A random binary order: a new model of random partial orders

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Abstract

We describe a family of models of random partial orders, called *classical sequential growth models*, and study a specific case, which is the simplest interesting model, called a *random binary growth model*. This model produces a random poset, called a *random binary order*, B_2, on the vertex set $\mathbb{N}$ by considering each vertex $n \geq 2$ in turn and placing it above 2 vertices chosen uniformly at random from the set $\{0, \ldots, n-1\}$ (with additional relations added to ensure transitivity). We show that B_2 has infinite dimension, almost surely. Using the differential equation method of Wormald, we can closely approximate the size of the up-set of an arbitrary vertex. We give an upper bound on the largest vertex incomparable with vertex n, which is polynomial in n, and using this bound we provide an example of a poset P, such that there is a positive probability that B_2 does not contain P.

1 Introduction

A *random binary growth model* is a specific case from a family of models of random partial orders, called *classical sequential growth models*, introduced by Rideout and Sorkin [7]. Each model is defined on the (labelled) vertex set $\mathbb{N}$, which we will always take to include 0. Any model can be restricted to $[n] = \{0, 1, 2, \ldots, n\}$ and regarded as a model of random finite posets. The model starts with a poset of one element (labelled 0), and grows in stages. At stage $n = 1, 2, \ldots$, vertex n is added to the existing poset, P_{n-1}, by placing n above some choice of vertices of P_{n-1}. The poset P_n is defined as the transitive closure of the existing and added relations. This is called a *transition* from P_{n-1} to P_n, written $P_{n-1} \to P_n$. The models are random, so each transition occurs with some probability. These *transition probabilities* are fixed and depend on the particular model. Let $\mathbb{P}(P_{n-1} \to P_n)$ denote the probability of transition $P_{n-1} \to P_n$ occurring.

Rideout and Sorkin then impose four conditions on this model with the aim of giving the model physical meaning. They call these conditions: internal temporality, discrete general covariance, Bell causality and Markov sum. The first and last conditions are implicit in the mathematical approach to random partial orders, namely that the labelling of a poset is natural (can be extended to the $<$ order on the natural numbers), and that the model is indeed "random" (at each stage n and for any fixed P_{n-1} the sum of probabilities over all possible transitions $P_{n-1} \to P_n$ must be equal to 1). *Discrete general covariance* states that the probability of producing a particular poset should not depend on the labelling of the poset, that is, given two different sequences of transitions, $(P_i \to P_{i+1})$ and $(Q_i \to Q_{i+1})$ which produce the isomorphic posets P_n and Q_n, the products

$$\prod_{i=0}^{n-1} \mathbb{P}(P_i \to P_{i+1}) \quad \text{and} \quad \prod_{i=0}^{n-1} \mathbb{P}(Q_i \to Q_{i+1})$$

must be equal. So, for example, discrete general covariance immediately implies that any two transitions from P_{n-1} to isomorphic posets P_n and P'_n have the same transition probability $\mathbb{P}(P_{n-1} \to P_n) = \mathbb{P}(P_{n-1} \to P'_n)$. *Bell causality* is a condition on ratios of transition probabilities.[1] Given a particular poset P, and any two transitions $P \to P'$, $P \to P''$ which add the new element n, let S be the set of all elements which are incomparable with n in both P' and P''. Let Q be the poset formed from P by removing all the elements of S (and obsolete relations), and define Q' and Q'' similarly. Then, Bell causality states that

$$\frac{\mathbb{P}(P \to P')}{\mathbb{P}(P \to P'')} = \frac{\mathbb{P}(Q \to Q')}{\mathbb{P}(Q \to Q'')},$$

the idea being that, since the new element is not placed above any of the elements of S in either transition, the presence of the set S should not affect the ratio of the transition probabilities.

A particular model is specified by a sequence $\mathbf{t} = (t_0, t_1, \ldots)$ of non-negative constants. The random poset is defined as the transitive closure of a directed random graph $G_{\mathbf{t}}$ on $\mathbb{N}$ in which all arcs go from a lower numbered vertex to a higher. The arcs are selected sequentially, considering each vertex n in turn and choosing the set $D_n \subseteq [n-1]$ of vertices sending an arc to n; the probability that D_n is equal to a set D being proportional to $t_{|D|}$, so that

$$\mathbb{P}(D_n = D) = \frac{t_{|D|}}{\sum_{j=0}^{n} \binom{n}{j} t_j}.$$

A model defined according to this description is called a *classical sequential growth*

[1] Actually, to avoid problems with division by zero we can think of the condition as an identity of products of probabilities.

model. Rideout and Sorkin show that these models are the only 'generic' models[2] satisfying their conditions. It is an easy exercise to check that these models do indeed satisfy the four conditions; for example, the internal temporality and Markov sum conditions are immediate as explained earlier.

The family of classical sequential growth models also contains models of *random graph orders*. A random graph order $P_{n,p}$ is defined as follows. The ground set of $P_{n,p}$ is the set $\{0, 1, \ldots, n-1\}$. For each pair of vertices $i < j$ the relation (i, j) is introduced with probability p. The poset $P_{n,p}$ is then the transitive closure of these relations. Random graph orders were introduced by Albert and Frieze [1] and have been studied further by Bollobás and Brightwell [2, 3, 4] and Simon, Crippa and Collenberg [8]. The area is covered in the survey of random partial orders by Brightwell [5]. A classical sequential growth model defined by sequence $\mathbf{t}$ where $t_i = t^i$ for all i, and $t = p/(1-p)$, will after stage $n-1$ produce a random graph order $P_{n,p}$.

In this paper, we concentrate on the model where the sequence $\mathbf{t}$ is $(0, 0, 1, 0, \ldots)$, i.e., where all t_i are zero except t_2. This means that $|D_n| = 2$ for each vertex n. We say that n *selects* the two vertices in D_n. So, in this model each vertex n selects two vertices chosen uniformly at random from the set $[n-1]$. We assume that we start with the vertices 0 and 1 incomparable with probability 1 and then add vertices $n = 2, 3, \ldots$ according to the model. (So, for example, $D_2 = \{0, 1\}$ with probability 1.) We call this model a *random binary growth model* and call the random poset it produces a *random binary order*.

This is the simplest interesting model; the model defined by $\mathbf{t}$ with t_0 non-zero and t_i equal to zero for $i \geq 1$ produces an infinite antichain ($D_n = \emptyset$ with probability 1, for all n), and the model defined by $\mathbf{t}$ with t_0 and t_1 non-zero and t_i equal to zero for $i \geq 2$ produces a forest of infinitely many infinite trees, where each vertex is an upper cover of exactly one other vertex and a lower cover of infinitely many other vertices. These are called the "dust universe" and "forest universe", respectively, in [7]. The random binary growth model is essentially the same as any other model with $t_3 = t_4 = \ldots = 0$ since for large n the number of 2-element subsets of $[n-1]$ is significantly greater than the number of 1-element subsets and so the probability of n selecting just one vertex (or no vertices) is very small in comparison to the probability of n selecting two vertices.

We will denote the *random binary growth model* by $\mathcal{B}_2$ and the *random binary order* it produces by B_2. We write $B_2[n]$ for the restriction of B_2 to $[n]$ and $B_2[n_1, n_2]$ for the restriction of B_2 to $[n_1, n_2] = \{x \in \mathbb{N} : n_1 \leq x \leq n_2\}$.

The random binary order B_2 is a sparse order; each vertex n has at most 2 lower covers since x is a lower cover of n if and only if it is selected by n and is not below the other vertex y selected by n. This means the Hasse diagram of $B_2[n]$ has at most $2n$ edges. Also, as we now show, the expected width of $B_2[n]$ increases with n. A vertex x in $B_2[n]$ is maximal if and only if all vertices $y = x+1, x+2, \ldots, n$ do not select x, so

$$\mathbb{P}(x \text{ is maximal in } B_2[n]) = \prod_{y=x+1}^{n} \left(1 - \frac{2}{y}\right) = \prod_{y=x+1}^{n} \frac{y-2}{y} = \frac{x(x-1)}{n(n-1)}$$

[2]Rideout and Sorkin use this term to mean that the model has no zero transition probabilities; in this way they resolved the problem of division by zero in the Bell causality condition.

and so the expected number of maximal elements is

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{x=2}^{n}x(x-1) &= \frac{1}{n(n-1)}\left(\sum_{x=1}^{n}x^2-\sum_{x=1}^{n}x\right)\\ &= \frac{1}{n(n-1)}\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right)=(n+1)/3.\end{aligned}$$

The maximal elements form an antichain, so the expected width of $B_2[n]$ is at least $(n+1)/3$. However, the number of minimal elements is always 2, since only 0 and 1 are minimal. Moreover, the expected number of minimal elements of $B_2[n_1, n_2]$, for $n_1 \geq 2$, is bounded above by n_1 as n_2 tends to infinity. A vertex x in $B_2[n_1, n_2]$ is minimal if and only if it selects both vertices from $[n_1 - 1]$ and the probability of this is $\binom{n_1}{2}/\binom{x}{2} = n_1(n_1-1)/x(x-1)$. Summing over x from n_1 to n_2 gives the expected number of minimal elements equal to $n_1 - n_1(n_1-1)/n_2$.

In Section 2 we study the dimension of B_2. Since B_2 is sparse, one might suppose there to be a relatively simple structure to B_2. However, we show this is not the case in so much as showing that B_2 has infinite dimension, almost surely. Using standard notation (see, e.g., [10]), we write $P(1, 2; m)$ for the subposet of the subset lattice formed by the 1-element and 2-element subsets of the m-element set $\{1, \ldots, m\}$ ordered by inclusion. Spencer [9] proved that the dimension of $P(1, 2; m)$ is greater than $\log_2 \log_2 m$, so we show that B_2 has infinite dimension, almost surely, by showing it contains a copy of $P(1, 2; m)$ as a subposet, for each m, almost surely. This is done by counting (and bounding the expected number of) certain "paths" in B_2 (the "paths" in B_2 are exactly the paths in $G_{\mathbf{t}}$).

In Section 3 we study the sizes of up-sets in $B_2[n]$ and, related to this, the number of elements in B_2 incomparable with an arbitrary element. Although B_2 is sparse, we show that for all but finitely many r the number of elements incomparable with r is finite. In particular, this implies that B_2 does not contain an infinite antichain, almost surely. Moreover, for any classical sequential growth model defined by sequence $\mathbf{t}$ where $t_i \neq 0$ for some $i \geq 2$, the same result is true, that the random poset produced does not contain an infinite antichain, almost surely.

We use the differential equation method of Wormald [11, 12] which specifies when and how a discrete Markov process can be closely approximated by the solution to a related differential equation. We prove a version of Wormald's theorem which makes explicit the errors in the approximation. We use this result to analyse the growth of the up-set of an arbitrary point. For a fixed point r, write $U_r^{[n]}$ for the set of elements above r in the finite poset $B_2[n]$. We can think of "growing the poset" by increasing n. Then $|U_r^{[n]}|$, which depends on n, can be considered as a Markov process. Using this "differential equation method", we give good estimates on $|U_r^{[n]}|$ for particular values of n, and show that there exists an $n = n(r)$ such that $I_r \subseteq [n]$. Here, I_r is the set of vertices greater than r which are incomparable with r. So, for fixed r, there are no vertices greater than n incomparable to r, and so the number of vertices incomparable with r is finite. We provide two similar proofs, one giving bounds for a typical r, and one giving bounds for all but finitely many r.

Is the fact that $P(1, 2, m)$ is almost surely contained in B_2 a special case of something more general? Is it possible, as in the case of random graph orders, that *every* finite poset is contained in B_2, almost surely? In Section 4 we show that this is not the case. We use our result from Section 3, that there is an $n = n(r)$ such that for all but finitely many r, there are no vertices greater than n incomparable with r. So, we know that if two elements in B_2 have labels with a large enough difference then they must be comparable. We construct a poset which if contained in B_2 must have two elements whose labels have large difference. Combining these two results, we provide an example of a poset not contained in B_2 (or rather, there is a positive probability that B_2 does not contain the poset).

2 B_2 has infinite dimension

We write $P(1,2;m)$ for the subposet of the subset lattice formed by the 1-element and 2-element subsets of the set $\{1,\ldots,m\}$ ordered by inclusion. For a particular vertex r, let U_r be the set of all vertices above r in B_2 and let $U_r^{[t]}$ be the set of all vertices above r in $B_2[t]$. Denote by T_k the hitting time of the event $|U_r| = k$, i.e., the smallest t such that $|U_r^{[t]}| = k$, and the waiting time between events $|U_r| = k-1$ and $|U_r| = k$ by W_k, so that $T_{k+1} = T_k + W_{k+1}$. We include the point r in U_r so that $T_1 = r$.

Theorem 1. *For every m, there exists a copy of $P(1,2;m)$ in B_2, almost surely.*

Proof. We will prove a stronger result; there is some r_0 such that the probability of there being a copy of $P(1,2;m)$ in $B_2[r, 2r^{7/5}]$ is greater than $3/5$ for all $r \geq r_0$. Note that r_0 depends on m.

Fix m. Throughout, we assume that r_0 is sufficiently large and that $r \geq r_0$. Consider the points $r, r+1, \ldots, r+m-1$. We attempt to find a copy of $P(1,2;m)$ in which these are the minimal elements. We have,

$$\mathbb{P}(r, r+1, \ldots, r+m-1 \text{ are incomparable}) = \prod_{i=1}^{m-1} \frac{\binom{r}{2}}{\binom{r+i}{2}} \geq 9/10 \quad \text{for } r_0 \geq 20m^2.$$

Now grow the poset by adding points up to $n = r^{7/5}$. We consider the growth of the set U_r. We calculate the expected waiting time $\mathbb{E}W_{k+1}$ as follows. Suppose $T_k = t$, then since W_{k+1} always takes integer values greater than or equal to 1 we have

$$\mathbb{E}W_{k+1} = 1 + \sum_{j=1}^{\infty} \mathbb{P}(W_{k+1} > j) = 1 + \sum_{j=1}^{\infty} \prod_{l=1}^{j} \frac{\binom{t+l-k}{2}}{\binom{t+l}{2}}$$

and using the inequalities $1 - x \leq e^{-x}$ and $\int_a^{b+1} f(x)dx \leq \sum_{j=a}^{b} f(j) \leq \int_{a-1}^{b} f(x)dx$, for f

decreasing, we have

$$\begin{aligned}
\mathbb{E}W_{k+1} = 1 + \sum_{j=1}^{\infty} \prod_{l=1}^{j} \frac{\binom{t+l-k}{2}}{\binom{t+l}{2}} &\leq 1 + \sum_{j=1}^{\infty} \left(\prod_{l=1}^{j} \left(\frac{t+l-k}{t+l} \right) \right)^2 \\
&\leq 1 + \sum_{j=1}^{\infty} \exp\left(-2 \sum_{l=1}^{j} \frac{k}{t+l} \right) \\
&\leq 1 + \sum_{j=1}^{\infty} \exp\left(-2k \int_{1}^{j+1} \frac{1}{t+l} dl \right) \\
&= 1 + \sum_{j=1}^{\infty} \left(\frac{t+1}{t+j+1} \right)^{2k} \\
&\leq 1 + (t+1)^{2k} \int_{0}^{\infty} \frac{1}{(t+j+1)^{2k}} dj \\
&= 1 + \frac{(t+1)^{2k}}{(t+1)^{2k-1}} \frac{1}{2k-1}.
\end{aligned}$$

That is,

$$\mathbb{E}(W_{k+1}|T_k) \leq 1 + \frac{T_k + 1}{2k-1}.$$

So, we have

$$\mathbb{E}T_{k+1} = \mathbb{E}T_k + \mathbb{E}W_{k+1} \leq \mathbb{E}T_k + \left(1 + \frac{\mathbb{E}T_k + 1}{2k-1} \right) = \frac{2k}{2k-1}(\mathbb{E}T_k + 1), \tag{1}$$

which by induction on k gives

$$\mathbb{E}T_{k+1} \leq \left(2^{2k} / \binom{2k}{k} \right) r + 2k. \tag{2}$$

Using Stirling's approximation we have

$$\binom{2k}{k} \geq \frac{\sqrt{2\pi}(2k)^{2k+1/2} e^{-2k+1/(24k+1)}}{(\sqrt{2\pi} k^{k+1/2} e^{-k+1/12k})^2} \geq \frac{2^{2k+1/2} e^{1/(24k+1)}}{\sqrt{2\pi} k^{1/2} e^{1/6k}}, \quad \text{for } k \geq 1,$$

so $\mathbb{E}T_{k+1} \leq \sqrt{\pi} e^{1/6k - 1/(24k+1)} \sqrt{k} r + 2k$, for $k \geq 1$. For $k \geq 2$, $\sqrt{\pi} e^{1/6k-1/(24k+1)} \leq 2$ and using (2) we have $\mathbb{E}T_2 \leq 2r + 2$, so $\mathbb{E}T_{k+1} \leq 2r\sqrt{k} + 2k$ and so

$$\mathbb{E}T_k \leq 2r\sqrt{k} + 2k. \tag{3}$$

If we similarly define $U_{r+i}, T_k^{(i)}, W_k^{(i)}$ for $r+i$, $i = 1, \ldots, m-1$ and write $T_k^{(0)}$ for T_k, then we have $T_1^{(i)} = r + i$, giving equations

$$\mathbb{E}T_{k+1}^{(i)} \leq \frac{2k}{2k-1}(\mathbb{E}T_k^{(i)} + 1), \tag{4}$$

$$\mathbb{E}T_{k+1}^{(i)} \leq \left(2^{2k} / \binom{2k}{k} \right)(r+i) + 2k, \tag{5}$$

$$\mathbb{E}T_k^{(i)} \leq 2(r+i)\sqrt{k} + 2k, \tag{6}$$

corresponding to equations (1),(2) and (3).

For $r_0 \geq m$ we have $r+i \leq r+m \leq 2r$, so (6) becomes

$$\mathbb{E}T_k^{(i)} \leq 4r\sqrt{k} + 2k, \quad i = 0, \ldots, m-1.$$

So, recalling that $n = r^{7/5}$, we have

$$\mathbb{P}(|U_r^{[n]}| < r^{3/4}) = \mathbb{P}(T_{r^{3/4}} > n) \leq \mathbb{E}T_{r^{3/4}}/n \leq (4r^{11/8} + 2r^{3/4})/r^{7/5} \leq 6/r^{1/40} \leq 1/10m$$

for $r_0 \geq (60m)^{40}$, and similarly for $|U_{r+i}^n|$, $i = 1, \ldots, m-1$.

Therefore, $\mathbb{P}(\text{all } |U_r^{[n]}|, \ldots, |U_{r+m-1}^{[n]}| \geq r^{3/4}) \geq 9/10$.

We say a point x selects a pair of sets (X_1, X_2) if $D_x = \{x_1, x_2\}$ for some $x_1 \in X_1$ and $x_2 \in X_2$, that is, if x selects a point from each set X_1 and X_2. Using the lower bounds on $U_{r+i}^{[n]}$ we can show that, with high probability, there exist points in $B_2[2n]$ selecting each pair $(U_{r+i}^{[n]}, U_{r+j}^{[n]})$. We might hope for these to form the maximal points of a copy of $P(1,2;m)$, since for each pair of minimal points $r+i, r+j$ we have a point above both. However, it is possible for these potential maximal points to be above more than 2 minimal points. We need to find points above exactly 2 of the minimal points. To do this we need to look at a subset of U_{r+i}, namely the set of points above $r+i$ but not above any other $r+j$ for $j \neq i$.

For points x, y in B_2, write $U_{x\,y}$ for the set of points above both x and y. Consider the restricted poset $B_2[n]$ and write $U_{x\,y}^{[n]}$ for the set of points in $B_2[n]$ above both x and y. We will show that $|U_{r\,r+1}^{[n]}|$ is small in comparison to $|U_r^{[n]}|$ and $|U_{r+1}^{[n]}|$. Call a sequence of integers $(i_j)_{j=1}^s$ from $[r, n]$ a *path* if i_j selects i_{j-1} in the poset, for $j = 2, \ldots, s$. So a path is necessarily a strictly increasing sequence. We say a path $(i_j)_{j=1}^s$ is from i_1 to i_s. Define a *forked path* with ends x, y, z and connection point w to be three paths, one from x and one from y both to w, and a third from w to z (so $x, y < w \leq z$), with w the only common point of the first two paths. Note that we allow the possibility that $w = z$, in which case the third path is the single point $w = z$.

For each point u in $U_{r\,r+1}^{[n]}$ there must be paths P_r from r to u and P_{r+1} from $r+1$ to u; if we set $v = \min\{j : j \text{ is a common point of } P_r \text{ and } P_{r+1}\}$ then by taking the subpath (subsequence of consecutive terms of a path) from r to v (of P_r), the subpath from $r+1$ to v (of P_{r+1}) and the subpath from v to u (of either P_r or P_{r+1}) we have a forked path with ends r, $r+1$ and u, and connection point v. This forked path is not necessarily unique, since P_r and P_{r+1} are not necessarily unique. Let $FP(r, r+1, v)$ be the total number of forked paths with ends r and $r+1$ and connection point v all fixed, and with arbitrary third end u, with $v \leq u \leq n$. Let $FP(r, r+1) = \sum_{v=r+2}^n FP(r, r+1, v)$. Then $|U_{r\,r+1}| \leq FP(r, r+1)$.

Now, the probability that a strictly increasing sequence $(i_j)_{j=1}^s$ is a path in $B_2[n]$ is $\mathbb{P}\big(\cap_{j=2}^s (i_j \text{ selects } i_{j-1})\big) = \prod_{j=2}^s (2/i_j)$, by independence.

We can also calculate the probability that the points $\{i_0, i_1, \ldots, i_s\}, i_0 < i_1 < \cdots < i_s$ form two disjoint paths in $B_2[n]$, one from i_0, the other from i_1, as follows. Start with two sequences $A = (i_0)$ and $B = (i_1)$, then taking each point $i_j, j = 2, \ldots, s$ in turn make it the next term in either sequence A or sequence B. (So, the resulting A and B are disjoint subsequences of $(i_j)_{j=0}^s$). The probability that we can make A and B paths is the probability that at each step i_j selects one of the current end terms of A or B. For step j this is at most $4/i_j$ so by independence the total probability is less than $\prod_{j=2}^s(4/i_j)$. We have inequality here because we are over-counting the case where i_j is above both of the current end terms of A and B.

The expected size of $FP(r,r+1,v)$ is the sum over all subsets I of $[r,n]$, with $r,r+1,v\in I$, of the probability that I forms a forked path with ends $r,r+1,\max I$ and connection point v. This is the probability that $I_{<v}=\{i\in I : i<v\}$ forms two disjoint paths from r and $r+1$; and v selects the end of both paths; and $I_{\geq v}=\{i\in I : i\geq v\}$ forms a path from v to $\max I$. So, for $I=\{r,r+1,i_2,\ldots,i_{s-1},v,i_{s+1},\ldots,i_{s+s'}\}$ with i_j increasing and $r+1<i_1$, $i_{s-1}<v<i_{s+1}$ this probability is less than $\prod_{j=2}^{s-1}(4/i_j)\times 1/\binom{v}{2}\times\prod_{j=1}^{s'}(2/i_{s+j})$.

So the sum over all such subsets I can be written as the following product, since the individual terms of the expanded product correspond exactly to the required probabilities for all subsets I,

$$\begin{aligned}
\mathbb{E}FP(r,r+1,v) &\leq \prod_{i=r+2}^{v-1}\left(1+\frac{4}{i}\right)\frac{1}{\binom{v}{2}}\prod_{i=v+1}^{n}\left(1+\frac{2}{i}\right)\\
&\leq \exp\left\{4\sum_{i=r+2}^{v-1}\frac{1}{i}\right\}\frac{2}{v(v-1)}\exp\left\{2\sum_{i=v+1}^{n}\frac{1}{i}\right\}\\
&\leq \left(\frac{v-1}{r+1}\right)^4\frac{2}{v(v-1)}\left(\frac{n}{v}\right)^2\\
&\leq \frac{2n^2}{r^4},
\end{aligned}$$

using the inequalities $1+x\leq e^x$ and $\sum_{i=a}^b f(i)\leq\int_{a-1}^b f(x)dx$ for f decreasing, so in particular $\sum_{i=a}^b 1/i\leq\log b-\log(a-1)$.

Therefore, $\mathbb{E}FP(r,r+1)\leq 2n^3/r^4$ and so $\mathbb{E}|U_{r\,r+1}^{[n]}|\leq\mathbb{E}FP(r,r+1)\leq 2r^{1/5}$. The same method gives the same upper bound on the expected size of $U_{x\,y}^{[n]}$ for all pairs (x,y) in $[r,r+m-1]^{(2)}$ so $\mathbb{P}(|U_{r\,r+1}^{[n]}|\geq(10m^2)r^{1/5})\leq 1/5m^2$ and $\mathbb{P}(\text{all }|U_{x\,y}^{[n]}|\leq(10m^2)r^{1/5})\geq 9/10$.

Let $A_r^{[n]}$ be the set of points above r but not above $r+1,\ldots,r+m-1$ in $B_2[n]$, then $A_r^{[n]}=U_r^{[n]}\setminus\bigcup_{i=1}^{m-1}U_{r\,r+i}^{[n]}$. Similarly define $A_x^{[n]}$, $x\in[r+1,r+m-1]$. Then, for $r\geq r_0\geq 400m^6$, we have $(10m^2)r^{1/5}<r^{3/4}/2m$ so with probability greater than $4/5$ we have all $|A_x^{[n]}|$, $x\in[r,r+m-1]$ at least $\frac{1}{2}r^{3/4}$.

We grow the poset by adding a further $n=r^{7/5}$ points, to find $M=\binom{m}{2}$ points $a_1,\ldots,a_M$, so that each pair of sets $(A_x^{[n]},A_y^{[n]})$, $(x,y)\in[r,r+m-1]^{(2)}$ is selected by some a_i.

Now,

$$\mathbb{P}\big(n+i \text{ selects } (A_r^{[n]}, A_{r+1}^{[n]})\big) = \frac{|A_r^{[n]}||A_{r+1}^{[n]}|}{\binom{n+i}{2}} \geq \frac{r^{3/2}}{2(n+i)^2} \geq \frac{r^{3/2}}{8n^2} \quad \text{for } i \leq n,$$

so

$$\begin{aligned}\mathbb{P}\big(\text{none of } n+1,\dots,2n \text{ selects } (A_r^{[n]}, A_{r+1}^{[n]})\big) &\leq \left(1-\frac{r^{3/2}}{8n^2}\right)^n \\ &\leq \exp\left\{\frac{-r^{3/2}}{8n}\right\} \\ &\leq \exp(-r^{1/10}/8),\end{aligned}$$

which is less than $1/10M$ for $r_0 \geq (8\log 10M)^{10}$. The same calculations give the same upper bound on the probability of failing to find a point in $[n+1,2n]$ which selects $(A_x^{[n]}, A_y^{[n]})$ for each $(x,y) \in [r, r+m-1]^{(2)}$, so the probability of failing to find points $a_1,\dots,a_M$ in $[n+1,2n]$ as desired is less than $1/10$.

So with probability at least $3/5$ we have a set $\{r, r+1, \dots, r+m-1, a_1, a_2, \dots, a_M\}$ with the following properties:

(i) the points $r, r+1, \dots, r+m-1$ are incomparable,

(ii) the points $a_1,\dots,a_M$ are incomparable,

(iii) for each pair of points $r+i, r+j$ there is exactly one a_l which is above only these points in the set.

So $\{r, r+1, \dots, r+m-1, a_1, a_2, \dots, a_M\}$ is a copy of $P(1,2;m)$ in $B_2[2n]$.

This proves the theorem, since if this method fails (it will with probability at most $2/5$), so we do not have a copy in $B_2[r,2n]$, then we repeat the method but starting from the point $r = 2n+1$. For there not to be a copy of $P(1,2;m)$ in the infinite poset B_2, the repeating method must perpetually fail. But since the outcome of the method for each repetition is independent, the probability of this is zero and so we have a copy of $P(1,2;m)$ in B_2, almost surely. □

Corollary 2. *B_2 has infinite dimension.*

Proof. This is immediate, since $\dim P(1,2;m) \geq \log_2 \log_2 m$. □

3 Up-sets of vertices in B_2

Brightwell [6] proved that, almost surely, each element of B_2 is comparable with all but finitely many others. This result is contained within what we prove here; we need a more refined version, providing an estimate of the number of elements in $B_2[n]$ that are incomparable with an element r, and an estimate of the largest element incomparable with r. Recall that $U_r^{[n]}$ is the up-set of r in $B_2[n]$ and that $I_r^{[n]} = [r, n] \setminus U_r^{[n]}$ is the set of points larger than r and incomparable with r. We study the size $|U_r^{[n]}|$ and give good estimates of how $|U_r^{[n]}|$ grows with n. We then use these estimates to provide estimates of the size $|I_r^{[n]}|$.

In [11, 12], Wormald presented a theorem which describes when and how a discrete time Markov process can be approximated by the solution to a related differential equation. However the approximation is only in terms of asymptotic bounds; here we state and prove a version of the theorem which gives explicit expressions for the approximation.

We begin with some definitions.

Definition 3. A function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies a *Lipschitz condition* on a domain $\mathcal{D} \subseteq \mathbb{R}^2$ if there exists a constant $L > 0$ with the property

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|) \tag{7}$$

for all (x_1, y_1) and (x_2, y_2) in $\mathcal{D}$.

Definition 4. For Y a real variable of a discrete time random process $G_0, G_1, \ldots$ which depends on a scale parameter n, we write $Y(t)$ for $Y(G_t)$, and for a domain $\mathcal{D} \subseteq \mathbb{R}^2$ define the *stopping time* $T_{\mathcal{D}} = T_{\mathcal{D}}(Y)$ to be the minimum t such that $(t/n, Y(t)/n) \notin \mathcal{D}$.

The following lemma will be used in the theorem and is an extension of Azuma's inequality to supermartingales.

Lemma 5. *Let $Y_0, Y_1, \ldots$ be a supermartingale with respect to a sequence of σ-algebras $\{\mathcal{F}_i\}$ with $\mathcal{F}_0$ empty, and suppose $Y_0 = 0$ and $|Y_{i+1} - Y_i| \leq c$ for $i \geq 0$ always. Then for all $\alpha > 0$,*

$$\mathbb{P}(Y_i \geq \alpha c) \leq \exp\left(-\alpha^2/2i\right).$$

The lemma follows from exactly the same proof as Azuma's inequality.

Theorem 6. *Let Y be a real-valued function of the components of a discrete time Markov process $\{G_t\}_{t \geq 0}$. Assume that $\mathcal{D} \subseteq \mathbb{R}^2$ is closed and bounded and contains the set*

$$\{(0, y) : \mathbb{P}(Y(0) = yn) \neq 0 \text{ for some non-negative integer } n\}$$

and

(i) for some constant β,

$$|Y(t+1) - Y(t)| \leq \beta$$

always for $t < T_{\mathcal{D}}$,

(ii) for some function $f : \mathbb{R}^2 \to \mathbb{R}$ which is Lipschitz with constant L on some open set $\mathcal{D}_0$ containing $\mathcal{D}$, and some constant λ,

$$|\mathbb{E}(Y(t+1) - Y(t) | G_t) - f(t/n, Y(t)/n)| \leq \lambda/n$$

for $t < T_{\mathcal{D}}$,

(iii) $f : \mathbb{R}^2 \to \mathbb{R}$ *is bounded on* $\mathcal{D}_0$*, i.e., there is a constant* γ *such that* $|f(x,y)| \leq \gamma$ *for all* $(x,y) \in \mathcal{D}_0$.

Let $w = w(n)$ *be a fixed integer-valued function. Then the following are true.*

(a) For $(0, \hat{y}) \in \mathcal{D}$ *the differential equation*

$$\frac{dy}{dx} = f(x,y)$$

has a unique solution $y = y(x)$ *in* $\mathcal{D}$ *passing through* $y(0) = \hat{y}$*, and which extends for some positive* x *past some point, at which* $x = \sigma$ *say, at the boundary of* $\mathcal{D}$*;*

(b) Writing $i_0 = \min\{\lfloor T_{\mathcal{D}}/w \rfloor, \lfloor \sigma n/w \rfloor\}$ *and* $k_i = iw$*, there exists some* $B > 0$ *such that*

$$\mathbb{P}\big(|Y(t) - ny(t/n)| \geq B_i + (\beta + \gamma)w\big) \leq 2ie^{-2w^3/n^2}$$

for all $i = 0, 1, \ldots, i_0 - 1$ *and all* t*,* $k_i \leq t \leq k_{i+1}$*, and for* $i = i_0$ *and* $k_{i_0} \leq t \leq \min\{T_{\mathcal{D}}, \sigma n\}$*, where* $B_i = \big((1 + Lw/n)^i - 1\big)Bw/L$*, and* $y(x)$ *and* σ *are as in (a) with* $\hat{y} = Y(0)/n$.

Proof. Following the proof in [11], we have part (a) from the theory of differential equations. Let $y(x)$ and σ be as in part (a).

Let $0 \leq t \leq T_{\mathcal{D}} - w$ and let $0 \leq k < w$. This implies that $t + k < T_{\mathcal{D}}$ and so $(\frac{t+k}{n}, \frac{Y(t+k)}{n}) \in \mathcal{D}$.

By (i), we have $|Y(t+k+1) - Y(t+k)| \leq \beta$. Also, by (ii),

$$\begin{aligned}
\mathbb{E}(Y(t+k+1) - Y(t+k)|G_{t+k}) &\leq f\left(\frac{t+k}{n}, \frac{Y(t+k)}{n}\right) + \frac{\lambda}{n} \\
&\leq f\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + L\left(\frac{k}{n} + \frac{|Y(t+k) - Y(t)|}{n}\right) + \frac{\lambda}{n} \\
&\leq f\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + \frac{L(w + \beta w) + \lambda}{n},
\end{aligned}$$

where the second inequality follows from (7). Writing $g(n)$ for $(L(w + \beta w) + \lambda)/n$, the inequality becomes

$$\mathbb{E}(Y(t+k+1) - Y(t+k)|G_{t+k}) \leq f\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + g(n).$$

Therefore, conditional on G_t,

$$Y(t+k) - Y(t) - kf\left(\frac{t}{n}, \frac{Y(t)}{n}\right) - kg(n)$$

is a supermartingale in k with respect to the sequence of σ-fields generated by $G_t, \ldots, G_{t+w}$. The differences of the supermartingale are, by (i) and (iii), at most

$$\beta + f\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + g(n) \leq \beta + \gamma + g(n).$$

So, by Lemma 5, for all $\alpha > 0$,

$$\mathbb{P}\Big(Y(t+w) - Y(t) - wf\left(\tfrac{t}{n}, \tfrac{Y(t)}{n}\right) - wg(n) \geq \alpha\big(\beta + \gamma + g(n)\big)\Big) \leq e^{-\alpha^2/2w}. \tag{8}$$

The same argument with

$$Y(t+k) - Y(t) - kf\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + kg(n)$$

a submartingale gives

$$\mathbb{P}\Big(Y(t+w) - Y(t) - wf\left(\tfrac{t}{n}, \tfrac{Y(t)}{n}\right) + wg(n) \leq -\alpha\big(\beta + \gamma + g(n)\big)\Big) \leq e^{-\alpha^2/2w}. \tag{9}$$

Setting $\alpha = 2w^2/n$ and combining (8) and (9) gives

$$\mathbb{P}\Big(|Y(t+w) - Y(t) - wf\left(\tfrac{t}{n}, \tfrac{Y(t)}{n}\right)| \geq 2(w^2/n)\big(\beta + \gamma + g(n)\big) + wg(n)\Big) \leq 2e^{-2w^3/n^2}. \tag{10}$$

Now, define $k_i = iw$, $i = 0, 1, \ldots, i_0$ where $i_0 = \min\{\lfloor T_{\mathcal{D}}/w \rfloor, \lfloor \sigma n/w \rfloor\}$. We show by induction that for each such i,

$$\mathbb{P}\big(|Y(k_i) - y(k_i/n)n| \geq B_i\big) \leq 2ie^{-2w^3/n^2} \tag{11}$$

where $B_i = \big((1 + Lw/n)^i - 1\big)Bw/L$ for some $B > 0$.

The induction begins by the fact that $y(0) = Y(0)/n$. (Take $\hat{y} = Y(0)/n$ and use part (a).)

So, assume (11) is true for i. Write

$$\begin{aligned}
A_1 &= Y(k_i) - y(k_i/n)n \\
A_2 &= Y(k_{i+1}) - Y(k_i) \\
A_3 &= y(k_i/n)n - y(k_{i+1}/n)n
\end{aligned}$$

The inductive hypothesis (11) gives $|A_1| < B_i$ with probability at least $1 - 2ie^{-2w^3/n^2}$. By (10) we have

$$|A_2 - wf(k_i/n, Y(k_i)/n)| < 2(w^2/n)\big(\beta + \gamma + g(n)\big) + wg(n)$$

with probability at least $1 - 2e^{-2w^3/n^2}$.

Since f satisfies the Lipschitz condition (because $k_{i+1} < T_{\mathcal{D}}$ so $(k_{i+1}/n, Y(k_{i+1})/n) \in \mathcal{D}$) we also have

$$\begin{aligned}
|A_3 + wy'(k_i/n)| &= |y(k_i/n)n - y(k_{i+1}/n)n + wy'(k_i/n)| \\
&= |-wy'(k/n) + wy'(k_i/n)| && \text{for some } k,\ k_i \leq k \leq k_{i+1} \\
&= w\big|f\big(k/n, y(k/n)\big) - f\big(k_i/n, y(k_i/n)\big)\big| && \text{since } y \text{ is solution to (a)} \\
&\leq wL\big[w/n + |y(k/n) - y(k_i/n)|\big] && \text{by (7)} \\
&\leq wL\big[w/n + (w/n)\big|f\big(k'/n, y(k'/n)\big)\big|\big] && \text{for some } k',\ k_i \leq k' \leq k \\
&\leq wL\big[w/n + (w/n)\gamma\big] && \text{by (iii)} \\
&= L(1+\gamma)w^2/n
\end{aligned}$$

where we have used the Mean Value Theorem (twice, to get lines 2 and 5). So,

$$|y'(k_i/n) - f(k_i/n, Y(k_i)/n)| = |f(k_i/n, y(k_i)/n) - f(k_i/n, Y(k_i)/n)| \le L|A_1|/n$$

and so assuming $|A_1| < B_i$, we have

$$|A_3 - (-wf(k_i/n, Y(k_i)/n))| \le \frac{L(1+\gamma)w^2}{n} + \frac{Lw}{n}|A_1| \le \frac{L(1+\gamma)w^2}{n} + \frac{Lw}{n}B_i.$$

So, we have

$$\begin{aligned} |Y(k_{i+1}) - y(k_{i+1}/n)n| &= |A_1 + A_2 + A_3| \\ &< B_i + 2(w^2/n)\big(\beta+\gamma+g(n)\big) + wg(n) + L(1+\gamma)w^2/n + B_iLw/n \\ &= [2(w^2/n)\big(\beta+\gamma+g(n)\big) + wg(n) + L(1+\gamma)w^2/n] + B_i(1+Lw/n) \end{aligned} \quad (12)$$

with probability at least $1 - 2(i+1)e^{-2w^3/n^2}$.

There exists $B > 0$ with

$$2(w^2/n)\big(\beta+\gamma+g(n)\big) + wg(n) + L(1+\gamma)w^2/n \le Bw^2/n \quad (13)$$

for all n, so the term on the right hand side of inequality (12) can be replaced with $B_i(1 + Lw/n) + Bw^2/n$, which is exactly B_{i+1}. So we have (11) for $i + 1$.

Finally, $k_{i+1} - k_i = w$ and the variation in $Y(t)$ when t changes by at most w is at most βw, by (i), and as before $|y(t_1/n)n - y(t_2/n)n|$ is less than $w|f(t/n, y(t/n))|$ for some t, $t_1 \le t \le t_2$ and this is less than γw. So

$$\mathbb{P}\big(|Y(t) - ny(t/n)| \ge B_i + (\beta+\gamma)w\big) \le 2ie^{-2w^3/n^2}$$

for all $i = 0, 1, \ldots, i_0 - 1$ and all t, $k_i \le t \le k_{i+1}$, and for $i = i_0$ and $k_{i_0} \le t \le \min\{T_{\mathcal{D}}, \sigma n\}$.
□

We can apply Theorem 6 to $|U_r^{[n]}|$ as follows. We take as the Markov process the random binary growth model, and as the real-valued function the size of the up-set of a fixed vertex r. We then find sets $\mathcal{D}$ and $\mathcal{D}_0$, a function f, and constants β, λ and γ satisfying the assumptions of the theorem. We obtain the following corollary, which shows fairly precisely how $|U_r^{[m]}|$ grows as m goes from some initial n to $(\sigma+1)n$, where σ is a large constant. Over this range, $|U_r^{[n]}|/n$ grows from a small value to a value near to 1.

Corollary 7. *For fixed r and any $n > r$, if $|U_r^{[n]}| = c(n)n$ for $c(n)$ an arbitrary function of n, then*

$$\mathbb{P}\left(\left|\frac{|U_r^{[n(\sigma+1)]}|}{n(\sigma+1)} - \frac{\sigma+1}{\sigma+1/c(n)}\right| \ge \left(\frac{10e^{2.1\sigma}+2.1}{\sigma+1}\right)\frac{1}{n^{1/3-\delta}}\right) \le 2\sigma n^{1/3-\delta}e^{-2n^{3\delta}}$$

for any constants $0 < \delta < 1/3$, $\sigma > 0$.

Proof. Fix a vertex r in $B_2[n]$. Let the Markov process $\{G_t\}_{t\geq 0}$ be the random binary growth model but starting at stage n, so that G_t corresponds to $B_2[n+t]$. Let $Y(t)$ be the size of the up-set of r in $B_2[n+t]$, i.e., $Y(t) = |U_r^{[n+t]}|$. For any constant σ, define $\mathcal{D}$ as the region $\{(x,y) : 0 \leq x \leq \sigma, 0 \leq y \leq x+1\}$. The region $\mathcal{D}$ contains the interval $\{(0,y) : 0 \leq y \leq 1\}$, and since $|U_r^{[n]}| = c(n)n$, we must have $c(n) \leq 1$ for all n. So, $\mathcal{D}$ satisfies the assumption in Theorem 6, since it contains all points $(0, c(n))$ for $n = 1, 2, \ldots$. We now find a set $\mathcal{D}_0$, a function f, and constants β, λ and γ satisfying assumptions (i)–(iii).

Since $Y(t) = |U_r^{[n+t]}| \leq n+t$ we have $Y(t)/n \leq t/n + 1$, and so $(t/n, Y(t)/n) \in \mathcal{D}$ as long as $t/n \leq \sigma$. This implies $T_{\mathcal{D}} = \lfloor \sigma n \rfloor + 1$.

Let $\beta = 1$, then (i) holds since $|Y(t+1) - Y(t)| = |U_r^{[n+t+1]}| - |U_r^{[n+t]}| \leq 1$ always for $t \leq \sigma n$.

Let $f(x,y) = 2y/(x+1) - y^2/(x+1)^2$. Let $L = 2.1$ and $\gamma = 1.1$. The function f is bounded on $\mathcal{D}$ by 1 (attained when $y = x+1$) and is continuous over the boundary of $\mathcal{D}$, so there exists an open set $\mathcal{D}'$ containing $\mathcal{D}$ on which f is bounded by $\gamma = 1.1$. Also, $\|\boldsymbol{\nabla} f\|$, the length of the gradient vector of f ($\boldsymbol{\nabla} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$), is bounded on $\mathcal{D}$ by 2 and is continuous over the boundary of $\mathcal{D}$, so there exists an open set $\mathcal{D}''$ containing $\mathcal{D}$ on which $\|\boldsymbol{\nabla} f\|$ is bounded by $L = 2.1$. But then

$$|f(\mathbf{u}) - f(\mathbf{v})| \leq L|\mathbf{u} - \mathbf{v}| \tag{14}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{D}''$, so f is Lipschitz with constant L on $\mathcal{D}''$ (this follows by applying the triangle inequality to the right hand side of (14)). Let $\mathcal{D}_0$ be the intersection of the two sets $\mathcal{D}', \mathcal{D}''$. So, (iii) holds, and (ii) holds with $\lambda = 1$, since

$$\begin{aligned}
\mathbb{E}(Y(t+1) - Y(t)|G_t) &= 0 \times \mathbb{P}(Y(t+1) = Y(t)|G_t) + 1 \times \mathbb{P}(Y(t+1) = Y(t)+1|G_t) \\
&= 1 - \tbinom{n+t+1-Y(t)}{2} / \tbinom{n+t+1}{2} \\
&= 1 - \frac{(n+t+1-Y(t))(n+t-Y(t))}{(n+t+1)(n+t)} \\
&= \frac{2Y(t)(n+t+1) - Y(t)(Y(t)+1)}{(n+t+1)(n+t)},
\end{aligned}$$

which differs from $f(t/n, Y(t)/n)$ by at most $1/n$ for $t \leq \sigma n$.

Now $T_{\mathcal{D}} = \lfloor \sigma n \rfloor + 1$ and so $T_{\mathcal{D}} > \sigma n$. So Theorem 6 gives the result (b) for $i = i_0$, $t = \sigma n$, namely that, for some $B > 0$,

$$\mathbb{P}(|Y(\sigma n) - ny(\sigma)| > B_{i_0} + 2.1w) \leq 2i_0 e^{-2w^3/n^2}. \tag{15}$$

Here $y(x)$ is the solution to the differential equation

$$\frac{dy}{dx} = 2\frac{y}{x+1} - \frac{y^2}{(x+1)^2}$$

with initial condition $y(0) = c(n)$. This is a homogeneous equation with solution

$$y(x) = \frac{(x+1)^2}{x + 1/c(n)}.$$

Also, $i_0 \le \sigma n/w$, so $B_{i_0} = ((1+Lw/n)^{i_0} - 1)\, Bw/L \le Bwe^{L\sigma}/L$, and (15) becomes

$$\mathbb{P}\left(\left| |U_r^{[n+\sigma n]}| - n\frac{(\sigma+1)^2}{\sigma+1/c(n)} \right| > Bwe^{L\sigma}/L + 2.1w \right) \le 2(\sigma n/w)e^{-2w^3/n^2}.$$

for some $B > 0$.

Choose δ with $0 < \delta < 1/3$ and set the arbitrary function $w(n)$ to $n^{2/3+\delta}$. Then $w(n) = o(n)$ and so using the particular values for L, β, γ and λ, we can satisfy equation (13) with $B = 21$ and this gives the required result. □

In the proof of Theorem 1 we bounded the expectation of the hitting time of the event $|U_r| = k$. We use this bound to show that U_r contains all but finitely many points of B_2, almost surely. In terms of I_r we have the following theorem.

Theorem 8. *For fixed r, $|I_r| < \widetilde{M} r^{2+4\varepsilon}$ and $I_r \subseteq [r, r^{4+8\varepsilon}]$ with probability at least $1-\eta$, where ε is an arbitrary constant with $0 < \varepsilon < 1/4$, η is an arbitrary constant with $0 < \eta < 1$ and $\widetilde{M}$ is a constant depending on ε and η.*

Proof. Fix r. As before, let T_k be the hitting time of event $|U_r| = k$, in terms of the growth model, i.e., the smallest t such that $|U_r^{[t]}| = k$. As in (3), we have $\mathbb{E}T_k \le 2r\sqrt{k} + 2k$. So $\mathbb{E}T_{r^2} \le 4r^2$ and Markov's inequality gives

$$\mathbb{P}\big(|U_r^{[(16/\eta)r^2]}| < r^2\big) = \mathbb{P}\big(T_{r^2} > (16/\eta)r^2\big) < \eta/4 \tag{16}$$

so that with suitably high probability the size of the up-set, $|U_r^{[(16/\eta)r^2]}|$, is at least fraction $\eta/16$ of the size of the poset, $(16/\eta)r^2$.

Set $n_0 = (16/\eta)r^2$. We can rewrite equation (16) as

$$\mathbb{P}\big(|U_r^{[n_0]}|/n_0 \ge \eta/16\big) > 1 - \eta/4. \tag{17}$$

Assume we have $|U_r^{[n_0]}|/n_0 \ge \eta/16$. Let ε be an arbitrary constant with $0 < \varepsilon < 1/4$. We will use Corollary 7 to show that as the size of the poset, n, increases from n_0 to $(\sigma+1)n_0$, for some constant σ, the ratio $|U_r^{[n]}|/n$ also increases, to a value that is at least $1 - \varepsilon/2$.

Claim 1. *There exists a constant σ_0 (dependent on ε and η) such that if $\dfrac{|U_r^{[n_0]}|}{n_0} \ge \eta/16$ then $\dfrac{|U_r^{[(\sigma_0+1)n_0]}|}{(\sigma_0+1)n_0} \ge 1 - \varepsilon/2$ with probability at least $2\sigma_0 n_0^{1/4} e^{-2n_0^{1/4}}$.*

Proof of Claim 1. Suppose $|U_r^{[n_0]}|/n_0 \ge \eta/16$. Applying Corollary 7 with $n = n_0$, $c(n_0) = \eta/16$ and $\delta = 1/12$ we have

$$\mathbb{P}\left(\left|\frac{|U_r^{[n_0(\sigma+1)]}|}{n_0(\sigma+1)} - \frac{\sigma+1}{\sigma+16/\eta}\right| \ge \left(\frac{10e^{2.1\sigma}+2.1}{\sigma+1}\right)\frac{1}{n_0^{1/4}}\right) \le 2\sigma n_0^{1/4} e^{-2n_0^{1/4}} \tag{18}$$

for any $\sigma > 0$. Set σ_0 so that

$$\frac{\sigma_0 + 1}{\sigma_0 + 16/\eta} = 1 - \varepsilon/4 \tag{19}$$

and then for sufficiently large r, $\left(\frac{10e^{2.1\sigma_0} + 2.1}{\sigma_0 + 1}\right)\frac{1}{n_0^{1/4}} \leq \varepsilon/4$. Combining this inequality with (18) and (19) and setting $\sigma = \sigma_0$ gives the result. $\square$

Let $M = (16/\eta)(\sigma_0 + 1)$, so that $(\sigma_0 + 1)n_0 = Mr^2$. We have shown that, with suitably high probability, $|U_r^{[n]}|/n \geq 1 - \varepsilon/2$ for $n = Mr^2$. We now show that $|U_r^{[n]}|/n$ remains close to 1 for all larger n. That is, that $|U_r^{[n]}|/n \geq 1 - \varepsilon$ for all $n \geq Mr^2$.

Let $n_1 = Mr^2$, and $n_i = (1 + \varepsilon/2)^{i-1} n_1$ for $i = 2, 3, \ldots$.

Claim 2. *If $|U_r^{[n_i]}|/n_i \geq 1 - \varepsilon/2$ then*

(a) $|U_r^{[n]}|/n \geq 1 - \varepsilon$ for $n = n_i + 1, n_i + 2, \ldots, \lfloor n_{i+1} \rfloor$ and

(b) $|U_r^{[n_{i+1}]}|/n_{i+1} \geq 1 - \varepsilon/2$ with probability at least $1 - \varepsilon n_i^{1/4} e^{2n_i^{1/4}}$.

Proof of Claim 2. Suppose we have $|U_r^{[n_i]}|/n_i \geq 1 - \varepsilon/2$.

For part (a) we use the fact that $|U_r^{[n]}|$ is increasing in n, so that

$$\frac{|U_r^{[n]}|}{n} \geq \frac{|U_r^{[n_i]}|}{n_{i+1}} = \frac{|U_r^{[n_i]}|}{(1 + \varepsilon/2)n_i} \geq \frac{1 - \varepsilon/2}{1 + \varepsilon/2} \geq 1 - \varepsilon$$

for all $n = n_i + 1, n_i + 2, \ldots, \lfloor n_{i+1} \rfloor$.

For part (b) we apply Corollary 7 with $n = n_i$, $\sigma = \varepsilon/2$ and $\delta = 1/12$. We have

$$\mathbb{P}\left(\left|\frac{|U_r^{[n_i(\varepsilon/2+1)]}|}{n_i(\varepsilon/2 + 1)} - \frac{\varepsilon/2 + 1}{\varepsilon/2 + 1/c(n_i)}\right| \geq \left(\frac{10e^{2.1\varepsilon/2} + 2.1}{\varepsilon/2 + 1}\right)\frac{1}{n_i^{1/4}}\right) \leq \varepsilon n_i^{1/4} e^{-2n_i^{1/4}} \tag{20}$$

with $c(n_i) \geq 1 - \varepsilon/2$. So,

$$\frac{\varepsilon/2 + 1}{\varepsilon/2 + 1/c(n_i)} \geq \frac{\varepsilon/2 + 1}{\varepsilon/2 + 1/(1 - \varepsilon/2)}$$

and for sufficiently large r,

$$\frac{\varepsilon/2 + 1}{\varepsilon/2 + 1/(1 - \varepsilon/2)} - \left(\frac{10e^{2.1\varepsilon/2} + 2.1}{\varepsilon/2 + 1}\right)\frac{1}{n_i^{1/4}} \geq 1 - \varepsilon/2. \tag{21}$$

Then, (20) becomes $\mathbb{P}\big(|U_r^{[n_{i+1}]}|/n_{i+1} \leq 1 - \varepsilon/2\big) \leq \varepsilon n_i^{1/4} e^{-2n_i^{1/4}}$. $\square$

Notice that, since $n_{i+1} > n_i$, if the inequality (21) is satisfied for $i = 1$, then it is automatically satisfied for all larger i. That is, if we have r sufficiently large to be able to apply Claim 2 once, then we can apply it repeatedly to get the following.

Assuming $|U_r^{[n_1]}|/n_1 \geq 1-\varepsilon/2$, we have $|U_r^{[n]}|/n \geq 1-\varepsilon$ for all integers $n \geq n_1 = Mr^2$ with probability at least $1-\sum_{i=1}^{\infty} \varepsilon n_i^{1/4} e^{-2n_i^{1/4}}$, for sufficiently large r.

Let r be sufficiently large so that $2\sigma_0 n_0^{1/4} e^{-2n_0^{1/4}} + \sum_{i=1}^{\infty} \varepsilon n_i^{1/4} e^{-2n_i^{1/4}} < \eta/4$. Then, we have $|U_r^{[n]}|/n \geq 1-\varepsilon$ for all integers $n \geq Mr^2$ with probability at least $1-\eta/2$. Once $|U_r^{[n]}|$ is always a large fraction of n, we can show that $U_r^{[n]}$ becomes almost all of the poset $B_2[n]$ for $n = r^{4+8\varepsilon}$. Rather, we now look at $I_r^{[n]}$, the set of points in $[r,n]$ incomparable with r in $B_2[n]$.

For $t \geq Mr^2$, set $s_t = |I_r^{[t]}|/\sqrt{t}$, and consider the sequence (s_t) as a stochastic process.

We have that

$$s_{t+1} = \begin{cases} s_t \frac{\sqrt{t}}{\sqrt{t+1}} & \text{with probability } 1 - \binom{|I_r^{[t]}|}{2} / \binom{t+1}{2} \\ \frac{s_t\sqrt{t+1}}{\sqrt{t+1}} & \text{with probability } \binom{|I_r^{[t]}|}{2} / \binom{t+1}{2} \end{cases}$$

Therefore

$$\mathbb{E}s_{t+1} = \frac{s_t\sqrt{t} + \binom{s_t\sqrt{t}}{2}/\binom{t+1}{2}}{\sqrt{t+1}} = s_t\sqrt{\frac{t}{t+1}}\left(1 + \frac{s_t\sqrt{t}-1}{t(t+1)}\right).$$

Now, provided $s_t \leq \varepsilon\sqrt{t}$ (which will be the case unless $|U_r^{[t]}|$ drops below $(1-\varepsilon)t$), we have

$$\mathbb{E}s_{t+1} \leq s_t \left(1 - \frac{1}{t+1}\right)^{1/2} \left(1 + \frac{\varepsilon}{t+1}\right) \leq s_t \left(1 - \frac{1/2-\varepsilon}{t+1}\right)$$

for all $t \geq Mr^2$. So

$$\begin{aligned} \mathbb{E}s_{Mr^2+k} \leq s_{Mr^2} \prod_{j=1}^{k} \left(1 - \frac{1/2-\varepsilon}{Mr^2+j}\right) &\leq s_{Mr^2} \exp\left(-(1/2-\varepsilon)\sum_{j=1}^{k} \frac{1}{Mr^2+j}\right) \\ &\leq \varepsilon\sqrt{Mr^2}\left(\frac{Mr^2+1}{Mr^2+k+1}\right)^{1/2-\varepsilon} \\ &\leq \frac{\sqrt{Mr^2}}{4}\left(\frac{Mr^2+1}{Mr^2+k+1}\right)^{1/2-\varepsilon}, \end{aligned}$$

where we have used the fact that $\varepsilon < 1/4$ to get the last line. So,

$$\mathbb{E}s_{r^{4+8\varepsilon}} \leq \sqrt{Mr^2}\left(\frac{Mr^2}{r^{4+8\varepsilon}}\right)^{1/2-\varepsilon} \leq M^{1-\varepsilon} r^{-2\varepsilon+8\varepsilon^2}.$$

Using Markov's inequality, we have $s_{r^{4+8\varepsilon}} \leq (4/\eta)M^{1-\varepsilon}r^{-2\varepsilon+8\varepsilon^2}$ with probability at least $1-\eta/4$.

Therefore $|I_r^{[r^{4+8\varepsilon}]}| \leq \sqrt{r^{4+8\varepsilon}}.(4/\eta)M^{1-\varepsilon}r^{-2\varepsilon+8\varepsilon^2} = \widetilde{M}r^{2+2\varepsilon+8\varepsilon^2}$ with probability at least $1-\eta/4$, where $\widetilde{M} = (4/\eta)M^{1-\varepsilon}$.

Finally, let us consider the probability that all vertices with a label higher than $r^{4+8\varepsilon}$ are comparable with r; in other words $I_r^{[s]} = I_r^{[r^{4+8\varepsilon}]}$ for $s \geq r^{4+8\varepsilon}$. Given the size $|I_r^{[r^{4+8\varepsilon}]}|$, this probability is exactly

$$\prod_{s=r^{4+8\varepsilon}+1}^{\infty} \left(1 - \frac{\binom{|I_r^{[r^{4+8\varepsilon}]}|}{2}}{\binom{s}{2}}\right),$$

which is at least

$$1 - \frac{|I_r^{[r^{4+8\varepsilon}]}|^2}{2} \sum_{s=r^{4+8\varepsilon}+1}^{\infty} \frac{1}{\binom{s}{2}} = 1 - \frac{|I_r^{[r^{4+8\varepsilon}]}|^2}{r^{4+8\varepsilon}} \geq 1 - \frac{\widetilde{M}^2}{r^{4\varepsilon-16\varepsilon^2}}.$$

Since $\varepsilon < 1/4$ we have $4\varepsilon - 16\varepsilon^2 > 0$ so that for sufficiently large r, $\widetilde{M}^2/r^{4\varepsilon-16\varepsilon^2} < \eta/4$. Also, $|I_r^{[r^{4+8\varepsilon}]}| \leq \widetilde{M}r^{2+2\varepsilon+8\varepsilon^2} \leq \widetilde{M}r^{2+4\varepsilon}$. So, combining all the probabilities, we have $|I_r| = |I_r^{[r^{4+8\varepsilon}]}| \leq \widetilde{M}r^{2+4\varepsilon}$ and $I_r \subseteq [r, r^{4+8\varepsilon}]$ with probability at least $1 - \eta$, as required. $\square$

This result is close to the best possible; we have that $\mathbb{E}|U_r^{[n]}| \leq n^2/r^2$, so for small $\varepsilon > 0$, $|I_r| \geq r^{2-\varepsilon}$ with high probability.

We have shown that for a typical r, the size $|U_r^{[n]}|$ is a constant fraction of n for $n = \Theta(r^2)$, and that the set I_r is contained in $[r^{4+8\varepsilon}]$, with $|I_r| = O(r^{2+4\varepsilon})$. What about for a worst case r? Can we say something about all but finitely many r?

Clearly, we cannot always expect $|U_r^{[n]}|$ to be a constant fraction of n for $n = \Theta(r^2)$. As we showed in Section 1,

$$\mathbb{P}(r \text{ is maximal in } B_2[n]) = \frac{r(r-1)}{n(n-1)}$$

which is approximately r^2/n^2. Setting $n = r^{3/2}$, we have that

$$\mathbb{P}(r \text{ is maximal in } B_2[r^{3/2}]) \approx \frac{1}{r}$$

which means there are infinitely many r with $|U_r^{[r^{3/2}]}| = 1$, that is, with $T_2 > r^{3/2}$. Then we have that $\mathbb{E}|U_r^{[n]}| \leq n^2/r^3$, so for such an r the expected size of $U_r^{[r^2]}$ is less than r, and we need $n = \Theta(r^3)$ before the expected size of $U_r^{[n]}$ is a constant fraction of n. We believe this is the worst case, that $|U_r^{[n]}|$ is a constant fraction of n for $n = \Theta(r^3)$, and then I_r is contained in $[r^{6+\varepsilon'}]$, with $|I_r| = O(r^{3+\varepsilon})$. Heuristically, it appears that the growth of $|U_r^{[n]}|$ is highly dependent on the values of the hitting times, T_k, for small k, which we have seen (for $k = 2$) are not concentrated near the mean values. Indeed, once $|U_r^{[n]}|/n$ is at least $1/n^{1/3}$ we can apply Corollary 7, to closely approximate the growth. However, it appears rather difficult to prove these statements in full, and we settle for the following polynomial bounds on the size $|I_r|$ and the value of the largest s incomparable with r.

Theorem 9. *For all but finitely many r, $|I_r| \leq r^{27/5}$ and $I_r \subseteq [r^{12}]$.*

The proof is naturally very similar to the proof of Theorem 8.

Proof. Fix r. As before, let T_k be the hitting time of event $|U_r| = k$, in terms of the growth model, i.e., the smallest t such that $|U_r^{[t]}| = k$. As (3), we have $\mathbb{E}T_k \le 2r\sqrt{k} + 2k$. So $\mathbb{E}T_{r^{13/6}} \le 4r^{13/6}$. Markov's inequality gives

$$\mathbb{P}\big(|U_r^{[r^{3+8/45}]}| < r^{13/6}\big) = \mathbb{P}\big(T_{r^{13/6}} > r^{3+8/45}\big) < 4/r^{91/90}. \tag{22}$$

Set $n_0 = r^{3+8/45}$. Equation (22) becomes

$$\mathbb{P}\big(|U_r^{[n_0]}|/n_0 \ge 1/n_0^{7/22}\big) > 1 - 4/r^{91/90}.$$

Assume we have $|U_r^{[n_0]}|/n_0 \ge 1/n_0^{7/22}$. We will use Corollary 7 to show that as we increase the size of the poset by a factor of 2, the fraction $|U_r^{[n]}|/n$ also increases by a factor that is only slightly smaller than 2. We can use this method repeatedly until $|U_r^{[n]}|/n$ is at least some constant fraction.

Let $n_i = 2^i n_0$ for $i = 1, 2, \dots$ and let $c(n) = |U_r^{[n]}|/n$ for all $n \ge n_0$.

Claim 1. *If* $1/n_0^{7/22} < c(n_i) < 1/300$ *then* $c(n_{i+1}) \ge (149/75)c(n_i)$ *with probability at least* $1 - 2n_i^{8/25}e^{-2n_i^{1/25}}$.

Proof of Claim 1. Suppose $1/n_0^{7/22} < c(n_i) < 1/300$. The upper bound on $c(n_i)$ implies

$$\frac{2}{1 + 1/c(n_i)} > (299/150)c(n_i) \tag{23}$$

and the lower bound implies

$$\left(\frac{10e^{2.1} + 2.1}{2}\right)\frac{1}{n_i^{8/25}} < (1/150)\frac{1}{n_0^{7/22}} < (1/150)c(n_i). \tag{24}$$

So applying Corollary 7 with $n = n_i, \delta = 1/75, \sigma = 1$, we have

$$\mathbb{P}\left(\left|\frac{|U_r^{[2n_i]}|}{2n_i} - \frac{2}{1 + 1/c(n_i)}\right| \ge \left(\frac{10e^{2.1} + 2.1}{2}\right)\frac{1}{n_i^{8/25}}\right) \le 2n_i^{8/25}e^{-2n_i^{1/25}}$$

which, using (23) and (24), gives the result. □

Using Claim 1 repeatedly we have that for $k = 0, 1, \dots$ either $c(n_l) \ge 1/300$ for some $l < k$, or

$$c(n_k) \ge (149/75)^k c(n_0) \ge (149/75)^k/n_0^{7/22}$$

with probability at least $1 - \sum_{i=0}^{k-1} 2n_i^{8/25}e^{-2n_i^{1/25}}$.

So, there exists a $k \le \dfrac{\log\left((1/300)n_0^{7/22}\right)}{\log(149/75)}$ such that $|U_r^{[n_k]}|/n_k \ge 1/300$ with probability at least $1 - (\log n_0)n_0^{1/2}e^{-2n_0^{1/25}}$.

We have

$$n_k \le 2^{\log(n_0^{7/22}/300)/\log(149/75)} n_0 = (n_0^{7/22}/300)^{\log 2/\log(149/75)} n_0. \tag{25}$$

Using $n_0 = r^{3+8/45}$ we get $n_k \le r^{21/5}/317$.

Assume we have $|U_r^{[n_k]}|/n_k \ge 1/300$. We will apply Corollary 7 once more to increase the fraction $|U_r^{[n]}|/n$ to a constant close to 1.

Claim 2. $\dfrac{|U_r^{[n]}|}{n} \ge 77/78$ *with probability at least* $1 - 10^5 n_k^{1/4} e^{-2n_k^{1/4}}$, *where* $n = 46345 n_k \le 150 r^{21/5}$.

Proof of Claim 2. We have $|U_r^{[n_k]}|/n_k \ge 1/300$. Applying Corollary 7, with $n = n_k$ and $\delta = 1/12$ we have

$$\mathbb{P}\left(\left| \frac{|U_r^{[n_k(\sigma+1)]}|}{n_k(\sigma+1)} - \frac{\sigma+1}{\sigma+1/c(n_k)} \right| \ge \left(\frac{10e^{2.1\sigma}+2.1}{\sigma+1} \right) \frac{1}{n_k^{1/4}} \right) \le 2\sigma n_k^{1/4} e^{-2n_k^{1/4}} \tag{26}$$

for any $\sigma > 0$. Set $\sigma = 46344$ so that

$$\frac{\sigma+1}{\sigma+1/c(n_k)} = \frac{46345}{46344+1/c(n_k)} \ge 155/156, \tag{27}$$

which is possible, since $c(n_k) \ge 1/300$. Then for sufficiently large r, $\dfrac{10e^{2.1\sigma}+2.1}{\sigma+1}\dfrac{1}{n_k^{1/4}} \le 1/156$. Combining with (26) and (27) and setting $\sigma = 46344$ gives the result. $\square$

By a similar method we can show that $\dfrac{|U_r^{[t]}|}{t} \ge 77/78$ for all $t \ge n$ with probability at least $1 - \sum_{t=n}^{\infty} t^{1/4} e^{-2t^{1/4}}$.

As before, for $t \ge n$, set $s_t = |I_r^{[t]}|/\sqrt{t}$, and consider the sequence (s_t) as a stochastic process. Again, we have

$$\mathbb{E}s_{t+1} = \frac{s_t\sqrt{t} + \binom{s_t\sqrt{t}}{2}/\binom{t+1}{2}}{\sqrt{t+1}} = s_t\sqrt{\frac{t}{t+1}}\left(1 + \frac{s_t\sqrt{t}-1}{t(t+1)}\right).$$

Now, provided $s_t \le \sqrt{t}/78$ (which will be the case unless $|U_r^{[t]}|$ drops below $(77/78)t$), we have

$$\mathbb{E}s_{t+1} \le s_t\left(1 - \frac{1}{t+1}\right)^{1/2}\left(1 + \frac{1}{78(t+1)}\right) \le s_t\left(1 - \frac{1/2 - 1/78}{(t+1)}\right)$$

which gives

$$\mathbb{E}s_{t+k} \le s_t \prod_{j=1}^{k}\left(1 - \frac{19}{39(t+j)}\right) \le s_t \exp\left(-(19/39)\sum_{j=1}^{k}\frac{1}{t+j}\right) \le \frac{\sqrt{t}}{78}\left(\frac{t+1}{t+k+1}\right)^{19/39}.$$

So, for example, $\mathbb{E}s_{t^{20/7}} \leq 1/(78t^{17/42})$, and for $t = r^{21/5}$ this gives $\mathbb{E}s_{r^{12}} \leq 1/r^{17/10}$. By Markov's inequality, we have $s_{r^{12}} \leq 1/r^{3/5}$ with probability at least $1 - 1/r^{11/10}$.

Therefore $|I_r^{[r^{12}]}| \leq \sqrt{r^{12}}/r^{3/5} = r^{27/5}$ with probability at least $1 - 1/r^{11/10}$.

Finally, let us consider the probability that all vertices with a label higher than r^{12} are comparable with r; in other words $I_r^{[s]} = I_r^{[r^{12}]}$ for $s \geq r^{12}$. Given the size $|I_r^{[r^{12}]}|$, this probability is exactly

$$\prod_{s=r^{12}+1}^{\infty} \left(1 - \frac{\binom{|I_r^{[r^{12}]}|}{2}}{\binom{s}{2}}\right),$$

which is at least

$$1 - \frac{|I_r^{[r^{12}]}|^2}{2} \sum_{s=r^{12}+1}^{\infty} \frac{1}{\binom{s}{2}} = 1 - \frac{|I_r^{[r^{12}]}|^2}{r^{12}} \geq 1 - \frac{1}{r^{6/5}}.$$

So, combining all the probabilities, we have $|I_r| = |I_r^{[r^{12}]}| \leq r^{27/5}$ and $I_r^{[s]} = I_r^{[r^{12}]}$ for $s \geq r^{12}$ with probability at least

$$1 - 4/r^{91/90} - (\log n_0) n_0^{1/2} e^{-2n_0^{1/25}} - 10^5 n_k^{1/4} e^{-2n_k^{1/4}} - \sum_{t=n}^{\infty} t^{1/4} e^{-2t^{1/4}} - 1/r^{11/10} - 1/r^{6/5}.$$

Since

$$\sum_{r=1}^{\infty} \left(4/r^{91/90} + (\log n_0) n_0^{1/2} e^{-2n_0^{1/25}} + 10^5 n_k^{1/4} e^{-2n_k^{1/4}} + \sum_{t=n}^{\infty} t^{1/4} e^{-2t^{1/4}} + 1/r^{11/10} + 1/r^{6/5}\right)$$

is finite, the first Borel-Cantelli Lemma gives us the required result. □

Notice that in this proof we use Markov's inequality twice, each time introducing a factor of r, which is why our bound is (essentially) $|I_r| \leq r^{5+\varepsilon}$ and not $|I_r| \leq r^{3+\varepsilon}$ as we believe.

4 A poset not contained in B_2

In Section 2 we have shown that B_2 contains $P(1, 2; m)$ almost surely. It is natural to ask whether this is typical: which posets are contained in B_2? For any poset P, $\mathbb{P}(B_2 \supseteq P)$ is positive, as P is a subposet of some possible binary order. So, is every finite poset contained, almost surely? This has been shown for random graph orders; here we show that it is not true for B_2.

Recall that we write $P(1, 2; 3)$ for the poset consisting of the 1-element and 2-element subsets of $\{1, 2, 3\}$ ordered by inclusion (Figure 1). Write $P(1, 2; 3)^{(k)}$ for a "tower" of k copies of $P(1, 2; 3)$ with the maximal elements of copy i identified with the minimal elements of copy $i + 1$, for $i = 1, \dots, k - 1$ (Figure 2).

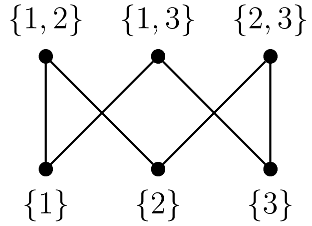


Figure 1: $P(1, 2; 3)$

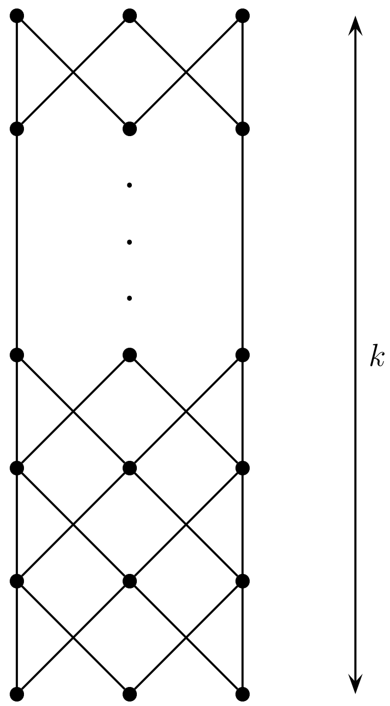


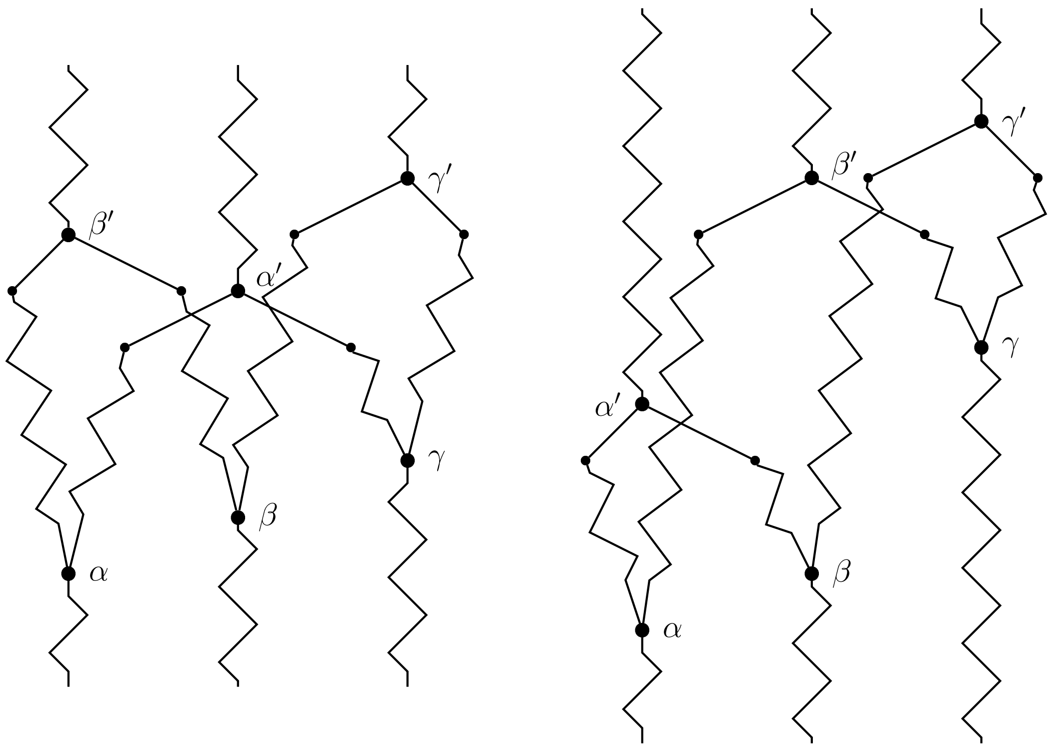
Figure 2: $P(1, 2; 3)^{(k)}$

The result from Theorem 1, for the case $m = 3$, is that a copy of $P(1, 2; 3)$ with minimal points $r, r+1, r+2$ is contained in $B_2[r, n]$, where $n = 2r^{7/5}$, with probability at least 3/5. The method used certainly requires $k^2 = |U_r|^2 > n = 2r\sqrt{k} + 2k$, i.e., $n \gtrsim r^{4/3}$. We now consider the probability that there exists any copy of $P(1, 2; 3)^{(k)}$ in $B_2[r, n]$, and show this is very small for $n = o(r^{(k+2)/3})$. (So for $k = 1$ this is a trivial result but, interestingly, if we restrict to only copies of $P(1, 2; 3)^{(k)}$ with minimal points $r, r+1, r+2$ then the result becomes that the probability that there exists such a copy in $B_2[r, n]$ is very small for $n = o(r^{k/3+1})$. This gives a certain justification to the method used to construct such a $P(1, 2; 3)$.) Using this result with Theorem 9 we provide an example of a poset that, with positive probability, is not contained in B_2.

Theorem 10. *The probability that there exists a $P(1, 2; 3)^{(k)}$ as a subposet of $B_2[r, n]$ is $O(n^9/r^{3k+6})$.*

Proof. Throughout we will write x is above (below) y to mean x is above (below) y in B_2, and write x is greater (less) than y to mean x is greater (less) than y in $\mathbb{N}$. Usually, we will reserve $<, \leq, >$ and $\geq$ for the order on $\mathbb{N}$.

Consider $P(1, 2; 3)$ as a subposet of B_2 and take a minimal point, a. It is below two maximal points, b_1, b_2, so there is at least one path from a to b_1 and at least one path from a to b_2. Choosing one path to b_1 and one to b_2, we can find the greatest point common to both paths, call this a *branching point.* We can do this for all three minimal points to obtain three branching points. The six chosen paths can also be paired according to which



(a) $\gamma < \alpha'$

(b) $\alpha' < \gamma$

Figure 3: $P(1,2;3)$ with branching and connection points

maximal point they go to, and taking the least point common to a pair of paths gives three *connection points*, one for each maximal point. Note that the branching and connection points are not unique if we had a choice of paths, but are distinct for any choice of paths. We label the branching points α, β, γ and the connection points α', β', γ', so that $\alpha < \beta < \gamma$ and $\alpha' < \beta' < \gamma'$. Each path contains both a branching point and a connection point, and since each connection point is contained in two paths, it must be greater than (at least) two branching points. In particular, α' must be greater than α and β. Similarly, each branching point is less than (at least) two connection points, so γ must be less than β' and γ'. So, we have the inequalities $\beta < \alpha'$ and $\gamma < \beta'$, which gives the order $\alpha < \beta < \gamma, \alpha' < \beta' < \gamma'$. It is not possible to order γ and α'. An example of the branching and connection points for the two cases $\gamma < \alpha'$ and $\alpha' < \gamma$ are shown in Figure 3. Note that in Fig. 3(a) α' can be above any pair of branching points, whereas in Fig. 3(b) α' has to be above α and β.

For a particular copy of $P(1,2;3)^{(k)}$ in B_2 we have k copies of $P(1,2;3)$ so we can find branching points and connection points for each copy. We label the branching points in copy i by $\alpha_i, \beta_i, \gamma_i$ and the connection points by $\alpha'_i, \beta'_i, \gamma'_i$. So, we have sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ of branching points and sequences $\boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ of connection points, where subscript i denotes the points in copy i. We have the order $\alpha_i < \beta_i < \gamma_i, \alpha'_i < \beta'_i < \gamma'_i$ for each i, as before. Call the points $\alpha_i, \beta_i, \gamma_i$, i-branching points, and the points $\alpha'_i, \beta'_i, \gamma'_i$, i-connection points.

Ideally, we would aim to separate the copies of $P(1,2;3)$ to analyse them individually (for example by assuming $\gamma'_i < \alpha_{i+1}$). Unfortunately this is not possible so we have more cases to consider.

Since $P(1,2;3)^{(k)}$ is formed by identifying maximal points in copy i of $P(1,2;3)$ to minimal points in copy $i+1$, we have that each $(i+1)$-branching point $\alpha_{i+1} < \beta_{i+1} < \gamma_{i+1}$ is above (and therefore greater than) a distinct i-connection point $\alpha'_i < \beta'_i < \gamma'_i$. This immediately gives the inequalities $\alpha_{i+1} > \alpha'_i$ and $\gamma'_i < \gamma_{i+1}$. Looking at β_{i+1}, either it is above β'_i or γ'_i which implies $\beta_{i+1} > \beta'_i$, or it is above α'_i in which case α_{i+1} is not above α'_i and so must be above β'_i or γ'_i. But this implies $\beta_{i+1} > \alpha_{i+1} > \beta'_i$. To summarise, we have

$$\alpha_i < \beta_i < \gamma_i, \alpha'_i < \beta'_i < \gamma'_i \qquad \text{for } i = 1, \ldots, k \tag{28}$$

$$\alpha_{i+1} > \alpha'_i \qquad \beta_{i+1} > \beta'_i \qquad \gamma_{i+1} > \gamma'_i \qquad \text{for } i = 1, \ldots, k-1 \tag{29}$$

which is all we can deduce about the order of branching and connection points.

Suppose we have a $P(1,2;3)^{(k)}$ in $B_2[r,n]$. We partition $[r,n]$ into sets of two types (plus two 'end' sets). A set of Type I is of the form $[\beta_i, \beta'_i]$ and a set of Type II of the form $[\beta'_i + 1, \beta_{i+1} - 1]$. The k sets of Type I and $k-1$ sets of Type II and the 'end' sets $[r, \beta_1 - 1]$ and $[\beta'_k + 1, n]$ form the partition of $[r,n]$. We investigate which parts can contain the branching and connection points. Clearly, β_i and β'_i are contained in the Type I sets. From (28) we have that $\gamma_i, \alpha'_i \in [\beta_i, \beta'_i]$ $(i = 1, \ldots, k)$. Also, (28) and (29) give the inequalities $\beta_{i-1} < \alpha_i < \beta_i$ and $\beta'_i < \gamma'_i < \beta'_{i+1}$ which implies that $\alpha_i \in [\beta_{i-1}, \beta'_{i-1}] \cup [\beta'_{i-1} + 1, \beta_i - 1]$ $(i = 2, \ldots, k)$ and $\gamma'_i \in [\beta'_i + 1, \beta_{i+1} - 1] \cup [\beta_{i+1}, \beta'_{i+1}]$ $(i = 1, \ldots, k-1)$. The end cases $\alpha_1 \in [r, \beta_1 - 1]$ and $\gamma'_k \in [\beta'_k + 1, n]$ are obvious. So, looking at a Type I set $[\beta_i, \beta'_i]$, it contains $\beta_i, \gamma_i, \alpha'_i$ and β'_i and possibly γ'_{i-1} and α_{i+1}. This gives four possibilities which are shown in Figure 4. Finally, we have that the points in the Type II sets are determined by the points in the two adjacent Type I sets. That is, $[\beta'_i + 1, \beta_{i+1} - 1]$ may contain γ'_i (but only if $\gamma'_i \notin [\beta_{i+1}, \beta'_{i+1}]$) and α_{i+1} (but only if $\alpha_{i+1} \notin [\beta_i, \beta'_i]$).

Fix $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$. We call a set $J \subseteq [r,n]$ an $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-*framework* in $B_2[r,n]$ if J contains all the points in the sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ and the remaining points in J form disjoint paths so that:

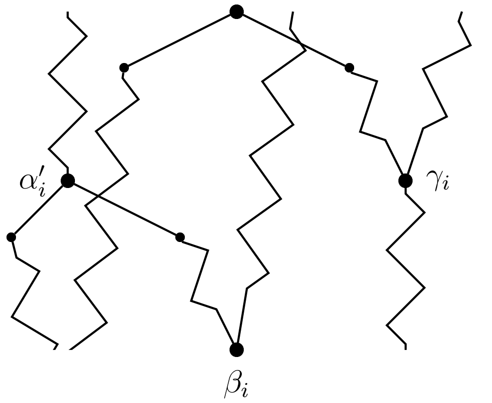
(a) there are two paths from each branching point,

(b) there are two paths to each i-connection point, which are from two i-branching points so that no two i-connection points have their paths from the same two i-branching points, for $i = 1, \ldots, k$,

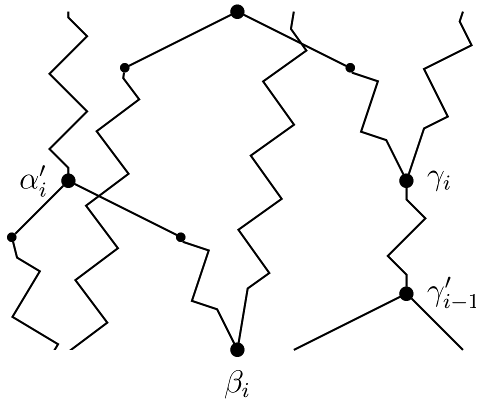
(c) there is one path from each connection point (except for the k-connection points),

(d) there is one path to each i-branching point, which is from a $(i-1)$-connection point, for $i = 2, \ldots, k$.

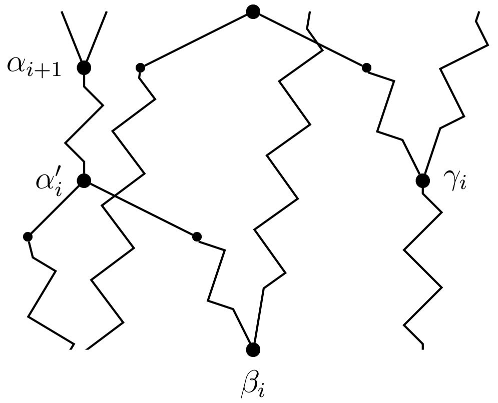
Note that these paths can just consist of start and end points, that is, it is possible for the set that only contains the points in $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ to be a $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework.



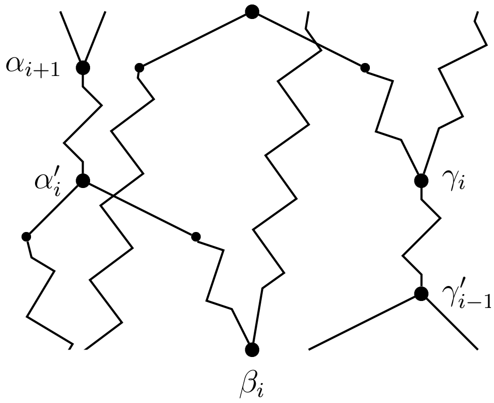
(a) Contains $\beta_i, \gamma_i, \alpha_i', \beta_i'$



(b) Also contains γ_{i-1}'



(c) Also contains α_{i+1}



(d) Also contains γ_{i-1}' and α_{i+1}

Figure 4: Points in $[\beta_i, \beta_i']$ — 4 possible cases

Indeed, for any set $J \subseteq [r,n]$ containing all the points in $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ there is a positive probability of J being a $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework.

For any copy of $P(1,2;3)^{(k)}$ with branching points given by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and connection points given by $\boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$, we can construct an $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework by taking the set of all the branching and connection points and the points of the paths that defined them (but not including those paths below 1-branching points, and those paths above the k-connection points). Calling a set $J \subseteq [r,n]$ a *framework* in $B_2[r,n]$ if it is an $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework in $B_2[r,n]$ for some $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$, we have that if $B_2[r,n]$ contains no frameworks then it also contains no copies of $P(1,2;3)^{(k)}$.

So, it is enough to show that the expected number of frameworks in $B_2[r,n]$ is small and we do this by showing that the expected number of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-frameworks in $B_2[r,n]$ is small for all sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ satisfying (28) and (29).

For fixed $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$, we count the number of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-frameworks in $B_2[r,n]$ by considering the event "J is a $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework" as a sequence of events in the sets of the partition of $[r,n]$. Label the partition

$$\begin{gathered} K_1 = [r, \beta_1 - 1], \quad K_{2k+1} = [\beta'_k + 1, n] \\ K_{2i} = [\beta_i, \beta'_i], i = 1, \ldots, k \\ K_{2i+1} = [\beta'_i + 1, \beta_{i+1} - 1], i = 1, \ldots, k-1. \end{gathered}$$

We write $\max K_j$ for the largest element of K_j. In a definition similar to that of an $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework, for $j = 1, \ldots, 2k+1$, we call a set $J \subseteq [r, \max K_j]$ a *j-framework* in $B_2[r, \max K_j]$ if J contains all the points in the sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ that are in $[r, \max K_j]$ and the remaining points in J form disjoint paths so that

(a) there are two paths from each branching point in J,

(b) there are two paths to each i-connection point in J, which are from two i-branching points in J so that no two i-connection points in J have their paths from the same two i-branching points in J, for $i = 1, \ldots, k$,

(c) there is one path from each connection point in J (except for the k-connection points),

(d) there is one path to each i-branching point in J, which is from a $(i-1)$-connection point in J, for $i = 2, \ldots, k$.

Again, for any set $J \subseteq [r, \max K_j]$ containing all the points in $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$ that are in $[r, \max K_j]$ there is a positive probability of J being a j-framework.

So, a $(2k+1)$-framework is the same as a $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework. Notice that, whereas in a $(2k+1)$-framework all paths are between branching and connection points, in a j-framework, for $j \neq 2k+1$, there can be paths from some branching and connection points that do not end at a branching or connection point (the paths from the branching and connection points that are not below any others in J). Call the end points of these paths the *end points* of the j-framework. We shall see that, although the end points of a j-framework can be different for different j-frameworks, what is important for our calculations is that the number of end points of a j-framework is the same for different j-frameworks, for fixed j.

Now, define an l-frame as follows:

- $l = 1$: A 1-*frame* is a set $J_1 \subseteq K_1$ which is a 1-framework in $B_2[r, \max K_1]$.
- $l \neq 1$: Given that J is an $(l-1)$-framework in $B_2[r, \max K_{l-1}]$, an l-*frame for* J is a set $J_l \subseteq K_l$ such that $J \cup J_l$ is an l-framework in $B_2[r, \max K_l]$.

So, for sets $J_j \subseteq K_j$, $j = 1, \dots, 2k+1$, we have

$$\mathbb{P}\left(\bigcup_{j=1}^{2k+1} J_j \text{ an } (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})\text{-framework}\right) =$$

$$\mathbb{P}\big(J_1 \text{ a 1-frame}\big)\mathbb{P}\big(J_2 \text{ a 2-frame for } J_1\big) \cdots \mathbb{P}\left(J_{2k+1} \text{ a } (2k+1)\text{-frame for } \bigcup_{j=1}^{2k} J_j\right). \quad (30)$$

Now, write $X(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$ for the number of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-frameworks. We have $X(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$ equal to the sum

$$\sum_{J_1 \subseteq K_1} \cdots \sum_{J_{2k+1} \subseteq K_{2k+1}} I\left(\bigcup_{j=1}^{2k+1} J_j \text{ is an } (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})\text{-framework}\right), \quad (31)$$

but $\bigcup_{j=1}^{2k+1} J_j$ is an $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$-framework only if $\bigcup_{j=1}^{2k+1} J_j$ contains all the points in $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$. So, writing $K_j(BC)$ for the set of branching and connection points that are in K_j, the sum (31) is equal to

$$\sum_{\substack{J_1 \subseteq K_1: \\ K_1(BC) \subseteq J_1}} \cdots \sum_{\substack{J_{2k+1} \subseteq K_{2k+1}: \\ K_{2k+1}(BC) \subseteq J_{2k+1}}} I\left(\bigcup_{j=1}^{2k+1} J_j \text{ is an } (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})\text{-framework}\right).$$

Taking expectations and using (30) gives

$$\mathbb{E}X(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}) =$$

$$\sum_{\substack{J_1 \subseteq K_1: \\ K_1(BC) \subseteq J_1}} \cdots \sum_{\substack{J_{2k+1} \subseteq K_{2k+1}: \\ K_{2k+1}(BC) \subseteq J_{2k+1}}} \mathbb{P}\big(J_1 \text{ a 1-frame}\big) \cdots \mathbb{P}\left(J_{2k+1} \text{ a } (2k+1)\text{-frame for } \bigcup_{j=1}^{2k} J_j\right) \quad (32)$$

But $\mathbb{P}\big(J_l \text{ an } l\text{-frame for } \bigcup_{j=1}^{l-1} J_j\big)$ does not depend on $J_1, \dots, J_{l-1}$; this is the conditional probability that $\bigcup_{j=1}^{l} J_j$ is an l-framework, given that $\bigcup_{j=1}^{l-1} J_j$ is an $(l-1)$-framework. Since

$K_l(BC) \subseteq J_l$, this is the probability that the points in $\bigcup_{j=1}^{l} J_j$ form paths satisfying (a)–(d). But we know that $\bigcup_{j=1}^{l-1} J_j$ is an $(l-1)$-framework, so $\bigcup_{j=1}^{l} J_j$ is an l-framework provided the points in J_l form paths that continue the paths in $\bigcup_{j=1}^{l-1} J_j$ in such a way that (a)–(d) are satisfied. That is, the points in J_l must either select other points in J_l, or one of the end points of the $(l-1)$-framework, $\bigcup_{j=1}^{l-1} J_j$. So the probability $\mathbb{P}\big(J_l$ an l-frame for $\bigcup_{j=1}^{l-1} J_j\big)$ can only depend on the set J_l and the number of end points of $\bigcup_{j=1}^{l-1} J_j$. However, the number of end points of a j-framework is determined by which branching and connection points are not below any others in the j-framework and these are fixed for particular sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$.

So, for $j = 2, \dots, 2k+1$ we write $\mathbb{P}_l(J_l)$ for $\mathbb{P}\big(J_l$ an l-frame for $\bigcup_{j=1}^{l-1} J_j\big)$, and we write $\mathbb{P}_1(J_1)$ for $\mathbb{P}\big(J_1$ a 1-frame$\big)$. Equation (32) becomes

$$\mathbb{E}X(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}) = \prod_{l=1}^{2k+1} \sum_{\substack{J_l \subseteq K_l: \\ K_l(BC) \subseteq J_l}} \mathbb{P}_l(J_l)$$

Writing X for the total number of frameworks and $\mathbb{E}_l$ for $\sum_{J_l \subseteq K_l : K_l(BC) \subseteq J_l} \mathbb{P}_l(J_l)$, we have that the expected number of frameworks is

$$\mathbb{E}X = \sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \\ \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'} : (28), (29)}} \prod_{l=1}^{2k+1} \mathbb{E}_l.$$

We shall calculate an upper bound for $\prod_{l=1}^{2k+1} \mathbb{E}_l$ for the case $\alpha_i < \beta_i < \gamma_i < \alpha'_i < \beta'_i < \gamma'_i < \alpha_{i+1}$, etc. (Figure 4(a)) and then show that this ordering is the worst case. That is, that the upper bound for the case $\alpha_i < \beta_i < \gamma_i < \alpha'_i < \beta'_i < \gamma'_i < \alpha_{i+1}$, etc, is an upper bound for any ordering of the branching and connection points subject to (28) and (29).

We again use the inequalities $1 + x \le e^x$ and $\sum_{j=a}^{b} f(j) \le \int_{a-1}^{b} f(x)dx$ for $f(x)$ decreasing, so that in particular

$$\prod_{j=a}^{b} \left(1 + \frac{c}{j}\right) \le \exp\left(\sum_{j=a}^{b} \frac{c}{j}\right) \le \exp\left(c \log \frac{b}{a-1}\right) = \left(\frac{b}{a-1}\right)^c.$$

For $l = 1$, $K_1(BC) = \{\alpha_1\}$, so we sum over $J_1 \subseteq K_1 = [r, \beta_1 - 1]$ containing $\{\alpha_1\}$. If $J_1 = \{\alpha_1, j_1, \dots, j_t\}$ with $\alpha_1 < j_1 < \cdots < j_t \le \beta_1 - 1$, then the probability $\mathbb{P}_1(J_1)$ is the probability that the points $j_s, s = 1, \dots, t$ form two disjoint paths from α_1, which is at most $\prod_{s=1}^{t} (4/j_s)$, by independence, and if $J_1 \not\subseteq [\alpha_1, \beta_1 - 1]$ then $\mathbb{P}_1(J_1) = 0$, so

$$\mathbb{E}_1 \le \prod_{j=\alpha_1+1}^{\beta_1 - 1} \left(1 + \frac{4}{j}\right) \le \left(\frac{\beta_1 - 1}{\alpha_1}\right)^4 \le \frac{\beta_1^4}{\alpha_1^4}.$$

For $l = 2$, $K_2 = \{\beta_1, \gamma_1, \alpha'_1, \beta'_1\}$, we sum over $J_2 \subseteq K_2 = [\beta_1, \beta'_1]$ containing $\{\beta_1, \gamma_1, \alpha'_1, \beta'_1\}$. So, $J_2 = \{\beta_1, j_1^{(1)}, \dots, j_{t_1}^{(1)}, \gamma_1, j_1^{(2)}, \dots, j_{t_2}^{(2)}, \alpha'_1, j_1^{(3)}, \dots, j_{t_3}^{(3)}, \beta'_1\}$ and the probability $\mathbb{P}_2(J_2)$ is the probability that

(i) the points $j_s^{(1)}, s = 1, \dots, t_1$ form four disjoint paths — two from β_1, two from the existing end points in the 1-frame J_1,

(ii) the points $j_s^{(2)}, s = 1, \dots, t_2$ form six disjoint paths — two from γ_1, four from the end points of the paths formed in (i),

(iii) the point α_1' is above two of the end points of the paths formed in (ii) (specifically, two paths with different starting points),

(iv) the points $j_s^{(3)}, s = 1, \dots, t_3$ form five disjoint paths — one from α_1', four from the end points of the remaining paths formed in (ii),

(v) the point β_1' is above two of the end points of the four "branching" paths formed in (iv) (i.e., not the path from α_1', and again specifically, two paths with different starting points).

All these events are independent of each other, and so this probability is at most

$$\left(\prod_{s=1}^{t_1} \frac{8}{j_s^{(1)}}\right) \left(\prod_{s=1}^{t_2} \frac{12}{j_s^{(2)}}\right) \frac{12}{\binom{\alpha_1'}{2}} \left(\prod_{s=1}^{t_3} \frac{10}{j_s^{(3)}}\right) \frac{4}{\binom{\beta_1'}{2}},$$

so the sum over all subsets of K_2 is

$$\begin{aligned}
\mathbb{E}_2 &\le \prod_{j=\beta_1+1}^{\gamma_1-1} \left(1 + \frac{8}{j}\right) \prod_{j=\gamma_1+1}^{\alpha_1'-1} \left(1 + \frac{12}{j}\right) \frac{12}{\binom{\alpha_1'}{2}} \prod_{j=\alpha_1'+1}^{\beta_1'-1} \left(1 + \frac{10}{j}\right) \frac{4}{\binom{\beta_1'}{2}} \\
&\le \left(\frac{\gamma_1 - 1}{\beta_1}\right)^8 \left(\frac{\alpha_1' - 1}{\gamma_1}\right)^{12} \frac{24}{\alpha_1'(\alpha_1' - 1)} \left(\frac{\beta_1' - 1}{\alpha_1'}\right)^{10} \frac{8}{\beta_1'(\beta_1' - 1)} \\
&\le 2^6 3 \frac{\beta_1'^8}{\beta_1^8 \gamma_1^4}.
\end{aligned}$$

For $l = 2i+1$, $K_{2i+1} = [\beta_i'+1, \beta_{i+1}-1]$, $K_{2i+1}(BC) = \{\gamma_i', \alpha_{i+1}\}$ and by a similar calculation we have

$$\begin{aligned}
\mathbb{E}_{2i+1} &\le \prod_{j=\beta_i'+1}^{\gamma_i'-1} \left(1 + \frac{8}{j}\right) \frac{1}{\binom{\gamma_i'}{2}} \prod_{j=\gamma_i'+1}^{\alpha_{i+1}-1} \left(1 + \frac{6}{j}\right) \frac{6}{\alpha_{i+1}} \prod_{j=\alpha_{i+1}+1}^{\beta_{i+1}-1} \left(1 + \frac{8}{j}\right) \\
&\le \left(\frac{\gamma_i' - 1}{\beta_i'}\right)^8 \frac{2}{\gamma_i'(\gamma_i' - 1)} \left(\frac{\alpha_{i+1} - 1}{\gamma_i'}\right)^6 \frac{6}{\alpha_{i+1}} \left(\frac{\beta_{i+1} - 1}{\alpha_{i+1}}\right)^8 \\
&\le 2^2 3 \frac{\beta_{i+1}^8}{\beta_i'^8 \alpha_{i+1}^3}
\end{aligned}$$

and for $l = 2i$ $(i = 2, \dots, k-1)$, $K_{2i} = [\beta_i, \beta'_i]$, $K_{2i}(BC) = \{\beta_i, \gamma_i, \alpha'_i, \beta'_i\}$ and

$$\begin{aligned}
\mathbb{E}_{2i} &\leq \frac{4}{\beta_i} \prod_{j=\beta_i+1}^{\gamma_i-1} \left(1 + \frac{10}{j}\right) \frac{2}{\gamma_i} \prod_{j=\gamma_i+1}^{\alpha'_i-1} \left(1 + \frac{12}{j}\right) \frac{12}{\binom{\alpha'_i}{2}} \prod_{j=\alpha'_1+1}^{\beta'_i-1} \left(1 + \frac{10}{j}\right) \frac{4}{\binom{\beta'_i}{2}} \\
&\leq \frac{4}{\beta_i} \left(\frac{\gamma_i - 1}{\beta_i}\right)^{10} \frac{2}{\gamma_i} \left(\frac{\alpha'_i - 1}{\gamma_i}\right)^{12} \frac{24}{\alpha'_i(\alpha'_i - 1)} \left(\frac{\beta'_i - 1}{\alpha'_i}\right)^{10} \frac{8}{\beta'_i(\beta'_i - 1)} \\
&\leq 2^9 3 \frac{\beta'^8_i}{\beta^{11}_i \gamma^3_i},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{2k} &\leq \frac{4}{\beta_k} \prod_{j=\beta_k+1}^{\gamma_k-1} \left(1 + \frac{10}{j}\right) \frac{2}{\gamma_k} \prod_{j=\gamma_k+1}^{\alpha'_k-1} \left(1 + \frac{12}{j}\right) \frac{12}{\binom{\alpha'_k}{2}} \prod_{j=\alpha'_k+1}^{\beta'_k-1} \left(1 + \frac{8}{j}\right) \frac{4}{\binom{\beta'_k}{2}} \\
&\leq \frac{4}{\beta_k} \left(\frac{\gamma_k - 1}{\beta_k}\right)^{10} \frac{2}{\gamma_k} \left(\frac{\alpha'_k - 1}{\gamma_k}\right)^{12} \frac{24}{\alpha'_k(\alpha'_k - 1)} \left(\frac{\beta'_k - 1}{\alpha'_k}\right)^{8} \frac{8}{\beta'_k(\beta'_k - 1)} \\
&\leq 2^9 3 \frac{\alpha'^2_k \beta'^6_k}{\beta^{11}_k \gamma^3_k}
\end{aligned}$$

and

$$\mathbb{E}_{2k+1} \leq \prod_{j=\beta'_k+1}^{\gamma'_k-1} \left(1 + \frac{4}{j}\right) \frac{1}{\binom{\gamma'_k}{2}} \leq \left(\frac{\gamma'_k - 1}{\beta'_k}\right)^4 \frac{2}{\gamma'_k(\gamma'_k - 1)} \leq 2 \frac{\gamma'^2_k}{\beta'^4_k}.$$

This gives the upper bound

$$\begin{aligned}
\prod_{l=1}^{2k+1} \mathbb{E}_l &\leq \frac{\beta^4_1}{\alpha^4_1} . 2^6 3 \frac{\beta'^8_1}{\beta^8_1 \gamma^4_1} \prod_{i=1}^{k-1} \left(2^2 3 \frac{\beta^8_{i+1}}{\beta'^8_i \alpha^3_{i+1}}\right) \prod_{i=2}^{k-1} \left(2^9 3 \frac{\beta'^8_i}{\beta^{11}_i \gamma^3_i}\right) 2^9 3 \frac{\alpha'^2_k \beta'^6_k}{\beta^{11}_k \gamma^3_k} . 2 \frac{\gamma'^2_k}{\beta'^4_k} \\
&= 2^7 3 \frac{\alpha'^2_k \beta'^2_k \gamma'^2_k}{\alpha^4_1 \beta^4_1 \gamma^4_1} \prod_{i=2}^{k} \frac{2^{11} 3^2}{\alpha^3_i \beta^3_i \gamma^3_i}.
\end{aligned}$$

We show that this is also an upper bound on $\mathbb{E}X(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'})$ for any ordering of the branching and connection points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$. For any other ordering, where some of the α_{i+1} and γ'_{i-1} fall into K_{2i}, we can carry out a similar calculation, and obtain an expression of a similar form, namely

$$A_k \prod_{i=1}^{k} (\alpha_i \beta_i \gamma_i)^{b_i} (\alpha'_i \beta'_i \gamma'_i)^{c_i}.$$

For any framework, from the conditions (a)–(d) in the definition, every i-branching point $(i \neq 1)$ must have one fewer path to it than from it (two fewer for $i = 1$), but b_i depends only on this difference, so b_i is independent of the ordering of the terms of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$.

Similarly, c_i is independent of the ordering since, for any framework, each i-connection point ($i \neq k$) has one more path to it than from it (two more for $i = k$). So we have

$$b_i = \begin{cases} 2 \times (-1) - 1 = -3 & \text{for } i \neq 1, \\ 2 \times (-2) \qquad = -4 & \text{for } i = 1, \end{cases} \qquad c_i = \begin{cases} 2 \times (+1) - 2 = 0 & \text{for } i \neq k, \\ 2 \times (+2) - 2 = 2 & \text{for } i = k, \end{cases}$$

for any ordering. The constant factor, A_k, does depend on the ordering of the terms of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha'}, \boldsymbol{\beta'}, \boldsymbol{\gamma'}$. In particular, it depends on the number of choices of end points (or pairs of end points) of paths that each branching (or connection) point can be above, respectively. It remains to show that this number is smaller for any ordering (satisfying (28) and (29)) other than $\alpha_i < \beta_i < \gamma_i < \alpha'_i < \beta'_i < \gamma'_i < \alpha_{i+1}$, etc.

Suppose we have an ordering where $\alpha'_i < \gamma_i$ for some i. Then there is only a choice of four pairs of end points of paths for α'_i to be above, rather than the twelve pairs of end points in the case $\gamma_i < \alpha'_i$. So we need only consider orderings with $\gamma_i < \alpha'_i$ for all i. This means events occurring below γ_i are independent of events occurring above α'_i. In particular we can consider the cases illustrated in Figures 4(b) and 4(c) separately (so the case in Figure 4(d) is a combination of the two). If we have an ordering with $\gamma'_{i-1} > \beta_i$, then there is only one end point for β_i to be above, rather than the two end points in the case $\gamma'_{i-1} < \beta_i$. If we have an ordering with $\alpha_{i+1} < \beta'_i$ then there is only one end point α_{i+1} can be above, rather than the two end points in the case $\alpha_{i+1} > \beta'_i$. This only leaves the case that $\alpha_i < \gamma'_{i-1}$, but then there is only a choice of two end points for α_i to be above, rather than the three end points in the case $\alpha_i > \gamma'_{i-1}$.

Therefore,

$$\mathbb{E}X \leq \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}, \\ \boldsymbol{\alpha'},\boldsymbol{\beta'},\boldsymbol{\gamma'}:(28),(29)}} 2^7 3 \frac{\alpha'^2_k \beta'^2_k \gamma'^2_k}{\alpha_1^4 \beta_1^4 \gamma_1^4} \prod_{i=2}^{k} \frac{2^{11} 3^2}{\alpha_i^3 \beta_i^3 \gamma_i^3},$$

and summing first over $\alpha'_i < \alpha_{i+1}$ for $i = 1, \ldots, k-1$ (and similarly for $\boldsymbol{\beta'}, \boldsymbol{\gamma'}$) and then relaxing all other constraints gives

$$\begin{aligned} \mathbb{E}X &\leq \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}, \\ \alpha'_k,\beta'_k,\gamma'_k}} 2^7 3 \frac{\alpha'^2_k \beta'^2_k \gamma'^2_k}{\alpha_1^4 \beta_1^4 \gamma_1^4} \prod_{i=2}^{k} \frac{2^{11} 3^2}{\alpha_i^2 \beta_i^2 \gamma_i^2} \\ &\leq \frac{2^7}{3^5} \frac{n^9}{r^9} \left(\frac{2^{11} 3^2}{r^3} \right)^{k-1} \\ &= (2^7/3^5)(2^{11}3^2)^{k-1} \frac{n^9}{r^{3k+6}}. \end{aligned}$$

So, the probability that there exists a copy of $P(1,2;3)^{(k)}$ in $B_2[r,n]$ is less than the probability that there exists a framework in $B_2[r,n]$, which is $O(n^9/r^{3k+6})$ by Markov's inequality. $\square$

We define the poset $Q(k)$ as the poset $P(1,2;3)^{(k)}$ with an additional point incomparable to all others. Write $B_2[r,\infty)$ for the random poset B_2 restricted to the set of points greater than or equal to r. We have the following corollary of Theorems 9 and 10.

Corollary 11. *For $k \geq 450$, the probability that $B_2[r, \infty)$ contains a copy of $Q(k-1)$ is $O(r^{-91/90})$.*

Proof. For there to be a copy of $Q(k-1)$ in $B_2[r, \infty)$ there must exist a copy P of $P(1, 2; 3)^{(k-1)}$ in $B_2[r, \infty)$, and some point b in $B_2[r, \infty)$ such that b is incomparable to all the points in P. Label the least point in P by m, and the greatest point by n, so that P is in $B_2[m, n]$, and b must be incomparable to m and n. So, the probability that there is a copy of $Q(k-1)$ in $B_2[r, \infty)$ is less than the probability that there both exists some P in $B_2[m, n]$, and some $b \geq r$ incomparable to m and n, for some $m, n \geq r$. If $n = \omega(m^{150})$ then the probability that there exists some b incomparable to both m and n is $O(r^{-91/90})$. Now taking $k \geq 450$, if $n = O(m^{150})$ then the probability there exists an P in $B_2[m, n]$ is $O(m^{-3}) = O(r^{-3})$, since $m \geq r$. So for fixed $k \geq 450$ the probability that $B_2[r, \infty)$ contains a copy of $Q(k-1)$ is $O(r^{-91/90})$. $\square$

Since events in $B_2[r]$ are independent of events in $B_2[r, \infty)$ we have the following corollary.

Corollary 12. *For $k \geq 450$, there is a positive probability that the random poset B_2 does not contain a copy of $Q(k)$.*

Proof. Fix $k \geq 450$. Fix r so that the probability that $B_2[r, \infty)$ does not contain a copy of $Q(k-1)$ is at least $1/2$. This is possible by Corollary 11.

With some positive probability p, the points $2, \ldots, r$ in B_2 form a chain. (For this to happen, each point $j = 3, \ldots, r$ must select point $j-1$, so $p = \prod_{j=3}^{r}(2/j) = 2^{r-1}/r!$.) Recall that points 0 and 1 are defined to be incomparable, and vertex 2 selects 0 and 1 with probability 1, so all points in $[r]$ are below r in $B_2[r]$.

Now, we can calculate the probability that B_2 contains a copy of $Q(k)$ given that the first r elements are as above. Suppose such a B_2 contains a copy Q of $Q(k)$. Because of the structure of $B_2[r]$ there can be at most one point of Q in $B_2[r]$. Either this is the incomparable element of Q, or one of the minimal points of the tower in Q. If the former, label this point b, and we have $b \leq r$ and so b is below r in B_2. The point b is incomparable with all points in Q, which implies that r is also incomparable with all points in Q. Since Q is a copy of $Q(k)$, so is $Q \cup \{r\} \setminus \{b\}$, and there is a copy of $Q(k)$ in $B_2[r, \infty)$. If the latter, then Q contains a copy of $Q(k-1)$ with all points greater than r, that is, a copy of $Q(k-1)$ in $B_2[r, \infty)$. If none of the points in Q are in $B_2[r]$, then Q, a copy of $Q(k)$, is contained in $B_2[r, \infty)$.

So, B_2 does not contain a copy of $Q(k)$ if $B_2[r, \infty)$ does not contain a copy of $Q(k-1)$. However, the probability of this is at least $1/2$, and is independent of the events in $B_2[r]$. Therefore the probability that B_2 does not contain a copy of $Q(k)$ is at least $p/2 > 0$. $\square$

We have shown that there is a positive probability that B_2 does not contain $Q(k)$, that is, that $Q(k)$ is not almost surely contained in B_2. So, which posets are almost surely contained in B_2? It seems ambitious to ask for a complete answer, but it may be possible to provide both families of posets almost surely contained in B_2, and families of posets not almost surely contained in B_2. We have already shown that $Q(k)$, $k \geq 450$ (and so, also, any posets containing $Q(k)$) are not almost surely contained in B_2. In fact, we can apply the argument used in Corollary 11 to any poset in place of $P(1,2;3)^{(k-1)}$, if we can show that it is not contained in $B_2[r, r^{150}]$ almost surely. This is one way to provide further examples of posets not almost surely contained in B_2.

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