\}^c}} 2^{l(T_{\{j\}})-1} + \sum_{2 \le |L| \le r/2} \alpha_{T_L}\alpha_{T_{L^c}}}{2^{l(T)-1} - 1} & r \ge 3 \end{cases} \tag{9}$$

*Moreover, if $d(T) > 0$ the coefficient $\beta_T$ of $n^{d(T)-1}$ satisfies the following equations.*

*If $T$ is the 3-element chain, then $\beta_T = -3$. Otherwise, if the root of $T$ has $r$ lower covers, then*

$$\beta_T = \begin{cases} \dfrac{\beta_{D[x_1]}}{d(T) - 1} - \dfrac{d(T)\alpha_T}{2} & r = 1 \text{ and } T \text{ a chain} \\[2ex] \dfrac{\beta_{D[x_1]} - d(T)\alpha_T 2^{l(T)-1}}{2^{l(T)-1} - 1} & r = 1 \text{ and } T \text{ not a chain} \\[2ex] \dfrac{(\alpha_{T_{\{1\}}}\beta_{T_{\{2\}}} + \alpha_{T_{\{2\}}}\beta_{T_{\{1\}}})2^{l(T)-2} - d(T)\alpha_T}{2^{l(T)-1} - 1} & r = 2 \\[2ex] \dfrac{\sum_{j=1}^{r}(\alpha_{T_{\{j\}}}\beta_{T_{\{j\}^c}} + \alpha_{T_{\{j\}^c}}\beta_{T_{\{j\}}} - d(T_{\{j\}^c})\alpha_{T_{\{j\}}}\alpha_{T_{\{j\}^c}})2^{l(T_{\{j\}})-1}}{2^{l(T)-1} - 1} \\ \quad + \dfrac{\sum_{2 \le |L| \le r/2}(\alpha_{T_L}\beta_{T_{L^c}} + \alpha_{T_{L^c}}\beta_{T_L} - d(T)\alpha_{T_L}\alpha_{T_{L^c}}) - d(T)\alpha_T}{2^{l(T)-1} - 1} & r \ge 3 \end{cases} \tag{10}$$

*where $\beta_S = 0$ for any subtree $S \subseteq T$ with $d(S) = 0$.*

**Proof.** We proceed by induction on $|T|$. For $|T| = 2$ we have already shown that $T$ is the 2-element chain and $A_T(n) = 2^n - 2$. For this tree $d(T) = 0$, $l(T) = 1$, so $q_l(n) = 1$ a polynomial of degree 0, with leading coefficient equal to 1. That is, $\alpha_T = 1$ as claimed.

Suppose the result is true for all $T$ with $|T| < k$ and let $T$ be any tree with $|T| = k$. As in the proof of Theorem 1, there are two cases to consider, depending on whether the root of $T$ has exactly one lower cover. If the root has exactly one lower cover, $x_1$, we have equation $A_T(n) = 2C_{D[x_1]}(n-1)$. But by Theorem 1, and our inductive hypothesis, we know that

$$A_{D[x_1]}(n) \sim \alpha_{D[x_1]} n^{d(D[x_1])} 2^{l(D[x_1])n}.$$

If $T$ is a chain, then $l(T) = l(D[x_1]) = 1$ and $d(T) = d(D[x_1]) + 1$ since the element $x_1$ contributes to $d(T)$ but not $d(D[x_1])$. So, by Lemma 3,

$$C_{D[x_1]}(n) \sim \frac{\alpha_{D[x_1]}}{d(D[x_1]) + 1} n^{d(D[x_1])+1} 2^n = \frac{\alpha_{D[x_1]}}{d(T)} n^{d(T)} 2^n.$$

So

$$A_T(n) = 2C_{D[x_1]}(n-1) \sim 2\frac{\alpha_{D[x_1]}}{d(T)}(n-1)^{d(T)} 2^{n-1} = \frac{\alpha_{D[x_1]}}{d(T)}(n-1)^{d(T)} 2^n.$$

Therefore $\alpha_T = \alpha_{D[x_1]}/d(T)$, as claimed. If $T$ is not a chain, then $l(T) = l(D[x_1]) > 1$ and $d(T) = d(D[x_1])$ since the element $x_1$ does not contribute to either $d(T)$ or $d(D[x_1])$. So, by Lemma 3,

$$C_{D[x_1]}(n) \sim \frac{2^{l(T)-1}}{2^{l(T)-1} - 1} \alpha_{D[x_1]} n^{d(T)} 2^{l(T)n}.$$

So

$$\begin{aligned} A_T(n) = 2C_{D[x_1]}(n-1) &\sim 2\frac{2^{l(T)-1}}{2^{l(T)-1} - 1} \alpha_{D[x_1]}(n-1)^{d(T)} 2^{l(T)(n-1)} \\ &= \frac{\alpha_{D[x_1]}}{2^{l(T)-1} - 1}(n-1)^{d(T)} 2^{l(T)n}. \end{aligned}$$

Therefore $\alpha_T = \alpha_{D[x_1]}/(2^{l(T)-1} - 1)$, as claimed.

If the root of $T$ has two lower covers $x_1, x_2$ then equations (6) and (4) give $A_T(n) - 2A_T(n-1) = A_{T_{\{1\}}}(n)A_{T_{\{2\}}}(n)/2$. So,

$$\begin{aligned} A_T(n) - 2A_T(n-1) &\sim \alpha_{T_{\{1\}}} n^{d(T_{\{1\}})} 2^{l(T_{\{1\}})n} \alpha_{T_{\{2\}}} n^{d(T_{\{2\}})} 2^{l(T_{\{2\}})n}/2 \\ &= \alpha_{T_{\{1\}}} \alpha_{T_{\{2\}}} n^{d(T)} 2^{l(T)n}/2 \end{aligned}$$

since $d(T_{\{1\}}) + d(T_{\{2\}}) = d(T)$ and $l(T_{\{1\}}) + l(T_{\{2\}}) = l(T)$. Since $l(T) > 1$ applying Lemma 3 gives

$$A_T(n) \sim \frac{2^{l(T)-1} \alpha_{T_{\{1\}}} \alpha_{T_{\{2\}}}}{(2^{l(T)-1} - 1)2} n^{d(T)} 2^{l(T)n}$$

and $\alpha_T$ is as claimed.

Finally, if the root of $T$ has $r \geq 3$ lower covers $x_1, \ldots, x_r$ we can write (6) as

$$A_T(n) - 2A_T(n-1) = 2\sum_{j=1}^{r} \frac{1}{2} A_{T_{\{j\}}}(n) A_{T_{\{j\}^c}}(n-1) + 2 \sum_{2 \leq |L| \leq r/2} A_{T_L}(n-1) A_{T_{L^c}}(n-1).$$

Terms in the first sum are of the form

$$\alpha_{T_{\{j\}}} n^{d(T_{\{j\}})} 2^{l(T_{\{j\}})n} \alpha_{T_{\{j\}^c}} (n-1)^{d(T_{\{j\}^c})} 2^{l(T_{\{j\}^c})(n-1)} \sim \frac{\alpha_{T_{\{j\}}} \alpha_{T_{\{j\}^c}}}{2^{l(T_{\{j\}^c})}} n^{d(T)} 2^{l(T)n}$$

and terms in the second sum are of the form

$$2\alpha_{T_L}(n-1)^{d(T_L)} 2^{l(T_L)(n-1)} \alpha_{T_{L^c}} (n-1)^{d(T_{L^c})} 2^{l(T_{L^c})(n-1)} \sim 2\frac{\alpha_{T_L} \alpha_{T_{L^c}}}{2^{l(T)}} n^{d(T)} 2^{l(T)n}$$

Applying Lemma 3 gives

$$A_T(n) \sim \frac{2^{l(T)-1}}{2^{l(T)-1}-1}\left(\sum_{j=1}^{r}\frac{\alpha_{T_{\{j\}}}\alpha_{T_{\{j\}^c}}}{2^{l(T_{\{j\}^c})}}n^{d(T)}2^{l(T)n}+\sum_{2\leq|L|\leq r/2}2\frac{\alpha_{T_L}\alpha_{T_{L^c}}}{2^{l(T)}}n^{d(T)}2^{l(T)n}\right)$$

$$= \frac{\sum_{j=1}^{r}\alpha_{T_{\{j\}}}\alpha_{T_{\{j\}^c}}2^{l(T_{\{j\}})-1}+\sum_{2\leq|L|\leq r/2}\alpha_{T_L}\alpha_{T_{L^c}}}{2^{l(T)-1}-1}n^{d(T)}2^{l(T)n}.$$

Therefore $\alpha_T$ is as claimed.

We omit the proof that $\beta_T$ is as claimed, which can be shown by also considering the coefficient of $n^{d(T)-1}2^{l(T)n}$ in the above calculations. □

We have that for $T$ a tree with $|T|>1$, $A_T(n)\sim\alpha_T n^{d(T)}2^{l(T)n}$, for $\alpha_T$ some constant that can be found. We can see that $A_T(n)=\Omega(n^{d(T)}2^{l(T)n})$ as follows. For any tree $T$, call the elements counted by $d(T)$ *lower bead elements* of $T$. So, a lower bead element of $T$ is an element $x$ such that $D[x]$ is a chain, and $x$ is not a leaf or the root. Call an element which has more than one lower cover a *branching element* of $T$. Call the remaining elements of $T$ *upper bead elements* of $T$. These are elements $x$ which have only one lower cover, but $D[x]$ is not a chain. Therefore, upper bead elements only occur on a chain above a branching element. Note that, depending on the tree $T$, the root can be either a branching element or an upper bead element.

So, if $T$ is a chain, then $T$ has a root and one leaf, joined by a chain of $d(T)$ lower bead elements. Otherwise, for $l(T)>1$, the tree $T$ has a root, the root and the branching elements are joined by (possibly empty) chains of upper bead elements, and some branching elements are joined by (possibly empty) chains of lower bead elements (of which there are $d(T)$) to the $l(T)$ leaves.

To see that $A_T(n)=\Omega(n^{d(T)}2^{l(T)n})$, first consider $T$ a chain. We count the embeddings that map the root of $T$ to $1_n$ and the leaf of $T$ to some leaf of $T^n$. We have $2^{n-1}$ choices for where to map the leaf. Once we have fixed the leaf of $T^n$, this defines a path from $1_n$ to the leaf of $T^n$. This gives a choice of $n-2$ elements of $T^n$ into which we can map the $d(T)$ lower bead elements of $T$. So, asymptotically we have $\Theta(n^{d(T)})$ choices for where to map the $d(T)$ lower bead elements. Therefore $A_T(n)=\Omega(n^{d(T)}2^n)$, and since $l(T)=1$ we have that $A_T(n)=\Omega(n^{d(T)}2^{l(T)n})$ for $T$ a chain.

For $T$ not a chain, so there exist branching elements of $T$, let $\phi$ be some embedding which maps the root of $T$ to $1_n$, and maps the branching elements of $T$ to as high a level of $T^n$ as possible. Consider, for large $n$, the number of embeddings of $T$ into $T^n$ that agree with this fixed $\phi$ on the root, branching elements and upper bead elements. Let us only consider those embeddings which map the leaves of $T$ to the leaves of $T^n$. Let $x$ be a branching element. Since the elements $\phi(x)$ will take up some constant number of the levels of $T^n$, $\phi(x)$ will have some constant fraction $c_x$ of leaves of $T^n$ below it. Now, as explained above, each leaf $y$ in $T$ is joined to a branching element, $x_y$ say, by a chain of lower bead elements. So, each leaf $y$ can be mapped to $c_{x_y}2^{n-1}$ leaves in $T^n$, and the total number of choices for all the leaves is asymptotically $\Theta(2^{l(T)n})$. (The over-counting due to the possibility that two leaves that are below the same branching point are mapped to the same leaf of $T^n$ is negligible for large $n$.) It remains to choose where to map the lower bead elements. However, in a similar way to the case where $T$ is a chain, a lower bead element on the chain between the branching point $x$ and the leaf $y$ must be mapped to an element on the path between the images of $x$ and $y$. Since $x$ is mapped to a high level, and $y$ to a leaf, the path has asymptotically $n - c'_x$ elements, for some constant $c'_x$. Since there are $d(T)$ lower bead elements, we have asymptotically $\Theta(n^{d(T)})$ choices for where to map the lower bead elements. So, the number of embeddings that agree with $\phi$ is asymptotically $\Omega(n^{d(T)}2^{l(T)n})$, and we have $A_T(n) = \Omega(n^{d(T)}2^{l(T)n})$ for $T$ not a chain.

By Lemma 3 we also have the asymptotic behaviour of $C_T(n)$, given in the following corollary.

**Corollary 5.** *For any tree $T$ with $l(T) = 1$ the number of embeddings of $T$ into $T^n$ is asymptotically*

$$C_T(n) \sim \frac{\alpha_T}{d(T)+1} n^{d(T)+1} 2^n$$

*and if $d(T) > 0$ then*

$$C_T(n) \sim \left( \frac{\alpha_T}{d(T)+1} n^{d(T)+1} + \left( \frac{\beta_T}{d(T)} + \frac{\alpha_T}{2} \right) n^{d(T)} \right) 2^n$$

*For any tree with $l(T) > 1$ the number of embeddings of $T$ into $T^n$ is asymptotically*

$$C_T(n) \sim \frac{2^{l(T)-1}}{2^{l(T)-1} - 1} \alpha_T n^{d(T)} 2^n$$

*and if $d(T) > 0$ then*

$$C_T(n) \sim \frac{2^{l(T)-1}}{2^{l(T)-1} - 1} \left( \alpha_T n^{d(T)} + \left( \beta_T - \frac{d(T)\alpha_T}{2^{l(T)-1} - 1} \right) n^{d(T)-1} \right) 2^{l(T)n}$$

**Proof.** We have that $A_T(n) \sim \alpha_T n^{d(T)} 2^{l(T)n}$, and if $d(T) > 0$ then $A_T(n) \sim (\alpha_T n^{d(T)} + \beta_T n^{d(T)-1}) 2^{l(T)n}$. So $C_T(n)$ satisfies the recurrence relation (1) which is of the form in Lemma 3. Applying Lemma 3 with $\alpha = \alpha_T$ and $\beta = \beta_T$ gives the result. □

This tells us that for a tree $T$ not a chain, a typical embedding of $T$ into $T^n$ maps the leaves of $T$ to the low levels of $T^n$, the branching points and upper bead elements of $T$ to the high levels of $T^n$, and the lower bead elements of $T$ will be mapped to elements spread roughly evenly along the paths in $T^n$ defined by the images of branching elements and leaves of $T$, as explained earlier. There are $\Theta(n^{d(T)}2^{l(T)n})$ of these embeddings.

For $T$ a chain, a typical embedding maps the leaf of $T$ to a low level of $T^n$, and the remaining elements of $T$ are mapped to elements spread roughly evenly on the path from $1_n$ to image of the leaf in $T^n$. Here the root is not necessarily mapped to $1_n$, and the root can be thought of as a lower bead element, so there are $d(T) + 1$ elements to position on this path. So, we get $\Theta(n^{d(T)+1}2^n)$ of these embeddings.

## 4 Asymptotics of the ratio $A_T(n)/C_T(n)$

In [3], Kubicki, Lehel and Morayne proved that $\lim_{n\to\infty} \frac{A(n;T_1)}{B(n;T_1)} \leq \lim_{n\to\infty} \frac{A(n;T_2)}{B(n;T_2)}$, by showing that $\lim_{n\to\infty} A(n;T)/B(n;T) = 2^{l(T)-1}-1$ (Proposition 2.3 in [3]). Here, using Theorem 4 and Corollary 5 we have

$$\lim_{n\to\infty} \frac{A_T(n)}{C_T(n)} = \frac{2^{l(T)-1}-1}{2^{l(T)-1}}$$

which is equivalent to Proposition 2.3 in [3], since $B_T(n)/A_T(n) = C_T(n)/A_T(n) - 1$. This tells us that for trees $T_1, T_2$ with $l(T_1) < l(T_2)$ there exists some $n_0$ such that $A_{T_1}(n)/C_{T_1}(n) < A_{T_2}(n)/C_{T_2}(n)$ for all $n \geq n_0$. Here, we show that there exist trees $T_1 \subseteq T_2$, with $l(T_1) = l(T_2)$, with the inequality the other way round. That is, there is an $n_0$ such that $A_{T_1}(n)/C_{T_1}(n) > A_{T_2}(n)/C_{T_2}(n)$ for all $n \geq n_0$. All such pairs $T_1, T_2$ are counterexamples to the conjecture, for all $n \geq n_0$.

**Theorem 6.** *For any tree $T$ with $l(T) > 1$ and $d(T) > 0$, we have*

$$\frac{A_T(n)}{C_T(n)} = 1 - \frac{1}{2^{l(T)-1}}\left(1 - \frac{d(T)}{n} + \frac{\binom{d(T)}{2}}{n^2} + \frac{b_T}{n^2}\right) + o(n^{-2}) \tag{11}$$

*where*

$$b_T = \frac{\beta_T}{\alpha_T} - \frac{d(T)}{2^{l(T)-1}-1}. \tag{12}$$

*For any tree $T$ with $l(T) > 1$ and $d(T) = 0$, we have*

$$\frac{A_T(n)}{C_T(n)} = 1 - \frac{1}{2^{l(T)-1}} + O(2^{-n}). \tag{13}$$

*For any tree $T$ with $l(T) = 1$, we have*

$$\frac{A_T(n)}{C_T(n)} = \frac{d(T)+1}{n} + o(n^{-1}). \tag{14}$$

**Proof.** Let $T$ be a tree with $l(T) > 1$ and $d(T) > 0$. By (1) it is sufficient to work with the ratio $C_T(n-1)/C_T(n)$. By Theorem 1 we have that $C_T(n) = \sum_{j=0}^{l(T)} q_j(n)2^{jn}$ and by Lemma 3 we have that

$$q_l(n) \sim \frac{2^{l(T)-1}}{2^{l(T)-1}-1}\left(\alpha_T n^d(T) + \left(\beta_T - \frac{\alpha_T d(T)}{2^{l(T)-1}-1}\right) n^{d(T)-1}\right).$$

So,

$$C_T(n) = 2^{l(T)n} a_T (n^{d(T)} + b_T n^{d(T)-1} + c_T n^{d(T)-2} + o(n^{d(T)-2}))$$

where

$$a_T = \frac{2^{l(T)-1}}{2^{l(T)-1}-1}\alpha_T, \qquad b_T = \frac{\beta_T}{\alpha_T} - \frac{d(T)}{2^{l(T)-1}-1}$$

and $c_T$ is an unspecified constant. Note that this equation is true for $d \geq 2$, and can be made true for $d = 1$ by setting $c_T$ to 0. We have

$$\begin{aligned}
\frac{C_T(n-1)}{C_T(n)} &= \frac{2^{l(T)(n-1)} a_T((n-1)^{d(T)} + b_T(n-1)^{d(T)-1} + c_T(n-1)^{d(T)-2} + o(n^{d(T)-2}))}{2^{l(T)n} a_T(n^{d(T)} + b_T n^{d(T)-1} + c_T n^{d(T)-2} + o(n^{d(T)-2}))} \\
&= \frac{1}{2^{l(T)}} \frac{(1-1/n)^{d(T)} + \frac{b_T}{n}(1-1/n)^{d(T)-1} + \frac{c_T}{n^2}(1-1/n)^{d(T)-2} + o(n^{-2})}{1 + b_T/n + c_T/n^2 + o(n^{-2})} \\
&= \frac{1}{2^{l(T)}} \left(1 - \frac{d(T)}{n} + \frac{\binom{d(T)}{2}}{n^2} + \frac{b_T}{n} - \frac{b_T(d(T)-1)}{n^2} + \frac{c_T}{n^2} + o(n^{-2})\right) \\
&\quad \times \left(1 - \frac{b_T}{n} - \frac{c_T}{n^2} + \frac{b_T^2}{n^2} + o(n^{-2})\right) \\
&= \frac{1}{2^{l(T)}} \left(1 - \frac{d(T)}{n} + \frac{\binom{d(T)}{2}}{n^2} + \frac{b_T}{n^2} + o(n^{-2})\right)
\end{aligned}$$

and, using (1), we have

$$\frac{A_T(n)}{C_T(n)} = 1 - \frac{1}{2^{l(T)-1}} \left(1 - \frac{d(T)}{n} + \frac{\binom{d(T)}{2}}{n^2} + \frac{b_T}{n^2} + o(n^{-2})\right)$$

as required.

Now, suppose $l(T) > 1$ and $d(T) = 0$. So, $C_T(n) = 2^{l(T)n} a_T(1 + O(2^{-n}))$ and

$$\frac{C_T(n-1)}{C_T(n)} = \frac{1}{2^l(T)}(1 + O(2^{-n}))$$

which by (1) gives the required result.

If $l(T) = 1$, then $A_T(n) = 2^n \alpha_T(n^{d(T)} + o(n^{d(T)}))$ and $C_T(n) = 2^n \frac{\alpha_T}{d(T)+1}(n^{d(T)+1} + o(n^{d(T)+1}))$. So,

$$\frac{A_T(n)}{C_T(n)} = \frac{2^n \alpha_T(n^{d(T)} + o(n^{d(T)}))}{2^n \frac{\alpha_T}{d(T)+1}(n^{d(T)+1} + o(n^{d(T)+1}))} = \frac{d(T)+1}{n}(1 + o(1))$$

□

**Corollary 7.** *For any two trees $T_1, T_2$, if either*

*(i)* $l(T_1) > l(T_2)$, *or*

*(ii)* $l(T_1) = l(T_2)$ *and* $d(T_1) > d(T_2)$, *or*

*(iii)* $l(T_1) = l(T_2)$, $d(T_1) = d(T_2) > 0$ *and* $\alpha_{T_1}/\beta_{T_1} > \alpha_{T_2}/\beta_{T_2}$,

*then there exists an integer $n_0$ such that*

$$\frac{A_{T_1}(n)}{C_{T_1}(n)} > \frac{A_{T_2}(n)}{C_{T_2}(n)}$$

*for all $n \geq n_0$.*

**Proof.** (i) If $l(T_1) > l(T_2)$ then we can just compare the limits of the ratios $A_{T_1}(n)/C_{T_1}(n)$ and $A_{T_2}(n)/C_{T_2}(n)$. By Theorem 6 (or from [3]) we have that

$$\lim_{n\to\infty} \frac{A_T(n)}{C_T(n)} = 1 - \frac{1}{2^{l(T)-1}}.$$

Note that this also holds for trees $T$ with $l(T) = 1$. Since the limit is increasing in $l(T)$ the result follows.

(ii) If $l(T_1) = l(T_2)$ and $d(T_1) > d(T_2)$ there are two cases to consider. If $l(T_1) = l(T_2) = 1$ then using equation (14) from Theorem 6 we have that

$$\frac{A_{T_1}(n)}{C_{T_1}(n)} = \frac{d(T_1)+1}{n} + o(n^{-1}) \qquad\qquad \frac{A_{T_2}(n)}{C_{T_2}(n)} = \frac{d(T_2)+1}{n} + o(n^{-1})$$

and since $d(T_1) > d(T_2)$ there exists an $n_0$ such that $A_{T_1}(n)/C_{T_1}(n) > A_{T_2}(n)/C_{T_2}(n)$ for all $n \geq n_0$.

If $l(T_1) = l(T_2) > 1$ then using equation (11) from Theorem 6, and considering only terms up to $n^{-1}$ we have

$$\frac{A_{T_1}(n)}{C_{T_1}(n)} = 1 - \frac{1}{2^{l(T_1)-1}}\left(1 - \frac{d(T_1)}{n}\right) + o(n^{-1}), \quad \frac{A_{T_2}(n)}{C_{T_2}(n)} = 1 - \frac{1}{2^{l(T_2)-1}}\left(1 - \frac{d(T_2)}{n}\right) + o(n^{-1}).$$

This is also true for $d(T_2) = 0$ by equation (13). Since $l(T_1) = l(T_2)$ and $d(T_1) > d(T_2)$ there exists an $n_0$ such that $A_{T_1}(n)/C_{T_1}(n) > A_{T_2}(n)/C_{T_2}(n)$ for all $n \geq n_0$.

(iii) If $l(T_1) = l(T_2)$ and $d(T_1) = d(T_2) > 0$ and $\alpha_{T_1}/\beta_{T_1} > \alpha_{T_2}/\beta_{T_2}$, we first note that $l(T_1)$ cannot be equal to 1. (If $l(T_1) = l(T_2) = 1$ then $d(T_1) = d(T_2)$ implies that $T_1$ and $T_2$ are the same tree, the $(d+2)$-element chain.) So we have $l(T_1) = l(T_2) > 1$ and using equation (11), we see that $A_{T_1}(n)/C_{T_1}(n)$ and $A_{T_2}(n)/C_{T_2}(n)$ differ only in the $n^{-2}$ term and in terms of lower order. Therefore, it is enough to show that $b_{T_1} < b_{T_2}$. But this follows immediately from the inequality $\alpha_{T_1}/\beta_{T_1} > \alpha_{T_2}/\beta_{T_2}$ and (12). □

Corollary 7 provides a simple method for comparing the asymptotics of the ratios $A_{T_1}(n)/C_{T_1}(n)$ and $A_{T_2}(n)/C_{T_2}(n)$. Firstly, we compare the number of leaves of the two trees, the tree with more leaves being the tree with the asymptotically larger ratio $A/C$. If the trees have the same number of leaves, then we compare the values of $d(T_1)$ and $d(T_2)$; the tree with the larger $d$ has the asymptotically larger ratio $A/C$. Both the number of leaves, $l(T)$, and $d(T)$ are very easily obtained from the Hasse diagram of the tree. If both of these are the same for the two trees, then we need to compare the ratios $\alpha_{T_1}/\beta_{T_1}$ and $\alpha_{T_2}/\beta_{T_2}$. The tree with the larger ratio $\alpha/\beta$ has the asymptotically larger ratio $A/C$. This comparison involes rather more calculation, using the algorithm provided by Theorem 4. These calculations can be simplified if the two trees have a very similar structure, for example, as we will see later, if the trees are identical except for the addition of one element to one of the trees.

Corollary 7 also guides our search for more counterexamples to the conjecture of Kubicki, Lehel and Morayne. The counterexample given in Section 1 has two important properties, namely that $l(T_1) = l(T_2)$ and $d(T_1) = d(T_2)$. That this is a necessary condition for a pair of trees to be an asymptotic counterexample follows from Corollary 7. Since we are only considering trees $T_1 \subseteq T_2$ we must have $l(T_1) \leq l(T_2)$. But we are looking for trees $T_1, T_2$ where the ratio $A/C$ is asymptotically larger for $T_1$ than for $T_2$, so we need to look at trees with $l(T_1) = l(T_2)$. If $T_1 \subseteq T_2$ and the trees have the same number of leaves we must have $d(T_1) \leq d(T_2)$. (Each element counted by $d(T_1)$ must also be counted by $d(T_2)$ otherwise $T_2$ would have more leaves than $T_1$.) So, to find our counterexamples we need to look at trees with $d(T_1) = d(T_2)$.

The following theorem gives an infinite family of pairs of trees which form counterexamples. We do not claim, or believe, that this is the only way to construct counterexamples. However, the construction is relatively simple, which makes the calculations much more manageable. Also, there are many ternary tree pairs in this family, including the counterexample given in Section 1, which shows that the conjecture does not just fail for trees with high branching numbers.

**Theorem 8.** *Let $T$ be a tree whose root $x$ has three lower covers $x_1, x_2, x_3$, and let $T'$ be formed from $T$ by adding a new element $y$ below $x$ and above $x_2$ and $x_3$ (see Figure 4). If $d(T) > 0$ and $d(D[y]) = 0$, then there exists $n_0$ such that $A_T(n)/C_T(n) > A_{T'}(n)/C_{T'}(n)$ for all $n \geq n_0$.*

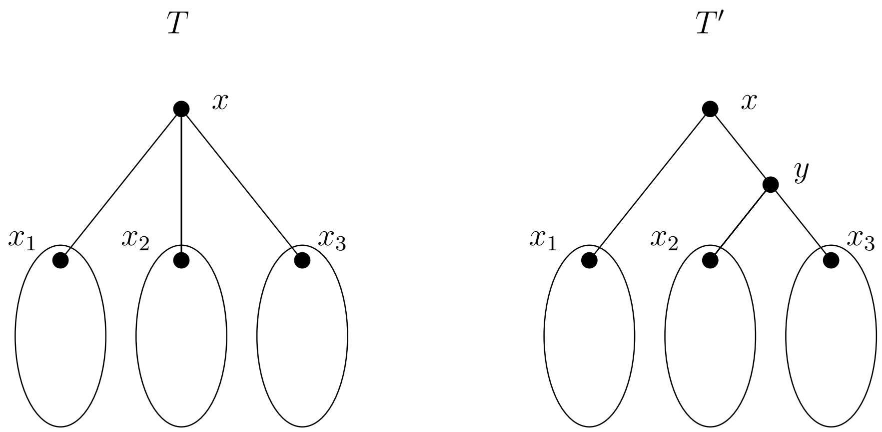



Figure 4: General counterexample for $d(T) > 0, d(D[y]) = 0$

**Proof.** We have $l(T) = l(T')$ and $d(T) = d(T') > 0$ so by Corollary 7 it is enough to show that $\alpha_T \beta_{T'} > \alpha_{T'} \beta_T$. We use equations (9) and (10) to express these $\alpha$ and $\beta$ in terms of some other $\alpha_S$ and $\beta_S$ for common subtrees $S$ of $T$ and $T'$. As before, for $L \subseteq [3]$ write $T_L$ for the subtree formed from $T$ by removing the elements in $D[x_j]$ for each $j \in L^c$. Write $T'_{\{1\}}$ for the subtree formed from $T'$ by removing elements in $D[y]$ and write $T_y$ for the subtree formed by removing elements in $D[x_1]$. We have that $T_{\{1\}} = T'_{\{1\}}$ and $T_{\{2,3\}} = D[y]$. By the assumption that $d(D[y]) = 0$ we have that $d(T) = d(T') = d(T_{\{1,2\}}) = d(T_{\{1,3\}}) = d(T_{\{1\}})$, and we denote this common value by $d$. We also have that $d(T_{\{2\}}) = d(T_{\{3\}}) = d(T_{\{y\}}) = d(D[y]) = 0$. For ease of notation, we write $l$ for the common value $l(T) = l(T')$, write $l_1$ for $l(T_{\{1\}})$, $l_{12}$ for $l(T_{\{1,2\}})$, etc., and we use a similar notation for $\alpha$ and $\beta$. For example, writing $\alpha_1$ for $\alpha_{T_{\{1\}}}$.

Using equation (10) to find $\beta_T$ and $\beta_{T'}$, we have

$$\beta_T = \frac{\alpha_{23}\beta_1 2^{l_1-1} + (\alpha_2\beta_{13} - d\alpha_2\alpha_{13})2^{l_2-1} + (\alpha_3\beta_{12} - d\alpha_3\alpha_{12})2^{l_3-1} - d\alpha_T}{2^{l-1}-1}$$

$$\beta_{T'} = \frac{\alpha_y\beta_1 2^{l-2} - d\alpha_{T'}}{2^{l-1}-1}$$

so

$$\alpha_T\beta_{T'} - \alpha_{T'}\beta_T = \frac{\alpha_T\alpha_y\beta_1 2^{l-2} - \alpha_{T'}\left(\alpha_{23}\beta_1 2^{l_1-1} + (\alpha_2\beta_{13} - d\alpha_2\alpha_{13})2^{l_2-1} + (\alpha_3\beta_{12} - d\alpha_3\alpha_{12})2^{l_3-1}\right)}{2^{l-1}-1}$$

and using (9) to find $\alpha_T$ and $\alpha_{T'}$ we have

$$\alpha_T = \frac{\alpha_1\alpha_{23}2^{l_1-1} + \alpha_2\alpha_{13}2^{l_2-1} + \alpha_3\alpha_{12}2^{l_3-1}}{2^{l-1}-1}$$

$$\alpha_{T'} = \frac{\alpha_1\alpha_y 2^{l-2}}{2^{l-1}-1}$$

This gives

$$\begin{aligned}\frac{(\alpha_T\beta_{T'} - \alpha_{T'}\beta_T)(2^{l-1}-1)^2}{\alpha_y 2^{l-2}} &= \left(\alpha_2\alpha_{13}2^{l_2-1} + \alpha_3\alpha_{12}2^{l_3-1}\right)\beta_1 \\ &\quad - \alpha_1\left((\alpha_2\beta_{13} - d\alpha_2\alpha_{13})2^{l_2-1} + (\alpha_3\beta_{12} - d\alpha_3\alpha_{12})2^{l_3-1}\right) \\ &= 2^{l_2-1}\alpha_2(\alpha_{13}\beta_1 - \alpha_1\beta_{13} + d\alpha_1\alpha_{13}) \\ &\quad + 2^{l_3-1}\alpha_3(\alpha_{12}\beta_1 - \alpha_1\beta_{12} + d\alpha_1\alpha_{12})\end{aligned}$$

Finally, we have

$$\beta_{13} = \frac{\beta_1\alpha_3 2^{l_{13}-2} - d\alpha_{13}}{2^{l_{13}-1}-1} \quad \text{and} \quad \alpha_{13} = \frac{\alpha_1\alpha_3 2^{l_{13}-2}}{2^{l_{13}-1}-1}$$

so

$$\begin{aligned}\alpha_{13}\beta_1 - \alpha_1\beta_{13} + d\alpha_1\alpha_{13} &= \frac{\alpha_1\alpha_3 2^{l_{13}-2}}{2^{l_{13}-1}-1}\beta_1 - \alpha_1\frac{\beta_1\alpha_3 2^{l_{13}-2} - d\alpha_{13}}{2^{l_{13}-1}-1} + d\alpha_1\alpha_{13} \\ &= \frac{d\alpha_1\alpha_{13}2^{l_{13}-1}}{2^{l_{13}-1}-1}\end{aligned}$$

and similarly

$$\alpha_{12}\beta_1 - \alpha_1\beta_{12} + d\alpha_1\alpha_{12} = \frac{d\alpha_1\alpha_{12}2^{l_{12}-1}}{2^{l_{12}-1}-1}$$

Therefore

$$\begin{aligned}\alpha_T\beta_{T'} - \alpha_{T'}\beta_T &= \frac{\alpha_y 2^{l-2}}{(2^{l-1}-1)^2}\left[2^{l_2-1}\alpha_2\frac{d\alpha_1\alpha_{13}2^{l_{13}-1}}{2^{l_{13}-1}-1} + 2^{l_3-1}\alpha_3\frac{d\alpha_1\alpha_{12}2^{l_{12}-1}}{2^{l_{12}-1}-1}\right] \\ &= \frac{\alpha_y\left(2^{l-2}\right)^2 d\alpha_1}{(2^{l-1}-1)^2}\left[\frac{\alpha_2\alpha_{13}}{2^{l_{13}-1}-1} + \frac{\alpha_3\alpha_{12}}{2^{l_{12}-1}-1}\right] \\ &= \frac{d\alpha_{T'}2^{l-2}}{2^{l-1}-1}\left[\frac{\alpha_2\alpha_{13}}{2^{l_{13}-1}-1} + \frac{\alpha_3\alpha_{12}}{2^{l_{12}-1}-1}\right]\end{aligned}$$

and since $\alpha_S > 0$ for all trees $S$, we have $\alpha_T\beta_{T'} - \alpha_{T'}\beta_T > 0$ as required. $\square$

# 5 Embeddings and other mappings into the complete $p$-ary tree

We have shown that the result of Kubicki, Lehel and Morayne, that

$$\frac{A(n;T_1)}{C(n;T_1)} \le \frac{A(n;T_2)}{C(n;T_2)}$$

for binary trees $T_1, T_2$ such that $T_2$ contains a subposet isomorphic to $T_1$, does not extend to arbitrary trees $T_1 \subseteq T_2$. Here, we look at generalisations of the result in other directions, for example by looking at embeddings into the complete $p$-ary tree, for any $p \ge 2$. We will also generalise the result to strict order-preserving maps of arbitrary trees into the complete $p$-ary tree.

Recall that $T_p^n$ is the complete $p$-ary tree with root $1_n$, $A_p(n;T) = |\{S \subseteq T_p^n : 1_n \in S, S \cong T\}|$ and $B_p(n;T) = |\{S \subseteq T_p^n : 1_n \notin S, S \cong T\}|$. Define $C_p(n;T)$ to be the sum $A_p(n;T) + B_p(n;T)$. Recall that in these definitions the isomorphisms are on unlabelled trees.

Let $A_T^{(p)}(n)$ be the number of embeddings of $T$ into $T_p^n$ that map the root $1_T$ of $T$ to $1_n$ and let $B_T^{(p)}(n)$ be the number of embeddings that do not map the root to $1_n$. Let $C_T^{(p)}(n) = A_T^{(p)}(n) + B_T^{(p)}(n)$ be the total number of embeddings of $T$ into $T_p^n$. As before, we have $A_T^{(p)}(n) = |G|A_p(n;T)$, $B_T^{(p)}(n) = |G|B_p(n;T)$ and $C_T^{(p)}(n) = |G|C_p(n;T)$ where $G$ is the group of symmetries of the (unlabelled) tree $T$.

We prove the result that

$$\frac{A_{T_1}^{(p)}(n)}{C_{T_1}^{(p)}(n)} \le \frac{A_{T_2}^{(p)}(n)}{C_{T_2}^{(p)}(n)}$$

for binary trees $T_1, T_2$ such that $T_2$ contains a subposet isomorphic to $T_1$. We do so by defining an appropriate distributive lattice and then applying the FKG-inequality. The FKG-inequality is a powerful corollary of the Four Functions Theorem by Ahlswede and Daykin. See, for example, [1] for a background to the FKG-inequality and examples of its use in probabilistic combinatorics. We state a form of the inequality that we will use repeatedly.

**Theorem 9 (Fortuin, Kasteleyn, Ginibre (1971)).** *If $(\mathcal{F}, <)$ is a finite distributive lattice and if $\alpha, \beta$ are both increasing (or both decreasing) non-negative functions on $\mathcal{F}$ and $\mu$ is a non-negative function on $\mathcal{F}$ such that $\mu(f)\mu(g) \le \mu(f \vee g)\mu(f \wedge g)$ for all $f, g \in \mathcal{F}$, then*

$$\sum_{f\in\mathcal{F}} \mu(f)\alpha(f) \sum_{f\in\mathcal{F}} \mu(f)\beta(f) \le \sum_{f\in\mathcal{F}} \mu(f) \sum_{f\in\mathcal{F}} \mu(f)\alpha(f)\beta(f) \tag{15}$$

A function $\mu$ on a lattice $\mathcal{F}$ is said to be *log-supermodular* if

$$\mu(f)\mu(g) \le \mu(f \vee g)\mu(f \wedge g) \quad \text{for all } f, g \in \mathcal{F}. \tag{16}$$

The power of this result means that Theorem 12 can be viewed as just one of many correlation inequalities for embeddings of binary trees into complete trees. We define an appropriate distributive lattice $\mathcal{F}$ and log-supermodular function $\mu$ so that $\sum_{f \in \mathcal{F}} \mu(f)$ equals the number of embeddings into $T_p^n$. Then we have the FKG-inequality (15) for *any* pair of increasing functions $\alpha, \beta$. As we will see, the definition of the lattice $\mathcal{F}$ means that the indicator functions of events like "the root of $T$ is mapped to $1_n$" or "element $x \in T$ is mapped to a high level of $T_p^n$" will be increasing on $\mathcal{F}$. The FKG-inequality then tells us that events like this are positively correlated, i.e., the probability that one event occurs increases if we condition on the other event occurring.

We only need consider the case where $T_1 \cong T_2 \setminus \{m\}$, since we can reduce to this case by the following lemmas. Lemma 10 is obvious, and the proof of Lemma 11 can be found in [2].

**Lemma 10.** *Given a binary tree, the following types of operation produce another binary tree with one element fewer.*

*(a) Removing a leaf,*

*(b) Removing the lower cover of an element that has exactly one lower cover.*

□

Note that if an element has exactly one lower cover and the lower cover is also a leaf, removing this leaf can be considered as an operation of both types. Also, note that we can think of operation (b) as contracting the edge between the element and its lower cover, that is, identifying them in the new tree.

**Lemma 11.** *If $T_1$ and $T_2$ are binary trees and $T_2$ contains a subposet isomorphic to $T_1$, then there is a sequence of operations of type (a) and (b) leading from $T_2$ to an isomorphic copy of $T_1$ through binary trees.* □

**Theorem 12.** *If $T_1$ and $T_2$ are binary trees such that $T_2$ contains a subposet isomorphic to $T_1$, then*

$$\frac{A_{T_1}^{(p)}(n)}{C_{T_1}^{(p)}(n)} \leq \frac{A_{T_2}^{(p)}(n)}{C_{T_2}^{(p)}(n)} \tag{17}$$

**Proof.** From Lemma 11 it is enough to show (17) for the particular cases where $T_1$ is isomorphic to the subposet $T_2 \setminus \{m\}$ produced from $T_2$ by exactly one operation of either type (a) or (b). For ease of notation we identify $T_1$ with the subposet $T_2 \setminus \{m\}$.

Firstly, we define a distributive lattice. Write $[n]$ for the chain on the $n$-element set $\{1, 2, \ldots, n\}$ with the natural ordering. For any binary tree $T$, write $\mathcal{F}_T = \mathcal{F}(n; T)$ for the lattice of strict order-preserving maps from $T$ to $[n]$. So $f \in \mathcal{F}_T$ is a function from $T$ to $[n]$ such that $x > y$ in $T$ implies $f(x) > f(y)$ in $[n]$. The ordering on $\mathcal{F}_T$ is $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in T$. The join, $f \vee g$, is the pointwise maximum of $f$ and $g$, and the meet, $f \wedge g$, is the pointwise minimum of $f$ and $g$. It is relatively simple to check that $\mathcal{F}_T$ is a distributive lattice.

We call a function in $\mathcal{F}_T$ a *level function.* If we have an embedding $\phi$ of $T$ into $T_p^n$, we can construct a function $f$ by setting $f(x)$ equal to the level of $\phi(x)$ in $T_p^n$. Since $\phi$ is an embedding, $x > y$ in $T$ implies that the level of $\phi(x)$ is greater than the level of $\phi(y)$, and so $f(x) > f(y)$. Therefore, $f$ is a level function and we say that $\phi$ *corresponds to* $f$. In fact, we can do this for any strict order-preserving map $\phi$ from $T$ to $T_p^n$. For each level function $f \in \mathcal{F}_T$ we can count the number of embeddings from $T$ to $T_p^n$ that correspond to $f$. This defines a function $\mu$ from $\mathcal{F}_T$ to $\mathbb{R}_+$: $\mu(f) = \mu_1(f)\mu_2(f)$ where $\mu_1, \mu_2$ are defined as

$$\mu_1(f) = p^{n-f(1_T)} \prod_{x>y,\text{ an edge in }T} p^{f(x)-f(y)},$$

$$\mu_2(f) = \prod_{\substack{y\in T,\\ y\text{ has 2 lower}\\ \text{covers, } z_1,z_2}} (1 - p^{\max\{f(z_1),f(z_2)\}-f(y)}).$$

Here, $\mu_1(f)$ counts the number of strict order-preserving maps from $T$ to $T_p^n$ that correspond to the level function $f$. However, a strict order-preserving map from $T$ to $T_p^n$ need not be an embedding of $T$ into $T_p^n$. The term $\mu_2(f)$ is exactly the fraction of those strict order-preserving maps from $T$ to $T_p^n$ corresponding to the level function $f$ that are also embeddings of $T$ into $T_p^n$. To see that $\mu_1(f)$ and $\mu_2(f)$ are as claimed, suppose we are constructing a strict order-preserving map $\phi$ that corresponds to $f$, by choosing the element $\phi(x)$ from level $f(x)$, for each $x$ from the root, $1_T$, downwards. We have $p^{n-f(1_T)}$ choices for $\phi(1_T)$, and then for each edge $x > y$ in $T$, once we have chosen $\phi(x)$ we have $p^{f(x)-f(y)}$ choices for $\phi(y)$. This gives a total of $\mu_1(f)$ strict order-preserving maps. Since we have $\phi(x) > \phi(y)$ for all $x > y$ in $T$, the map $\phi$ is an embedding if $\phi(z_1)$ and $\phi(z_2)$ are incomparable for all elements $z_1$, $z_2$ with a common upper cover in $T$. Let $y$ be some element of $T$ which has two lower covers $z_1, z_2$ and, without loss of generality, suppose that $f(z_1) \geq f(z_2)$. When constructing $\phi$, once we have chosen $\phi(y)$ and $\phi(z_2)$ (elements in the levels $f(y)$ and $f(z_2)$ respectively), there are $p^{f(y)-f(z_1)}$ choices for $\phi(z_1)$. One of these choices (the element on the path between $\phi(y)$ and $\phi(z_2)$) will give $\phi(z_1) > \phi(z_2)$ in $T_p^n$, meaning that $\phi$ is not an embedding. The other choices mean $\phi(z_1)$ and $\phi(z_2)$ are incomparable as required for $\phi$ to be an embedding. Because of the regularity of $T_p^n$, these numbers are independent of the choice of $\phi(z_2)$, so the fraction of choices which allow $\phi$ to be an embedding is $1 - p^{-(f(y)-f(z_1))}$. So, taking the product over all such $y$ gives the expression $\mu_2(f)$ as the fraction of strict order-preserving maps (corresponding to $f$) that are also embeddings.

**Claim 1.** *$\mu$ is log-supermodular on $\mathcal{F}_T$.*

**Proof of Claim.** Since $(f \wedge g)(x) + (f \vee g)(x) = \min(f(x), g(x)) + \max(f(x), g(x)) = f(x) + g(x)$ for all $x \in T$, we have that $\mu_1(f)\mu_1(g) = \mu_1(f \wedge g)\mu_1(f \vee g)$. So, it is enough to prove (16) for $\mu_2$. For each $y \in T$ with two lower covers, $z_1, z_2$, write $\sigma(f) = \max(f(z_1), f(z_2)) - f(y)$. Since $\mu_2$ is a product of terms indexed by such $y$, it is sufficient to prove that

$$(1 - p^{\sigma(f)})(1 - p^{\sigma(g)}) \leq (1 - p^{\sigma(f\wedge g)})(1 - p^{\sigma(f\vee g)}) \tag{18}$$

for all $y \in T$ with two lower covers.

Without loss of generality, we can assume that $f(z_1) \geq f(z_2), g(z_1), g(z_2)$. So

$$\begin{aligned}
\sigma(f \wedge g) + \sigma(f \vee g) &= \max\{\min(f(z_1), g(z_1)), \min(f(z_2), g(z_2))\} - \min\{f(y), g(y)\} \\
&\quad + \max\{\max(f(z_1), g(z_1)), \max(f(z_2), g(z_2))\} - \max\{f(y), g(y)\} \\
&= \max\{g(z_1), \min(f(z_2), g(z_2))\} + f(z_1) - f(y) - g(y) \\
&\leq \max\{g(z_1), g(z_2)\} + f(z_1) - f(y) - g(y) \\
&= \sigma(f) + \sigma(g)
\end{aligned}$$

(with equality unless both $g(z_1) < g(z_2)$ and $f(z_2) < g(z_2)$). Moreover, since $\sigma(f \vee g) = f(z_1) - \max\{f(y), g(y)\}$, if $f(y) \geq g(y)$ then $\sigma(f \vee g) = \sigma(f)$ and so $\sigma(f \wedge g) \leq \sigma(g)$ and then (18) follows. Otherwise, $f(y) < g(y)$. Set $s = g(y) - f(y) > 0$. Then $\sigma(f \vee g) = f(z_1) - g(y) = \sigma(f) - s$ and $\sigma(f \wedge g) = \max\{g(z_1), \min(f(z_2), g(z_2))\} - f(y) \leq \sigma(g) + s$. Also, $\sigma(g) + s = \max\{g(z_1), g(z_2)\} - g(y) + s \leq f(z_1) - f(y) = \sigma(f)$. So,

$$\begin{aligned}
(1 - p^{\sigma(f \wedge g)})(1 - p^{\sigma(f \vee g)}) &\geq (1 - p^{\sigma(g)+s})(1 - p^{\sigma(f)-s}) \\
&= 1 - p^{\sigma(g)+s} - p^{\sigma(f)-s} + p^{\sigma(f)+\sigma(g)} \\
&\geq 1 - p^{\sigma(g)} - p^{\sigma(f)} + p^{\sigma(f)+\sigma(g)} \\
&= (1 - p^{\sigma(f)})(1 - p^{\sigma(g)}),
\end{aligned}$$

where the second inequality holds since the function $\chi : x \mapsto p^x$ is convex for all $x \in \mathbb{R}$, and $\sigma(g) \leq \sigma(g) + s, \sigma(f) - s \leq \sigma(f)$ with $s > 0$. □

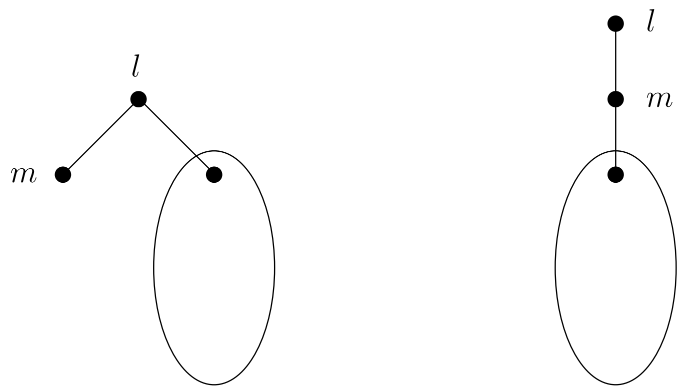
So, we have that $\mu$ is log-supermodular on $\mathcal{F}_T$, and therefore the restriction $\mu'$ of $\mu$ to any sublattice $\mathcal{F}'$ of $\mathcal{F}_T$ is log-supermodular on $\mathcal{F}'$.

We have that the number of embeddings of $T$ into $T_p^n$ is $\sum_{f \in \mathcal{F}_T} \mu(f)$. Also, we can split a tree $T$ at any point and perform similar sums on the two subtrees. Let $x$ be an element of $T$ and define subtrees $S_1 = T \setminus D(x)$ and $S_2 = D[x]$ and consider two lattices $\mathcal{F}_1(k) = \{f \in \mathcal{F}(n; S_1) : f(x) = k\}$ and $\mathcal{F}_2(k) = \{f \in \mathcal{F}(k; S_2) : f(x) = k\}$, where $1 \leq k \leq n$. The $\sum_{f \in \mathcal{F}_1(k)} \mu(f)$ is the number of embeddings of $S_1$ into $T_p^n$ that map $x$ to an element of $T_p^n$ in level $k$, and $\sum_{f \in \mathcal{F}_2(k)} \mu(f)$ is the number of embeddings of $S_2$ into $T_p^k$ that map $x$ to the root (the only element in level $k$ of $T_p^k$). Consider any pair of embeddings $(\phi_1, \phi_2)$ where $\phi_1$ is an embedding of $S_1$ into $T_p^n$ that maps $x$ to an element in level $k$, and $\phi_2$ is an embedding of $S_2$ into $T_p^k$ that maps $x$ to the root of $T_p^k$. We can construct an embedding $\phi$ of $T$ into $T_p^n$ as follows. For any point $y \in S_1$, define $\phi(y)$ to be $\phi_1(y)$. So, the point $x \in S_1$ is mapped to $\phi(x) = \phi_1(x)$, an element in level $k$. So, $\phi_1$ defines a unique copy of $T_p^k$ in $T_p^n$, namely the down-set of $\phi_1(x)$ in $T_p^n$. So, for elements $y \in S_2$ define $\phi(y)$ to be the element corresponding to $\phi_2(y)$ in this copy of $T_p^k$. Since the only element in $S_1 \cap S_2$ is $x$ and $\phi_2(x)$ is by definition the root of $T_p^k$, we have a well defined function $\phi$. It is easy to see that $\phi$ is indeed an embedding of $T$ into $T_p^n$. Therefore, $\phi$ is an embedding of $T$ into $T_p^n$ that maps $x$ to an element in level $k$. Since any embedding of $T$ into $T_p^n$ that maps $x$ to an element in level $k$ can be split into two embeddings by reversing this process, we have that the number of embeddings of $T$ into $T_p^n$ that map $x$ to an element in level $k$ is $\sum_{f \in \mathcal{F}_1(k)} \mu(f) \sum_{g \in \mathcal{F}_2(k)} \mu(g)$ and therefore the total number of embeddings of $T$ into $T^n$ is

$$\sum_{k=1}^{n} \sum_{f \in \mathcal{F}_1(k)} \mu(f) \sum_{g \in \mathcal{F}_2(k)} \mu(g). \tag{19}$$

Note that this holds for any element $x$ in $T$.

Recall that $m$ is the point removed from $T_2$ to obtain $T_1$. Let $l$ be the upper cover of $m$ in $T_2$. Write $T_t$ for the subtree $T_1 \setminus D(l)$, and $T_b$ for $D[l]$ as a subtree of $T_1$. Note that we have split $T_1$ into two trees $T_t$ and $T_b$ as explained earlier. Write $T_b+$ for the tree $D[l]$ as a subtree of $T_2$, so that $T_b+ = T_b \cup \{m\}$. Therefore, we have split $T_2$ into two trees $T_t$ and $T_b+$. So, $T_t$ is common to both trees $T_1, T_2$ and $T_b$ and $T_b+$ differ by only one element. Furthermore, since we have that $T_1$ is obtained from $T_2$ either by (a) removing a leaf, or (b) removing the lower cover of an element with exactly one lower cover, we know that either (a) $T_b+$ has the extra element $m$ as a leaf, directly below the root $l$ of $T_b+$, or (b) $T_b+$ has the extra element $m$ as the only lower cover of $l$. (See Figure 5.)




(a) $m$ is a leaf

(b) $m$ is the only lower cover of $l$

Figure 5: The two cases for $T_b+$

Let us look at the sublattice $\mathcal{F}'$ of $\mathcal{F}(n;T_t)$ defined by $\mathcal{F}' = \{f \in \mathcal{F}(n;T_t) : f(l) = k \text{ or } f(l) = k+1\}$, for $1 \leq k < n$. We have $\mu$ defined on $\mathcal{F}'$ as described earlier, and $\mu$ is log-supermodular. Define $\alpha(f) = I\{f(1_{T_t}) = n\}$ as the indicator function of the event $f(1_{T_t}) = n$ and define $\beta(f) = I\{f(l) = k+1\}$ as the indicator of the event $f(l) = k+1$. Both $\alpha$ and $\beta$ are increasing functions, since the sets $\{f : f(1_{T_t}) = n\}$ and $\{f : f(l) = k+1\}$ are both up-sets of $\mathcal{F}'$.

For $k = 1, \ldots, n$, let $a_k$ be the number of embeddings of $T_t$ into $T_p^n$ that map $l$ to an element in level $k$, and let $b_k$ be the number of embeddings of $T_t$ into $T_p^n$ that map $l$ to an element in level $k$ and map the root $1_{T_t}$ to the root $1_n$. Then,

$$\sum_{f \in \mathcal{F}'} \mu(f)\alpha(f) = b_k + b_{k+1}, \qquad \sum_{f \in \mathcal{F}'} \mu(f) = a_k + a_{k+1},$$

$$\sum_{f \in \mathcal{F}'} \mu(f)\beta(f) = a_{k+1}, \qquad \sum_{f \in \mathcal{F}'} \mu(f)\alpha(f)\beta(f) = b_{k+1},$$

and applying Theorem 9 to $\mathcal{F}', \mu, \alpha, \beta$ gives $(b_k + b_{k+1})a_{k+1} \leq (a_k + a_{k+1})b_{k+1}$ or

$$\frac{b_k}{a_k} \leq \frac{b_{k+1}}{a_{k+1}}$$

for all $k$, $1 \leq k < n$.

Now let us look at the trees $T_b$ and $T_b+$. Let $c_k$ be the number of embeddings of $T_b$ into $T_p^k$ that map $l$ to $1_k$, and let $d_k$ be the number of embeddings of $T_b+$ into $T_p^k$ that map $l$ to $1_k$, for $k = 1, \ldots, n$. First consider case (a), where $m$ is a leaf of $T_b+$.

Each embedding of $T_b+$ with $l$ mapped to $1_k$ can be thought of as an extension of an embedding of $T_b$ with $l$ mapped to $1_k$. Moreover, since $l$ has at most two lower covers in $T_b+$, one of which is $m$, every embedding of $T_b$ with $l$ mapped to $1_k$ can be extended to at least $p^{k-1}-1$ distinct embeddings of $T_b+$ with $l$ mapped to $1_k$, but to at most $(p^k-1)/(p-1)-1$ distinct embeddings of $T_b+$ with $l$ mapped to $1_k$. Therefore,

$$\frac{d_k}{c_k} \leq \frac{p^k-1}{p-1} - 1 < p^k - 1 \leq \frac{d_{k+1}}{c_{k+1}}.$$

We now show that $d_k/c_k \leq d_{k+1}/c_{k+1}$ also holds in case (b), again using Theorem 9. Let $\mathcal{F}''$ be the sublattice of $\mathcal{F}(k+1; T_b)$ defined as $\mathcal{F}'' = \{f \in \mathcal{F}(k+1; T_b) : f(l) = k \text{ or } f(l) = k+1\}$, for $1 \leq k < n$. Take $\mu$ defined on this sublattice as before, so that $\mu$ is log-supermodular. Define $\alpha(f) = I\{f(l) = k+1\}$ and define $\beta(f) = (p^{f_{min}}-1)/(p-1)-1$, where $f_{min} = \min_{x \in T_b} f(x)$. We have that $\alpha$ is increasing on $\mathcal{F}''$, and $f_{min}$ is increasing on $\mathcal{F}''$ therefore $\beta$ is also increasing on $\mathcal{F}''$. Before applying Theorem 9 we show what each of the terms in (15) is.

Since there are $p$ elements in level $k$ of $T_p^{k+1}$ each of the $c_k$ embeddings corresponds to $p$ embeddings in the sum $\sum_{f \in \mathcal{F}''} \mu(f)$, so this equals $pc_k + c_{k+1}$. The sum $\sum_{f \in \mathcal{F}''} \mu(f)\alpha(f)$ equals $c_{k+1}$. The sum $\sum_{f \in \mathcal{F}''} \mu(f)\beta(f)$ counts the number of embeddings of $T_b+$ into $T_p^{k+1}$ that map $l$ to an element in level $k$ or $k+1$. To see this, fix $f$ in $\mathcal{F}''$ and let $\phi$ be an embedding of $T_b$ into $T_p^{k+1}$ that corresponds to $f$. By definition the lowest level mapped to by $\phi$ is $f_{min}$, so $\phi$ maps the elements of $T_b$ to elements of $T_p^{k+1}$ between levels $f_{min}$ and $f(l)$ inclusive. In fact, it maps $T_b$ into a copy of $T_p^{f(l)-f_{min}+1}$ defined as the elements in the down-set of $\phi(l)$ that are in levels $f_{min}$ to $f(l)$ of $T^{k+1}$, inclusive. Call this copy $T_f$. We can construct an embedding $\psi$ of $T_b+$ into $T_p^{k+1}$ as follows. Choose some integer $i$ between 1 and $f_{min}-1$, this is the number of levels by which we will "push down" the embedding $\phi$ so as to "fit in" the element $m$. (So, if $f_{min} = 1$ this construction does not yield an embedding of $T_b+$, which agrees with $\mu(f)\beta(f) = 0$ for $f_{min} = 1$.) Define $\psi(l)$ to be $\phi(l)$ and define $\psi(m)$ to be any element in level $f(l) - i$ that is below $\psi(l)$. Once this choice is made $\psi$ is then determined. Consider the copy of $T_p^{f(l)-i}$ that is the down-set of $\psi(m)$. By the choice of $i$, this has at least as many levels as $T_f$, so just considering the top $f(l) - f_{min} + 1$ levels, we have a copy of $T_f$. Then, for all $x \in T_b+$ with $x \neq l, m$, define $\psi(x)$ to be the element in this copy of $T_f$ that corresponds to the element $\phi(x)$ in the original $T_f$. Since for each $i$ we have a choice of $p^i$ elements for $\psi(m)$, the total number of distinct embeddings this construction yields for a particular $\phi$ that corresponds to $f$ is

$$\sum_{i=1}^{f_{min}} p^i = \frac{p^{f_{min}} - 1}{p - 1} - 1 = \beta(f)$$

Since there are $\mu(f)$ distinct embeddings that correspond to $f$, this construction yields $\sum_{f \in \mathcal{F}''} \mu(f)\beta(f)$ distinct embeddings of $T_b+$ into $T_p^{k+1}$ that map $l$ to an element in level $k$ or $k+1$.

Since each embedding of $T_b+$ into $T_p^{k+1}$ that maps $l$ to level $k$ or $k+1$ can be converted to an embedding of $T_b$ into $T_p^{k+1}$ that maps $l$ to level $k$ or $k+1$ by reversing the above construction, we have that the total number of embeddings of $T_b+$ into $T_p^{k+1}$ that map $l$ to an element in level $k$ or $k+1$ is exactly $\sum_{f \in \mathcal{F}''} \mu(f)\beta(f)$. Therefore, $\sum_{f \in \mathcal{F}''} \mu(f)\beta(f) = pd_k + d_{k+1}$ and $\sum_{f \in \mathcal{F}''} \mu(f)\alpha(f)\beta(f) = d_{k+1}$. So, applying Theorem 9 gives $c_{k+1}(pd_k + d_{k+1}) \leq (pc_k + c_{k+1})d_{k+1}$ which is equivalent to the inequality $d_k/c_k \leq d_{k+1}/c_{k+1}$.

So, we have two increasing sequences $(b_k/a_k)$ and $(d_k/c_k)$ for $k = 1, \ldots, n$. We need to apply Theorem 9 once more to a very simple lattice, namely the $n$-element chain, $[n]$. A chain is obviously a distributive lattice, and moreover *any* function $\mu$ is log-supermodular, since $\{k, k'\} = \{k \wedge k', k \vee k'\}$ for all $k, k' \in [n]$. Define $\mu(k) = a_k c_k$, define $\alpha(k) = b_k/a_k$, and define $\beta(k) = d_k/c_k$. Then $\alpha$ and $\beta$ are increasing on $[n]$, and applying Theorem 9 gives

$$\sum_{k=1}^{n} b_k c_k \sum_{k=1}^{n} a_k d_k \leq \sum_{k=1}^{n} a_k c_k \sum_{k=1}^{n} b_k d_k. \tag{20}$$

But $\sum_{k=1}^{n} a_k c_k$ is the total number of embeddings of $T_1$ into $T_p^n$, as we split $T_1$ into $T_t$ and $T_b$. Similarly, $\sum_{k=1}^{n} a_k d_k$ is the total number of embeddings of $T_2$ into $T_p^n$, as we split $T_2$ into $T_t$ and $T_b+$. Since $b_k$ only counts those embeddings counted by $a_k$ that also map the root of $T_t$ to $1_n$, we have that $\sum_{k=1}^{n} b_k c_k$ is the number of embeddings of $T_1$ into $T_p^n$ that map the root of $T_1$ to $1_n$, and $\sum_{k=1}^{n} b_k d_k$ is the number of embeddings of $T_2$ into $T_p^n$ that map the root of $T_2$ to $1_n$.

Therefore equation (20) becomes

$$A_{T_1}^{(p)}(n) C_{T_2}^{(p)}(n) \leq C_{T_1}^{(p)}(n) A_{T_2}^{(p)}(n)$$

as required. □

Note that the proof is similar in its approach to the original proof by Kubicki, Lehel and Morayne, however in the set-up where we can apply the FKG-inequality we can view this result as one of many possible correlation inequalities on the lattice $\mathcal{F}(n;T)$, for $T$ some binary tree. Informally, in the proof of Theorem 12 we first show that the events "the root of $T_t$ is mapped to a high level of $T_p^n$" and "the element $l$ is mapped to a high level of $T_p^n$ are positively correlated on the lattice $\mathcal{F}(n;T_t)$. We then show that in the lattice $\mathcal{F}(k;T_b)$ having "$l$ mapped to a high level of $T_p^k$" means "the number of ways to embed an extra element" increases. We combine these correlations to show that if the root of $T_1$ is embedded "higher up" in $T_p^n$, then there are more embeddings of an extra element into $T_p^n$.

We can use the lattice $\mathcal{F}(n;T)$ and the function $\mu$ and other pairs of increasing functions on $\mathcal{F}$, to find other correlation inequalities. For example, we have the following result, which informally says that for any binary tree $T$ and any two elements $x, y$ in $T$, the events "$x$ is mapped to a high level of $T_p^n$" and "$y$ is mapped to a high level of $T_p^n$" are positively correlated.

**Theorem 13.** *For any binary tree $T$, and any elements $x, y \in T$, and for any $k$ and $l$ with $1 \leq k, l < n$, we have*

$$\frac{E(k+1,l)}{E(k,l)} \leq \frac{E(k+1,l+1)}{E(k,l+1)},$$

*where $E(i,j)$ is the number of embeddings of $T$ into $T_p^n$ that map $x$ into level $i$, and $y$ into level $j$.*

**Proof.** Consider the sublattice $\mathcal{F}'$ of $\mathcal{F}(n;T)$ defined by $\mathcal{F}' = \{f \in \mathcal{F}(n;T) : f(x) = k, k+1 \text{ and } f(y) = l, l+1\}$. We take $\mu$ to be our log-supermodular function as described above, so that $\sum_{f\in\mathcal{F}'} \mu(f)$ is exactly $E(k,l) + E(k+1,l) + E(k,l+1) + E(k+1,l+1)$. Define $\alpha(f) = I\{f(x) = k+1\}$ as the indicator of the event $f(x) = k+1$, and define $\beta(f) = I\{f(y) = l+1\}$ as the indicator of the event $f(y) = l+1$. Both $\alpha$ and $\beta$ are increasing on $\mathcal{F}'$ and so we can apply Theorem 9. This gives the inequality

$$\begin{aligned}&\{E(k+1,l) + E(k+1,l+1)\}\{E(k,l+1) + E(k+1,l+1)\}\\ &\qquad \leq \{E(k,l) + E(k+1,l) + E(k,l+1) + E(k+1,l+1)\}\, E(k+1,l+1)\end{aligned}$$

which is equivalent to the required inequality. □

This statement is not true if $T$ is allowed to be arbitrary, as illustrated by the following example. Let $T$ be a tree with 4 elements, the root $x$ and its three lowers covers $x_1, x_2, x_3$. Suppose we are embedding $T$ into $T^4$, the complete binary tree on 4 levels. We can calculate the different number of embeddings that map the elements $x_1$ and $x_2$ into particular levels. There are 12 embeddings that map $x_1$ to level 3 and $x_2$ to level 2, there are 32 embeddings that map $x_1$ to level 3 and $x_2$ to level 1, there are 76 embeddings that map $x_1$ to level 2 and $x_2$ to level 2 and there are 184 embeddings that map $x_1$ to level 2 and $x_2$ to level 1. So, if we consider a uniform probability distribution over all embeddings of $T$ into $T^n$, we have that the conditional probability that an embedding maps $x_2$ into level 2, given that it maps $x_2$ into either level 1 or 2 and maps $x_1$ into level 3, is $12/32 = 3/8$. However, the conditional probability that an embedding maps $x_2$ into level 2, given that it maps $x_2$ into either level 1 or 2 and maps $x_1$ into level 2, is $76/184 = 19/46$ which is greater than $3/8$. In other words, it is more likely for $x_2$ to be in the higher of the two levels 1 and 2, if $x_1$ is in the lower of the two levels 2 and 3. This is still true for embeddings of $T$ into $T_p^4$ for $p > 2$. This means that we are unable to use this approach even for embeddings of $p$-ary trees into the complete $p$-ary tree.

In this sense the case of $T$ being binary is special. For arbitrary $T$ we cannot define a log-supermodular function $\mu$ on $\mathcal{F}(n;T)$ so that $\sum_{f\in\mathcal{F}(n;T)} \mu(f)$ is the number of embeddings of $T$ into $T_p^n$. However, we can look at other types of mapping from $T$ into $T_p^n$, for example strict order-preserving maps. In this case, the situation is very much simplified; as we have seen in the proof of Theorem 12 the function $\mu_1$, which counts the number of strict order-preserving maps, is log-supermodular on $\mathcal{F}$. Moreover, if we allow $T$ to be arbitrary, the function $\mu_1$ still counts the number of strict order-preserving maps. This is essentially because a strict order-preserving map only needs to preserve edges and not incomparability between elements. Therefore we can generalise the correlation inequalities for embeddings of binary trees to correlation inequalities for strict-order preserving maps of arbitrary trees.

For example, if we define $\tilde{A}_T^{(p)}(n)$ to be the number of strict order-preserving maps of $T$ into $T_p^n$ that map the root of $T$ to $1_n$, and define $\tilde{C}_T^{(p)}(n)$ to be the total number of strict order-preserving maps of $T$ into $T_p^n$, then we have the following result, corresponding to the inequality of Theorem 12.

**Theorem 14.** *If $T_1$ and $T_2$ are trees such that $T_2$ contains a subposet isomorphic to $T_1$, then*

$$\frac{\tilde{A}_{T_1}^{(p)}(n)}{\tilde{C}_{T_1}^{(p)}(n)} \leq \frac{\tilde{A}_{T_2}^{(p)}(n)}{\tilde{C}_{T_2}^{(p)}(n)}$$

**Proof.** We follow through the proof of Theorem 12, making the following necessary changes for strict order-preserving maps of arbitrary trees.

Firstly, note that we can define a distributive lattice of level functions $\mathcal{F}(n;T)$ when $T$ is an arbitrary tree. We take $\mu_1$ as our log-supermodular function. This satisfies log-supermodularity with equality (as noted in the proof of Theorem 12). Also, for any tree $T$, the sum $\sum_{f\in\mathcal{F}(n;T)} \mu_1(f)$ is the number of strict order-preserving maps of $T$ into $T_p^n$, as explained above.

If we define $\tilde{a}_k$ to be the number of strict order-preserving maps of $T_t$ into $T_p^n$ that map $l$ to an element of level $k$, and define $\tilde{b}_k$ to be the number of strict order-preserving maps of $T_t$ into $T_p^n$ that map $l$ to an element of level $k$ and map the root of $T_t$ to the root $1_n$, then

$$\frac{\tilde{b}_k}{\tilde{a}_k} \leq \frac{\tilde{b}_{k+1}}{\tilde{a}_{k+1}},$$

as in the proof of Theorem 12.

Now when comparing the trees $T_b$, $T_b+$, define $\tilde{c}_k$ to be the number of strict order-preserving maps of $T_b$ into $T_p^k$ that map $l$ to $1_k$, and define $\tilde{d}_k$ to be the number of strict order-preserving maps of $T_b+$ into $T_p^k$ that map $l$ to $1_k$. Whereas in the proof of Theorem 12 we had two cases to consider, here we just need that $m$ is the lower cover of $l$ in $T_b+$, where $l$ is the root of $T_b+$.

We use a similar construction to the one in the proof of Theorem 12 when considering case (b). However, since we are counting strict order-preserving maps and not embeddings, we can position $m$ as if it were the only lower cover of $l$. Let $D(m)$ be the elements in $T_b$ that are below $m$ in the tree $T_b+$. Define $\tilde{f}_{min} = \min_{x\in D(m)} f(x)$, unless $D(m)$ is empty, in which case let $\tilde{f}_{min} = f(l)$. Define $\beta(f) = (p^{\tilde{f}_{min}} - 1)/(p-1) - 1$. We have that each strict order-preserving map of $T_b$ into $T_p^{k+1}$ that corresponds to $f$ yields $\beta(f)$ strict order-preserving maps of $T_b+$ into $T_p^{k+1}$, and so applying Theorem 9 yields

$$\frac{\tilde{d}_k}{\tilde{c}_k} \leq \frac{\tilde{d}_{k+1}}{\tilde{c}_{k+1}}.$$

Finally, the last part of the proof is identical to the proof of Theorem 12 and we have

$$\tilde{A}_{T_1}^{(p)}(n)\tilde{C}_{T_2}^{(p)}(n) \leq \tilde{C}_{T_1}^{(p)}(n)\tilde{A}_{T_2}^{(p)}(n)$$

as required. □

As with embeddings of binary trees, by applying the FKG-inequality to different increasing functions, versions of this proof can be used to establish other correlation inequalities for strict order-preserving maps of arbitrary trees into the complete $p$-ary tree.

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