# Bilateral Street Searching in Manhattan (Line-Of-Sight Rendezvous on a Planar Lattice)

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**Abstract**

We consider the rendezvous problem faced by two mobile agents, initially placed according to a known distribution on intersections in Manhatten (nodes of the integer lattice $Z^2$). We assume they can distinguish streets from avenues (the two axes) but have no common notion of North or East (positive directions along axes). How should they move, from node to adjacent node, so as to minimize the expected time required to 'see' each other, to be on a common street or avenue. This problem can be viewed either as a bilateral form (with two players) of the *street searching* problems of computer science, or a 'line-of-sight' version of the *rendezvous* problem studied in operations research.

We show how this problem can be reduced to a *Double Alternating Search* (DAS) problem in which a single searcher minimizes the time required to find one of two objects hidden according to known distributions in distinct regions (e.g. a datum held on multiple disks), and we develop a theory for solving the latter problem. The DAS problem generalizes a related one introduced earlier by the author and J. V. Howard.

We solve the original rendezvous problem in the case that the searchers are initially no more than four streets or avenues apart.

**Keywords: rendezvous, mobile agents, street searching, search**

# 1 Introduction

The rendezvous search problem [1] asks how two unit-speed searchers (mobile agents) can minimize the expected time to 'find each other', after they are randomly placed in a known search region $Q$. Traditionally, they are said to have found each other at the first time when their distance is within a given 'detection radius', which we call proximity detection. The detection radius is usually taken to be zero when $Q$ is the line or a network. However, critics of rendezvous theory have sometimes argued that in reality mutual detection is often achieved without close proximity. Consider, for example, the query of D. Kafkewitz sent to the *New Scientist* [26, p. 36] (italics are mine):

> Two people lose each other while wandering through the aisles of a large supermarket. The height of the shelves precludes aisle-to-aisle visibility. One person wishes to find the other. Should that person stop moving and remain in a single visible site while the other person continues to move through the aisles? Or would an encounter or *sighting* occur sooner if both were moving through the aisles?

The answer given by the author [26, p. 37-38] interpreted the problem as rendezvous search on a network ([3][6]), and assumed proximity detection, which would be realistic in a crowded supermarket with very limited visibility. However in an empty supermarket a line-of-sight detection would be more appropriate. This is what we attempt to model in this paper. The notion of line-of-sight detection is well known in computer science in the context of 'street searching' (e.g. [25]), usually with the search region ('street') $Q$ a polygon. So in those terms this paper could be called *bilateral street searching.*

Stepping back for a moment to a more abstract perspective, we could posit a general 'detection relation' $z \sim w$ on the search region $Q$, so that the meeting time $T$ would be given by

$$T = \min \{t : f(t) \sim h(t)\}, \tag{1}$$

where $f(t)$ and $h(t)$ are the locations of players I and II at time $t$. The proximity model is then be defined by the metric $\rho$ on $Q$ as $z \sim w \Leftrightarrow \rho(z, w) \leq$ detection radius, and the line-of-sight model would be defined by $z \sim w \Leftrightarrow \overline{pq} \subset Q$, where $\overline{pq}$ is the straight line connecting $p$ and $q$.

This paper is the first to adopt line-of-sight detection in a rendezvous problem. We take the search region $Q$ to be the planar lattice (graph paper) consisting of all points in the plane with at least one integer coordinate. Moreover, we use a discrete model of Anderson and Fekete [12] in which at times $t = 0, 1, 2, \ldots$ both players are at the nodes of the lattice $Z^2$ consisting of 2-tuples with both coordinates integers, and adjacent nodes having one coordinate identical and the other differing by one. In this setting, we will say that two nodes $z$ and $w$ are mutually detectable if they satisfy the *lattice line-of-sight* relation,

$$z \sim w \ \Leftrightarrow \ z_1 = w_1 \text{ or } z_2 = w_2. \tag{2}$$

This particular notion of detection is identical to that of the so-called 'CNN Problem' studied in [22] where a mobile camera crew can film an event taking place on the same street or avenue as their van.

The problem we analyze here is called the Line-of-Sight Planar Rendezvous (LSPR) problem, denoted $\Gamma(D_1, D_2)$. Two agents are initially placed at nodes $z$ and $w$ whose vector difference $z - w = (2d_1, 2d_2)$ where the $d_i$ are positive integers drawn independently from given distributions $D_i$, $i = 1, 2$. Equivalently, we place Player I at the origin and Player II equiprobably at one of the four nodes $(\pm 2d_1, \pm 2d_2)$. The reason for requiring the initial coordinate differences to be *even* will be explained, together with some related assumptions, in Section 3. We assume that the players do not know the initial location of the other, and have no common notion of locations (e.g. no street or avenue names) or even of directions (e.g. North) in the plane. (The general significance of sense of direction for distributed computing problems was first analyzed in [19].) However we will assume that they can distinguish the two axes (e.g. streets are narrow and avenues are wide). Thus the symmetry notions we adopt here are the same as those of the proximity rendezvous problem $\Gamma_1'$ of [5].

The paper is organized as follows. In Section 2, we introduce the Double Alternating Search (DAS) Problem, whose solution will be a stepping stone to the solution of $\Gamma(D_1, D_2)$. The DAS problem asks how to minimize the time required to find one of two objects (say two identical car keys), each hidden in a distinct region (two rooms of a house). The probability that the object in region $i$ will be found after searching there for time $t$ is a given function $F_i(t)$, $i = 1, 2$. However only one region can be searched at a time. The problem is to find the optimal method of alternating the search between the two regions, that is, the method which finds one of the objects (either one) in least expected time. The DAS problem is of independent interest, and generalizes a similar search problem for a *single* object hidden at one of two regions (ASP) studied by the author and John Howard [10].

In Section 3 we give a rigorous definition of the LSPR problem $\Gamma(D_1, D_2)$ and establish its formal equivalence to a particular version of the DAS problem in which each 'region' consists of two lines, and is called the Search On Two Linepair (SOTL) problem. In Section 4 we give a complete solution to the LSPR on $Z^2$ for the case where the initial nodes $z$ and $w$ of the two players satisfy $|z_i - w_i| \leq 4$, $i = 1, 2$. Section 5 summarizes our results and suggests some extensions.

Other work on the rendezvous problem for the line or plane in the operations research literature can be found in [4], [5], [7], [8], [11], [12], [15], [20], and [24], with overviews in [2] and [9]. Related work on search and rendezvous in the computer science literature can be found in [13], [14], [18], [23], and [27].

## 2 Double Alternating Search (DAS) Problem

In order to solve the line of sight rendezvous problem on the planar lattice, it is necessary to introduce a new single-sided search problem. In this problem there are two objects hidden in distinct nearby regions, and the Searcher can alternate searching in the two regions, until he finds one of them. There is no switching cost.

It is observed in [22] that the CNN Problem discussed in the Introduction is related to that of retrieving information which resides on multiple disks, compared with the $2-$server problem (e.g. [17]) which related to multiple heads on a disk. Not surprisingly, it is the former problem which in its search theory form is a DAS problem.

We assume at first that there is a fixed method of searching each region, and that if he searches at region $i$ for time $t$ according to this method, he will find the object hidden there with known probability $F_i(t)$, $i = 1, 2$. If by time $t$ he has searched in region 1 for time $\tau(t)$ and region 2 for the remaining time $t - \tau(t)$, the probability that he has found at least one object (his aim) will be given by

$$F^{\tau}(t) = F_1(\tau(t)) + F_2(t - \tau(t)) - F_1(\tau(t)) F_2(t - \tau(t)). \tag{3}$$

For simplicity of exposition (and since this case is sufficient for the application to rendezvous), we will first assume that there are finite maximum search times $m$ and $n$ in each region, that is, $F_1(m) = 1$ and $F_2(n) = 1$. We first assume that the *alternation rule* $\tau(t)$ is (what we will call *simple*) a continuous nondecreasing function with $\tau' = 1$ on certain time intervals (during which region 1 is being searched) and $\tau' = 0$ on others (during which region 2 is being searched). We naturally assume $\tau(0) = 0$. A more geometric way of viewing the situation is that one object is hidden on a ray $I_1 = [0, m]$ according to distribution $F_1$ and the other on on a distinct ray $I_2 = [0, n]$ according to $F_2$. We use the term ray rather than interval to cover the cases where $m$ or $n$ might be infinite. If the searcher is following the alternation rule $\tau$, then he has searched the interval $[0, \tau(t)]$ on $I_1$ and the interval $[0, t - \tau(t)]$ on $I_2$ by time $t$. When position $m$ on $I_1$ or position $n$ on $I_2$ has been reached, the alternation rule will certainly have found an object. We call this time

$$\tau_{\max} = \min\{t : \tau(t) = m \text{ or } t - \tau(t) = n\}. \tag{4}$$

We can classify an alternation strategy $\tau$ by where it ends, that is, which region it searches exhaustively. We say that it ends in region $e(\tau)$. More precisely we define

$$e(\tau) = \begin{cases} 1, & \text{if } \tau(\tau_{\max}) = m, \\ 2, & \text{if } \tau_{\max} - \tau(\tau_{\max}) = n. \end{cases} \tag{5}$$

So for any alternation strategy, $e(\tau)$ is either 1 or 2. In some cases an alternation strategy $\tau$ will search a single region $i$ *exclusively*, which we emphasize by writing $e(\tau) = i^*$, that is,

$$e(\tau) = \begin{cases} 1^*, & \text{if } \tau(m) = m, \\ 2^*, & \text{if } \tau(n) = 0. \end{cases} \tag{6}$$

The problem for the searcher is to find the alternation rule $\tau$ which finds an object in least least expected time, such a rule will be called *optimal* and the least expected time will be denoted $V$,

$$V = V\left(F_1, F_2\right) = \min_{\tau} \int_0^{\tau_{\max}} t \; dF^{\tau}\left(t\right). \tag{7}$$

In the case that only one region, say region 1, is searched, we will write the expected search time as

$$V = V\left(F_1, \mathbf{0}\right) = \int_0^{\infty} t \; dF_1\left(t\right) = \int_0^{\infty} \left(1 - F\left(t\right)\right) \, d\left(t\right), \tag{8}$$

where $\mathbf{0}$ represents the identically zero distribution $\mathbf{0}\left(t\right) \equiv 0$. To insure the existence of the minimum we will in general have to consider all $\tau$ in the class $\Upsilon$ defined by the Lipshitz condition

$$0 \leq \tau\left(t\right) - \tau\left(t^*\right) \leq t - t^*, \text{ for all pairs } t^* < t. \tag{9}$$

Under the topology of uniform convergence on compact intervals, the set $\Upsilon$ is compact. And since the integral $\int_0^{\tau_{\max}} t \; dF^{\tau}\left(t\right)$ is lower semicontinuous in $\tau$, the minimum is guaranteed. For general alternation rules $\tau \in \Upsilon$, the derivative $\tau'\left(t\right)$ exists almost everywhere and is interpreted as the speed with which interval $I_1$ is being searched, with $I_2$ being searched at the speed $1 - \tau'\left(t\right)$. We can always approximate a general alternation rule $\tau \in \Upsilon$

A more general problem of this type occurs when there are many ways to search region 1 and many ways to search region 2. That is, many distribution pairs $F_1, F_2$. In this case the searcher has first to decide which ways to search each region (to choose particular $F_i$) and then to decide how to alternate searching the two regions. If the sets of distributions of 'finding times' in each region corresponding to allowable search methods are denoted $\mathcal{F}_1$ and $\mathcal{F}_2$, then optimal time for this problem is also denoted $V$, with

$$V\left(\mathcal{F}_1, \mathcal{F}_2\right) = \inf_{F_i \in \mathcal{F}_i} V\left(F_1, F_2\right). \tag{10}$$

We now give a story which illustrates the DAS problem in both its basic and general forms. Suppose you have just inherited some land under which there is definitely a large oil pool. There are already two drilling rigs, rig 1 and rig 2, with an electric generator able to drive one of them (or both at half speed). You are also given a geological survey which says that the probability of rig i reaching the oil if used for time $t$ at full speed is $F_i\left(t\right)$. Furthermore, assume that the oil that can be withdrawn from a single rig is more than you can transport, so striking oil with both rigs is no better than with just one. This problem can be modeled by the formula (7) for the DAS problem Now suppose furthermore that each rig can be set to drill at certain angles, each giving a different distribution of time required to reach the oil. The distributions obtainable for rig $i$ can be labelled $\mathcal{F}_i$ and the resulting problem can be modeled as the generalized

DAS problem (10). The generalized DAS problem is two-fold; first one must fix the drilling angles, and then one must decide how to alternate between drilling between the two rigs.

Another example of the DAS problem is the problem of finding a datum which is known to occur on two lists which can each only be read in a specified order. Suppose I am looking for a picture of a red cow, which I know is in my album of cows and also my album of red objects. I could start going through the list of cows, then switch for a while to the reds, then go back to the cows, etc. As soon as I find the picture in one list, I no longer need to find it in the other. If the lists don't have to be read in a specified order, then the problem is a generalized DAS problem. The author admits to finding himself in this problem often when looking for a phone number, as he has several phone lists, none of which is properly alphabetized.

One final example which may appeal to academic researchers is the question of how to attack a problem for which there are two (or more) methods which may be applied. Of course one may simply decide on a method and persevere until it reaches a solution. However if it is going slowly one may be tempted to switch to the other method. Here again, once the problem is solved by one method, there may be no need (at least in the short term) to pursue the other one.

We now specialize to the important special case where the distributions $F_1$ and $F_2$ are concentrated on a finite number of integers. We assume that one '$\alpha$' object has been hidden in one of a family of $m$ '$\alpha$-boxs' $\alpha_1, \ldots, \alpha_m$ and that another '$\beta$' object has been hidden in one of the boxes $\beta_1, \ldots, \beta_n$. The probabilities that the $\alpha$ object is in box $\alpha_i$ is given by $p_i$ (and similarly $q_j$ of being in $\beta_j$). Each family of boxes must be searched in order ( e.g. $\alpha_5$ before $\alpha_6$). It takes one unit of time to search a box, and if the object is there it is said to be found at the end of that time unit. To illustrate these ideas, suppose that $m = 3$, $n = 4$, and the associated probability density vectors are $p = (.8, .1, .1)$ and $q = (.7, 0, 0, .3)$. Then it can be shown (as outlined below) that the least expected time to find one of the two objects is given by

$$V = 1.29, \tag{11}$$

and the uniquely optimal alternation rule searches the boxes in the order

$$\alpha_1, \beta_1, \alpha_2, \alpha_3. \tag{12}$$

We will denote this discrete alternation rule as $[1, 2, 1, 1]$ where a 1 indicates the next $\alpha$ box is searched, a 2 indicates a $\beta$ box ($\alpha$ is considered region 1, $\beta$

region 2).

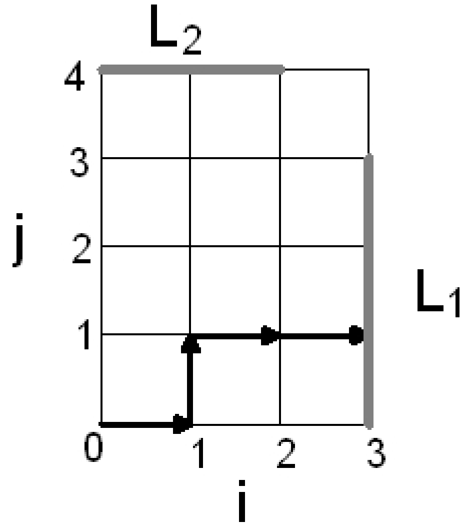


Figure 1: Alternation rule $[1, 2, 1, 1]$ .

Figure 1 illustrates the alternation rule $[1, 2, 1, 1]$. Observe that it is a directed path, which moves either up or right, from $(0, 0)$ to the termination set $L_T = L_1 \cup L_2$, where $L_1 = \{(i, j) : i = m = 3,\ j < n = 4\}$ and $L_2 = \{(i, j) : i < m = 3,\ j = n = 4\}$ . A vertical edge from $(i, j)$ to $(i, j + 1)$ represents a search of box $\beta_{j+1}$ after boxes $\alpha_1$ to $\alpha_i$ (and of course $\beta_1$ to $\beta_j$) have been searched. Such a search will always end at time $i + j + 1$ . If an alternation rule $\tau$ ends in $L_i$, then $e(\tau) = i$. If in addition it ends on the $i-$axis (that is, at $(3, 0)$ or $(0, 4)$) then $e(\tau) = i^*$.

The *optimal* path will be the shortest path from $(0, 0)$ to $L_T$ if we assign a length to each edge corresponding to its contribution to the expected finding time integral (sum) (7). To derive this, let $P_i$ and $Q_j$ denote the respective probabilities that the $\alpha$ object lies in the first $i$ of the $\alpha-$boxes or the first $j$ of the $\beta-$boxes (so $P_i = F_1(i)$ , $Q_j = F_2(j)$),

$$\begin{aligned} P_i &= p_1 + \cdots + p_i, \\ Q_j &= q_1 + \cdots + q_j. \end{aligned}$$

We now consider how to assign a 'length' to the edge from $(2, 1)$ to $(3, 1)$ in Figure 1 (the answer, .12, is the underlined number above the edge in Figure 2 For example, the probability that the first object to be found will be found at time $t = 4$ when box $\alpha_3$ is searched is the probability that it has not been found in boxes $\alpha_1$ or $\alpha_2$ (this is $(1 - P_2)$) times the probability that it has not been found in box $\beta_1$ (this is $(1 - Q_1)$) times the conditional probability that it will be found in box $\alpha_3$ (this is $p_3/(1 - P_2)$). This product simplifies to $(1 - Q_1)\, p_3$. Since in this case the time the object in $\alpha_3$ would be found is 4, we give the edge from (2,1) to (3,1) the 'length' $4\,(1-Q_1)\,p_3 = 4\,(1-.7)\,(.1) = .12$, which is the length (non-underlined number) assigned to the edge from $(2,1)$ to $(3,1)$ in Figure 2. In general, the probability that is assigned to the horizontal edge from $(i,j)$ to $(i+1,j)$ is $(1-Q_j)\,p_{i+1}$, so its contribution to the expected time (and its 'length') is

$$(i+j+1)\,(1-Q_j)\,(p_{i+1})$$

and similarly the 'length' assigned to the vertical edge form $(i,j)$ to $(i,j+1)$ is

$$(i+j+1)\,(1-P_j)\,(q_{i+1})\,.$$

Once this is done it is an easy recursive process to calculate the shortest directed path from $(0,0)$ to the termination set $L_T$. We have to calculate the (shortest) directed distance from each of the $mn$ lattice points to this set, starting with the point $(m-1,n-1)$ and then going to the left and down; then calculating this for $(m-2,n-2)$, etc.. We need only compare the total distance if we go up or right to the set already calculated. The distance from $(0,0)$ is the least expected time $V\,(P,Q) = V\,(F_1,F_2)$ of (7) and the shortest path corresponds to the optimal alternation rule. The full calculation, with the optimal search (right for $\alpha$, up for $\beta$) in bold arrow, is shown below. The edge lengths are indicated below or to the right of the edge, and the distance of a node $(i,j)$ to $L_T$ is given in the underlined number just above and to the right of the node. Note that if we start at the origin and follow the bold arrows, we obtain the optimal alternation rule of Figure 1.

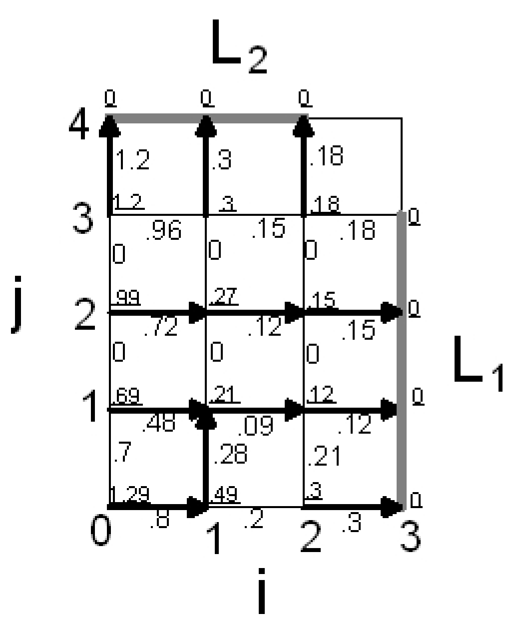



Figure 2: 'Shortest line' algorithm.

We now investigate the circumstances under which an optimal alternation rule can search an interval $A$ on ray 1 immediately before an interval $B$ on ray 2, both at maximum speed. In the corresponding *Alternating Search Problem* introduced in [10], where there is only one hidden object (equally likely on either ray) a necessary condition was that the average density (probability divided by length) of $A$ was at least as large as the average density of $B$. We now derive a similar but weaker condition for the DAS problem, which involves the *conditional density* of $A$, the probability that object 1 is in $A$ given that it has not been found before $A$ is reached, divided by the length of $A$. It also involves the *offset* of $A$, which measures the expected distance of the object from the right endpoint of $A$, assuming it lies in $A$. Of course for discrete distributions and intervals where the object can only be at the right endpoint, the offset will be zero.

Suppose that the interval $A = (a, a')$ on ray 1 has the following characteristics (with similar notation for the interval B on ray 2):

$$
\begin{aligned}
\mu_A &= F_1(a') - F_1(a) > 0 \,\text{(the mass of } A,\text{ probability object 1 is in } A) \\
c_A &= (1/\mu_A) \int_A t\, dF_1(t) - a \text{ (distance of center of gravity of } A \text{ from left endpoint)} \\
L_A &= a' - a = \text{length of } A > 0 \\
\rho_A &= \frac{\mu_A / (1 - F_1(a))}{L_A} \text{ conditional density of } A \\
O_A &= L_A - c_A = \text{offset of } A \text{ (distance of center of gravity from right endpoint).}
\end{aligned}
$$

**Proposition 1** *A necessary condition that an optimal alternation rule $\tau$ can search the interval $A$ on ray 1 immediately before the interval $B$ on ray 2 is that either:*

**(i)** $\rho_A \geq \rho_B$ or

**(ii)** $O_A \geq O_B$.

**Proof.** Consider the alternation rule $\hat{\tau}$ which is the same as $\tau$ except that when $\tau$ searches $A$ and then $B$, it searches $B$ first and then $A$. We will establish the claim by showing that if both conditions fail, then

$$\int_0^{\tau_{\max}} t\ dF^{\hat{\tau}}(t) < \int_0^{\tau_{\max}} t\ dF^{\tau}(t).$$

Since the two strategies differ only on some interval $(x, y)$, with $y - x = L_A + L_B$, during which $A$ and $B$ are searched, it is enough to show that

$$\hat{I} \equiv \int_x^y t\ dF^{\hat{\tau}}(t) < \int_x^y t\ dF^{\tau}(t) \equiv I.$$

To calculate $I$ observe that the probability of alternation rule $\tau$ finding an object for the first time when searching $A$ is

$$[(1 - F_1(a))(1 - F_2(b))] \left[\frac{\mu_A}{1 - F_1(a)}\right] = (1 - F_2(b))\, \mu_A,$$

and when searching $B$ is

$$[(1 - F_1(a) - \mu_A)(1 - F_2(b))]\left[\frac{\mu_B}{1 - F_2(b)}\right] = (1 - F_1(a) - \mu_A)\,\mu_B.$$

On each interval we can assume that the object will be found at its center of gravity, so that

$$I = (1 - F_2(b))\,\mu_A(x + c_A) + (1 - F_1(a) - \mu_A)\,\mu_B(x + L_A + c_B).$$

Similarly we have

$$\hat{I} = (1 - F_1(a))\,\mu_B(x + c_B) + (1 - F_2(b) - \mu_B)\,\mu_A(x + L_B + c_A).$$

We now evaluate $I - \hat{I}$. Observe that the coefficient of $x$ in the expansion of $I - \hat{I}$ is zero, so ignoring the $x$ terms we get

$$\begin{aligned} I - \hat{I} &= [(1 - F_2(b))\,\mu_A c_A] - [(1 - F_2(b) - \mu_B)\,\mu_A(L_B + c_A)] \\ &\quad + [(1 - F_1(a) - \mu_A)\,\mu_B(L_A + c_B)] - [(1 - F_1(a))\,\mu_B c_B] \\ &= \mu_A\mu_B(L_B + c_A) - \mu_A L_B(1 - F_2(b)) \\ &\quad - \mu_A\mu_B(L_A + c_B) + \mu_B L_A(1 - F_1(a)) \\ &= \mu_A\mu_B[[(L_B - c_B) - (L_A - c_A)] + [(L_A/\mu_A)(1 - F_1(a)) - (L_B/\mu_B)(1 - F_2(b))]] \\ &= \mu_A\mu_B([O_B - O_A] + [1/\rho_A - 1/\rho_B]) \end{aligned}$$

Hence $I > \hat{I}$ if both conditions $(i)$ and $(ii)$ fail, in which case the alternation rule $\tau$ of searching $A$ just before $B$ cannot be optimal. ■

When dealing with the generalized DAS problem (10) it is useful to know which distributions $F_1$ in $\mathcal{F}_1$ are candidates for optimality. We first observe that the distribution in $\mathcal{F}_1$ with the smallest mean (which finds object 1 in least expected time when there is only one ray and one object) need not be part of the optimal pair in (10). For example, suppose we have distribution sets given by discrete probability vectors

$$\mathcal{F}_1 = \{(.7, 0, 0, .3), (.6, 0, .4)\},\ \mathcal{F}_2 = \{(.8, .1, .1)\}.$$

Note that an object in four boxes with probabilities $(.7, 0, 0, .3)$ will be found in expected time $.7 + 4(.3) = 1.9$, but in the case of $(.6, 0, .4)$ will be found in shorter expected time $.6 + 3(.4) = 1.8$. However when these vectors are combined in a DAS problem with $(.8, .1, .1)$, we find (using the algorithm outlined in Figure 2) that the first vector $(.7, 0, 0, .3)$ gives the shorter expected time to find one of the objects, of

$$\begin{aligned} V((.7, 0, 0, .3), (.8, .1, .1)) &= 1.29,\ ([2, 1, 2, 2] \text{ is optimal) compared with} \\ V((.6, 0, .4), (.8, .1, .1)) &= 1.3, \text{(where [2,2,2] is optimal).} \end{aligned}$$

So we cannot simply ignore the probability vector $(.6, 0, .4)$. However if it had been instead the vector $(.6, 0, 0, .4)$ then we could could have ignored it, because $(.7, 0, 0, .3)$ dominates it in the following sense.

**Definition 2** *For distributions $G$ and $\hat{G}$, we say that $\hat{G}$* ***dominates*** *$G$ if $\hat{G}(t) \geq G(t)$ for all $t$.*

**Proposition 3** *If $\hat{F}_1$ dominates $F_1$, then for any $F_2$ we have*

$$V\left(\hat{F}_1, F_2\right) \leq V(F_1, F_2). \tag{13}$$

**Proof.** Let $\tau$ be an optimal alternation rule for the pair $F_1, F_2$ in the DAS problem, so that

$$V(F_1, F_2) = \int_0^{\tau_{\max}} t \ dF^\tau(t)$$

by (7), where $F^\tau$ is defined by (3). The alternation rule $\tau$ need not be optimal for the pair $\hat{F}_1, F_2$ but in any case we have

$$V\left(\hat{F}_1, F_2\right) \leq \int_0^{\tau_{\max}} t \ d\hat{F}^\tau(t)$$

where

$$\hat{F}^\tau(t) = \hat{F}_1(\tau(t)) + F_2(t - \tau(t)) - \hat{F}_1(\tau(t)) F_2(t - \tau(t)).$$

Consequently

$$\begin{aligned} V(F_1, F_2) - V\left(\hat{F}_1, F_2\right) &\geq \int_0^{\tau_{\max}} t \ dF^\tau(t) - \int_0^{\tau_{\max}} t \ d\hat{F}^\tau(t) \\ &= \int_0^{\tau_{\max}} (1 - F^\tau(t)) \, dt - \int_0^{\tau_{\max}} \left(1 - \hat{F}^\tau(t)\right) d(t) \\ &= \int_0^{\tau_{\max}} \left(\hat{F}^\tau(t) - F^\tau(t)\right) d(t) \geq 0, \text{because for all } t \end{aligned}$$

$$\left(F^\tau - \hat{F}^\tau\right)(t) = \left[\left(\hat{F}_1 - F_1\right)(\tau(t))\right] \left[(1 - F_2(t - \tau(t)))\right] \geq 0. \tag{14}$$

■

It is important to note that even if $\hat{F}_1$ dominates $F_1$ and $\hat{F}_1(t) > F_1(t)$ for some $t$, we still may not have strict inequality in (13). For example (reversing the coordinates in (11)) the density $(.7, 0, 0, .3)$ dominates $(.7, 0, 0, .2, .1)$ and has a higher cumulative probability (1 compared with .9) at $\bar{t} = 4$, however $V((.7, 0, 0, .3), (.8, .1, .1)) = V((.7, 0, 0, .2, .1), (.8, .1, .1)) = 1.29$. The reason for the equality is simply that the optimal alternation rule $\tau$ in the above proof is $[2, 1, 2, 2]$, which never reaches the $\bar{t}$'th (fourth) $\alpha-$box. However if this box is reached, that is, if $\tau(\tau_{\max}) > \bar{t}$ , then the inequality (14) will be strict for $t = \tau^{-1}(\bar{t}) < \tau_{\max}$, and consequently so will the value inequality (13). Hence we have.

**Corollary 4** *Let $\tau$ be an optimal alternation rule for the distribution pair $F_1, F_2$. If $\hat{F}_1$ dominates $F_1$, and $\hat{F}_1(\bar{t}) > F_1(\bar{t})$ for some $\bar{t} < \tau(\tau_{\max})$, then for any $F_2$ we have*

$$V\left(\hat{F}_1, F_2\right) < V(F_1, F_2). \tag{15}$$

*(Similarly if $\hat{F}_2$ dominates $F_2$ and $\hat{F}_2(\bar{t}) > F_2(\bar{t})$ for some $\bar{t} < \tau_{\max} - \tau(\tau_{\max})$ then $V\left(F_1, \hat{F}_2\right) < V(F_1, F_2)$.)*

# 3 Formal Definition of LSPR problem on $Z^2$

We now return to the Line of Sight Planar Rendezvous (LSPR) Problem $\Gamma(D_1, D_2)$ described in the Introduction. We think of $Z^2$ as the grid of a city, with streets and avenues distinguished. That is, we assume that the two players have a common numbering of the coordinate axes (1 and 2). However they do not have a common notion of a positive direction along either axis (e.g. N or E), or a common ordering of the axes. For those familiar with the general formulation of rendezvous search given in [1], this informational system corresponds to the four element symmetry group generated by the reflection $\psi(z_1, z_2) = (-z_1, z_2)$ and the inversion $\phi(z) = -z$. (The same symmetry group is analyzed for proximity rendezvous on $Z^2$ in [5]) We allow the two players to agree on their two strategies (the so called *player-asymmetric* version of rendezvous search), for example using mobile phones. Corresponding to any strategy pair there are rendezvous times (depending on whether player II has the same N or E as player I), and the average of these is called the rendezvous time for the strategy. The aim of the players (and of this article) is to find a method of deriving the (optimal) strategy pair which has minimal rendezvous time. This minimum is called the *rendezvous value* $R = R(D_1, D_2)$ of the Line-of-Sight Planar Rendezvous (LSPR) problem.

It is customary in (spatial proximity) network rendezvous to avoid the problem that occurs when the two players transpose their locations at adjacent nodes at consecutive times $t$ and $t+1$. Has rendezvous occurred at time $t$, at time $t+1$, at $t + 1/2$, or not at all. In the area called rendezvous on *graphs* [6], one says there is no rendezvous at all. In *network* rendezvous, following Howard [21], we make certain simplifying assumptions that avoid the problem. We start the players an even distance apart, and require that they move to adjacent nodes (and cannot remain stationary) in every period. This insures, in the lattice $Z^n$ or in any bipartite graph, that their distance is always even, and so they can never simultaneously occupy adjacent nodes. In the present line-of-sight setting, we require that their difference is always even in *both* coordinates. So we assume that their initial locations $z$ and $w$ have $z_1 - w_1$ and $z_2 - w_2$ *both* even, that they can distinguish between the two axes (the so called 'streets and avenues' assumption of [5]), and that they both move in the same coordinate in each period. We do not, however, assume that they have a common notion of a positive direction along either axis. That is, they agree which is the N-S axis but not which direction is N and which is S.

From the point of view of an observer, the initial configuration can be obtained by placing agent I at the origin with his E direction to the right and his N direction up. Then agent II is placed at the node $(2d_1, 2d_2)$ (with $d_i$ drawn randomly from $D_i$) and given a random direction to call positive in each axis. (Alternatively, we could say each player randomly picks a direction along each axis to call positive.) The full set of all 16 initial configurations, for say initial differences $d_1 = 2$, $d_2 = 1$, is illustrated in Figure 3. Player I starts and the origin, while Player II is initially placed equiprobably at one of the four nodes $(\pm 4, \pm 2)$, and randomly given a positive direction on each axis. If his positive direction on the vertical axis is South and on the horizontal axis is East, we indicate this by drawing an arrow in the Southeast direction. We can assume that I has the usual orientation, as indicated by a Northeast arrow at the origin.

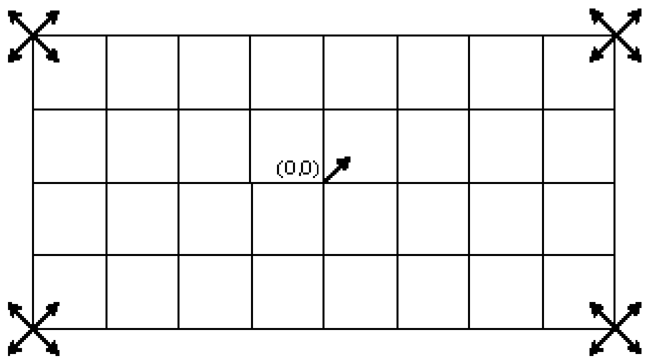



Figure 3: Initial configurations for $d_1 = 2, d_2 = 1$.

A strategy for a player is a function $f(t)$ that specifies, for each non-negative integer $t$, his net displacement $f(t)$ from his initial location, in terms of the two directions that he calls 'forward' in each coordinate. Clearly $f(0) = (0,0)$ for any strategy. For example, if Player I's strategy begins $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$, $(0,0)$, and his positive directions are North and West (from the observer's perspective) then he starts by going in an anticlockwise direction around the square that is Northwest of his starting location.

Suppose each agent faces along the horizontal axis in the direction he calls positive. There are four cases (all have subscript 1 to indicate the horizontal axis). They may be facing in opposite directions (call this case $O_1$) or in the same direction ($S_1$). If $O_1$ obtains, they are either facing each other (call this subcase $O_1^+$) or away from each other ($O_1^-$). If $S_1$ holds, then either agent $I$ is in front $\left(S_1^+\right)$ or agent $II$ is in front ($S_1^-$). Similar cases apply in the vertical axis, indicated by the same notation but a subscript of 2. The cases for the horizontal axis are illustrated in Figure 4, for various initial differences $d_1$ and $d_2$. In each case, the base of the thick arrow designates the Player's starting node and the direction of the arrow indicates his positive orientation (East) of the horizontal axis. Note the the two starting points for the second case, $O_1^-$, lie on the same vertical line, so the meeting time in this case is trivially $T = T_2 = 0$. (It was drawn in this way simply to fit in the box.)

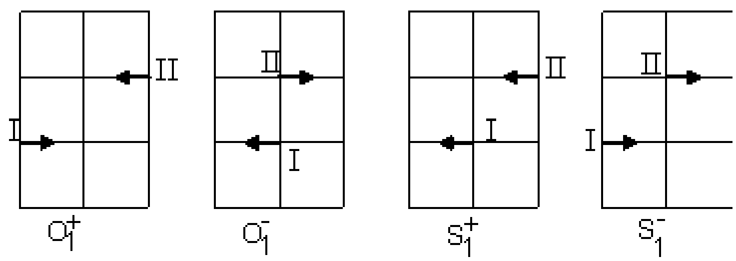


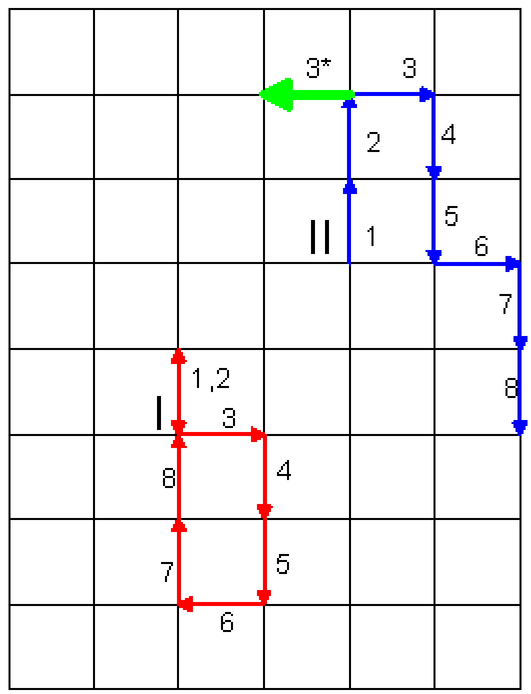

Figure 4: Four horizontal (1) start types.

As an example, we consider the rendezvous strategy pair given in the following form, where each player chooses among the four possible moves $1+, 1-, 2+, 2-$ in each time period. A move $i+$ goes in the player's positive direction along the $i$ axis, $i-$ in the negative direction (more formally, $i+$ could be written as $+e_i$,where $e_i$ is the $i$'th unit vector).

| t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\Delta f$ | 2+ | 2- | 1+ | 2- | 2- | 1- | 2+ | 2+ |
| $\Delta g$ | 2+ | 2+ | 1+ | 2- | 2- | 1+ | 2- | 2- |

(16)

We draw below in Figure 5 the paths followed by players using these strategies when Player I starts at $(0, 0)$, Player II starts at $(2, 2)$, Player I has North and East as his positive directions, and the two cases where player II has North and East or North and West, as his positive directions. The arrows mark the moves of the players. The first (thin) path for II corresponds to the cases $(S_1^-, S_2^-)$ and the second (thick line) to the cases $(O_1^+, S_2^-)$. In the first case we have $T = T_2 = 8$, as they then have the same vertical coordinate 0; and in the second we have $T = T_1 = 3$, as they then have the same horizontal coordinate 1.(Note that the second move for player I retraces his first move in the opposite direction.) There are 16 (pairs of) cases in all, but they correspond to only four possibilities for $T_1$ and four for $T_2$. For given initial differences $d_1, d_2$, the time $T_1$ depends only the four horizontal cases $\left\{S_1^{\pm}, O_1^{\pm}\right\}$, while $T_2$ depends only on the four cases $S_2^{\pm}, O_2^{\pm}$




[2+,2-,1+,2-,2-,1-,2+,2+], [2+,2+,1+,2-,2-,1+,2-,2-]

Figure 5: Plot of strategy pair.

If the initial displacement in the horizontal axis is given by $2d_1$, and the players adopt strategies $f = (f_1, f_2)$ and $g = (g_1, g_2)$, at what time $t = T_1$ will they first have the same horizontal component (be able to see each other along the common vertical axis)? If they are facing each other, case $O_1^+$, they will have the same horizontal component when the sum of their net forward motions in this axis equals their initial displacement in this axis. That is, at time

$$T_1 = T_1^1 (d_1, f, g) = \min \{t : f_1 (t) + g_1 (t) = 2d_1\} \text{ (in case } O_1^+), \tag{17}$$

If we adopt a change of coordinate from $f_i, g_i$ to new coordinates $z_i, w_i$, this and the following definitions will be simpler to express.

$$z_i = \frac{f_i + g_i}{2}, \ w_1 = \frac{= -f_i + g_i}{2}, \ i = 1, 2. \tag{18}$$

Thus in case $O_1^+$, we can write $T_1^1$ more simply as

$$T_1^1 (d_1, f, g) = \min \{t : z_1 = d_1\} \text{ (in case } O_1^+). \tag{19}$$

Similarly, they will first have the same horizontal (1) coordinate at the following times $T_1^k$ for the remaining cases.

$$\begin{aligned} T_1^2(d_1,f,g) &= \min\{t : z_1 = -d_1\} \text{ (in case } O_1^-). \\ T_1^3(d_1,f,g) &= \min\{t : w_1 = d_1\} \text{ (in case } S_1^+). \\ T_1^4(d_1,f,g) &= \min\{t : w_1 = -d_1\} \text{ (in case } S_1^-). \end{aligned} \tag{20}$$

Similarly, there are four times $T_2 = T_2^l(d_2,f,g)$ when the players have the same vertical component, using the same definitions with the subscript 1 replaced by 2. Since the initial configurations are equiprobable, for fixed values $d_1$ and $d_2$ the expected time $\hat{T}$ required for the players to see each other is given by the average of the sixteen values of $\min(T_1,T_2)$, that is

$$\hat{T} = \hat{T}(f,g,d_1,d_2) = \frac{1}{16}\sum_{k,l=1}^{4} \min\left[T_1^k, T_2^l\right]. \tag{21}$$

For the strategy (16) illustrated in Figure 4, we observed (translating to the notation of (20)) that for fixed initial differences $d_1 = d_2 = 2$, we have first sighting times $T_1^1 = 3$ and $T_2^4 = 8$. The remaining times $T_i^k$ are given around the table below (an easy way to derive these is given in the next section), which has the $k,l$ entry of first sighting time $\min\left[T_1^k, T_2^l\right]$ :

| $T_1^K \setminus T_2^l$ | $T_2^1 = 1$ | $T_2^2 = 5$ | $T_2^3 = 2$ | $T_2^4 = 8$ |
|---|---|---|---|---|
| $T_1^1 = 3$ | 1 | 3 | 2 | 3 |
| $T_1^2 > 8$ | 1 | 5 | 2 | 8 |
| $T_1^3 = 6$ | 1 | 5 | 2 | 6 |
| $T_1^4 > 8$ | 1 | 5 | 2 | 8 |

(22)

$$\text{so } \hat{T} = \frac{55}{16}.$$

Note that in this case ($d_1 = d_2 = 1$ with probability 1), it has been shown [8] that if the players restrict their entire motion to a fixed (and common) direction, the least expected meeting time is 52/16. (The optimal strategy pair given there is the A-G strategy $[1+,1+,1-,1-,1-,1-],[1+,1-,1+,1+,1-,1-]$.) We show in Section 5 that if $24/25 < \Pr[d_1 = 1] < \Pr[d_2 = 1]$, then the optimal strategy (called $h_{12}$) has its first six moves all in coordinate 2 and agreeing with the A-G strategy. So we know that if the distributions $D_1$ and $D_2$ are concentrated on fixed values $d_1 \geq d_2$, then the Players should adopt the A-G strategy in coordinate 2.

For general distributions $D_1$ and $D_2$, where the probability that $d_1 = i$ is denoted $p_i$ and the probability that $d_2 = j$ is denoted $q_j$, we have

$$\hat{T} = \hat{T}(f,g) = \sum_{i,j=1}^{\infty} \hat{T}(f,g,i,j)\, p_i q_j. \tag{23}$$

The *rendezvous value* $R = R(D_1,D_2)$ is given by

$$R = \min_{f,g} \hat{T}(f,g), \tag{24}$$

and any pair achieving the minimum is called *optimal*. We will be mainly concerned with bounded distributions $D_1$ and $D_2$, so there are only finitely many strategies and the minimum will exist. For unbounded distributions, a sufficient condition for $R$ to be finite is that $D_1$ or $D_2$ have finite mean, because in that case (say $D_1$ has finite mean) the players could restrict themselves to motion in the horizontal directions and achieve a finite expected value of $T_1$ (see Theorem 16.3 of [9]).

## 4 Search On Two Linepairs (SOTL)

In this section we show that if we view the four $z_i, w_i$ variables as the strategic ones, we arrive at a problem of search for stationary objects which is equivalent to the LSPRP defined in the previous section. We view the four variables $z_i, w_i$ , $i = 1, 2$, as the locations of four searchers on four distinct lines called $Z_1$ and $W_1$ (together called linepair 1) and $Z_2$ and $W_2$ (together called *linepair* 2). At time 0 all the searchers are at the origins of their lines (all the variables are 0). The following list shows how any moves $\Delta f, \Delta g$ of the rendezvousers in the LSPRP produce a unit move by exactly one of the searchers (here $1' = 2$, $2' = 1$).

$$\begin{aligned} \text{if} &= e_i \text{ then } \Delta z_i = 1, \Delta w_i = 0, \Delta z_{i'} = \Delta w_{i'} = 0, \\ \text{if } \Delta f &= \Delta g = -e_i \text{ then } \Delta z_i = -1, \Delta w_i = 0, \Delta z_{i'} = \Delta w_{i'} = 0, \\ \text{if } \Delta f &= e_i \text{ , } \Delta g = -e_i \text{then } \Delta z_i = 0, \Delta w_i = -1, \Delta z_{i'} = \Delta w_{i'} = 0, \\ \text{if } \Delta f &= -e_i \text{ , } \Delta g = e_i \text{then } \Delta z_i = 0, \Delta w_i = +1, \Delta z_{i'} = \Delta w_{i'} = 0. \end{aligned} \tag{25}$$

Using this table we can convert the rendezvous strategy pair (16) drawn in Figure 1 to a sequence of moves on the four lines $Z_1, W_1$ (*linepair 1*) and $Z_2, W_2$ (*linepair* 2).

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\Delta f$ | 2+ | 2− | 1+ | 2− | 2− | 1− | 2+ | 2+ |
| $\Delta g$ | 2+ | 2+ | 1+ | 2− | 2− | 1+ | 2− | 2− |
| $\Delta z_1$ | | | + | | | | | |
| $\Delta w_1$ | | | | | | + | | |
| $\Delta z_2$ | + | | | − | − | | | |
| $\Delta w_2$ | | + | | | | | − | − |

(26)

So the corresponding sequence in the $z_i, w_i$ coordinates is given in the following table, with occasions when one of the four variables reaches either +1 or -1 (taking $d_1 = d_2 = 1$,as in Figure 4) for the first time highlighted with a *.

| t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| move | $z_2+$ | $w_2+$ | $z_1+$ | $z_2-$ | $z_2-$ | $w_1+$ | $w_2-$ | $w_2-$ |
| $z_1$ | 0 | 0 | $1^*$ | 1 | 1 | 1 | 1 | 1 |
| $w_1$ | 0 | 0 | 0 | 0 | 0 | $1^*$ | 1 | 1 |
| $z_2$ | $1^*$ | 1 | 1 | 0 | $-1^*$ | $-1$ | $-1$ | $-1$ |
| $w_2$ | 0 | $1^*$ | 1 | 1 | 1 | 1 | 0 | $-1^*$ |

(27)

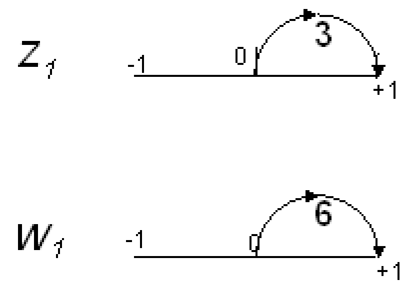


Figure 6: Strategy induced on linepair 1

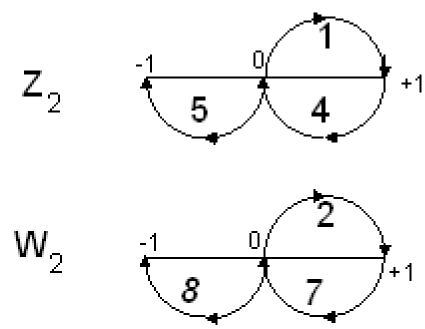


Figure 7: Strategy induced on linepair 2

Note that the six starred entries in the table (27) correspond to the six values of $T_i^k$ not greater than 8 which are used in the table(22). It is easy to

see that every change in the $z_i, w_i$ variables in which exactly one of the four variable changes, and changes by $\pm 1$, produces a pair of moves $\Delta f, \Delta g$, for the rendezvousers in the LSPRP with both moving one unit in the same coordinate.

Hence the LSPR problem is equivalent (via the invertible formula (18) to the following problem, which we call the (SOTL) Problem.

**Definition 5** *Given two independent distributions $D_1$ and $D_2$ for positive integer variables $d_1$ and $d_2$, the* ***Search On Two Linepair problem SOTL***$[D_1, D_2]$ *is defined as follows: Searchers $Z_1$ and $W_1$ are placed at the origin of two distinct lines (called linepair 1), and two other searchers $Z_2$ and $W_2$ are placed at the origin of a similar linepair 2. At each integer time $t = 1, 2, \ldots$ exactly one of the four searchers may move one unit in either direction on his line. For $i = 1, 2$, Nature picks an integer $d_i$ according to the distribution $D_i$ and places a (type $i$) object equiprobably at one of the four locations at distance $d_i$ from one of the origins of linepair $i$. A strategy for this problem is a rule for moving one of the four Searchers one unit at each time $t$ and the aim is to minimize the expected time for one of the Searchers to reach EITHER of the two hidden objects. That is, the objective function is the same as (23), except that the primary variable are taken as $z_i, w_i$, and $f$ and $g$ are obtained from the formula (18).*

With this definition, we have shown.

**Theorem 6** *Let $D_1$ and $D_2$ be two independent distributions over the positive integers. Then the Line-of-Sight Planar Rendezvous Problem $LSPR\,[D_1, D_2]$ is equivalent to the Search on Two Linepairs problem $SOTL[D_1, D_2]$ in the following sense: If the rendezvous strategy pair $(f, g)$ is related to the linepair strategy $(z, w)$ by the equations (18), then the expected rendezvous time $T\,(f, g)$ is the same as the expected time to find one of the hidden objects using $(z, w)$ in the SOTL problem .*

The equivalence of a rendezvous problem on a line to a search problem on a *single* linepair was established in [7], so the equivalence established here between a form of planar rendezvous and search on two linepairs can be seen as a natural extension. The idea that only *one* of two objects needs to be found is however new, and relates the problem to the Double Alternating Search Problem proposed in the previous section. Of course the reason for this is that the rendezvousers need only agree in a single coordinate (either one) in order to have locations that 'see' each other.

# 5 Solution of $LSPR$ problem for $d_1, d_2 \leq 2$.

In this section we will use the results of Section 2 on the Double Alternating Search (DAS) Problem to give a complete solution to the Line-Of-Sight Planar Rendezvous (LSPR) problem when the initial differences between the players do not exceed 4 in either coordinate, that is, for independent distributions $D_1, D_2$ satisfying $\Pr[d_1 = 1] = p$, $\Pr[d_1 = 2] = 1-p$, $\Pr[d_2 = 1] = q$, $\Pr[d_2 = 2] = 1-q$. In fact, we will solve the equivalent problem $SOTL\,[D_1, D_2]$ . We will view the latter problem as one of generalized alternating search, where the two regions are linepair 1 and linepair 2. Following the theory of the DAS problem derived in Section 2, it will be first necessary to restrict the search methods in each linepair to the set of undominated strategies. Then we can use the 'shortest line' algorithm of Figure 5 to derive the optimal method of alternating the search between the two linepairs.

We denote this problem as $\Gamma(p,q)$ and its optimal (or rendezvous) value as $R(p,q)$. Fix any method of searching a single linepair (call this region 1) and any positive integer $t$. Suppose that of the four rays going out of the two origins, $x$ of them have been searched a maximum distance 1 and $y$ $(\leq x)$ of them have been searched to the end (distance 2). Suppose that on this linepair we have $\Pr[d_1 = 1] = r$, $\Pr[d_1 = 2] = 1 - r$. Then the probability that the object has been found is given by

$$F_1(t) = F_1^r(t) = x\left(\frac{r}{4}\right) + y\left(\frac{1-r}{4}\right) = \frac{x-y}{4}r + \frac{y}{4}.$$

Since a search cannot reach the end of a ray without having previously passed its middle, it follows that for all $t$ we have $x \geq y$ and for some $t$ (in particular, for $\bar{t} = 1$ when $x = 1 > 0 = y$) we have $x > y$. Hence $F_1^r$ is nondecreasing in $r$ for all $t$ and increasing for some $t$. In particular, we have.

**Lemma 7** *Let $F_i^r$ be as above, and let $r' > r$. Then $F_i^{r'}$ dominates $F_i^r$ and $F_i^{r'}(1) > F_i^r(1)$.*

**Proposition 8** *For $0 \leq p \leq q \leq 1$, $R(p,q)$ is nonincreasing in $p$ and decreasing in $q$. If $p < q$, every optimal strategy must search at least part of linepair 2 (in rendezvous terms, must have some vertical moves).*

**Proof.** We prove the second sentence first. Suppose an optimal strategy $s_1$ for the two linepairs searches only linepair 1, and has cumulative distribution of finding the object there of $F_1 = F_1^p$. An identical search strategy $s_2$ of linepair 2 will have distribution $F_2 = F_1^q$. But since $p < q$ and by symmetry of $V$ (equivalence of the two linepairs), we have

$$V(s_2) = V(\mathbf{O}, F_2) = V(F_2, \mathbf{O}) = V(F_1^q, \mathbf{O}) < V(F_1^p, \mathbf{O}) = V(s_1), \qquad (28)$$

by the previous lemma and contradicting the assumed optimality of strategy $s_1$.

To obtain the first part, suppose that for some $p \leq q$ we have

$$R(p,q) = V(F_1, F_2) = V(F_1^p, F_2^q). \qquad (29)$$

Since for $p' > p$ Lemma 7 implies that $F_1^{p'}$ dominates $F_1^p$, we have

$$R(p',q) \leq V\left(F_1^{p'}, F_2^q\right) \leq V(F_1^p, F_2^q) = R(p,q). \qquad (30)$$

■

We show in the last part of this section that for certain regions of $p, q$ space the rendezvous value $R(p,q)$ is constant in $p$ because it is optimal to search only linepair 2, so $p$ is irrelevant.

## 5.1 Undominated strategies for a linepair

In this subsection, we consider the problem of searching a single linepair (say linepair 1) to find the single object hidden with probabilities $r$ and $1-r$ at distance 1 or 2, respectively, from the origin of one of the lines. Note that all the constituent lines are of the form $[-2,+2]$, with all searchers starting at 0. Any undominated method of searching two will induce undominated methods of searching each lines. So we may first consider undominated methods of searching a *single* line and then look at ways to combine these to search two lines.

Ignoring symmetries, there are only two undominated ways of searching a single line with these probabilities, namely

$$J = ++----, \text{ and } K = +----++++. \tag{31}$$

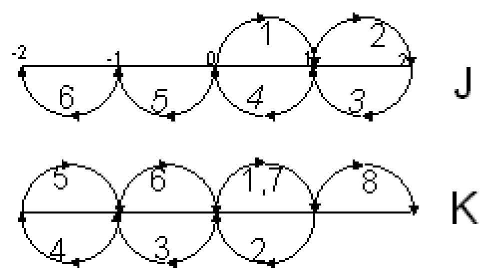



Figure 8: Undominated searches of a line

These are shown in Figure 8 (which is NOT a way of searching a linepair, but two different ways of searching a single line). Note that $J$ has length 6 and $K$ has length 8. Since the object is at $\pm 1$ with probability $r/2$ for each and at $\pm 2$ with equal probabilities $(1-r)/2$, the cumulative distributions for the two methods (times 2 to eliminate fractions ) are

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ J & r & 1 & 1 & 1 & 1+r & 2 & 2 & 2 \\ K & r & r & 2r & 1+r & 1+r & 1+r & 1+r & 2 \end{array} \tag{32}$$

Note that neither distribution dominates the other, as $J$ is larger at time 6, while $K$ is larger at time 4. Any undominated way of searching the two lines must search each of the lines with one of these methods.

We now consider how to search a linepair $Z, W$ by an undominated method which induces $J$ or $K$ on each line. There are four undominated ways of searching a linepair using $J$ on both lines, which we call $JJ_1$ to $JJ_4$; three using $K$ on both lines ($KK_1$ to $KK_3$); and three using $J$ on one (say line $Z$) and $K$ on the other (say line $W$) ($JK_1$ to $JK_3$). These strategies can be designated unambiguously by indicating the alternation strategy.

For example, the strategy $JJ_1$ begins by going right on the top line $Z$, then right on the bottom line $W$, then right again on $Z$, then right on $W$, then left four times (to $-2$) on $Z$, then left four times on $W$, ending at time 12.

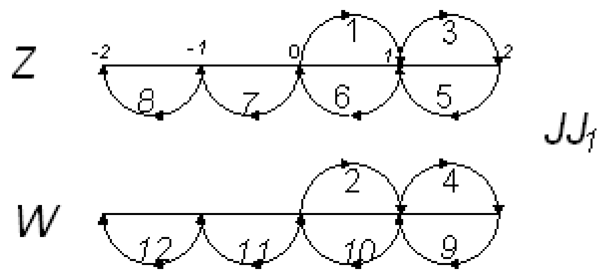



Figure 9: The strategy $JJ_1$ on a linepair

In a similar manner, the strategy $JK_2$ searches the top line $Z$ according to the route $J$ and the bottom line $W$ according to the rule $K$, combined as indicated below.

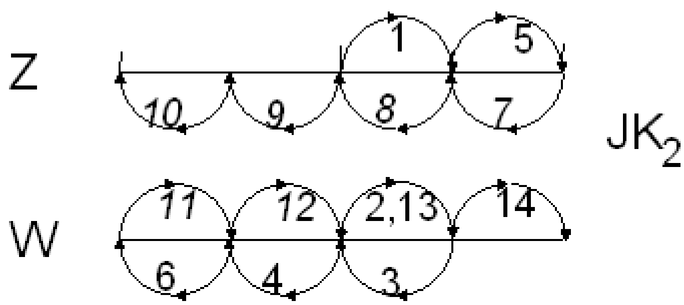



Figure 10: The strategy $JK_2$ for searching a linepair.

The ten undominated strategies for searching a single linepair are

$$\begin{aligned}
JJ_1 &= [121211112222] \\
JJ_2 &= [112211112222] \\
JJ_3 &= [121211122212] \\
JJ_4 &= [112211122212] \\
KK_1 &= [1211122211112222] \\
KK_2 &= [1211221211112222] \\
KK_3 &= [1111222211112222] \\
JK_1 &= [12122211112222] \\
JK_2 &= [12221211112222] \\
JK_3 &= [11222211112222]
\end{aligned} \tag{33}$$

The following table lists (4 times) the cumulative distribution functions for the ten strategies for search both lines, that is the probability that the object has been found by time $t$. We may view the lines as containing four close objects at distance 1, each with mass $r$, and four far objects at distance 2, each with mass 1-p. An entry of $x + yr = (x + y)\, r + x\, (1 - r)$ at time $t$ indicates that the strategy has reached $x + y$ of the close objects and $x$ of the distant objects. The search finds the object, at latest, when the element 4 appears in the table.

| | $JJ_1$ | $JJ_2$ | $JJ_3$ | $JJ_4$ | $KK_1$ | $KK_2$ | $KK_3$ | $JK_1$ | $JK_2$ | $JK_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $r$ | $r$ | $r$ | $r$ | $r$ | $r$ | $r$ | $r$ | $r$ | $r$ |
| 2 | $2r$ | $1$ | $2r$ | $1$ | $2r$ | $2r$ | $r$ | $2r$ | $2r$ | $1$ |
| 3 | $1+r$ | $1+r$ | $1+r$ | $1+r$ | $2r$ | $2r$ | $2r$ | $1+r$ | $2r$ | $1+r$ |
| 4 | $2$ | $2$ | $2$ | $2$ | $3r$ | $3r$ | $1+r$ | $1+r$ | $3r$ | $1+r$ |
| 5 | $2$ | $2$ | $2$ | $2$ | $1+2r$ | $3r$ | $1+2r$ | $1+2r$ | $1+2r$ | $1+2r$ |
| 6 | $2$ | $2$ | $2$ | $2$ | $1+2r$ | $4r$ | $1+2r$ | $2+r$ | $2+r$ | $2+r$ |
| 7 | $2+r$ | $2+r$ | $2+r$ | $2+r$ | $1+3r$ | $1+3r$ | $1+3r$ | $2+r$ | $2+r$ | $2+r$ |
| 8 | $3$ | $3$ | $2+r$ | $2+r$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+r$ | $2+r$ | $2+r$ |
| 9 | $3$ | $3$ | $2+r$ | $2+r$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+2r$ |
| 10 | $3$ | $3$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+2r$ | $2+2r$ | $3+r$ | $3+r$ | $3+r$ |
| 11 | $3+r$ | $3+r$ | $3+r$ | $3+r$ | $2+2r$ | $2+2r$ | $2+2r$ | $3+r$ | $3+r$ | $3+r$ |
| 12 | $4$ | $4$ | $4$ | $4$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ |
| 13 | $4$ | $4$ | $4$ | $4$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ | $3+r$ |
| 14 | $4$ | $4$ | $4$ | $4$ | $3+r$ | $3+r$ | $3+r$ | $4$ | $4$ | $4$ |
| 15 | $4$ | $4$ | $4$ | $4$ | $3+r$ | $3+r$ | $3+r$ | $4$ | $4$ | $4$ |
| 16 | $4$ | $4$ | $4$ | $4$ | $4$ | $4$ | $4$ | $4$ | $4$ | $4$ |

(34)

We now make a few observations regarding strategies that are dominated by others in the two cases $r < 1/2$ and $r > 1/2$. For $r < 1/2$, $JJ_1, JJ_3$, $JJ_4$, $KK_1$ and $KK_2$ are all dominated by $JJ_2$; $JK_1$ and $JK_2$ are dominated by $JK_3$. Hence for $r < 1/2$ every distribution corresponding to an allowable strategy for searching the two lines is dominated by one in the family

$$\mathcal{F}_{<1/2}\,(r) = \{JJ_2, JK_3\}\,.$$

Similarly, for $r > 1/2$ every allowable distribution is dominated by one in the family

$$\mathcal{F}_{>1/2}\,(r) = \{JJ_1, JJ_3, KK_1, KK_2, JK_1, JK_2\}\,.$$

## 5.2 Optimal search of two linepairs

We now determine the *optimal* method of searching *two* linepairs. The idea is that we consider each linepair simply as a region in the DAS problem analyzed in Section 2, forgetting its structure. Using our program to calculate the joint optimum and the optimal joint strategy we find that in the region $1/2 < p < q < 1$ there are 12 optimal strategies. (These are also optimal on the boundary of this region, but other strategies may also be optimal on the boundary.) A strategy is determined by stating which of the local strategies in $\mathcal{F}_{>1/2}\,(p)$ is used on linepair 1 (with $r = p$) and on line 2 (with $r = q$). Then, for given $p$ and $q$, the optimal alternation strategy can be calculated by the algorithm illustrated in Figure 4. For example, the strategy $h_1$ is determined by $JJ_1$ on linepair 1 and $JK_2$ on linepair 2, with the alternation rule $[2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]$. This alternation rule has 14 symbol 2's, which is the length of the strategy $JK_2$. To avoid long strings like this, we code this (and other alternation rules, all starting with 2, by indicating the lengths of the successive strings. Thus, the alternation rule for $h_1$ can be given by either of the forms,

$$[2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] = \langle 2, 2, 12 \rangle ,$$

which means a string of 2 symbol 2's, followed by 2 symbol 1's, followed by 12 symbol 2's. Note that the sum of the entries in the $\langle 2, 2, 12 \rangle$ form is 16, and equals the maximum rendezvous (or search) time required by the strategy. The strategy $h_1$ searches the two linepairs as shown in Figures 12 and 12 below, followed by a formulation of $h_1$ as a sequence of moves on the two linepairs..

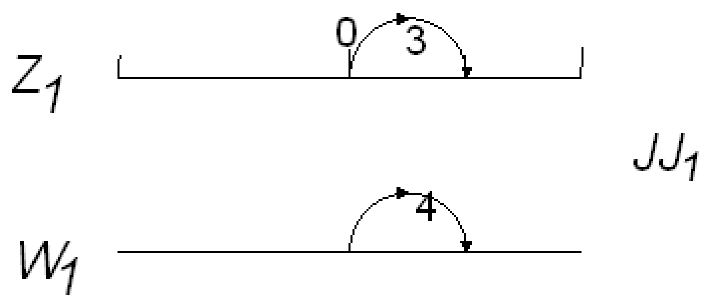



Figure 11: Action of strategy $h_1$ on linepair 1

(35)

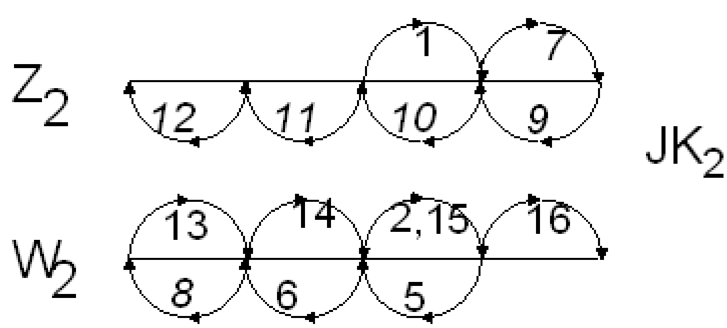



Figure 12: Action of strategy $h_1$ on linepair 2

(36)

$$[z_2+, w_2+, z_1+, w_1+, w_2-, w_2-, z_2+, w_2-, z_2-, z_2-, z_2-, z_2-, w_2+, w_2+, w_2+, w_2+]$$

## 5.3 Solution for $1/2 < p < q$.

The full set of strategies which are optimal for some $1/2 < p < q$ are listed as $h_1$ to $h_{12}$ in the table below, with $h_i(p,q)$ denoted the expected meeting time.

| strategy | 1 | 2 | $\langle \dots \rangle$, $e(h_i)$ | expected search time |
|---|---|---|---|---|
| $h_1$ | $JJ_1$ | $JK_2$ | $\langle 2,2,12 \rangle$, 2 | $\frac{43}{4} - \frac{29p}{8} - \frac{23q}{4} + \frac{19pq}{8}$ |
| $h_2$ | $KK_2$ | $KK_2$ | $\langle 8,8,8 \rangle$, 2 | $\frac{105}{8} - \frac{15p}{4} - \frac{79q}{8} + \frac{15pq}{4}$ |
| $h_3$ | $KK_2$ | $KK_2$ | $\langle 2,2,6,6,8 \rangle$, 2 | $\frac{113}{8} - \frac{27p}{4} - \frac{79p}{8} + \frac{47pq}{8}$ |
| $h_4$ | | $JK_1$ | exhaustive, $2^*$ | $\frac{33}{4} - 4q$ |
| $h_5$ | | $JJ_1$ | exhaustive, $2^*$ | $\frac{27}{4} - \frac{3q}{2}$ |
| $h_6$ | $JK_2$ | $KK_2$ | $\langle 8,14 \rangle$, 1 | $\frac{97}{8} - \frac{19p}{8} - \frac{71q}{8} + \frac{19pq}{8}$ |
| $h_7$ | $JJ_1$ | $KK_2$ | $\langle 8,12 \rangle$, 1 | $\frac{89}{8} - \frac{3p}{4} - \frac{63q}{8} + \frac{3pq}{4}$ |
| $h_8$ | $JJ_1$ | $JK_1$ | $\langle 2,2,12 \rangle$, 2 | $\frac{41}{4} - \frac{27p}{8} - 5q + 2pq$ |
| $h_9$ | | $JK_2$ | exhaustive, $2^*$ | $\frac{35}{4} - \frac{19q}{4}$ |
| $h_{10}$ | $JK_1$ | $KK_2$ | $\langle 8,14 \rangle$, 1 | $\frac{95}{8} - 2p - \frac{69q}{8} + 2pq$ |
| $h_{11}$ | $JK_2$ | $KK_2$ | $\langle 2,2,6,12 \rangle$, 1 | $\frac{105}{8} - \frac{43p}{8} - \frac{71q}{8} + \frac{9pq}{2}$ |
| $h_{12}$ | $KK_2$ | $KK_2$ | $\langle 6,8,10 \rangle$, 2 | $\frac{129}{8} - \frac{55p}{8} - \frac{103q}{8} + \frac{55pq}{8}$ |

Note that strategies $h_4$, $h_5$ and $h_7$ never search linepair 1 at all, have $e = 2^*$. (The equivalent LSPR strategies would have both rendezvousers moving vertically at all times.) Areas in $q \times p$ space, $1/2 < p < q < 1$, where the 12 strategies are optimal are indicated in the following table. The 1/2 by 1/2 square of $q \times p$ is divided into a $20 \times 20$ grid, and the index (e.g. 9 for $h_9$) of a strategy which is optimal in each $1/80 \times 1/80$ small square is indicated.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | 12 |
| | | | | | | | | | | | | | | | | | | 2 | 2 |
| | | | | | | | | | | | | | | | | | 2 | 2 | 2 |
| | | | | | | | | | | | | | | | | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | | | | | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | | | | 3 | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | | | 3 | 2 | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| | | | | | | | | | 11 | 11 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| | | | | | | | | 11 | 11 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| | | | | | | | 1 | 11 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| | | | | | | 8 | 9 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| | | | | | 4 | 4 | 9 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| | | | | 4 | 4 | 4 | 9 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| | | | 5 | 4 | 4 | 4 | 9 | 9 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| | | 5 | 5 | 4 | 4 | 4 | 9 | 9 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| | 5 | 5 | 5 | 4 | 4 | 4 | 9 | 9 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 5 | 5 | 5 | 5 | 4 | 4 | 4 | 9 | 9 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |

The boundaries between the regions can be calculated by setting the expected sighting times equal. These regions are indicated in Figure 13.

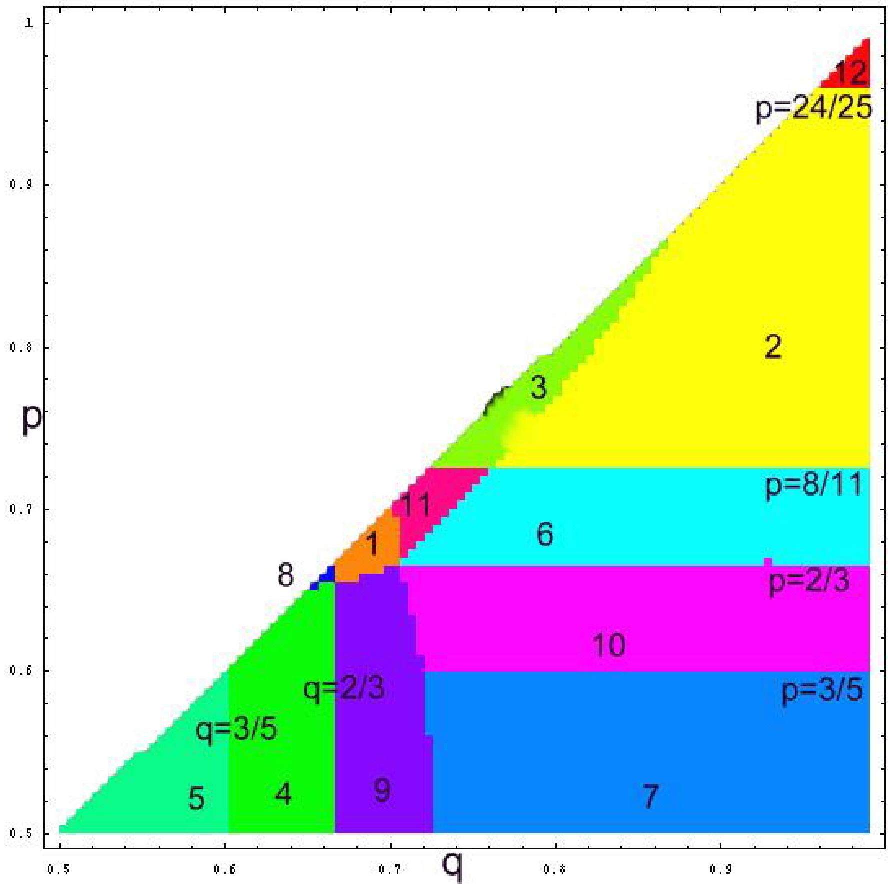



Figure 13: Regions $i$ where $h_i$ is optimal, $.5 < p < q$

## 5.4 Solution for $p \leq \min[1/2, q]$

The evaluation of $R(p, q)$ (and the determination of optimal strategies) for this region is relatively easy, based on the assumption that on the upper boundary $p = \min[1/2, q]$ there is always an optimal strategy $h$ which searches only linepair 2 (with $e(h) = 2^*$). Then a variational argument establishes that this strategy remains optimal for smaller $p$.

**Lemma 9** *If an optimal strategy $h$ for the SOTL problem with probabilities $p, q$ searches only linepair 2 ($e(h) = 2^*$) then $h$ is also optimal for probabilities $p', q$, $p' < p$.*

**Proof.** The optimality of $h$ may be rephrased as $R(p,q) = h(p,q)$. If $h$ is not optimal for $(p',q)$ then we have

$$h(p',q) > R(p',q) \geq R(p,q) = h(p,q).$$

However this is impossible, because $h$ never searches linepair 1 and hence $h(p,q)$ does not depend on $p$. ■

Observe that $h_5, h_4,$ and $h_9$ satisfy the hypothesis. The strategy $h_7$ does not; however $h_7(p,q) = h_{13}(p,q)$ for $p = 1/2$, where $h_{13}$ is the exclusive search of linepair 2 according to $KK_2$, with $h_{13}(p,q) = (86 - 60q)/8$. Hence for $p \leq 1/2 \leq q$ there is an optimal strategy which searches linepair 2 exclusively. For $p = q < 1/2$, an exclusive search of linepair 2 using $JJ_2$ is optimal, and hence by the above lemma this is also true for $p < q < 1/4$. Hence in the region $p < \min[1/2, q]$ it is always optimal to search linepair 2 exclusively, and the method of search depends only on $q$. (In the planar rendezvous version, it is always optimal for the players to move exclusively in the vertical direction.)

## 5.5 Interpretation in planar rendezvous terms

It has been simpler to carry out the investigations of this Section in terms of the SOTL problem. However it may be of interest to see what some of these results look like in the equivalent terms of planar rendezvous strategies, using the equivalence equations (18). For example, we may look at the formulation of the linepair search strategy $h_1$ and convert it to a rendezvous strategy pair $f, g$. The conversion is drawn in Figure 14 with Player I starting at (0,0) with the usual positive directions in each coordinate and with Player II starting at $(2,-2)$ and having the same directions as I. It is drawn with initial difference $2d_i = 4$ in both coordinates, so that the starting categories are $S_1^-$ and $S_2^+$. Hence the 'sighting' times are $T_1^4$ when they have the same horizontal coordinate (never) and $T_2^3$ when they have the same vertical coordinate (time 16).

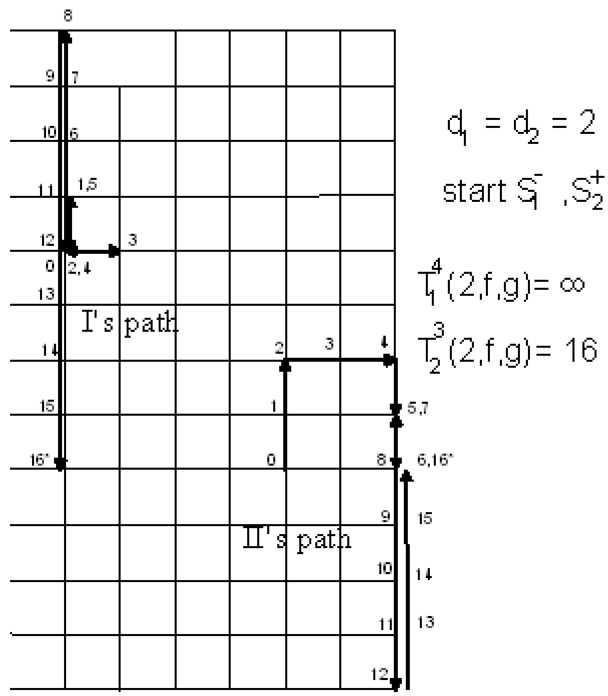



Figure 14: Planar rendezvous version of $h_1$.

## 6 Conclusions

In this paper we have analyzed the Line-Of-Sight Planar Rendezvous (LSPR) Problem under certain assumptions regarding initial locations, common-coordinate moves, and common notions of ‘streets and avenues’. We showed how this problem is formally equivalent to a pure search problem called the Search On Two Linepairs (SOTL) problem, where a team of searchers tries to find one of two objects, each hidden according to a known distribution on a ‘linepair’. The SOTL problem is of a type where two regions are to be searched to find one of two objects, and thus can be analyzed according to an algorithm and some optimality theorems for the Double Alternating Search (DAS) problem. This method of finding a search problem (for stationary hidden objects) equivalent to a rendezvous problem, and analyzing the former by dynamic programming (or other ) methods, is similar in spirit to the attack on the Linear Rendezvous Problem in [10] and [7]. There are no general methods known for the LSPR problem without the assumptions made in this paper, though individual results for special cases are not difficult, and can be better for the players than the results obtained with our assumptions. It would be useful for line-of-sight rendezvous theory to develop general methods that work without our assumptions.

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