Abstract. The Lemke–Howson algorithm is the classical algorithm for finding one equilibrium of a bimatrix game. In this paper we present a class of square bimatrix games for which the length of the shortest Lemke–Howson path grows exponentially in the dimension $d$ of the game. We construct the games using pairs of dual cyclic polytopes with $2d$ facets in $d$-space.

1. Introduction

A bimatrix game $(A, B)$ is a two-player game specified by two matrices $A$ and $B$ of equal dimension (but not necessarily square). The matrix entries represent the players’ payoffs if player 1 chooses a row and simultaneously player 2 a column as his strategy. The rows are the pure strategies of player 1 and the columns are the pure strategies of player 2. A (Nash) equilibrium [14] is a pair of randomized strategies, one for each player, that maximize the expected payoff against the respective other strategy. Here we are interested in the problem of finding one Nash equilibrium of a bimatrix game. The computational complexity of this problem is an open and interesting question [15].

The problem of computing an equilibrium of a bimatrix game is that of finding a solution to a Linear Complementarity Problem (LCP) with a special product structure [1]. The main algorithm for finding one solution to such an LCP is the Lemke–Howson algorithm. It is a pivoting (path-following) algorithm related to Lemke’s algorithm for general LCP’s [1], which is similar to the simplex algorithm for linear programming. Klee and Minty [5] showed, using a certain geometric realization of the $d$-cube (Klee-Minty cubes), that various standard pivot rules for the simplex algorithm may require exponentially many pivot steps in the worst case. Later, Murty [13] showed that similar exponential lower bounds exist for Lemke’s algorithm.
Morris [11] used dual cyclic polytopes to construct examples where the length of the shortest Lemke path grows exponentially. Both the simplex algorithm and Lemke’s algorithm work with a single polyhedron. The Lemke–Howson algorithm works with the product of two polytopes derived from the payoff matrices. It does not seem likely that one can construct games using cubes that make the Lemke–Howson algorithm perform badly for all possible paths. In this paper we adapt Morris’s construction; we use pairs of dual cyclic polytopes to make the Lemke–Howson algorithm take exponentially many steps for all paths. Von Stengel [17] used dual cyclic polytopes to construct $d \times d$ bimatrix games with more than $2^d$ equilibria. This paper blends the approaches of Morris and von Stengel.

For each player, the upper envelope of his expected payoffs for his own strategies (against the randomized strategy of his opponent) defines a polyhedron [6, 8, 18]. These polyhedra are given directly by their halfspace representations, that is, defined by linear inequalities. After a suitable projective transformation, these are polytopes (bounded polyhedra), which are simple if the game is nondegenerate. A polytope $P$ in dimension $d$ is simple if each vertex belongs to exactly $d$ facets of $P$. For each polytope, every pure strategy of either player corresponds to an inequality in the halfspace representation.

It is useful to give the pure strategies of the players distinct integer labels [16]. That is, each inequality has an associated label. If the game is nondegenerate, each binding inequality of the polytopes defines a facet or the empty set, but no face of lower dimension. Here we are only concerned with nondegenerate games and therefore simple polytopes. The facets of the polytopes are labeled according to the defining inequalities. A vertex of a polytope has the labels of the facets it lies on. A vertex pair (which is a vertex of the product polytope) is complementary if every label appears as a label of one of the vertices (or possibly both in cases of degeneracy). A complementary vertex pair corresponds to an equilibrium of the game.

Cyclic polytopes [10, 12, 22] are defined as the convex hull of a set of points, that is, by a vertex representation. They are remarkable objects which, among all polytopes, attain the maximal number of facets for a given number of vertices. Their duals, which are given by their halfspace representation, attain the maximal number of vertices for a given number of facets.

Both cyclic polytopes and their duals have convenient combinatorial representations. These representations allow us to investigate Lemke–Howson paths on dual cyclic polytopes in a purely combinatorial manner. We use dual cyclic polytopes to construct a class of square bimatrix games for which the length of the shortest Lemke–Howson path grows exponentially in the dimension of the game.

The correspondence between polytope pairs and equilibria of bimatrix games is explained in Section 2, to some extent taken from [17]. In Section 3 we outline the Lemke–
Howson algorithm, following [18]. Dual cyclic polytopes are recalled in Section 4, and in Sections 5, 6, 7, and 8 we discuss the construction and the results.

2. Game equilibria and polytopes

We use the following notation. The transpose of a matrix $B$ is $B^\top$. All vectors are column vectors. The zero vector is $\mathbf{0}$, the vector of all ones is $\mathbf{1}$, their dimension depending on the context. Inequalities like $x \geq \mathbf{0}$ between two vectors hold for all components.

Let $(A, B)$ be a bimatrix game, where $A$ and $B$ are $m \times n$ matrices of payoffs to player 1 and player 2, respectively. The rows are the pure strategies of player 1 and the columns are the pure strategies of player 2. Let $M$ be the set of the $m$ pure strategies of player 1 and let $N$ be the set of the $n$ pure strategies of player 2. We assume, for the purposes of identifying equilibria, that these sets are disjoint, as in

$$M = \{1, \ldots, m\}, \quad N = \{m + 1, \ldots, m + n\}. \quad (2.1)$$

A mixed strategy $\bar{x}$ for player 1 (or $\bar{y}$ for player 2) is a probability distribution on rows (respectively, columns), written as a vector of probabilities. Then the sets $X$ and $Y$ of mixed strategies are

$$X = \{\bar{x} \in \mathbb{R}^M \mid \mathbf{1}^\top \bar{x} = 1, \ \bar{x} \geq \mathbf{0}\}, \quad Y = \{\bar{y} \in \mathbb{R}^N \mid \mathbf{1}^\top \bar{y} = 1, \ \bar{y} \geq \mathbf{0}\}. \quad (2.2)$$

An equilibrium of the game is a pair $(\bar{x}, \bar{y})$ of mixed strategies so that $\bar{x}^\top A \bar{y} \geq x^\top A \bar{y}$ and $\bar{x}^\top B \bar{y} \geq \bar{x}^\top B \bar{y}$ for all other mixed strategies $x$ and $y$, respectively.

In equilibrium, player 1 (and similarly player 2) maximizes his expected payoff $\bar{x}^\top A \bar{y}$ against $\bar{y}$. Equivalently [14], only those rows $i$ that have maximum payoff $u$ can have positive probability $\bar{x}_i$. This combinatorial condition can be expressed using the following polyhedra. Let

$$\mathcal{P} = \{(\bar{x}, \nu) \in \mathbb{R}^m \times \mathbb{R} \mid B^\top \bar{x} \leq \mathbf{1} \nu, \ \bar{x} \in X\}, \quad \mathcal{Q} = \{(\bar{y}, u) \in \mathbb{R}^n \times \mathbb{R} \mid A \bar{y} \leq \mathbf{1} u, \ \bar{y} \in Y\}. \quad (2.3)$$

In $\mathcal{Q}$, for example, the smallest value of $u$ for all $\bar{y}$ in $Y$ defines the upper envelope of the expected payoffs for all pure strategies of player 1, given by the rows of $A \bar{y}$. In equilibrium, only optimal pure strategies $i$ may have positive probability, so that either the $i$th inequality in $A \bar{y} \leq \mathbf{1} u$ in the definition of $\mathcal{Q}$ is binding ($i$ is optimal), or the $i$th inequality in $\bar{x} \geq \mathbf{0}$ in the definition of $\mathcal{P}$ is binding ($\bar{x}_i = 0$), or both. Similarly, a pure strategy $j$ of player 2 is optimal or not played, represented by the $j$th inequality in $B^\top \bar{x} \leq \mathbf{1} \nu$ or in $\bar{y} \geq \mathbf{0}$ in the definition of $\mathcal{P}$ or $\mathcal{Q}$ that holds as an equality.
For identifying equilibria, it is therefore useful to consider the pure strategies of the two players as *labels* \(1, \ldots, m + n\) numbering the \(m + n\) inequalities in the definitions of \(\overline{P}\) and \(\overline{Q}\) in (2.3). The first \(m\) of these labels represent the pure strategies of player 1, the second \(n\) those of player 2. Then an equilibrium is a pair \((x, y)\) so that \((x, v) \in \overline{P}\) and \((y, u) \in \overline{Q}\) for suitable payoffs \(v\) and \(u\), and for each label \(1, \ldots, m + n\) the corresponding inequality in \(\overline{P}\) or in \(\overline{Q}\) is binding.

The polyhedra in (2.3) can be simplified by normalizing the payoffs to one and replacing probabilities by arbitrary nonnegative numbers. Let

\[
\begin{align*}
P &= \{x \in \mathbb{R}^m \mid x \geq 0, B^\top x \leq 1\}, \\
Q &= \{y \in \mathbb{R}^n \mid Ay \leq 1, y \geq 0\}.
\end{align*}
\tag{2.4}
\]

Then

\[
\begin{align*}
\overline{P} &\rightarrow P, \quad (x, v) \mapsto x/v, \\
\overline{Q} &\rightarrow Q, \quad (y, u) \mapsto y/u
\end{align*}
\tag{2.5}
\]

are projective transformations [22] with bounded \(P\) and \(Q\) if the payoffs \(v\) and \(u\) are always positive. For that purpose, we assume

\[
A \text{ and } B^\top \text{ are nonnegative and have no zero column.} \tag{2.6}
\]

This assumption can be made without loss of generality since a constant can be added to all payoffs without changing the game in a material way. By (2.6), \(P\) and \(Q\) are polytopes. The projective transformations (2.5) are one-to-one correspondences between \(\overline{P}\) and \(P - \{0\}\) and \(\overline{Q}\) and \(Q - \{0\}\), respectively, that preserve binding inequalities [18]. The extra vertex \(0\) of \(P\) and \(Q\) arises as projection “from infinity”.

As for \(\overline{P}\) and \(\overline{Q}\), a *label* of a point in \(P\) or \(Q\) is a number in \(1, \ldots, m + n\) so that the corresponding inequality in (2.4) is binding. A pair \((x, y)\) of points in \(P \times Q\) is called *complementary* if every label \(1, \ldots, m + n\) appears as a label of \(x\) or of \(y\). With the exception of \((0, 0)\), complementary pairs define the equilibria of the bimatrix game \((A, B)\) by renormalizing \(x\) and \(y\) to be vectors of probabilities.

Any complementary pair is the convex combination of *extreme* complementary pairs \((x, y)\) where \(x\) is a vertex of \(P\) and \(y\) is a vertex of \(Q\) \([8, 21, 4, 18]\). We consider only *nondegenerate* (or “generic”) games where only pairs of vertices can be complementary. Otherwise, the game may have infinitely many equilibria (as convex combinations of extreme equilibria).

A game is called nondegenerate if against every mixed strategy \(z\) of a player, there are at most \(|\{i \mid z_i > 0\}|\) pure strategies of the opponent that are optimal. This means that every point in \(P\) has at most \(m\) labels and every point in \(Q\) has at most \(n\) labels. It is easy to see that this is equivalent to the following [18]: A binding inequality for \(P\) or
Q defines either a facet of that polytope or the empty set, but no other lower-dimensional face; and P and Q are simple polytopes. Inequalities that are never binding represent strictly dominated strategies which are never played in equilibrium, so they can be omitted from the game. Hence, we assume that P and Q in (2.3) are simple polytopes with facets labeled 1, . . . , m + n. For complementary vertex pairs (x, y), only the combinatorial structure of these polytopes matters. The special structure of the first m inequalities x ≥ 0 of P and of the second n inequalities y ≥ 0 of Q is not a restriction, since this can be achieved by a suitable affine transformation for each polytope, as follows [17]:

**Proposition 2.1.** Let P′ be a simple m-polytope and Q′ be a simple n-polytope, both with m + n labeled facets, which have at least one complementary pair (x′, y′) of vertices. Then there are m × n matrices A and B defining P and Q in (2.4), a permutation of the labels 1, . . . , m + n of P′ and Q′ yielding the labels of P and Q, and invertible affine transformations from P′ to P and from Q′ to Q that map (x′, y′) to (0, 0). Furthermore, every complementary vertex pair of (P′, Q′) except (x′, y′) represents a Nash equilibrium of the bimatrix game (A, B).

**Proof.** Permute the labels 1, . . . , m + n in the same way for P′ and Q′ such that x′ has labels 1, . . . , m and y′ has labels m + 1, . . . , m + n. This does not change the complementary pairs of (P′, Q′). Let

\[ P′ = \{ z \in \mathbb{R}^m \mid Cz \leq p, \ Dz \leq q \} \]

where Cz ≤ p represents the m inequalities for the facets 1, . . . , m and Dz ≤ q the remaining n inequalities. For the vertex x′, we have Cx′ = p and Dx′ < q since P′ is simple. The m binding inequalities for x′ are linearly independent since x′ is a vertex, so C is invertible and \( z \mapsto x = -Cz + p \) is an affine transformation with inverse \( z = -C^{-1}(x - p) \). Let \( P = \{ x \in \mathbb{R}^m \mid -C^{-1}(x - p) \in P′ \} \). Then, with \( r = q - DC^{-1}p \),

\[ P = \{ x \in \mathbb{R}^m \mid -x \leq 0, \ -DC^{-1}x \leq r \}. \]

Corresponding points of P and P′ have the same labels. Since the vertex 0 of P corresponds to x′ in P′, 0 < r. Thus, the jth row of \(-DC^{-1}x \leq r\) can be normalized by multiplication with the scalar 1/rj, so we can assume r = 1. Then P is defined as in (2.4) with the n × m transposed payoff matrix \( B^\top = -DC^{-1} \). Similarly, we can find an m × n matrix A so that Q in (2.4) is an affine transform of Q′. The complementary vertex pairs of (P′, Q′) except (x′, y′) correspond to the Nash equilibria of (A, B) by construction. If desired, a constant can be added to the entries of A and B to obtain (2.6), which does not change the combinatorial structure of P and Q.

Polytopes P′ and Q′ with general labeling may have no complementary pairs at all, so this case is explicitly excluded in Proposition 2.1. The number of complementary pairs
of \((P, Q)\) in Proposition 2.1 is always even, since the Lemke–Howson algorithm connects complementary pairs in \(P \times Q\) by paths where a given label is missing.

Proposition 2.1 allows us to consider simple polytopes \(P\) and \(Q\) that do not have the special structure (2.4). We can label the facets of each polytope \(1, \ldots, m + n\) as we wish. Then, provided that \(P \times Q\) has at least one complementary vertex pair, say \(e\), the polytopes correspond to a bimatrix game \((A, B)\) which has Nash equilibria corresponding to each complementary vertex pair of \(P \times Q\) except \(e\).

3. The Lemke–Howson algorithm

Here we are only concerned with applying the Lemke–Howson to dual cyclic polytopes, which have a nice combinatorial description. We can apply the algorithm using only this combinatorial structure. We describe Shapley’s [16] graph-theoretic exposition of the algorithm, as opposed to the algebraic complementary pivoting method used with the polyhedra given by general bimatrix games, see [18].

We require disjoint pure strategy sets \(M\) and \(N\) of the two players as in (2.1). Recall that any pair of points in \(P \times Q\) is labeled with certain elements of \(M \cup N\). These labels denote the unplayed pure strategies of the player and the pure best responses of his or her opponent. Then a Nash equilibrium is a completely labeled pair \((x, y)\) since then any pure strategy \(k\) of a player is either a best response or played with probability zero, so it appears as a label of \(x\) or \(y\). Furthermore [18]:

**Proposition 3.1.** In a nondegenerate \(m \times n\) bimatrix game \((A, B)\), only finitely many points in \(P\) have \(m\) labels and only finitely many points in \(Q\) have \(n\) labels.

These finitely many points are the vertices of \(P\) and \(Q\). Let \(G_1\) be the graph of \(P\) whose vertices have \(m\) labels. Similarly, let \(G_2\) be the graph of \(Q\). Then the vertices of \(G_1\) have \(m\) labels and the vertices of \(G_2\) have \(n\) labels. Then any two vertices \(x\) and \(x'\) in \(P\) are joined by an edge if they differ in one label, that is, if they have \(m - 1\) labels in common and similarly the edges of \(G_2\) join those vertices that have \(n - 1\) labels in common. Note the vertex \(0\) in \(P\) has all labels \(i\) in \(M\) and the vertex \(0\) in \(Q\) has all labels \(j\) in \(N\). However the point \((0, 0)\) in \(P \times Q\) does not correspond to a Nash equilibrium of the game. This point is called the artificial equilibrium.

The *product graph* \(G_1 \times G_2\) of \(G_1\) and \(G_2\) has vertices \((x, y)\) where \(x\) is a vertex of \(G_1\), and \(y\) is a vertex of \(G_2\). It is identical to the graph of the product polytope \(P \times Q\). Its edges are given by \(\{x\} \times \{y, y'\}\) for vertices \(x\) of \(G_1\) and edges \(\{y, y'\}\) of \(G_2\), or by \(\{x, x'\} \times \{y\}\) for edges \(\{x, x'\}\) of \(G_1\) and vertices \(y\) of \(G_2\).

The Lemke–Howson algorithm can be defined combinatorially in terms of these graphs. Let \(k\) \(\in M \cup N\), and call a vertex pair \((x, y)\) of \(G_1 \times G_2\) \(k\)-almost completely
labeled if any $l$ in $M \cup N - \{k\}$ is either a label of $x$ or of $y$. Since two adjacent vertices $x, x'$ in $G_1$, say, have $m - 1$ labels in common, the edge $\{x, x'\} \times \{y\}$ of $G_1 \times G_2$ is also called $k$-almost completely labeled if $y$ has the remaining $n$ labels except $k$. The same applies to edges $\{x\} \times \{y, y'\}$ of $G_1 \times G_2$.

Then any equilibrium $(x, y)$ is in $G_1 \times G_2$ adjacent to exactly one vertex pair $(x', y')$ that is $k$-almost completely labeled. Namely, if $k$ is the label of $x$, then $x$ is joined to the vertex $x'$ in $G_1$ sharing the remaining $m - 1$ labels, and $y = y'$. If $k$ is the label of $y$, then $y$ is similarly joined to $y'$ in $G_2$ and $x = x'$. In the same manner, a $k$-almost completely labeled pair $(x, y)$ that is not completely labeled has exactly two neighbours in $G_1 \times G_2$. These are obtained by dropping the unique duplicate label that $x$ and $y$ have in common, joining to an adjacent vertex either in $G_1$ and keeping $y$ fixed, or in $G_2$ and keeping $x$ fixed. This defines a unique $k$-almost completely labeled path in $G_1 \times G_2$ connecting one equilibrium to another. The algorithm is started from the artificial equilibrium $(0, 0)$ that has all labels, follows the path where label $k$ is missing, and terminates at a Nash equilibrium of the game.

Since the polytopes in the construction of Section 5 will not necessarily have the special structure of (2.4), the artificial equilibrium is defined purely in terms of labels (and not as the point $(0, 0)$). It is defined as a vertex in $P$ with all labels $i$ in $M$ and a vertex in $Q$ with all labels $j$ in $N$.

### 4. Dual cyclic polytopes

Our examples are based on the duals of cyclic polytopes, which have the maximum number of vertices for a given dimension and number of facets. Every polytope admits a dual [22], that is, a polytope with the face lattice reversed. We construct a dual cyclic polytope from the cyclic polytope using the following polar construction. The polar [22] of a polytope $P$ that is the convex hull of the $d$-vectors $c_1, \ldots, c_n$ is given by

$$P^\Delta = \{x \in \mathbb{R}^d \mid c_i^\top x \leq 1, 1 \leq i \leq n\}, \quad (4.1)$$

provided $P$ (and then also $P^\Delta$) has $0$ in its interior, which can always be achieved by translating $P$. Any face of $P^\Delta$ of dimension $d - k$ is defined by $k$ binding inequalities in (3.2), and corresponds to a face of dimension $k - 1$ of $P$, given by the convex hull of the corresponding $k$ vertices of $P$.

A cyclic polytope $C_d(n)$ [10, 22] in dimension $d$ with $n$ vertices is defined as the convex hull of any $n$ points on the moment curve $\{\mu(t) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^d$, $\mu(t) = (t, t^2, \ldots, t^d)^\top$. Any $d + 1$ points on this curve are affinely independent, so $C_d(n)$ is simplicial (no facet contains more than $d$ vertices). The particular choice of the points $\mu(t_1), \ldots, \mu(t_n)$ on the moment curve does not affect the combinatorial structure (that is,
the face incidences) of $C_d(n)$. Assume $t_1 < \cdots < t_n$. A set $S$ of $d$ vertices corresponds to a 0-1 string $s = s_1s_2\ldots s_n$ with $s_i = 1$ if $\mu(t_i) \in S$ and $s_i = 0$ otherwise. The hyperplane $H$ through the points in $S$ defines a facet of $C_d(n)$ if and only if the string $s$ fulfills the \textit{Gale evenness} condition [2], that is, it contains no substring $s_i\ldots s_j = 01\ldots 10$ with an odd number $i - j - 1$ of 1’s (like 01110). Otherwise, the two vertices $\mu(t_i)$ and $\mu(t_j)$ would be on opposite sides of $H$, since the moment curve changes from one side of $H$ to the other at the points $\mu(t_i)$, $i \in S$.

Under duality, vertices correspond to facets and vice versa. We define a \textit{dual cyclic polytope} $C_d(n)^A$ as the polar of $C_d(n)$. The polytope $C_d(n)^A$ is simple and has $n$ facets. Each facet has, by polarity, the normal vector $\mu(t_i)$ which is a vertex of $C_d(n)$. Each vertex of $C_d(n)^A$ corresponds to a facet of $C_d(n)$ and can be identified with a 0-1 string $s = s_1s_2\ldots s_n$ (with exactly $d$ 1’s) that fulfills the Gale evenness condition. Each of the $d$ positions where $s_i = 1$ stands for a facet this vertex lies on. We will refer to vertices by the sets $S$ of facets containing them via the representation of $S$ as a 0-1 string.

Some of our proofs (as well as Morris’s [11]) use the cyclic symmetry of the Gale evenness condition in even dimension $d$. Namely, if $\psi$ is a permutation of the numbers $i = 1, \ldots, n$ which is a cyclic shift, where $\psi(i)$ equals $i + k$ (respectively, $i + k - n$) for some $k$, or a reversal where $\psi(i) = n + 1 - i$, or a composition of these, then a string $s_{\psi(1)}s_{\psi(2)}\cdots s_{\psi(n)}$ fulfills Gale evenness if and only if $s_1s_2\cdots s_n$ does. We call such a permutation $\psi$ of $\{1, \ldots, n\}$ a \textit{polytope symmetry}, which is of course to be understood combinatorially.

Pivoting algorithms like the Lemke–Howson algorithm traverse edges of polytopes. The edges of $C_d(n)^A$ connecting two vertices can be identified from the 0-1 string representations, namely they have to agree in $d - 1$ positions where there is a 1 in the string. These positions represent the $d - 1$ facets whose intersection is the edge in question. In general, if we consider a dual cyclic polytope $P = C_d(n)^A$, the \textit{position} $P_k(u)$ of a vertex $u$ for $k = 1, \ldots, n$ is either a 0 or a 1, and stands for the $k$th bit of the string representing the vertex, so this string is $P_1(u)P_2(u)\cdots P_n(u)$. In other words, $P_k(u) = 1$ if $u$ lies on the $k$th facet of $C_d(n)^A$ (with normal vector $\mu(t_k)$ as above). Similarly, if we consider a second cyclic polytope $Q = C_d(n)^A$, we identify a vertex $v$ of $Q$ with the 0-1 string $Q_1(v)Q_2(v)\cdots Q_n(v)$.

The following detail about pivoting steps on $C_d(n)^A$ is useful. Let $P = C_d(n)^A$ and let $u$ be a vertex of $P$ given by $P_1(u)P_2(u)\cdots P_n(u)$, and consider the edge connecting $u$ to vertex $v$ leaving the $j$th facet, that is, $P_j(u) = 1$ and $P_j(v) = 0$. Suppose $v$ is on the $k$th facet and $u$ is not, that is, $P_k(u) = 0$ and $P_k(v) = 1$. All other positions of $u$ and $v$ are identical. Assume that $j < k$; the case $j > k$ is symmetric. By Gale evenness, in both $u$ and $v$ either all positions between $j$ and $k$ (of which there is an odd number) are 1, so $P_{j+1}(u) = \cdots = P_{k-1}(u) = 1$ and $P_{j+1}(v) = \cdots = P_{k-1}(v) = 1$, or similarly.
\[ P_{k+1}(u) = \cdots = P_n(u) = P_1(u) = \cdots = P_{l-1}(u) = 1 \text{ and } P_{k+1}(v) = \cdots = P_n(v) = P_1(v) = \cdots = P_{l-1}(v) = 1. \]

In the latter case, we say that the edge connecting \( u \) and \( v \) wraps around (the end of the 0-1 string) since then the pivoting step representing the edge preserves Gale evenness by a cyclic “move” of the 1-position around the end of the string.

5. Lemke–Howson paths and Gale evenness

We construct \( d \times d \) games of even dimension \( d \). These correspond to polytopes in dimension \( d \) with \( 2d \) facets. The set of pure strategies for player 1 is \( M = \{1, \ldots, d\} \), and for player 2 it is \( N = \{d + 1, \ldots, 2d\} \). Let \( P \) and \( Q \) be dual cyclic polytopes of dimension \( d \) with \( 2d \) facets. We label the \( 2d \) facets of each polytope with distinct elements of \( M \cup N \), so a labeling of \( P \), say, is a bijection \( l : \{1, \ldots, 2d\} \rightarrow M \cup N \) (which is just a permutation of \( \{1, \ldots, 2d\} \)). The labels of a vertex \( u \) of \( P \) are the labels of the \( d \) facets it lies on. In the string representation \( P_1(u)P_2(u)\cdots P_n(u) \) of \( u \), the set of labels of \( u \) under the labeling \( l \) is \( \{l(k) \mid P_k(u) = 1, 1 \leq k \leq 2d\} \). Then a vertex pair \((u, v)\) in \( P \times Q \) (which is a vertex of this product polytope) is complementary if \( P_k(u) = 0 \iff Q_k(v) = 1 \) for \( k = 1, \ldots, 2d \). Note that the complementarity condition applies to the labels, whereas the Gale evenness applies to the positions of the strings representing \( u \) and \( v \).

The complementarity condition is not affected if the labels in \( M \cup N \) are permuted arbitrarily in the same way for \( P \) and \( Q \). We call such a permutation of \( \{1, \ldots, 2d\} \) a relabeling.

**Definition 5.1.** Let \( P = Q = C_d(2d)\Delta \). An artificial equilibrium is a vertex pair \((u, v)\) given by a vertex \( u \) in \( P \) with labels \( 1, \ldots, d \) and a vertex \( v \) in \( Q \) with labels \( d+1, \ldots, 2d \). Denote such a vertex pair in \( P \times Q \) by \( e_0 \). (It exists if the strings with these labels satisfy Gale evenness.)

**Definition 5.2.** Let \( P = Q = C_d(2d)\Delta \). A completely mixed equilibrium is given by both players playing all their pure strategies with positive probability. It is given by a vertex \( u \) in \( P \) with labels \( d+1, \ldots, 2d \) and a vertex \( v \) in \( Q \) with labels \( 1, \ldots, d \). Denote such a vertex pair \((u, v)\) by \( e_1 \).

The Lemke–Howson algorithm uses only the combinatorial structure of the polytopes. For the duals of the cyclic polytopes, the algorithm can therefore be expressed solely in terms of the 0-1 strings representing the vertices. At each step we maintain a 0-1 string for each polytope. We start at \( e_0 \) and choose a label which is dropped in the relevant polytope, say \( P \). This means we make the position corresponding to this label 0. The Gale evenness condition uniquely determines the adjacent vertex, that is, it tells us the new position that becomes 1 (we say we pick up this position), as explained at the end of the previous section. The label corresponding to this position is either the missing label...
or it is a duplicate label. If it is the missing label we are done. If not, we drop this label in the other polytope $Q$, and continue in this way, alternating between $Q$ and $P$, until the missing label is found.

As an example, suppose that $P$ and $Q$ are both equal to $C_d(2d)^A$ and the positions of $P$ and $Q$ are both labeled with $l(k) = k$ for $k = 1, \ldots, 2d$. We illustrate the path given by dropping label 1 when $d = 4$ (the facet that is left is underlined in the table below). This path has length two since it consists of two edges (pivoting steps), with three vertices including the endpoints. The path terminates at a pure strategy equilibrium where player 1 plays strategy 1 and player 2 plays strategy 5.

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<tr>
<th>Facet Label</th>
<th>$P$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_0$</td>
<td>1 1 1 1 0 0 0 0</td>
<td>0 0 0 0 1 1 1 1</td>
</tr>
<tr>
<td></td>
<td>0 1 1 1</td>
<td>0 0 0 0 1 1 1 1</td>
</tr>
<tr>
<td></td>
<td>0 1 1 1</td>
<td>0 0 0 0 1 1 1 1</td>
</tr>
</tbody>
</table>

6. Facet labels

We will label the facets of $P$ and $Q$, given as above by $C_d(2d)^A$, so that there are only two complementary vertex pairs, the artificial equilibrium $e_0$ and the completely mixed equilibrium $e_1$. Then, according to Proposition 2.1, $P$ and $Q$ can be transformed into a bimatrix game $(A, B)$ with a single completely mixed equilibrium.

We define the following labeling of the facets of $P$ and $Q$. The facets of $P$ are labeled by $l(k) = k$ for $k = 1, \ldots, 2d$. The facets of $Q$ are labeled by

$$l'(k) = \begin{cases} 
1, & k = 1 \\
k + 1, & k \text{ even}, \quad 2 \leq k \leq d - 1 \\
k - 1, & k \text{ odd}, \quad 2 \leq k \leq d - 1 \\
d, & k = d \\
k + 1, & k \text{ odd}, \quad d + 1 \leq k \leq 2d \\
k - 1, & k \text{ even}, \quad d + 1 \leq k \leq 2d.
\end{cases} \quad (6.1)$$

For $d = 4$, this labeling is as follows:
The labeling of facets 1, . . . , d of Q is similar to the one used by Morris [11]. If a single polytope is labeled as done by Morris, Lemke’s algorithm (which works with a single polytope where each label 1, . . . , d appears twice) takes exponentially many steps. This can be interpreted as finding a symmetric equilibrium of a symmetric game (specified by a single square matrix). However, Morris’s labeling also gives rise to non-symmetric equilibria, which the Lemke–Howson algorithm finds in only a few steps. The labeling of facets d + 1, . . . , 2d of Q is similar to the one used by von Stengel [17], who used such a labeling to create games with many equilibria. The labeling we employ here creates many almost-complementary (rather than complementary) vertex pairs, which make up the long Lemke–Howson paths.

**Proposition 6.1.** If P and Q are labeled as above, then e₀ and e₁ are the only complementary vertex pairs of P × Q.

**Proof.** Firstly note that, as the example above illustrates, e₀ and e₁ are complementary vertex pairs of P × Q for any d. We will try to construct a complementary vertex pair (u, v) other than e₀ or e₁. Assume that P_d(u) = 0, so u is not on the dth facet of P, which has label d. (The case P_d(u) = 1 will be treated below.) We will show that this implies the following property:

\begin{align*}
\text{(*)} & \quad Q_{d+1}(v) = 1, \text{ and this } (d + 1)\text{st position of } v \text{ is preceded by an odd number of } 1\text{'s,} \\
& \quad \text{that is, for the largest } j < d \text{ so that } Q_j(v) = 0, \text{ the number of positions } Q_i(v) \text{ for } j < i \leq d \text{ (which are all } 1\text{'s)} \text{ is odd.}
\end{align*}

To show (*), consider the largest k with 1 ≤ k < d so that position k of u is 1, which exists since otherwise all positions 1, . . . , d of u would be 0’s, which defines the completely mixed equilibrium e₁. We distinguish several cases.

Suppose that k = 1, as in the following example for d = 8:

<table>
<thead>
<tr>
<th>u</th>
<th>1 0 0 0 0 0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label of P</td>
<td>1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16</td>
</tr>
<tr>
<td>Facet</td>
<td>1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16</td>
</tr>
<tr>
<td>Label of Q</td>
<td>1 3 2 5 4 7 6 8 10 9 12 11 14 13 16 15</td>
</tr>
</tbody>
</table>
| v  | 0 1 1 1 1 1 1 1 1  

11
Then \(Q_1(v) = 0\), and all positions \(Q_2(v), \ldots, Q_d(v)\) are 1, by complementarity. Then, by Gale evenness, \(Q_{d+1}(v) = 1\). This shows (*) with \(j = k\).

Next, suppose that \(k\) is even, as in the following example where \(d = 8, k = 4\):

\[
\begin{array}{cccccccccccc}
\text{u} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\text{Label of P} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Facet} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Label of Q} & 1 & 3 & 2 & 5 & 4 & 7 & 6 & 8 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\
\text{v} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Then, since \(l'(d+1) = k\), complementarity implies that \(Q_{k+1}(v) = 0\), and \(Q_{k+2}(v) = \cdots = Q_d(v) = 1\). Then (*) holds with \(j = k + 1\) by Gale evenness.

Finally, suppose that \(k\) is odd, \(k > 1\), as in the following example where \(d = 8, k = 5\):

\[
\begin{array}{cccccccccccc}
\text{u} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\text{Label of P} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Facet} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Label of Q} & 1 & 3 & 2 & 5 & 4 & 7 & 6 & 8 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\
\text{v} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

By Gale evenness, \(P_{k-1}(u) = 1\). By complementarity, \(Q_{k-1}(v) = Q_k(v) = 0\) since these positions have corresponding labels \(k\) and \(k - 1\). Furthermore, complementarity implies that \(Q_{k+1}(v) = \cdots = Q_d(v) = 1\), and since this is an odd number of positions, Gale evenness implies \(Q_{d+1}(v) = 1\), which shows (*) for \(j = k\).

Condition (*), \(l'(d+1) = d + 2\), and complementarity, imply \(P_{d+2}(u) = 0\). Since \(P_d(u) = P_{d+2}(u) = 0\), Gale evenness implies \(P_{d+1}(u) = 0\). Next, \(l'(d+2) = d + 1\), so by complementarity, \(Q_{d+2}(v) = 1\). As in the considerations leading to (*), Gale evenness implies \(Q_{d+3}(v) = 1\). Since for this new position of \(v\) we have the label \(l'(d+3) = d+4\), complementarity implies \(P_{d+4}(u) = 0\). In the last example where \(d = 8, k = 5\), we therefore have the following situation:

\[
\begin{array}{cccccccccccc}
\text{u} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\text{Label of P} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Facet} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Label of Q} & 1 & 3 & 2 & 5 & 4 & 7 & 6 & 8 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\
\text{v} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

The 0-positions of \(u\) thus “ripple on”, since now \(P_{d+3}(u) = 0\) as before. Continuing in this manner, this shows \(P_d(u) = P_{d+1}(u) = \cdots = P_{2d} = 0\). But then more than \(d\)
positions of \( u \) are 0, so \( u \) does not define a vertex, contrary to our assumption. We have thus shown that the only complementary vertex pair \((u, v)\) with \( P_d(u) = 0 \) is \( e_1 \).

Analogously, we show that the only equilibrium with \( P_d(u) = 1 \) is \( e_0 \). This is true by symmetry when we exchange the players and rename the labels 1, \ldots, 2d given to the facets of the two polytopes, which does not affect the complementarity condition for an equilibrium. This relabeling is simply \( l' \) as defined in (6.1), applied to the labels of both polytopes. Then \( P \) is labeled with \( l' \), which corresponds to the original labeling of \( Q \), and \( Q \) is labeled with the identity (since \( l' \) is its own inverse), which corresponds to the original labeling \( l \) of \( P \). Then, a 1 in position \( d \) of \( P \) means that in equilibrium there is a 0 in position \( d \) of \( Q \), which corresponds to the situation above.

\[ \square \]

7. Paths for the first \( d \) labels

The Lemke–Howson algorithm is easy to implement combinatorially, using the labeling (6.1) and the Gale evenness condition. This gives empirical evidence of the lengths of the different paths, as shown in Table 1. From these data, recurrence relations among the path lengths are readily observed. The path lengths grow exponentially in the dimension of the polytopes.

As Table 1 illustrates, the paths are longest for dropped labels 1 and \( d \), and second longest for labels \( d + 1 \), \( d + 2 \), \( 2d - 1 \) and \( 2d \). The path lengths for other labels correspond to sums of path lengths in lower dimension. The relationships between the paths are similar, but different, for the first \( d \) and the last \( d \) labels. In this section, we study the first \( d \) labels. Very similar considerations hold for the last \( d \) labels, which we outline in the next section.

The most laborious part of the proof is a recurrence for the lengths of the longest path when label 1 is dropped, given in Proposition 7.5 below. It is proved by showing how such a path is composed repeatedly, with some connecting edges, from paths (or half-paths) in lower dimension. Lemmas 7.1 to 7.4 establish mainly certain symmetries to prove that recurrence. Subsequent lemmas express the relationships between the lengths of paths for different labels, using polytope symmetries and relabelings.

Similar to the notation of Morris [11], let \( \pi(d, i) \) denote the \( i \)th Lemke–Howson path (given by dropping label \( i \)) on \( P \times Q \) starting from \( e_0 \), and let \( L(d, i) \) denote its length. Define \( \pi^{-1}(d, i) \) to be the path \( \pi(d, i) \) traversed in the opposite direction. Since \( e_0 \) and \( e_1 \) are the only complementary vertex pairs, all Lemke–Howson paths starting at \( e_0 \) end at \( e_1 \), and vice versa.

The longest path results when label 1 is dropped. The following lemma applies to the edges of this path \( \pi(d, 1) \). It is used in a similar fashion, but not stated explicitly, by
Morris [11]. The lemma states that the edges on the path do not use the cyclic symmetry of the Gale evenness strings, that is, they do not “wrap around” the end of these strings as defined at the end of Section 4.

**Lemma 7.1.** No edge of $\pi(d, 1)$ wraps around.

**Proof.** Consider the first vertex pair $(u_0, v_0)$ (which is $e_0$) of $\pi(d, 1)$:

<table>
<thead>
<tr>
<th>label</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
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<tr>
<td>2</td>
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<td>24</td>
<td></td>
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</tbody>
</table>

Table 1. Path lengths for different dropped labels.
Since label 1 is dropped, the first edge leaves facet 1 in $P$, underlined in the preceding table. Gale evenness means that the vertex $u_1$ that is adjacent to $u_0$ in $P$ is on the new facet $d + 1$, giving the next vertex pair $(u_1, v_0)$ on the path:

\[
\begin{array}{cccccccc}
& 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\hline
u_0 & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Label of } P & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Facet} & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Label of } Q & 1 & 3 & 2 & \cdots & d & d + 2 & d + 1 & \cdots & 2d - 1 \\
v_0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\end{array}
\]

From then on, all vertex pairs $(u, v)$ on the path except for its endpoint $e_1$ are missing label 1, that is, $P_1(u) = Q_1(v) = 0$. The last two vertex pairs of the path are $(u_k, v_{k-1})$ and $(u_k, v_k) = e_1$ where $k = L(d, 1)$, since label 1 is eventually picked up in $Q$, as the path cannot return to $e_0$. These vertex pairs are

\[
\begin{array}{cccccccc}
& 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\hline
u_k & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Label of } P & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Facet} & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Label of } Q & 1 & 3 & 2 & \cdots & d & d + 2 & d + 1 & \cdots & 2d - 1 \\
v_{k-1} & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
& 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
\hline
u_k & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\text{Label of } P & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Facet} & 1 & 2 & 3 & \cdots & d & d + 1 & d + 2 & \cdots & 2d \\
\text{Label of } Q & 1 & 3 & 2 & \cdots & d & d + 2 & d + 1 & \cdots & 2d - 1 \\
v_k & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
\end{array}
\]

None of the edges on $\pi(d, 1)$ wrap around because that would mean that the vertex on either $P$ or $Q$ enters facet 1, which has the missing label 1 in both polytopes. Since that missing label is picked up in $Q$, this occurs only in the last step. Even that last edge does not wrap around, as the corresponding edge of $Q$ joins $v_{k-1}$ to $v_k$ where the facet that is left is facet $d + 1$ of $Q$ (with label $d + 2$) by Gale evenness, which is obtained as above since the last edge of $\pi(d, 1)$ is the first edge of the reverse path $\pi^{-1}(d, 1)$.  

\[\square\]
The following observations are useful to establish recursive relationships between paths of different dimensions.

**Lemma 7.2.** For a Lemke–Howson path on \(P \times Q\) (with given missing label \(k\)), no vertex \(u\) of \(P\) is revisited, in the sense that there are at most two vertices \(v\) and \(v'\) of \(Q\) so that \((u, v)\) and \((u, v')\) are \(k\)-almost complementary or complementary. Similarly, on a \(k\)-almost completely labeled path, no vertex \(v\) of \(Q\) is revisited.

**Proof.** Let \(P\) and \(Q\) have dimension \(d\) as in our setting (the claim holds generally). By nondegeneracy, \(u\) has \(d\) labels, so that \(d - 1\) labels are missing to get all labels except for label \(k\). The intersection of the facets in \(Q\) with these missing labels defines an edge in \(Q\) (or the empty face), whose endpoints are \(v\) and \(v'\). The claim for \(Q\) is analogous. □

A consequence of the preceding claim is that a Lemke–Howson path on \(P \times Q\) induces a path (without repetitions of vertices) on \(P\), and similarly a path on \(Q\). The following claim states that for \(\pi(d, 1)\), the path on \(P\) is the same as the path on \(Q\) backwards.

**Lemma 7.3.** Let \(L = L(d, 1)\) and \((u^i, v^i)\) be the \(i\)th vertex pair of the path \(\pi(d, 1)\) for \(0 \leq i \leq L\). Then for \(0 \leq i \leq L\),

\[
(u^i, v^i) = (v^{L-i}, u^{L-i}).
\]

**Proof.** As at the end of the proof of Proposition 6.1, apply the relabeling \(l'\) in (6.1) to both \(P\) and \(Q\). Then \(P\) and \(Q\) switch roles, \(e_0\) is exchanged with \(e_1\), label \(1\) stays the same, and \(\pi(d, 1)\) corresponds to \(\pi^{-1}(d, 1)\). □

**Lemma 7.4.** The path \(\pi(d, 1)\) contains at most one vertex pair of the form \((u, u)\), which is its midpoint. The vertex \(u\) of this midpoint is, as a string of bits, of the form

\[
u = \begin{cases} 
0(0110)^k1 | 10(0110)^k & \text{if } d = 4k + 2 \\
0(0110)^k011 | (0110)^{k+1} & \text{if } d = 4k + 4,
\end{cases}
\]

where the first \(d\) bits in \(u\) are separated from the last \(d\) bits by \(|\) and \(s^k\) denotes the \(k\)-fold repetition of a string \(s\). If the length of the path \(\pi(d, 1)\) is divisible by four, then the path continues from \((u, u)\) by dropping the duplicate label \(d\) in the first polytope \(P\).

**Proof.** The length \(L = L(d, 1)\) of the path \(\pi(d, 1)\) is even, because the missing label \(1\) is dropped from \(e_0\) in \(P\) and picked up when reaching \(e_1\) in \(Q\), and the duplicate label is alternately dropped in \(P\) and \(Q\). Therefore, the path has a middle vertex pair \((u^{L/2}, v^{L/2})\), using the notation in Lemma 7.3. By Lemma 7.3, it is equal to \((v^{L/2}, u^{L/2})\), so \(u^{L/2} = \ldots\)
\(v^{L/2} = u\). Furthermore, if there was any other vertex \((v, v) = (u^i, v^i)\) on the path for \(i \neq L/2\), then it would appear a second time as \((v, v) = (v^{L-i}, u^{L-i})\), contradicting Lemma 7.2.

In the midpoint \((u, u)\), label 1 is missing and label d is duplicate because \(P_d(u) = Q_d(u)\) which cannot also be zero, so that position is one. Consider position 2: If \(P_2(u) = 1\), then \(Q_3(u) = 0\) (these positions have label 2) and thus \(P_3(u) = 0\) which violates Gale evenness. Hence, \(P_2(u) = 0\) and we conclude in the same manner that the first \(d\) bits of \(u\) are of form 0(0110)\(k\) 1 where the last 1 is \(P_d(u)\) in the case that \(d = 4k + 2\), or of the form 0(0110)\(k\) 011 if \(d = 4k + 4\). Similarly, the last \(d\) bits of \(u\) are as described, by Gale evenness.

Finally, if \(L\) is a multiple of four (which we will later prove by induction on \(d\), so this is the only case that matters), then \(L/2\) is even, and after an even number of steps the label that is dropped is from the vertex in \(P\), which applies when considering the next step on the path from \((u^{L/2}, v^{L/2})\).

The length \(L(d, 1)\) of the path \(\pi(d, 1)\) is given by the following main recurrence.

**Proposition 7.5.**  \(L(2, 1) = 4\), \(L(4, 1) = 20\), and for \(d \geq 6\)

\[
L(d, 1) = 4L(d-2, 1) + L(d-4, 1) + 4.
\]

Consequently,

\[
L(d, 1) = \left(\frac{5 + 3\sqrt{5}}{10}\right)(\sqrt{5} + 2)^{d/2} + \left(\frac{5 - 3\sqrt{5}}{10}\right)(-\sqrt{5} + 2)^{d/2} - 1. \tag{7.1}
\]

**Proof.** It is easily verified that \(L(2, 1) = 4\) and \(L(4, 1) = 20\). Assume, as inductive hypothesis, that the length \(L(d, 1)\) of \(\pi(d, 1)\) is divisible by four, which is true for \(d = 2\) and \(d = 4\), and then holds by induction if the recurrence is proved.

We use the following notation. For any path \(\pi(d, 1)\), let \(\frac{1}{2}\pi(d, 1)\) be the first half of that path (consisting of the first \(L(d, 1)/2\) edges). Then Lemma 7.4 describes the last vertex pair \((u, u)\) of \(\frac{1}{2}\pi(d, 1)\), which we call \(m^d\). Let \(e_0^d\) and \(e_1^d\) denote \(e_0\) and \(e_1\), respectively, in dimension \(d\). We denote by \(P^d\) and \(Q^d\) the dual cyclic polytope \(C_d(2d)^A\), which we will use in particular for \(d - 2\) and \(d - 4\) instead of \(d\). Polytope \(P^d\) is labeled with the identity, the labels of the facets of \(Q^d\) are, for each \(d\), as in (6.1).

Consider any paths \(A\), \(B\), and \(C\) on \(P^4 \times Q^d\), considered as sequences of vertices. Then \(AB\) is the path \(A\) joined to path \(B\), which requires that the endpoint of \(A\) is equal to the starting point of \(B\). The length (number of edges) of \(AB\) is then the sum of the lengths of \(A\) and \(B\). If the endpoint of \(B\) is joined to the starting point of \(C\) by an edge (which in our case will be 1-almost complementary), then \(B + C\) denotes the path obtained by
joining B to C via that edge. The length of B + C is the sum of the lengths of B and C plus one.

For even \( d \), where \( d > 4 \), we show the following:

\[
\frac{1}{2}\pi(d, 1) = AB + C + D
\]

(7.2)

where

\[
A = \alpha\left(\frac{1}{2}\pi(d - 2, 1)\right), \\
B^{-1} = \beta\left(\frac{1}{2}\pi(d - 4, 1)\right), \\
C = \gamma\left(\pi(d - 2, 1)\right), \\
D = \delta\left(\frac{1}{2}\pi(d - 2, 1)\right).
\]

(7.3)

Here, the functions \( \alpha, \gamma, \delta \) map \( P^{d-2} \times Q^{d-2} \) to \( P^d \times Q^d \), and \( \beta \) maps \( P^{d-4} \times Q^{d-4} \) to \( P^d \times Q^d \), and these functions are then extended to paths, regarded as sequences of vertex pairs, as in (7.3). These functions are defined as follows, regarding the respective argument \((u, v)\) as a bitstring of suitable length; let \( b = 2(d - 4) \) and \( c = 2(d - 2) \):

\[
\alpha : (u_1 u_2 \cdots u_c, v_1 v_2 \cdots v_c) \mapsto (u_1 11 u_2 \cdots u_c 00, v_1 00 v_2 \cdots v_c 11)
\]

\[
\beta : (u_1 u_2 \cdots u_b, v_1 v_2 \cdots v_b) \mapsto (011 v_b \cdots v_2 v_1 11000, 000 u_b \cdots u_2 u_1 11011)
\]

\[
\gamma : (u_1 u_2 \cdots u_c, v_1 v_2 \cdots v_c) \mapsto (0 v_c \cdots v_3 v_2 v_1 11 v_1 0, 0 u_c \cdots u_3 u_2 u_1 011)
\]

\[
\delta : (u_1 u_2 \cdots u_c, v_1 v_2 \cdots v_c) \mapsto (0 u_c \cdots u_3 u_2 u_1 011, 0 v_c \cdots v_3 v_2 v_1 110)
\]

We consider each of the paths in (7.3) in turn.

The starting point of \( A \) is \( e_0^{d-2} \), which is equal to \( \alpha(e_0^{d-2}) \). For \( d = 8 \), it is shown in the following table, where \( P \) is shown in the top half and \( Q \) in the bottom half, writing \( \alpha(u, v) = (\alpha_P(u, v), \alpha_Q(u, v)) \) (and later similarly for \( \beta, \gamma, \delta \)).

<table>
<thead>
<tr>
<th>( \alpha_P(e_0^{d-2}) )</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
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<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>label of ( P^d )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>label of ( P^{d-2} )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>label of ( Q^{d-2} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>10</td>
<td>9</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>label of ( Q^d )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In this table, the positions with labels 2, 3, 2d − 1, and 2d in \( P^d \) and \( Q^d \) (outside the boxes) are fixed throughout \( A \), and are obviously complementary in \( P^d \) and \( Q^d \). The other positions (inside the boxes) are obtained via \( \alpha \) from the lower dimension \( d - 2 \). The function \( \alpha \) also defines an injection, call it also \( \alpha \), from the labels of \( P^{d-2} \) and \( Q^{d-2} \) to the labels 1, ..., 2d of \( P^d \) and \( Q^d \) according to \( \alpha(1) = 1 \) and \( \alpha(k) = k + 2 \) for \( 2 \leq k \leq 2(d - 2) \); this is done consistently for both \( P \) and \( Q \).
By Lemma 7.1, $\pi(d - 2, 1)$ does not wrap around. For the first $\frac{1}{2}L(d - 2, 1)$ steps of that path, only the labels $k$ of $P^{d-2}$ and $Q^{d-2}$ for $2 \leq k \leq 2(d - 2)$ are picked up or dropped, which are mapped to $\alpha(k)$ as described. The path $A$ is initiated by dropping label 1 in $P^d$ (underlined in the table above), which is equal to $\alpha(1)$, and 1 is also the dropped label in $\pi(d - 2, 1)$. Therefore, the first $\frac{1}{2}L(d - 2, 1)$ steps of of $\pi(d, 1)$ are given by $\alpha(\pi(d - 2, 1))$. (In fact, all steps of $\pi(d - 2, 1)$ except the last one are mapped to $\pi(d, 1)$ by $\alpha$ as well, but we do not use this here.)

The path $A$ ends after the first $\frac{1}{2}L(d - 2, 1)$ steps of $\pi(d, 1)$ at the vertex pair $\alpha(m^{d-2})$, shown in the following table for $d = 8$, where $m^6$ is given according to Lemma 7.4.

<table>
<thead>
<tr>
<th>$\alpha_P(m^{d-2})$</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>label of $P^d$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>label of $P^{d-2}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>label of $Q^{d-2}$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>label of $Q^d$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>$\alpha_Q(m^{d-2})$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

By the inductive hypothesis, the path length $L(d - 2, 1)$ is divisible by 4, so by Lemma 7.4, the duplicate label $d - 2$ at the midpoint of $\pi(d - 2, 1)$ is dropped in $P^{d-2}$. The corresponding label $\alpha(d - 2)$ in $P^d$ is $d$, so the next label to be dropped on $\pi(d, 1)$ is $d$, underlined in the preceding table. We have to make sure that the continuation of the path, at the start of path $B$, drops the label in this manner.

As claimed in (7.2), the endpoint of path $A$ is the starting point of $B$. In (7.3), the path $B$ is claimed to be the first half of $\pi(d - 4, 1)$ \textit{backwards}, mapped via $\beta$ to $P^d \times Q^d$. That is, the starting point of path $B$ (which is the endpoint of $B^{-1}$), is equal to $\beta(m^{d-4})$ since $m^{d-4}$ is the endpoint of $\frac{1}{2}\pi(d - 4, 1)$. The following table shows $\beta(m^{d-4})$ for $d = 8$, already given before as the endpoint of $A$.

<table>
<thead>
<tr>
<th>$\beta_P(m^{d-4})$</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>label of $P^d$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>label of $Q^{d-4}$</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>label of $Q^d$</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_Q(m^{d-4})$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that $\beta$ reverses the bitstrings and exchanges the polytopes, which is why in the table we first show $v$ in $Q^{d-4}$ and then $u$ in $P^{d-4}$ for $\beta(u, v)$; here, $(u, v) = m^{d-4}$. The endpoint $m^{d-4}$ of $\frac{1}{2}\pi(d - 4, 1)$ is reached by picking up the label $d - 4$ in $Q^{d-4}$ (by
Lemma 7.4, since \( L(d - 4, 1) \) is divisible by four. Via \( \beta \) (in the same way as via \( \alpha \) above), this label \( d - 2 \) is mapped to the label \( d \) in \( P^d \), which is indeed the label to be dropped in \( \pi(d, 1) \) as described before since \( \frac{1}{2} \pi(d - 4, 1) \) is considered backwards. Our example demonstrates the injection of labels from \( P^{d-4} \) to \( P^d \) and from \( Q^{d-4} \) to \( Q^d \) via \( \beta \), which in general is given by

\[
\beta(k) = \begin{cases} 
2d - 5 - k, & k \text{ even}, \ 2 \leq k < d - 4 \\
2d - 3 - k, & k \text{ odd}, \ 2 \leq k < d - 4 \\
d, & k = d - 4 \\
2d - 5 - k, & k \text{ odd}, \ d - 3 \leq k \leq 2(d - 4) \\
2d - 3 - k, & k \text{ even}, \ d - 3 \leq k \leq 2(d - 4) 
\end{cases}
\]

This mapping of labels \( k \) is consistent for both \( P \) and \( Q \) except for label 1. However, since label 1 is missing on all vertex pairs of \( \frac{1}{2} \pi(d - 4, 1) \) except for the starting point \( e_0^{d-4} \), there is no need to specify how label 1 is mapped via \( \beta \). Since there is no wrap-around on \( \pi(d - 4, 1) \), path \( B \) is indeed given as in (7.3).

Consider now \( \beta(e_0^{d-1}) \), the endpoint of \( B \), which for \( d = 8 \) is illustrated in the following table.

<table>
<thead>
<tr>
<th>( \beta_P(e_0^{d-1}) )</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label of ( P )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>label of ( P^{d-4} )</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>label of ( Q^{d-4} )</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Label of ( Q )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>9</td>
<td>12</td>
<td>11</td>
<td>14</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>Label of ( Q )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>[1]</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The label that has just been picked up is \( 2d - 5 \) in \( Q \), indicated by a small box in the preceding table. This label corresponds to label 1 in \( P^{d-4} \) which is dropped as the first label in the path \( \pi(d - 4, 1) \), which we consider backwards and map via \( \beta \).

The next step, in accordance with (7.2) and (7.3), is an additional edge on the path which connects the above endpoint of \( B \) to the first vertex pair of the path \( C \). In that step, the duplicate label \( 2d - 5 \), underlined above, is dropped in \( P \). The result is illustrated in the following table for \( d = 8 \).

<table>
<thead>
<tr>
<th>( \gamma_P(e_0^{d-2}) )</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label of ( P )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>label of ( Q^{d-2} )</td>
<td>11</td>
<td>12</td>
<td>9</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>[1]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>label of ( P^{d-2} )</td>
<td>12</td>
<td>11</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Label of ( Q )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>9</td>
<td>12</td>
<td>11</td>
<td>14</td>
<td>13</td>
<td>16</td>
<td>15</td>
</tr>
</tbody>
</table>
| Label of \( Q \)          | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | \[1\] | 0 | 1 | 1 | 1 
This table shows also the first vertex pair of the path $C$, given by $\gamma(e_0^{d-2})$. The corresponding mapping for the labels is

$$
\gamma(k) = \begin{cases} 
2d - 3 - k, & k \text{ even}, \ 2 \leq k < d - 2 \\
2d - 1 - k, & k \text{ odd}, \ 2 \leq k < d - 2 \\
d, & k = d - 2 \\
2d - 3 - k, & k \text{ odd}, \ d - 1 \leq k \leq 2(d - 2) \\
2d - 1 - k, & k \text{ even}, \ d - 1 \leq k \leq 2(d - 2).
\end{cases}
$$

Again, this mapping is consistent for both $P$ and $Q$, except that $\gamma$ is not specified for the label 1. However, that is the label that is dropped in $P$ at the start of $\pi(d - 2, 1)$. At the start of $C$ this label corresponds to the duplicate label $2d - 2$, to be dropped in $Q$, underlined above.

By Lemma 7.1, no edge on the path $\pi(d - 2, 1)$ wraps around, so that this path is mapped via $\gamma$ to the path $C$ in dimension $d$. The path $\pi(d - 2, 1)$ terminates after $L(d - 2)$ steps at $e_0^{d-2}$, picking up label 1 in $P^{d-2}$ in the last step. This agrees with the behavior at the endpoint $(x, y) = \gamma(e_0^{d-2})$ of the path $C$: Since the two positions $P_{2d-3}(x)$ and $P_{2d-2}(x)$ are fixed at 1, the Gale evenness condition for $x$ means that label $2d - 1$ has just been picked up in $P^d$, indicated by a small box in the following table.

The duplicate label at the endpoint of $C$ is $2d - 1$. It was picked up in $P$ in the last edge of $C$. So in the next step, label $2d - 1$ is dropped in $Q$ (underlined above) and label $2d - 3$ is picked up. This is the edge that joins $C$ and $D$. The first vertex of $D$ is given by $\delta(e_0^{d-2})$, which is illustrated in the table below for $d = 8$. 

<table>
<thead>
<tr>
<th>$\gamma_P(e_0^{d-2})$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Label of $P$</td>
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<td>2</td>
<td>3</td>
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<td>5</td>
<td>6</td>
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</tr>
<tr>
<td>label of $Q^{d-2}$</td>
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<td>8</td>
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<tr>
<td>label of $P^{d-2}$</td>
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<td>11</td>
<td>10</td>
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<td>8</td>
<td>7</td>
<td>6</td>
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</tr>
<tr>
<td>Label of $Q$</td>
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<td>3</td>
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<td>4</td>
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<td>14</td>
<td>13</td>
<td>16</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>$\gamma_Q(e_0^{d-2})$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</table>

<table>
<thead>
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<th>$\delta_P(e_0^{d-2})$</th>
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<th>0</th>
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<th>0</th>
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<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label of $P$</td>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>label of $P^{d-2}$</td>
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<td>11</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
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<td>2</td>
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<td></td>
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<tr>
<td>label of $Q^{d-2}$</td>
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<td>8</td>
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</tr>
<tr>
<td>Label of $Q$</td>
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<td>2</td>
<td>5</td>
<td>4</td>
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<td>13</td>
<td>16</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>$\delta_Q(e_0^{d-2})$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
At the starting point of $D$ the duplicate label is $2d - 3$, which is dropped in $P$ in the first step of $D$. This label results from label 1 in $P^{d-2}$ under the mapping of labels defined by $\delta$, given by $\delta(k) = 2d - 2 - k$ for $k \geq 2$. As before, label 1 is immediately dropped in $P^{d-2}$, so that $\delta(1)$ does not have to be specified. Since the edges on the path $\pi(d - 2, 1)$ do not wrap around, after $\frac{1}{2}L(d - 2, 1)$ steps we reach $\delta(m^{d-2})$. This is illustrated for $d = 8$ in the table below.

\[
\begin{array}{ccccccccccccc}
\delta_P(m^{d-2}) & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
\text{Label of } P & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{Label of } P^{d-2} & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\text{Label of } Q^{d-2} & 11 & 12 & 9 & 10 & 7 & 8 & 6 & 4 & 5 & 2 & 3 & 1 \\
\text{Label of } Q & 1 & 3 & 2 & 5 & 4 & 7 & 6 & 8 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\
\delta_Q(m^{d-2}) & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Since $\delta(m^{d-2}) = m^d$, we have proved (7.2) and (7.3). The second half of the path $\pi(d, 1)$ is given by Lemma 7.3. This implies the recurrence, which in turn completes the induction that the path length is divisible by four.

In order to derive the explicit formula (7.1), we first homogenize the linear recurrence, letting $x_{n} = L(2n, 1) + 1$, so that $x_1 = 5, x_2 = 21, x_n = 4x_{n-1} + x_{n-2}$. This gives the stated recurrence (7.1) for $L(d, 1)$ by a standard formula (see, for example, [3]).

All path lengths appear two or four times, depending on the dimension, which is a consequence of the following two easy lemmas. They are proved using polytope symmetries (see Section 4) and relabelings.

**Lemma 7.6.** For $1 \leq k \leq d/2$, $L(d, k) = L(d, d - k + 1)$.

*Proof.* Let $\psi$ be defined by $\psi(k) = d - k + 1$ and $\psi(d + k) = 2d - k + 1$ for $k = 1, \ldots, d$. This is a cyclic shift by $d$ followed by a reversal of positions, which is a polytope symmetry. Furthermore, $\psi$ leaves the labelings $l$ and $l'$ of $P$ and $Q$ invariant, so the Lemke–Howson algorithm proceeds in the same manner. That is to say, the vertex pairs on the path $\pi(d, k)$, seen as a pairs of 0-1 strings, are step by step the same when the positions in each string are permuted according to $\psi$. Under $\psi$, the first vertex pair $e_0$ (and similarly the last pair $e_1$) is mapped to itself, but the positions are changed as above. This means that under the polytope symmetry $\psi$, the path $\pi(d, k)$ is mapped to $\pi(d, d - k + 1)$, so these two paths have the same length. □

**Lemma 7.7.** For $2 \leq k \leq d - 2$ and $k$ even, $L(d, k) = L(d, k + 1)$.

*Proof.* Let $2 \leq k \leq d - 2$. When we apply the relabeling $l'$ in (6.1) to both $P$ and $Q$ we see that $\pi(d, k)$ corresponds to $\pi^{-1}(d, k + 1)$. □
The next lemma is analogous to Lemma 3.4 of Morris [11], using Lemma 7.1 (which is not spelled out in [11]). It expresses the lengths of paths where label $k$ is dropped, where $1 < k < d$, in terms of path lengths in lower dimensions. By Lemma 7.7, it suffices to consider the case that $k$ is even.

**Lemma 7.8.** For $2 \leq k \leq d - 2$ and $k$ even, $L(d, k) = L(k, 1) + L(d - k, 1)$.

**Proof.** We claim that the vertex pair $(u, v)$ on $\pi(d, k)$ after $L(k, 1)$ many pivoting steps has the following form:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$0 \cdots 0$</th>
<th>$1 \cdots 1$</th>
<th>$0 \cdots 0$</th>
<th>$1 \cdots 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label of $P$</td>
<td>$1 \cdots k$</td>
<td>$k+1 \cdots d$</td>
<td>$d+1 \cdots 2d-k$</td>
<td>$2d-k+1 \cdots 2d$</td>
</tr>
<tr>
<td>Label of $Q$</td>
<td>$1 \cdots k+1$</td>
<td>$k \cdots d$</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>$v$</td>
<td>$1 \cdots 1$</td>
<td>$0 \cdots 0$</td>
<td>$1 \cdots 1$</td>
<td>$0 \cdots 0$</td>
</tr>
</tbody>
</table>

Furthermore, the facet that is left at the next step from this vertex pair $(u, v)$ is the $(k+1)$st facet of $P$ (underlined in the table), since label $k+1$ is duplicate, and the second part of the path $\pi(d, k)$ starting at $(u, v)$ has $L(d - k, 1)$ many steps until it terminates at $e_1$. This last statement of our claim is easier to prove, so we do that first. Namely, observe that after the duplicate label $k+1$ is dropped in $P$, the path does not visit the $(k+1)$th facet in either polytope, since the “inner” positions $k+1, \ldots, 2d-k$ of both vertices correspond to the Gale evenness strings of $C_{d-k}(2(d-k))^\Lambda$, and the facet labels are the same as on $\pi(d - k, 1)$, after subtracting $k$ from each label and facet number. By Lemma 7.1, that path $\pi(d - k, 1)$ never wraps around. Thus, this Lemke–Howson path (in dimension $d - k$) becomes the path in the higher dimension $d$ if each 0-1 string representing the vertex in $P$ (of dimension $d - k$) is prefixed with $k$ 0’s and suffixed with $k$ 1’s, and similarly each string for the vertex in $Q$ is prefixed with $k$ 1’s and suffixed with $k$ 0’s. In that way, $\pi(d - k, 1)$ becomes the second part of the higher-dimensional path $\pi(d, k)$ starting at $(u, v)$.

The first $L(k, 1)$ steps of $\pi(d, k)$ mimic the path $\pi(k, 1)$ (which applies to the cyclic polytopes in dimension $k$) by an analogous argument, except that we have to consider the original “outer” $2k$ facets $2d - k + 1, \ldots, 2d, 1, \ldots k$ with a cyclic shift of $k$ positions, and then reverse their order. That is, consider these facets in dimension $d$ as facets of $C_{k}(2k)^\Lambda$ (by omitting the $2(d-k)$ “inner” positions of the vertices of the polytope of dimension $d$), and then applying the polytope symmetry $\psi$ used in the proof of Lemma 7.6 to the polytope of dimension $k$. Then the missing label $k$ in $\pi(d, k)$ mimicks the path $\pi(k, 1)$ in the lower-dimensional polytope after the application of its polytope symmetry $\psi$. Lemma 7.1, applied to this lower-dimensional polytope with its symmetry, then implies that the initial $L(k, 1)$ steps of $\pi(d, k)$ mimic the path $\pi(k, 1)$ as they do not affect the “inner” positions $k+1, \ldots, 2d-k$ in the $d$-dimensional polytope. The last
step of this initial path ends at \((u, v)\), which in \(\pi(k, 1)\) terminates as it picks up label \(k\) in \(Q\). This is the first time that the corresponding labels of the paths in dimension \(k\) and \(d\) differ, since in dimension \(d\), the \(k\)th facet of \(Q\) has label \(k + 1\). Therefore, \(k + 1\) is the duplicate label, which is then dropped in \(P\), which starts the second part of the path treated above.

The shortest of these paths is obtained for \(k = d/2\), so that the shortest Lemke–Howson path in our construction has a length that is only on the order of the square root of the length of the longest path. (This is summarized in Theorem 8.4 below.)

8. Paths for the last \(d\) labels

This section treats the paths where label \(k\) for \(k > d\) is dropped. For a dropped label that is larger than \(d\), the longest paths, all of the same length, are obtained when that label is \(d + 1\), \(d + 2\), \(2d - 1\), or \(2d\). For ease of exposition, we choose \(2d\) as the missing label. The proofs follow the same pattern as for the first \(d\) labels, treated in the previous section. Unfortunately, we have not succeeded in finding a labeling that is more regular than \(l'\) in (6.1) which would require only one reasoning of this kind. One gets very similar recurrences by considering odd rather than even dimension \(d\) (as does Morris [11]; we have disregarded odd dimensional games altogether, which obviously affects the longest path lengths only by a constant factor). For odd \(d\), a suitable labeling is \(l''\) where \(l''(1) = 1\) and \(l''(d) = d\) and \(l''(k) = k + 1\) for all other even labels \(k\), and \(l''(k) = k - 1\) for all other odd labels \(k\). Even in this case, however, the path lengths seem to require different proofs for the first \(d\) and the last \(d\) labels.

In analogy to Lemma 7.3, the path \(\pi(d, 2d)\) exhibits a certain symmetry when considered in reverse. In this case, however, that symmetry applies to each polytope \(P\) and \(Q\) separately. It amounts to writing each vertex (rather than the entire vertex pair) backwards as a bitstring, omitting the dropped label \(2d\), which is in position \(2d - 1\) in \(Q\). Furthermore, we have to disregard the first vertex pair (and thus edge) and the last two vertex pairs (and thus edges) of the paths, so only \(L(2, 2d) - 3\) edges are considered.

\textbf{Lemma 8.1.} Let \(L = L(d, 2d)\) and \((u^i, v^i)\) be the \(i\)th vertex pair of the path \(\pi(d, 2d)\) for \(0 \leq i \leq L\). We write \(u^i\) as the sequence of bits \(u^i_1 u^i_2 \cdots u^i_{2d}\), and \(v^i\) in a similar fashion. Then for \(1 \leq i \leq L - 2\),

\[
\begin{align*}
u^i_k &= u^i_{2d-k}^i & (1 \leq k \leq 2d - 1), \\
v^i_k &= v^i_{2d-k}^i & (2 \leq k \leq 2d - 1).
\end{align*}
\]

\textbf{Proof.} (Omitted in this preliminary version of the paper.)
The following is the main recurrence for the length of paths $\pi(d, 2d)$, in analogy to Proposition 7.5. It is proved in a similar way by observing repetitions of lower-dimensional paths. As noted before Lemma 8.1, certain initial and final edges are thereby disregarded, which accounts for the subtraction of 8 in the recurrence. The resulting exponential growth factor is the same as for the paths when label 1 is dropped, but the constant factor is slightly smaller.

**Proposition 8.2.** $L(2, 4) = 4$, $L(4, 8) = 10$, and for $d \geq 6$

$$L(d, 2d) = 4L(d - 2, 2(d - 2)) + L(d - 4, 2(d - 4)) - 8.$$  

*Proof.* (Omitted in this preliminary version of the paper.)

**Lemma 8.3.** For $1 \leq k \leq d/2$, $L(d, d + k) = L(d, 2d - k + 1)$.

*Proof.* The proof is analogous to the proof of Lemma 7.6. Using the same polytope symmetry $\psi$, the path $\pi(d, d + k)$ corresponds to $\pi(d, 2d - k + 1)$.

The next lemma is analogous to Lemma 7.7 above.

**Lemma 8.4.** For $1 \leq k \leq d - 1$ and $k$ odd,

$$L(d, d + k) = L(d, d + k + 1).$$

*Proof.* The argument is identical to the proof of Lemma 7.7. When we apply the relabeling $l'$ to $P$ and $Q$ we see that $\pi(d, d + k)$ corresponds to $\pi^{-1}(d, d + k + 1)$.

**Lemma 8.5.** For $2 \leq k \leq d - 2$ and $k$ even,

$$L(d, d + k) = L(k, 2k) + L(d - k + 2, 2(d - k + 2)) - 4.$$  

*Proof.* (Omitted in this preliminary version of the paper.)

**Theorem 8.6.** For any label $k = 1, \ldots, 2d$ that is dropped, $L(d, k) = \Omega((\sqrt{5} + 2)^{d/4})$.

This shows the exponential length of the Lemke–Howson paths.
References


