A Common Notion of Clockwise Can Help in Planar Rendezvous

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Abstract

Two players are lost in a grid of city streets and wish to meet as soon as possible. Knowing only the distribution of the other's initial location (two nodes away in one of the four compass directions), how do they move from intersection to intersection (between nodes of the lattice Z^2) to achieve this? We assume that they do not have common compass directions to coordinate on, but that they can use their common notion of clockwise. We show that the latter, realistic assumption, can aid them in expediting their meeting (relative to a previous rendezvous problem which did not allow this). We also solve the easier 'streets and avenues' version of the problem, in which the players can distinguish between the axes (between streets and avenues). We discover several new phenomenae which have not been seen before in planar rendezvous.

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1 Introduction

The rendezvous search problem [1] asks how two (or more) players, lost in a known region Q, can move so as to meet in least expected time. The work in this field up to about a year ago is outlined in the survey article [2] and presented in greater detail in the second part of the monograph [7]. Planar rendezvous was first studied by Thomas and Hulme [15], and the discretization to the planar integer lattice $Q = Z^2$ was initiated by Anderson and Fekete [9] (in a different, 'diagonal start', version than studied here). Their work on diagonal start rendezvous has been extended by the authors [4]. Related work on rendezvous theory can be found in [8], [6], [14], [10],[11], [13], [12], [3]. Other work on rendezvous, of less relevance to the present paper, is cited in [2]. This paper is primarily concerned with rendezvous on the rectangular lattice Z^2 when the players have initial locations on a common line and have a common notion of clockwise.

In this paper we consider the rendezvous problem faced by two players who are lost in a city with a Manhattan–like grid structure which we model as the integer lattice Z^2 (graph paper). It is common knowledge that they are initially placed at nodes ('intersections' in the city) so that their vector difference is of length 2 (measured by edges) and parallel to one of the coordinate axes. (The even initial distance, combined with the requirement that they move in each period to an adjacent node, ensures that the distance between them is always even, and that they cannot pass each other without meeting at a node.) Their common aim is to minimize the expected number of periods required to occupy a common node. They can communicate before or during the game (say with mobile phones) to agree on a strategy pair, for example that Player I would exhaustively search the possible starting points of Player II, while the latter would remain stationary (the so called Wait For Mommy Strategy). This is called the player-asymmetric version of the rendezvous problem. Our main results concern the game we call Γ_2 in which the players have no common sense of compass directions, but they *can* use their common notion of clockwise. We show that this ability does reduce their expected meeting time (the so called *rendezvous value* of the game), compared with the no-common-clockwise version Γ_2' which we analyzed in a previous article [5]. The rendezvous value *with* common clockwise is $R(\Gamma_2) = 194/32 = 6.0625$, compared with the larger time of $R(\Gamma_2) = 197/32 = 6.15625$ required *without* common clockwise. This improvement is in sharp contrast to the Anderson-Fekete [9] version of planar grid rendezvous, where the players start at opposite corners of one of the city blocks (diagonally opposite nodes of Z^2) and the rendezvous value is the same, namely $138/32 = 4.312\,5$, *with or without* the assumption of a common notion of clockwise.

As observers, we adopt Player I's coordinate system, and take his initial node as the origin $(0,0)$, and his North direction in the usual way. From this perspective, Player II starts equiprobably at one of the four starting nodes $V = \left\{v^1 = (0,2)\,, v^2 = (2,0)\,, v^3 = (0,-2)\,, v^4 = (-2,0)\right\}$ and facing (in the direction he calls North) equiprobably in one of the four directions (relative to Player I's system) $j = 0$ (N), $j = 1$ (E), $j = 2$ (S), $j = 3$ (W). (The direction j stands for the rotation $\mathcal{R}^j$ of N, where $\mathcal{R}$ is the clockwise rotation by $90°$.) The 16 initial configurations, with II starting at node v^i and facing in direction R^j are depicted in Figure 1, with Player I starting at the origin O.

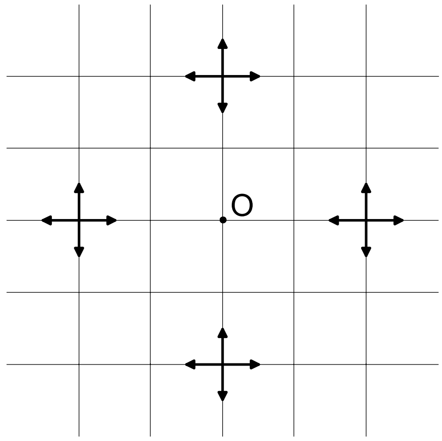
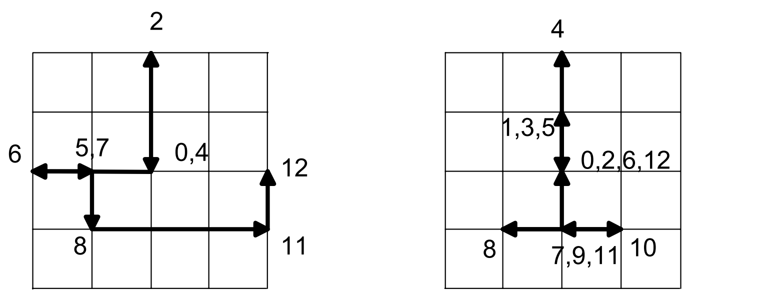


Figure 1: 16 initial configurations in Γ_2.

A *strategy* for a player is a sequence of compass directions that describe his consecutive moves, from node to node, over time. They are carried out with respect to his own coordinate system. Our main result, Theorem 19, says that the strategy pair drawn in Figure 2, which we denote $\left(\hat{f}, \hat{g}\right)$, is optimal for the game Γ_2.



(N,N,S,S,W,W,E,S,E,E,E,N) (N,S,N,N,S,S,S,W,E,E,W,N)

Figure 2: The optimal strategy $\left(\hat{f}, \hat{g}\right)$ for Γ_2.

Player I's strategy, drawn on the left, takes him from his start $(0,0)$ at the center of the drawing, to three out of the four possible starting nodes of II (all but $v^3 = (0,-2)$, at times 2, 6, and 12. Note that Player II is back at his start (the center of the drawing on the right, drawn in II's system) at these times, hoping to be found. At time 4, II searches out one of Player I's possible starting nodes, and I is back at the origin. The meeting time when II starts at node v^i in direction $\mathcal{R}^j$ is denoted $\boldsymbol{\omega}_{i,j}$ and these sixteen times are listed in the table below. Since both strategies start with N, if II starts at $v^1 = (0,2)$ $(i = 1)$ facing South $(j = 2)$, then his first instruction N in fact takes him South to $(0,1)$ where he meets Player I going N . Hence $\omega_{1,2} = 1$, and the other entries are similarly determined, with the average of the sixteen times giving the expected meeting time $T = 97/16$.

$\boldsymbol{\omega}_{i,j}\left(\hat{f},\hat{g}\right)$	N	E	S	W
j	0	1	2	3
$i=1,(0,2)$	2	2	1	2
$i=2,(2,0)$	11	10	12	4
$i=3,(0,-2)$	4	8	9	10
$i=4,(-2,0)$	6	4	6	6

(1)

Although the strategy $\left(\hat{f},\hat{g}\right)$ is optimal, it does not have the stronger property found in [4] for optimal strategies of the diagonal start game, namely that they maximize the probability of meeting by time t, for all t. From the above table we see that five of the entries exceed 8, so the probability that players using $\left(\hat{f},\hat{g}\right)$ will have met by time 8 is 11/16. However if they use the strategy $\left(\tilde{f},\tilde{g}\right)$ (drawn in Figure 8) which is optimal for the no-common-clockwise problem Γ_2', then it can be seen from the left part of the table in (31) that they meet by time 8 with higher probability 12/16.

As formalized in a more general setting in [1], the lack of a common initial orientation in these problems can be described in terms of a given subgroup G of the full symmetry group $G^*$of the search space $Q = Z^2$ fixing the origin. For the lattice Z^2, G^* is the 8-element group generated by the reflection ϕ about the vertical axis and the 90° clockwise rotation $\mathcal{R}$. Problems without (with) common clockwise have (don't have) ϕ in their group. Thus the main problem Γ_2 we consider here is $\Gamma(G_2)$, where G_2 is the four element group generated by $\mathcal{R}$ (hence the four possible orientations at each starting point, as drawn in Figure 1). The no-common-clockwise problem Γ_2' studied in [5] is $\Gamma(G_2')$ where G_2' is the full symmetry group G^*. As a sidelight to our main results on $\Gamma_2 = \Gamma(G_2)$, we also study the easier problems we call $\Gamma_1 = \Gamma(G_1)$ and $\Gamma_1' = \Gamma(G_1')$, in which the players have a common notion of the horizontal (E-W) and vertical (N-S) axes (though not of a positive direction along them), and also a common notion of clockwise in Γ_1 but not Γ_1'. The common notion of axes corresponds to search in a city with streets and avenues distinguished in some way, e.g. avenues are wider. The associated groups are generated by the inversion $\mathcal{R}^2$ (for the 2-element group G_1) and $\mathcal{R}^2$ and ϕ (for the 4-element group G_1'). We derive the optimal strategy for the 'streets and avenues' problems in Section 4, to give the reader an easy introduction to the more difficult problem Γ_2 in Section 5. We find that in this version of planar rendezvous, having a common notion of clockwise does *not* help the players (rendezvous value does not decrease).

Furthermore there is a *uniformly* optimal strategy in this version.

It is easy to show (as done in [6] for the line, using a technique which applies equally well to the plane) that if the players have common notions of directions (e.g., both have compasses), then the players should always move in opposite directions. The problem reduces to a one player search problem in which Player I starts at the origin and tries to minimize the expected time to find a stationary Player II whose initial locations are in $\frac{1}{2}V$, that is, the four nodes at distance 1 from the origin. Such a path trivially reaches these nodes at the times 1, 3, 5, 7, with expected meeting time 4. In the original formulation Player I follows that path while II moves towards the origin at odd times and back to his start at even times (always taking the opposite direction to that of I). Thus 4 is a useful lower bound on all the other rendezvous times.

The paper is organized as follows. Section 2 gives a formal definition of the problems we study, with rigorous definitions of strategies, agents, optimality and uniform optimality. Section 3 analyses how distinct 'agents' of Player II can coincide at a common node, so that Player I can meet several of them at the same time (explaining the multiple occurrences of certain numbers, e.g. 2, in the meeting time table (1)). Section 3 presents the fairly easy optimality proofs for Γ_1 and Γ_1', which provides a gentle introduction to the more sophisticated techniques used to solve Γ_2 in Section 4.

2 Formal Definitions of Γ_1, Γ_2, and Γ_1'.

In this section we first give a simultaneous formal definition of the common-clockwise planar rendezvous games $\Gamma = \Gamma_1$ or Γ_2, where they can and can't, respectively, distinguish between the axes. Then at the end of the section we show how Γ_1 can be modified to a no–common–clockwise version Γ_1'.

In all versions the players move on the search space $Q = Z^2$, the integer lattice (network) whose nodes $z = (z_1, z_2) \in Z^2$ are adjacent if they have one coordinate identical and the remaining coordinate differs by 1. This is just the familiar lattice of graph paper. The distance d between two nodes is defined as the sum of the edges in a shortest connecting path, or equivalently $d((z_1, z_2), (w_1, w_2)) = |z_1 - w_1| + |z_2 - w_2|$. At time $t = 0$ Nature places the two players on even nodes with the vector from I to II drawn from a given distribution. (A node $z \in Z^2$ is called *even* if the sum of its coordinates is even; otherwise it is called *odd*.) In every time period each player must move to an adjacent node. This 'even distance' initial placement (originating in the interval network of Howard [13]) ensures that the two players will always have the same parity, and cannot pass each other on an edge without meeting at a node. The players both wish to minimize the expected number of periods required for them to be at the same node.

We analyze the progress of the game in terms of Player I's coordinate system (and sense of clockwise). In this perspective, the initial random placement is achieved by Nature placing I at the origin facing North and placing Player II equiprobably at one of the four nodes $v^1 = (0, 2)$, $v^2 = (2, 0)$, $v^3 = (0, -2)$,

$v^4 = (-2, 0)$. The set of possible starting nodes for Player II (in I's coordinate system) is denoted by

$$V = \left\{ v^i,\ i = 1, \ldots, 4 \right\}. \tag{2}$$

Player II will have an initial orientation in the plane that is determined by the direction he calls North, which determines all the rest by the usual method of labeling them East, South, and West, going clockwise. (If the players did not have a common notion of clockwise, then there would be two orientations consistent with the given North direction.) In the game Γ_2, the direction that II calls North will be equiprobably any of the four compass directions. However in the game Γ_1, where the players have a common notion of the N-S and E-W axes, the direction that II calls North will be equiprobably the direction that I calls North or the direction that I calls South. At the end of the section we will define another game Γ_2', which is the same as Γ_2 except that the players do not have a common notion of clockwise.

The orientations of player II can be seen as transformations (or rigid motions, or symmetries) of the 'standard orientation' of Player I. In the game Γ_2, the four orientations correspond to the four orientation preserving symmetries (preserving the origin) of the planar lattice Z^2 given by

$$\begin{aligned} \mathcal{R}^j,\ j &\in J_2 = \{0, 1, 2, 3\}, \text{ where} && (3) \\ \mathcal{R}(z_1, z_2) &= (z_2, -z_1) \text{ is the clockwise rotation by } 90^\circ. && (4) \end{aligned}$$

The four rotations $\mathcal{R}^j$ correspond to the four possible choices of a North direction by Player II, and the set of these four rotations describes the information symmetry group $G_2 = \left\{ \mathcal{R}^j,\, j \in J_2 \right\}$ in the sense of Alpern [1]. In the game Γ_1, where the players have a common notion of the two axes, say because they can distinguish between streets and avenues, Player II either has the same orientation as I ($\mathcal{R}^j$, $j = 0$) or the opposite one ($\mathcal{R}^j$, $j = 2$). Thus the relevant group in this case is $G_1 = \left\{ \mathcal{R}^j,\, j \in J_1 = \{0, 2\} \right\}$. We call the general indexing set J to cover both cases.

We can now define a strategy and show how a pair of strategies determines the meeting times of the two players, one for each initial configuration.

Definition 1 *A* ***strategy*** *for a player is a sequence of directions $D_i \in \{N = (1, 0),\ E = (1, 0),\ S = (0, -1), W = (-1, 0)\}$, $i = 1, 2, \ldots$. A player pursuing this strategy moves successively one unit in his direction $D_1, D_2, \ldots$, according to his initial orientation. Equivalently, it can be seen as his net displacement $f(t)$ at time t from his initial location, given by $f(0) = (0, 0)$ and for $t \geq 1$,*

$$f(t) = \sum_{k=1}^{t} D_k. \tag{5}$$

So for example the strategy beginning N, E, E, corresponds to a net displacement function f with

$$[f(0), f(1), f(2), f(3)] = [(0, 0), (0, 1), (1, 1), (2, 1)]. \tag{6}$$

We shall deal with strategy pairs (f, g) where Player I adopts f and II adopts g. In this setting, the location of Player I at time t is simply $f(t)$, which we will denote by z_t when f is a fixed strategy under discussion. The location of II (in I's coordinate system) depends on his initial configuration (starting node v^i and orientation $\mathcal{R}^j$), as described below.

If the initial configuration gives Player II initial location v_i and orientation $\mathcal{R}^j$ then the location of Player II at time t under strategy g is given by

$$g_{i,j}(t) = v_i + \mathcal{R}^j(g(t)). \tag{7}$$

Note that the number of initial configurations is $4(\#J)$, which is 8 for Γ_1 and 16 for Γ_2.

Definition 2 *The* $4(\#J)$ *paths* $g_{i,j}$ *are called the* ***agents*** *of Player II. We call* g_{ij} *the* ***agent starting at*** v_i ***in direction*** $j \in \mathbf{J}$.

The time taken for agent $g_{i,j}$ to be met by Player I is called its *meeting time*, and denoted by

$$\omega_{i,j}(f, g) = \min\{t : f(t) = g_{i,j}(t)\}, \tag{8}$$

and the time required to meet *all* the agents is called $M(f, g)$, where

$$M = M(f, g) = \max_{i,j} \omega_{i,j}(f, g). \tag{9}$$

Figure 2 shows the strategy pair starting with WS for I and NE for II, with the paths of I (thick line) and of all 16 agents of II for $t = 0, 1, 2$. Observe that at time $t = 1$ Player I meets the agent $g_{4,1}$ of II starting at $v_4 = (-2, 0)$ and facing E (whose North is the direction that I calls East ($\mathcal{R}^j$, $j = 1$)), hence $\omega_{4,1} = 1$. Similarly at time $t = 2$ he meets the agent $g_{3,3}$ who started at $v_3 = (0, -2)$ and whose North is what I calls West ($\mathcal{R}^3$), so that $\omega_{3,3} = 2$. All other meeting times are greater than 2.

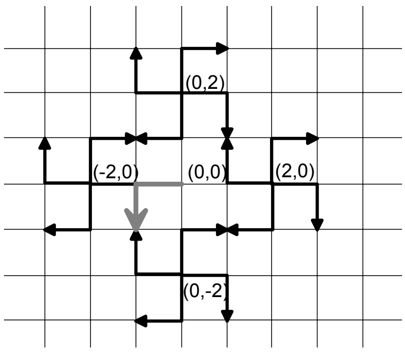


Figure 3: Strategy $(W, S), (N, E)$

Given a strategy pair (f, g), the expected meeting time is called $T(f, g)$. Thus

$$T(f, g) = \frac{1}{4(\#J)} \sum_{i,j} \omega_{i,j}(f, g). \tag{10}$$

In fact it will be easier in practice to calculate the expected meeting time by considering the number of agents x_t met (for the first time) at time t,

$$x_t = x_t(f, g) = \#\{(i, j) : \omega_{i,j}(f, g) = t\}. \tag{11}$$

With this notation we have an alternative definition of T as

$$T(f, g) = \frac{1}{4(\#J)} \sum_{t=1}^{M} t \cdot x_t. \tag{12}$$

The full vector $x = [x_1, x_2, \ldots, x_M]$ is called the *agent number profile*, or sometimes just the *profile*. Inequalities regarding elements of the profile will be the main tool in finding solving the games Γ_1 and Γ_2.

The *rendezvous value* R for Γ is the least expected time,

$$R(\Gamma) = \min_{f,g} T(f, g), \tag{13}$$

and any pair f, g achieving the minimum is called *optimal* for Γ^C.

In some cases (in fact for Γ_1 but not for Γ_2) a strategy pair may have a stronger type of optimality.

Definition 3 *A strategy pair is called* ***uniformly optimal*** *if for all t it maximizes the probability that the players have met by time t. (Note that if there is a uniformly optimal strategy, then all optimal strategies must be uniformly optimal.)*

A uniformly optimal strategy maximizes the expected utility of the meeting time ω as long as the utility function is non-increasing in ω (earlier meetings are preferred to later ones); an optimal strategy is only required to accomplish this for the particular utility function $-\omega$. For example if one player has all the water, it may be essential that they meet within say two days. Meeting in three days is no better than meeting in four. If the players are adopting a uniformly optimal strategy, it is certainly the best they can do; this is not necessarily true for a strategy which is merely optimal.

Most of this paper is concerned with rendezvous where the players have a common notion of clockwise, which would be the case if they are on a surface such as city streets where they know what 'up' is. However the game could also be presented (e.g. [5]) in a version where the grid Z^2 is a vertical 'fence' which the players might (or might not) approach from opposite sides. Or the grid might be a two dimensional array without any geometrical structure at all. In these cases the common clockwise assumption might not be justifiable. In particular, if the players have a common notion of axes (as in the game Γ_1), but not common clockwise, then this defines a game Γ_1'. The associated group is denoted $G_1' = \left\{\mathcal{R}^0 = identity, \mathcal{R}^2, \phi\right\}$, where ϕ is the reflection about the vertical axis defined by $\phi(w_1, w_2) = (-w_1, w_2)$. To every agent $g_{i,j}$ of Player II in Γ_1, there is an additional agent in Γ_1' denoted $g_{i,j}'$ whose motion when II is using strategy g is given by

$$g_{i,j}'(t) = v^i + \phi\mathcal{R}^j(g(t)). \tag{14}$$

Thus in the game Γ_1' there are 16 agents. If T denotes the expected meeting time in Γ_1, then the expected meeting time in Γ_1' is given by

$$T'(f, g) = \frac{1}{2}(T(f, g) + T(f, \phi g)), \text{ with} \tag{15}$$

$$R(\Gamma_1') = \min_{f,g} T'(f, g). \tag{16}$$

where ϕ reverses the E and W moves of a strategy, e.g. $\phi(N, S, E, W, N, \dots) = (N, S, W, E, N, \dots)$. In particular, a strategy is invariant under the reflection ϕ if and only if is has no E or W (moves only in the vertical direction). Since the players have less common information in Γ_1' than in Γ_1, the rendezvous value cannot be smaller in Γ_1'. In the following case, they can be equal. (We will show in Corollary 11 that in fact they are equal.)

Lemma 4 *If there is a strategy (f, g) which is (uniformly) optimal for Γ_1 and has a component which is invariant under ϕ, e.g. $\phi g = g$, then it is also (uniformly) optimal for Γ_1', and hence $R(\Gamma_1') = R(\Gamma_1)$.*

Proof. If $\phi g = g$ then $T'(f, g) = T(f, g)$ and the probability that the players have met by time t in the game Γ_1 is the same as in Γ_1'. The first gives optimality, the second uniform optimality. If $\phi f = f$ we use the symmetry property $T(f, g) = T(g, f)$ to reduce the problem to the solved case. ■

3 Nodes with multiple agents

If for a certain time t, Player I meets $x_t > 1$ agent of Player II, then in particular those agents will have to be at the same node z_t. This section analyses how that situation can arise. The results in this section apply equally to both games Γ_1 and Γ_2.

Lemma 5 *For any two agents, there is at most one node where they can meet. (In particular, agents with the same starting node will either both be at that node or they will be at different nodes.)*

Proof. Suppose there two distinct agents at a common node c at time t. They must have different initial directions k: if they have the same starting node this is what makes them distinct; if they have different starting nodes and the same direction, their vector difference always equal to the difference of their starting nodes, so they could never meet. Suppose one of the agents is not at node c at time t', so that $w = g(t') - g(t) \neq (0,0)$. The locations at time t' of agents at node c at time t are by (7) the vectors $c + \mathcal{R}^k(w)$, which are distinct for distinct k for $w \neq (0,0)$. ■

If the two agents have the same starting node, the only place they can coincide is at that node. Otherwise, as in Figure 2, the agents from a common node are at distinct locations.

We now determine the nodes z where agents from *distinct* starting nodes a and b can meet, at some time t. By (7) we have

$$\begin{aligned} a + \mathcal{R}^j(g(t)) &= z = b + \mathcal{R}^{j'}(g(t)), \text{ so} \\ \mathcal{R}^j(g(t)) &= z - a \text{ and} \\ R^{j'}(g(t)) &= z - b \text{ so that } g(t) = \mathcal{R}^{-j'}(z-b). \end{aligned}$$

Therefore

$$z - a = \mathcal{R}^j(g(t)) = \mathcal{R}^j\left(\mathcal{R}^{-j'}(z-b)\right) = \mathcal{R}^k(z-b), \text{ some } k = 0, \ldots, 3. \quad (17)$$

This motivates the definition of the following equivalence relation (actually two, one for J_1 and one for J_2) and the set B^*.

Definition 6 *We say that two vectors v and $w \in Z^2$ are* ***rotationally equivalent****, denoted* $\mathbf{v} \sim \mathbf{w}$, *if for some $j \in J$ we have $v = \mathcal{R}^j(w)$. (For Γ_1 this is simply $v = \pm w$ and for Γ_2 this is $(v_1, v_2) \in \{(w_1, w_2), (w_2, -w_1), (-w_1, -w_2), (-w_2, w_1)\}$. The* ***restricted perpendicular bisector*** *$B^*(a,b)$ of two vectors $a, b \in Z^2$ is defined by*

$$B^*(a,b) = \left\{z \in Z^2 : z - a \sim z - b\right\}.$$

(For Γ_1, this is simply $B^*(a,b) = (a+b)/2$; for Γ_2 it is more complicated - see Figure 4.)

Since rotationally equivalent vectors have the same Euclidean length, it follows that $B^*(a,b)$ is a subset of the Euclidean perpendicular bisector $B(a,b)$ of a and b. For several vectors, we have the obvious extension,

$$B^*\left(a^1, \ldots, a^K\right) = \left\{z \in Z^2 : z - a^i \sim z - a^{i'}, i, i' = 1, \ldots, K\right\}.$$

The main application of these ideas will be to the case when the nodes a^i are in the starting point set V (see (2)) for II as drawn below in Figure 4 for Γ_2. For Γ_1, $B^*(a,b,c) = B^*(a,b) \cap B^*(b,c)$ is empty, because one of these pairs has opposite starting points and is the singleton $(0,0)$; while the other has adjacent starting points, and belongs to the 'diagonal' set D defined later in (21).

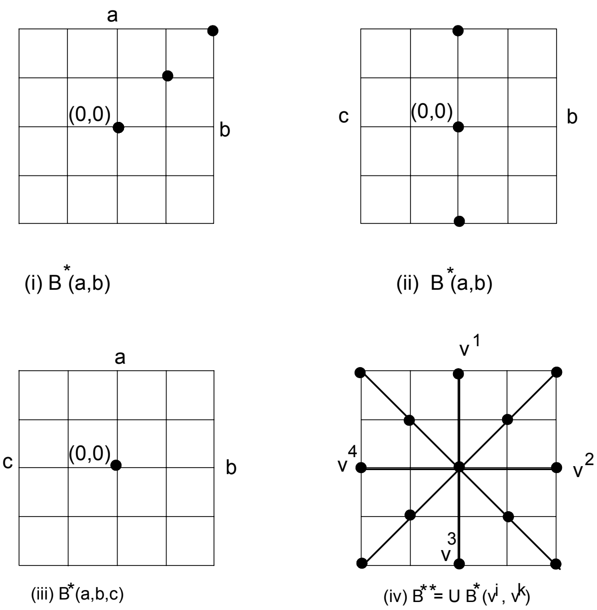


Figure 4: $B^* \left(v^i, v^{i'}\right)$ in Γ_2.

Note that for two starting points a and b, $B^*(a,b)$ consists of exactly three nodes if the starting points are (i) adjacent or (ii) opposite. In the case of any three starting points a, b, c, $B^*(a,b,c)$ contains only the origin. The full set of nodes of the type $B^*(a,b)$ for starting distinct nodes a and b consists exactly of the thirteen nodes drawn in (iv) lying on the bisector lines, which we denote by $B^{**} = \cup_{i \neq i'} B^* \left(v^i, v^{i'}\right)$. Since two agents from the same starting node can only be met in the Player II starting node set V, and $V \subset B^{**}$, it follows that multiple meetings can only take place in B^{**} ,that is

$$x_t \geq 2 \text{ implies } z_t \in B^{**} \text{ and hence } t \text{ even, so} \tag{18}$$

$$x_t \leq 1 \text{ for } t \text{ odd} \tag{19}$$

Of nodes B^{**} (with possible multiple meetings) drawn in Figure 4 (iv), the four corner nodes $(\pm 2, \pm 2)$ will not play an important role, but the nine remaining ones, grouped as shown in Figure 5 into two sets S (for *starting* nodes of I – $(0,0)$, and of II – V) and D (for diagonal) will be very important (see Definition 8):

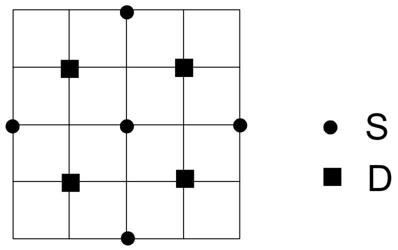


Figure 5: Special sets S and D.

$$S = \{(0,0)\} \cup V = \{(0,2),(2,0),(0,-2),(-2,0),(0,0)\}. \quad (20)$$
$$D = (\pm 1, \pm 1) = \{(1,1),(1,-1),(-1,-1),(-1,1)\}. \quad (21)$$

We can now summarize the discussion leading to the equation (17) and the calculation of B^* (Figure 4) in our new notation:

Lemma 7 *Suppose that agents from $m > 1$ distinct starting points $a, b, \dots \in V$ are at a common node $z = z_t$ at time t.*

(i) If $m = 2$, $z \in B^*(a,b)$. If $z \in D$, then $z - a$, $z - b$, $g(t) \sim (1,1)$ and $z = (a+b)/2$, as in Figure 4 (i).

(ii) If $m > 2$, $z = (0,0)$ $(= B^{**}(a,b,c))$ in Γ_2. In Γ_1 at most two agents can be at any node, so $x_t \leq 2$.

Note that actually this lemma applies as well (trivially) to the common starting point case $a = b$, because in this case we have $B^*(a,b) = \{a\}$, which says that agents from a common starting point can only meet at that point.

Certain kinds of meeting of Player I with agents of Player II taking place in the sets S or D will be very important in the subsequent analysis, so we make the following definitions:

Definition 8 *Suppose that Player I meets (for the first time) an agent $g_{i,j}$ of Player II at time t at node $z = z_t$ (so $x_t \geq 1$).*

S(t) *If $z = (0,0)$ or $z = v^i$ (note that in either case $z \in S$) we say there is an **S-Meeting,** for 'Starting Point Meeting', and denote this as **S(t)**. (In the latter case note that the agent g_{ij} met at v^i started there.) If $z = (0,0)$ we say the S-Meeting is of Type I; if $z = v^i$ we say it is Type II, the Type being the name of the player with z as a starting point.*

D(t) *If $z \in D$ and $g(t) \sim (1,1)$, we say there is a **D-Meeting** (for diagonal point meeting), and denote this as **D(t)**. Note the second part of Lemma 7 (i) implies that if $z \in D$ and $x_t = m \geq 2$ then we have $D(t)$ (see Figure 4 (i)).*

Note that since S and D contain only even nodes, S(t) or D(t) imply that t is even, that is, S or D-Meetings can occur only at even times.

A note of warning: a meeting which takes place in S is not necessarily an S-Meeting. For example if Player I meets the agent starting at $(2,0)$ at $z_t = (0,2)$, then we do not have S(t). Similarly if he meets an agent from $(-2,0)$ at $z_t = (1,1)$ we do not have D(t).

4 Analysis with 'Streets and Avenues': Γ_1, Γ_1'

In this section we assume the 'streets and avenues' scenario, in which the players can distinguish between the two axes. We show that the strategy pair

$$\left(\check{f}, \check{g}\right) = (N, N, S, S, S, S, N, N), (N, S, W, W, E, E, E, E) \tag{22}$$

is uniformly optimal for *both* rendezvous problems Γ_1 and Γ_1' where the players have a common notion both axes (but not of directions along those axes) and either have (Γ_1) or do not have (Γ_1') a common notion of clockwise. We analyze Γ_1 first and then use Lemma 4 to extend the optimality result of Γ_1' based on the observation that the first component $\check{f}$ is reflection-invariant (has only N's and S's). We first calculate the meeting times $\omega_{i,j}$ with the eight agents, $i = 1, 2, 3, 4$, $j = 0, 2$, of Player II. The calculation $\omega_{4,2} = 4$ is illustrated below in Figure 6. Player I's route from $(0,0)$ to $(0,2)$ and back is drawn in a thick line, together with the route of agent $g_{4,2}$ starting from $v^4 = (-2,0)$ and following the instructions (N, S, W, W) by taking the opposite directions (S, N, E, E) because $j = 2$ reverses all directions. They meet at the origin at time 4, as entered in the table below in bottom right position.

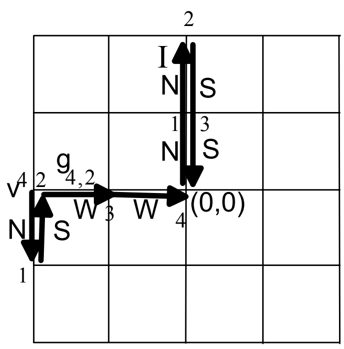


Figure 6: $\omega_{4,2} = 4$ for $(N, N, S, S,)$, (N, S, W, W)

The Eight of Values of $\omega_{i,j}$ for $\left(\bar{f},\bar{g}\right)$, $i=1,2,3,4, j=0,2.$ (23)

starting node\orientation	$\uparrow N$, $j=0$	$\downarrow N$, $j=2$
v^1, $i=1$	1	2
$v^2, i=2$	4	8
v^3, $i=3$	6	6
v^4, $i=4$	8	4

To put this into words, Player I goes up to v^1 at time 2, meeting one of the two agents at time 1 and the other at time 2. He then returns to the origin at time 4, meeting an agent each from the two horizontal starting nodes v^2 and v^4 (the latter illustrated in Figure 6). At time 6 he meets both agents who started at v^3 (since all agents are back at their starts, as $N+S+W+W+E+E=(0,0)$, finally meeting the remaining agents from v^2 and v^4 at time 8.

If we count the number of t's in the table (23) for $\left(\bar{f},\bar{g}\right)$, we see there is a single '1', so $\bar{x}_1=1$, and so on. The full agent number profile $\bar{x}$ is given by

$$\begin{aligned} \bar{x} &= [1,1,0,2,0,2,0,2], \\ \text{with } T &= (1\times 1+1\times 2+2\times(4+6+8))/8 = 39/8 = 4.875. \end{aligned}$$

Furthermore, the cumulative vector $\bar{y}$ defined for any profile by $y_k=\sum_{t\leq k}x_t$, with $y_k/8$ giving the probability that a meeting has occurred by time k is given by

$$\bar{y}=[1,2,2,4,4,6,6,8]$$

We will show that for any strategy and any t, we have $y_t\leq\bar{y}_t$, so that $\left(\bar{f},\bar{g}\right)$ maximizes the probability of a meeting by time t, for all t, and is therefore uniformly optimal.

Recall that $z_t=f(t)$ is the location of Player I at time t, and hence the location of any meeting at time t, if $x_t\geq 1$. Before reading the next lemma it is advisable to review the definitions of the sets S and D, and the definitions of S(t) and D(t). Also recall that for Γ_1 we have $x_t\leq 2$ for all t by the last part of Lemma 7 (ii)).

Lemma 9 *For any profile* $[x_1,x_2,\ldots]$ *in* Γ_1, *we have*

1. If $x_t=2$, then D(t) or S(t) and t is even .
2. If D(t) or S(t) then $x_{t-1}+x_t+x_{t+1}\leq 2$.
3. For t even, $x_t+x_{t+1}\leq 2$.
4. $x_1\leq 1$ and $x_1+x_2+x_3\leq 2$.

Proof.

1. If $x_t = 2$ then two distinct agents coincide at node z_t. If the two agents have the same starting node, then by Lemma 5, they can coincide only at that node $z_t \in V \subset S$, and we have S(t). If the agents have distinct starting nodes a and b then, by Lemma 7, $z_t \in B^* (a, b) = (a + b)/2$. So if a and b are opposite $(a + b = (0, 0))$ then $z_t = (0, 0)$ and so we again have S(t). If a and b are adjacent then $z \in D$ (for example $[(2, 0) + (0, 2)]/2 = (1, 1)$) and since $x_t \geq 2$ we have D(t).

2. If S(t) or D(t) then at time t there are two agents (including those already met) at z_t and all others are at (lattice) distance d at least 4 from z_t. To see this consider the three cases: $z_t \in V$, $z_t = (0, 0)$ (the two cases corresponding to S(t)), and $z_t \in D$. If $z_t \in V$, then d is just the distance between distinct starting nodes, 4. The remaining two cases are shown in Figure 7. Hence at times $t - 1$ and $t + 1$ the only agents that can be met for the first time are the two at z_t at time t. Hence $x_{t-1} + x_t + x_{t+1} \leq 2$, $x_{t+1} = 0$ because any agent at z_{t+1} has already been met at time t at z_t.

3. If $x_t = 2$, the result follows from parts (1) and (2). Suppose $x_t = 1$. Since $t + 1$ is odd it follows from part (1) that $x_{t+1} \leq 1$, and so we have $x_t + x_{t+1} \leq 1 + 1 = 2$.

4. Since 1 is odd, it follows from part (1) that x_1 is 0 or 1. If $x_1 = 0$, the result follows from part (3). Suppose $x_1 = 1$. If $x_2 = 0$ then $x_1 + x_2 + x_3 \leq 1 + 0 + 1 = 2$. If $x_2 \geq 1$ we can have: $z_t \in D$, in which case D(t), $z_t = (0, 0)$, in which case we have S(t), or $z_t = v^i \in V$, in which case the agents met must come from v^i (agents from other starting nodes cannot reach v^i in time 2), and hence S(t). Thus D(t) or S(t), and the result follows from part (2).

■

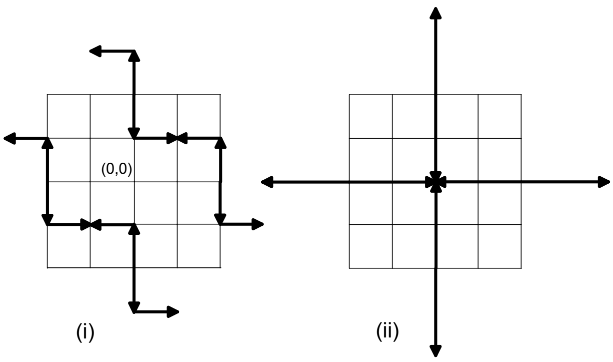


Figure 7: Unmet agents for (i) $z_t = (1, 1)$, (ii) $z_t = (0, 0)$.

Theorem 10 *The rendezvous value of the common-axes, common-clockwise game* Γ_1 *is* $39/8 = 4.875$ *and the strategy pair* $(\check{f}, \check{g})$ *is uniformly optimal.*

Proof. It is sufficient to show that for any strategy pair, the associated cumulative vectors y satisfies $y_t \leq \bar{y}_t =$ where $\bar{y} = [1,2,2,4,4,6,6,8]$ corresponds to the pair $(\check{f}, \check{g})$. For $t \leq 3$ this follows from Lemma 9 (4). By the result for $t = 3$ and Lemma 9 (3), we have for $j = 1, 2, 3$,

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_{2+2j} &\leq x_1 + x_2 + x_3 + \cdots + x_{2+2j+1} \\ &= (x_1 + x_2 + x_3) + \sum_{i=1,j} (x_{2+2i} + x_{2+2i+1}) \\ &\leq 2 + 2j, \text{ as required.} \end{aligned}$$

■

Since the strategy $\check{f}$ moves only in the vertical direction (has no E or W), it is invariant under the reflection ϕ (that is $\phi\check{f} = \check{f}$) and hence does equally well it the no common clockwise variant Γ_1' (where it is also optimal). Hence by Lemma 4 we have the following.

Corollary 11 *The strategy pair* $(\check{f}, \check{g})$ *is uniformly optimal in the game* Γ_1'*, where the players can distinguish between the axes but do not have a common notion of clockwise. In particular*

$$R(\Gamma_1') = R(\Gamma) = 39/8.$$

Thus having a common notion of clockwise does not help players who can distinguish between the vertical and horizontal axes.

This last result is in stark contrast to our main finding, Theorem 19, which shows that when players cannot distinguish between the axes, it certainly *does* help to have a common notion of clockwise (the least expected meeting time goes down from $197/32 = 6.15625$ to $97/16 = 6.0625$, an improvement of about 1.5%.

5 Analysis of Γ_2

This is the most important section of the paper, where we solve the game Γ_2. The results in this section concern a meeting number profile $x = [x_1, x_2, \ldots,]$ for this game. Recall that $z_t = f(t)$ denotes I's location at time t and that the net displacement of each agent of II at time t is rotationally equivalent to $g(t)$.

5.1 Properties of S- and D-Meetings in Γ_2.

We now obtain some consequences of Definition 8 for the game Γ_2. First, we group together some elementary properties of S-meetings in the following lemma.

Lemma 12 ***(S-Meetings)***

(i) *S(t) implies* $x_t \leq 3$ *and* $x_{t+1} = 0$.

(ii) $x_t \geq 3$ *implies S(t).*

(iii) *Hence for even* t, $x_t + x_{t+1} \leq 3$.

(iv) *If S(t) and S(t+2) the Types are different.*

Proof. (i) Assume S(t) and let $z = z_t$. Note that for either Type S-Meeting, there will be four agents of II (including perhaps some already met) at z, and they will have distinct directions j. This implies that at time $t-1$ they will occupy all four nodes adjacent to z and, since Player I will be at one of these, only at most 3 can be met for the first met at time t. Hence $x_t \leq 3$. At time t, the unmet agents of II will be at distance 4 from z, so the next meeting cannot occur before time $t+2$, and hence $x_{t+1} = 0$.

(ii) Suppose $x_t \geq 3$. If $z = z_t \notin V$, then by Lemma 5, only one agent from any starting point can be met; hence there must be agents from $m = x_t \geq 3$ distinct starting points a, b, c.. So $z = (0,0) = B^* (a, b, c)$ by Lemma 7 (ii), and hence S(t) (Type I). If $z \in V$ and $g(t) \neq (0,0)$ the same reasoning applies. If $z \in V$ and $g(t) = (0,0)$ we have S(t) (of type II). So in any case we have S(t).

(iii) Next, suppose $x_t + x_{t+1} > 3$, t even. Since $x_{t+1} \leq 1$ by (19) we have $x_t \geq 3$, and hence by part (ii) we have S(t). So by part (i) we have $x_t + x_{t+1} \leq x_t \leq 3$.

(iv) If S(t) is Type I, that is, $z_t = (0,0)$, then $g(t) \sim (2,0)$ and all unmet agents of II have distance 4 from $(0,0)$ at time t. So they cannot reach $(0,0)$ by time $t+2$, and we cannot have another Type I S-meeting at time $t+2$. If S(t) is of type II, then $z_t = v^i \in S$ so we cannot have another Type II S-meeting at $z_{t+2} = v^{i'}$, $i' \neq i$, because $d\left(v^i, v^{i'}\right) = 4$. (If $z_{t+2} = z_t = v^i$, then $x_{t+2} = 0$.) ■

Lemma 13 *If S(t) but not S(t+2) then* $x_{t+2} \leq 1$.

Proof. Suppose S(t) is of Type I. So I is at $z_t = (0,0)$ at time t and at time $t+2$ he is at some node $z = z_{t+2}$ of one of the types (i) $z = (0,0)$, (ii) $z \in D$, say $(1,1)$ by symmetry, or (iii) $z \in V$, say $(0,2)$. We consider each case separately:

(i) $z = (0,0)$ At time t, we have $g(t) \sim (2,0)$, and hence all unmet agents are at distance 4 from $(0,0)$. So in this case the next meeting cannot occur before time $t+3$, so $x_{t+2} = 0$.

(ii) $z = (1,1)$ Only agents who are at $(2,2)$ at time t can reach $(1,1)$ by time $t+2$. By Lemma 5, at most one of them can be there at time $t+2$.

(iii) $z = (0,2)$ Since there is no S-Meeting at time $t+2$, two agents reaching $(0,2)$ at that time must come, one each, from $(-2,0)$ via $(-2,2)$ and from$(2,0)$ via $(2,2)$ (as agents going via the origin would have already been met at time t). The first implies a right turn at time $t+1$, while the second implies a left turn, so only one of these can get there on time.

Next suppose S(t) is of Type II, with say $z_t = (0, 2)$. So at time t all unmet agents are at the remaining nodes of V, and of these the only nodes $z = z_{t+2}$ where he can meet agents by time $t + 2$ are of the type $(\pm 1, 1)$ or $(\pm 2, 2)$ (we can take the '+' one by symmetry), as $(0, 0)$ is excluded because it would imply S(t+2). If $z = (1, 1)$, then the agent(s) must come from $(2, 0)$ (since those from $(0, 2)$ have all been met at time t), and by Lemma 5 there can be only one, so $x_{t+2} \leq 1$. Similarly I can meet only one agent (starting at $(2, 0)$) at $z = (2, 2)$. ■

The following result will be used later to show that it is not possible to have the profile

$$[1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1] .$$

Lemma 14 *If $x_t = 2$ and $x_{t+1} = 1$ then D(t).*

Proof. Since $x_{t+1} \neq 0$, Lemma 12 says we cannot have S(t), so $z_t \neq (0, 0)$. Furthermore if $z_t \in V$ without S(t), we must have $g(t) \sim (2, 2)$ for agents from two other starting points to reach z_t. But if $g(t) \sim (2, 2)$ and $z_t \in V$ then all agents unmet by time t would be at distance at least 4 from z_t, and hence could not be met before two more periods, so x_{t+1} would be 0. Hence $z_t \notin S$, and the only points not in S where more than one agent can be met (see Figure 4 (iv)) are in D or of type $(\pm 2, \pm 2)$. But if $z_t = (\pm 2, \pm 2)$ then $g(t) \sim (2, 0)$ and by the same reasoning all agents unmet by time t cannot be met before time $t + 2$. Hence $z_t \in D$ and since $x_t \geq 2$ this implies $D(t)$. ■

5.2 Optimal strategies

We now determine some properties of profiles x that correspond to *optimal* strategies, and use these to establish that no such profile can have an expected meeting time T which is less than that of the strategy $\left(\hat{f}, \hat{g}\right)$. To avoid the division by 16 in the definition (10) of T we define $T^* = 16\,T$, so that $T^*\left(\hat{f}, \hat{g}\right) = 97$.

Lemma 15 *For an optimal strategy pair, D(t) implies $x_{t+2} \leq 2$.*

Proof. Assume, on the contrary, that $x_{t+2} = 3$, so that S(t+2) (at some node $w = z_{t+2} \in S$) by Lemma 12 (ii). By D(t), we may take $z = z_t = (1, 1)$ (since the other cases are symmetric), so that w is one of the nodes $(2, 0)$, $(0, 2)$, and $(0, 0)$. Note that in all three cases for w, Player I's location $z = (1, 1)$ at time t is adjacent to exactly two of the four nodes adjacent to w, call these nodes w' and w'' (For example, if $w = (2, 0)$, then $w'$and w'' are $(1, 0)$ and $(2, 1)$.)

First assume that $w \in V$, so that one of the agents met at $z_t = (1, 1)$ started at w. Now at time $t + 1$, the three agents I will meet at w are at three distinct nodes adjacent to w, and the fourth such node has only the agent from w met at $(1, 1)$. Hence either w' or w'' contains an unmet agent (say w'), and by our assumption $x_{t+2} = 3$ the supposedly optimal strategy must go via w'' (with

$x_{t+1} = 0$) If we maintain II's strategy but modify I's strategy so that he goes to w via w', then we will keep all the x_i the same except for $x_{t+1} = 1$ and $x_{t+2} = 2$, contradicting the assumed optimality.

Next assume $w = (0, 0)$. In this case the four agents located at $(0, 0)$ at time $t + 2$ will be at the four adjacent nodes at time $t + 1$, and *alone* at that node, since none of these are in B^{**} (Figure 4 (iv), the only nodes where more than one agent can be located). One of these four agents (at w') was met at $(1, 1)$ at time t, so the strategy with $x_{t+2} = 3$ must go to w via w'. If instead we go via w'', the expected meeting time improves as in the previous paragraph, again contradicting optimality. ■

Lemma 16 *If for some strategy $T^* < 97$, then the corresponding profile satisfies*

$$\sum_{t=1}^{7} x_t = 10. \tag{24}$$

Proof. For any strategy, we have $\sum_{t=1}^{7} x_t \leq 10$ because x_1 is always 1 and for even t, we have $x_t + x_{t+1} \leq 3$ by Lemma 12 (iii). If $\sum_{t=1}^{7} x_t < 10$ then the same inequality applied to $t = 2, 4$ implies that the best profile is (with brackets for emphasis)

$$\left[\overbrace{1, 3, 0, 3, 0, 2, 0}, \overbrace{3, 0, 3, 0, 1} \right],$$

with $T^* = 97$. ■

Since we always have $x_1 = 1$, the next result shows that no strategy pair achieves the agent number profile

$$[1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1].$$

Lemma 17 *There is no strategy pair with $x_t = 2$ and $x_{t+1} = 1$ for all even $t \leq 10$.*

Proof. Assume there is such a strategy. Then Lemma 14 implies D(t), which by definition means $z_t \in D$ and $g(t) \sim (1, 1)$. So Player II at distance 2 from his start at even times, and hence distance at most 3 at odd times.

Suppose that II is at distance 3 from his start at some odd time $t + 1 \leq 9$. Then II must be (all agents must be) at the same node of D at even times t and $t + 2$. Since I meets new agents at time $t + 2$, he cannot also be at the same node, so $z_{t+2} \neq z_t$. Without loss of generality, we may assume I moves $(1, 1) \to (0, 1) \to (-1, 1)$ starting at time t. At time $t + 1$, the only unmet agent who can be at $(0, 1)$ and at distance 3 from his start is one from $(-2, 0)$ who is at node $(-1, 1)$ at even times t and $t + 2$. (Note that any agent from $(2, 0)$ who is at $(1, 1)$ at time t has been met by that time.) So the agent from $(-2, 0)$ who I meets at $z_{t+2} = (-1, 1)$ has already been met, so $x_{t+2} < 2$, which contradicts our hypothesis. So we have shown that

$$\text{Player II is within distance 2 of his start at all times } t \leq 9. \tag{25}$$

Without loss of generality, we may suppose $z_1 = (1,0)$ and $z_2 = (1,1)$ Because $x_1 = 1$ and $x_2 = 2$, in order for I to meet different agents from $(2,0)$ at times 1 and 2, the agent met at time 2 must be at $(2,1)$ at time 1, making a left turn at time 2. Hence the unmet agents from $(2,0)$ are at nodes $(3,\pm 1)$ at time 2, and cannot reach node $(1,0)$ at time 3, and agents from other nodes surely can't by (25). Hence z_3 cannot be $(1,0)$, so we must have $z_3 = (0,1)$ where he meets an agent from $(0,2)$. The agent from $(0,2)$ met at time 2 is therefore at $(1,2)$ at time 3 so I would meet at most 1 new agent if he returned to $(1,1)$ at time 4. Hence he must go to node $(-1,1)$ at time 4 and meet agents from $(0,2)$ and $(-2,0)$ there. We can now apply the same argument another three times to show z_t, $t = 1, \ldots, 10$ is

$$(1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1), (1,-1), (1,0), (1,1).$$

However this would imply that *five* different agents from $v^2 = (2,0)$ had been met at the times $1, 2, 8, 9,$ and 10, which is impossible and so the lemma follows. ■

The next lemma says that as long as consecutive even-odd elements $[x_{2i}, x_{2i+1}]$ of the agent profile are maximal (sum to 3), whichever of the two possible forms $[2,1]$ and $[3,0]$ occurs, it will be repeated. Hence a profile beginning $[1,2,1,]$ will continue for a while as $[1,2,1,2,1,2,1,\ldots]$ and one beginning $[1,3,0]$ will continue for a while as $[1,3,0,3,0,3,0\ldots]$. This observation will be useful in limiting the possibilities for the beginning of an optimal profile.

Lemma 18 *For any optimal agent profile x,*

$$\text{if } x_{2i+2} + x_{2i+3} = x_{2i} + x_{2i+1} = 3, \tag{26}$$
$$\text{then } [x_{2i+2}, x_{2i+3}] = [x_{2i}, x_{2i+1}]. \tag{27}$$

Proof. Since $x_{2i+1}, x_{2i+3} \leq 1$ by Lemma 9 (i), equations (26) imply that the only two possibilities for the $2-$tuples in (27) are $[3,0]$ and $[2,1]$. If $[x_{2i}, x_{2i+1}] = [3,0]$ then Lemma 12 (ii) implies S(2i). If not S(2i+2) then by Lemma 13 we have $x_{2i+2} \leq 1$. But then $x_{2i+2} + x_{2i+3} \leq 1 + 1 < 3$, contrary to assumption. So we have S(2i+2), and consequently $x_{2i+3} = 0$ by Lemma 12 (i), so $[x_{2i+2}, x_{2i+3}] = [3,0] = [x_{2i}, x_{2i+1}]$. If $[x_{2i}, x_{2i+1}] = [2,1]$ then by Lemma 14 we have D(2i) and then by Lemma 15 we have $x_{2i+2} \leq 2$. Hence $[x_{2i+2}, x_{2i+3}] = [2,1] = [x_{2i}, x_{2i+1}]$. ■

5.3 Rendezvous Value of Γ_2

In [5] we showed that without a common notion of clockwise (that is, for the game Γ_2') the least expected meeting time (rendezvous value) that the players can achieve is $197/32$. We can now prove our main result, which establishes just how much better the players can do if they have a common notion of clockwise (that is, in Γ_2).

Theorem 19 *The rendezvous value of common-clockwise parallel start problem* Γ *is 97/16, and the strategy pair* $\left(\hat{f}, \hat{g}\right)$ *is optimal.*

Proof. Since $T^*\left(\hat{f}, \hat{g}\right) = 97$, and hence $T\left(\hat{f}, \hat{g}\right) = 97/16$, it is sufficient to show that no strategy has a T^* less than 97. So assume that for some optimal strategy the corresponding profile x has $T^* < 97$. Hence by Lemma 16 we have

$$\sum_{t=1}^{7} x_t = 10. \tag{28}$$

But since we always have $x_1 = 1$ and Lemma 5.3 says $x_{2i} + x_{2i+1} \leq 3$, it follows from (28) that we must have the equalities

$$x_{2i} + x_{2i+1} = 3, \ i = 1, 2, 3. \tag{29}$$

Hence by (19) $[x_2, x_3]$ is either $[2,1]$ or $[3,0]$. First assume $[2,1]$. Since we have (26) for $i = 1, 2$, and we may apply Lemma 18 for $i = 1$ and then 2, to obtain $[x_6, x_7] = [x_4, x_3] = [x_2, x_1] = [2,1]$. If $x_8 + x_9 \leq 2$, then by Lemma 12 (ii) the best possible profile would be $[1,2,1,2,1,2,1,2,0,3,0,1]$, with $T^* = 98$, contrary to assumption. Hence $x_8 + x_9 = 3$, so we may apply Lemma 18 again to obtain $[x_8, x_9] = [x_6, x_7] = [2,1]$. Then Lemma 17 implies D(8) and Lemma 15 implies $x_{8+2} = x_{10} \leq 2$. But by Lemma 17 the best profile with these constraints is $[1,2,1,2,1,2,1,2,1,2,0,1]$, with $T^* = 97$, contrary to assumption.

So we may assume $[x_2, x_3] = [3,0]$, and since our hypothesis is (26) for $i = 1, 2$, and we may apply Lemma 18 for $i = 1$ and then 2, to obtain

$$[x_6, x_7] = [x_4, x_3] = [x_2, x_1] = [3,0]. \tag{30}$$

Since $x_2 = x_4 = x_6 = 3$, Lemma 12 (i) says there are S-Meetings at the consecutive even times 2, 4, and 6. By renaming the players, if necessary, and by the alternation of Types (Lemma 12 (iv)), we may assume the respective Types are II, I, and II. Thus $z_4 = (0,0)$ and z_2 and z_6 are distinct nodes of S. Hence at time 6 all the unmet agents are at their starting nodes, the remaining two nodes of S, say a and b. Hence $x_8 \leq 2$ and $= 2$ only if $z_8 = (0,0)$, in which case S(8) and hence $x_9 = 0$.

Suppose $x_{10} = 3$. Then there is necessarily an S(10) of Type II, say at a, because an S(10) of Type I gives $x_{10} < 3$. Furthermore player I can meet only agents from b at times 7,8,9. Since I moves from a member V at time 6 to node a at time 10 and II is at his starting point at these times, I can meet an agent from b only at time 8 and at most one at this time. The remaining agents of b can be met at time 12 at the earliest so the profile can be no better than [1,3,0,3,0,3,0,1,0,3,0,2] which gives $T^* = 99$. Thus we may assume $x_{10} < 3$. Hence if $x_8 \leq 1$ then no profile is better than $[1,3,0,3,0,3,0,1,1,2,1,1]$, with $T^* = 97$.

So we may assume that $x_8 = 2$, $x_9 = 0$, and S(8). Thus there are two unmet agents from both a and b at time 8. If $x_{10} \leq 1$,then the best profile is

$[1,3,0,3,0,3,0,2,0,1,1,2]$, with $T^* = 98$, so we may suppose that $x_{10} = 2$. This can only be achieved at $z_{10} = a$, b, or (if $a \neq -b$) $(a+b)/2$. If z_{10} is a or b, then S(10) and so by Lemma 12 (i) we have $x_{11} = 0$. Hence the best profile is $[1,3,0,3,0,3,0,2,0,2,0,2]$, with $T^* = 97$. If $z_{10} = (a+b)/2$ we may suppose by symmetry that $a = (0,2)$, $b = (2,0)$, and $z_{10} = (1,1)$. In this case the two agents comprising $x_{10} = 2$ come from a and b, and are both at node $(2,2)$ at time 8. But this is impossible, as Lemma 5 says they cannot coincide at any node other than $(2,2)$. Hence our assumption of the existence of a strategy (and corresponding profile x) with $T^* < 97$ is false. ■

While the strategy $\left(\hat{f}, \hat{g}\right)$ is optimal for the problem $\Gamma = \Gamma_2$, it does have two drawbacks. First it requires a common notion of clockwise, and second it is not uniformly optimal. By the first remark we mean that if the players have a different notion of clockwise, and end up using the strategy $\left(\hat{f}, \hat{g}'\right) = \left(\hat{f}, \phi\hat{g}\right)$, they do not do as well. In particular, the strategy pair $\left(\hat{f}, \hat{g}'\right)$ does not have all its agents met by time 12, namely the E ($j = 1$) and W $(j = 3)$ agents from $v^3 = (0,-2)$. If they were both met at time 13 (so $\omega'_{3,1} = \omega'_{3,3} = 13$), the expected meeting time of the augmented strategy (of length 13) would be 105/16, as shown below.

$\boldsymbol{\omega}_{i,j}\left(\hat{f}, \hat{g}'\right) = \omega'_{i,j}$				
j	0	1	2	3
$i = 1, (0,2)$	2	2	1	2
$i = 2, (2,0)$	10	12	12	4
$i = 3, (0,-2)$	4	13^*	8	13^*
$i = 4, (-2,0)$	6	4	6	6

So if $\left(\hat{f}, \hat{g}\right)$ is used in the no common clockwise game Γ'_2, the expected meeting time satisfies the inequality

$$\begin{aligned} T'\left(\hat{f}, \hat{g}\right) &= \frac{1}{2}\left(T\left(\hat{f}, \hat{g}\right) + T\left(\hat{f}, \hat{g}'\right)\right) \\ &\geq \frac{1}{2}\left(\frac{97}{16} + \frac{105}{16}\right) = \frac{202}{32}. \end{aligned}$$

However this is not the best possible expected meeting time for the game Γ'_2. To see this, consider the strategy $\left(\tilde{f}, \tilde{g}\right)$ drawn in Figure 8. Note that, unlike $\left(\hat{f}, \hat{g}\right)$, Player I carries out an exhaustive search which visits all the possible starting points of II (at $t = 2, 6, 10$, and 14) in a time minimizing path, while II is back at his start (hoping to found) at these times.

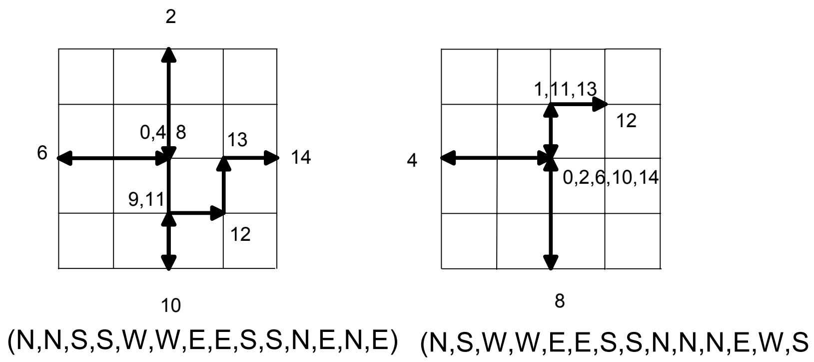


Figure 8: The strategy $\left(\tilde{f},\tilde{g}\right)$

$\omega_{i,j}\left(\tilde{f},\tilde{g}\right)$	$\omega_{i,j}$				$\omega'_{i,j}$			
j	0	1	2	3	0	1	2	3
$i=1,(0,2)$	2	1	1	2	2	2	1	2
$i=2,(2,0)$	4	8	12	13	14	8	4	12
$i=3,(0,-2)$	10	4	8	10	10	10	8	4
$i=4,(-2,0)$	6	6	4	6	4	6	6	6

(31)

We have

$$\begin{aligned} T\left(\tilde{f},\tilde{g}\right) &= \frac{98}{16}, T\left(\tilde{f},\tilde{g}'\right)=\frac{99}{16}, \text{ so} \\ T\left(\tilde{f},\tilde{g}\right) &= \frac{197}{32}. \end{aligned}$$

In fact, it is shown in [4] that $\left(\tilde{f},\tilde{g}\right)$ is optimal in Γ_2', and so $R\left(\Gamma_2'\right)=\frac{197}{32}$.

The argument that $\left(\hat{f},\hat{g}\right)$ is not uniformly optimal was given in the Introduction, and rests on the observation from its meeting time table (1) that it ensures a meeting by time 8 of 11/16. However (31) shows that $\left(\tilde{f},\tilde{g}\right)$ ensures a meeting (in the game Γ_2) with the higher probability of 12/16.

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