Wheel Facets of the OLS Polytope

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Abstract

Wheel structures of the Orthogonal Latin Squares (OLS) polytope (P_I) are presented in [2]. The current work focuses on the families of valid inequalities arising from wheels and proves that certain among them are facet-defining for P_I. For two of these families we provide efficient separation procedures. We also present results regarding odd-hole inequalities, which essentially form a larger class encompassing that of wheel inequalities.

1 Introduction

The Orthogonal Latin Squares (OLS) problem is the second member of the family of *planar* assignment problems, the first being the Latin Squares (LS) problem ([8]). As noted in [2], not many classes of facet-defining inequalities are known for the polytopes of either problem. For the LS polytope two families of odd-hole facets are described in [3] and [5], respectively. For the OLS polytope (P_I) all clique facets are presented in [1]. The current work adds to our knowledge of the facial structure of planar assignment polytopes by identifying wheel facets for P_I. It is easy to see that there are no wheel-induced inequalities for the LS polytope. Thus, P_I is the "simplest" among planar assignment polytopes having facet-defining inequalities of this type.

The families of the inequalities presented here are induced by some of the wheel classes presented in [2]. In fact, much of the ground work for the current paper has been laid out in [2]. Although we give some definitions, for the self-sufficiency of the current work, we refer to that paper for a complete presentation of concepts and conventions to be used throughout.

The OLS problem is defined in terms of four disjoint $n-$sets, namely I, J, K, L (see [1, 2] for a formulation). Let $G_A(C, E_C)$ denote the *column intersection* graph of the A matrix of the OLS problem.

Because $C = I \times J \times K \times L$ ([1]), $s \in C$, equivalently, denotes the tuple (i_s, j_s, k_s, l_s). For any $s, t \in C$ there exists the edge $(s,t) \in E_C$ if and only if nodes (s,t) have at least two indices in common. Hence, if $(s,t) \in E_C$ the sets of the common indices form the *ground* set of the edge. If s, t have two (three) indices in common ($|s \cap t| = 2(3)$) then the ground set is called a *double (triple)* set. The ground set of the edge (s,t) also denoted as $g((s,t))$, is defined in terms of the $\otimes$ operator. Thus, if $i_s = i_t$, $j_s = j_t$, $k_s \neq k_t$, $l_s \neq l_t$ then $g((s,t)) = I \otimes J$, where $I \otimes J = (I \times J) \cup (J \times I)$. If $i_s = i_t$, $j_s = j_t$, $k_s = k_t$, $l_s \neq l_t$ then $g((s,t)) = I \otimes J \otimes K$, where $I \otimes J \otimes K = (I \otimes J) \cup (J \otimes K) \cup (I \otimes K)$.

Few additional definitions are introduced in Section 2 where properties of the odd-hole and wheel inequalities are examined. Section 3 discusses the relation between the inequalities induced by cliques ([1]) and wheels. In Section 4 families of wheel-induced facets are presented. For two of these families polynomial time separation algorithms are given in Section 5.

2 Odd-hole and wheel inequalities

Let $H \subset C$ denote the node set of an induced odd hole ($|H| = 2p+1, p \geq 2$). Then, the odd-hole inequality is

$$\sum \{x_q : q \in H\} \leq p \tag{2.1}$$

A special case of odd-hole inequalities are the wheel inequalities. A wheel is an induced subgraph consisting of an odd hole, called the *rim*, and a node connected to all nodes of the rim called the *hub*. Let $c \in C$ denote the hub of a wheel and $H(c)$ the set of its rim nodes where $|H(c)| = 2p+1$, $p \geq 2$. Then, the *wheel* inequality ([6]) is

$$px_c + \sum (x_q : q \in H(c)) \leq p \tag{2.2}$$

Maximally lifted odd-hole (and consequently wheel) inequalities are known to be facet-inducing for the set packing relaxation of P_I ([9, 10]), denoted as $\tilde{P}_I$ (see [2]). An important issue is that of calculating the largest coefficient for the variables belonging to such an inequality. In other words, let x_s be any variable added to the left-hand side of (2.1) and let a_s denote its coefficient. We want to determine the maximum value of a_s such that (2.1) is not violated. First we need to show the following auxiliary result:

Lemma 2.1 *Let $H_m \subset H$ such that $|H_m| = m$. Then,*

$$\left\lceil \frac{m}{2} \right\rceil \leq \max \sum \{x_q : q \in H_m\} \leq m \tag{2.3}$$

Proof. For the upper bound consider H_m to consist only of non-adjacent nodes of H. Then all variables indexed by the elements of H_m can simultaneously take the value 1, yielding $\sum\{x_q : q \in H_m\} = m$. One the other hand, if each node of H_m is adjacent to at most two other nodes of this set, and since the elements of H_m do not induce a cycle ($H_m \subset H$), we have $\max \sum\{x_q : q \in H_m\} = \frac{m}{2}$, for m even, and $\max \sum\{x_q : q \in H_m\} = \lceil \frac{m}{2} \rceil$, for m odd. ■

Proposition 2.2 *No odd-hole inequality valid for OLS can have a left-hand side integer coefficient greater than* $\min\{5, p\}$.

Proof. Consider that (2.1) is augmented (lifted) by introducing the variable x_s with a coefficient a_s:

$$a_s x_s + \sum\{x_q : q \in H\} \leq p$$

Let $H_s \subset H$ be such that $H_s = \{q \in H : |q \cap s| \geq 2\}$. Therefore, if variable x_s is set to one the above inequality becomes

$$a_s + \sum\{x_q : q \in H \setminus H_s\} \leq p \tag{2.4}$$

We observe that a_s is maximized when $\max \sum\{x_q : q \in H \setminus H_s\}$ is minimized. To achieve this, $|H \setminus H_s|$ must be as small as possible. Thus, $|H_s|$ must be as large as possible. For each pair of indices of s there can be at most two nodes of H_s having the same pair of index values. There are six distinct double sets, therefore $|H_s| \leq \min\{12, 2p+1\}$. If $2p+1 > 12$ then $|H \setminus H_s| \geq 2p+1-12 = 2(p-6)+1$. So the minimum number of variables participating in the summand is $|H \setminus H_s| = 2(p-6)+1$. By inequality (2.3)

$$p - 6 + 1 \leq \max \sum\{x_q : q \in H \setminus H_s\}$$

Hence by (2.4), we have

$$p - 5 + a \leq p$$

yielding $a_s \leq 5$.

If $2p+1 < 12$ then there might be a node $c \in C$ such that $(c, q) \in E_C \ \forall q \in H$. In this case, node c is the hub of a wheel while H is the set of nodes of the rim. The wheel inequality states $a_s = p$. ■

Specifically for the lifted inequalities generated by wheels, we have the following proposition.

Proposition 2.3

$$\begin{aligned} a_s &= p, & \textit{if} & \quad s \equiv c, \\ a_s &\leq 2, & \textit{if} & \quad 3 \geq |c \cap s| \geq 2 \end{aligned}$$

Proof. The maximum value for a_s is achieved only if s is connected to all nodes of the wheel. In this case $s \equiv c$ and $a_s = p$ (Prop. 2.2).

Assume that $c = (n, n, n, n)$. If $s \in C \setminus (H(c) \cup \{c\})$ there are two cases.

Case 2.3.1 $|c \cap s| = 3$.

Without loss of generality assume that $s = (i_0, n, n, n)$, $i_0 \in I \setminus \{n\}$. First consider wheels with double-set spokes only. Observe that there can be none, one, or two rim nodes having i_0 as one of their

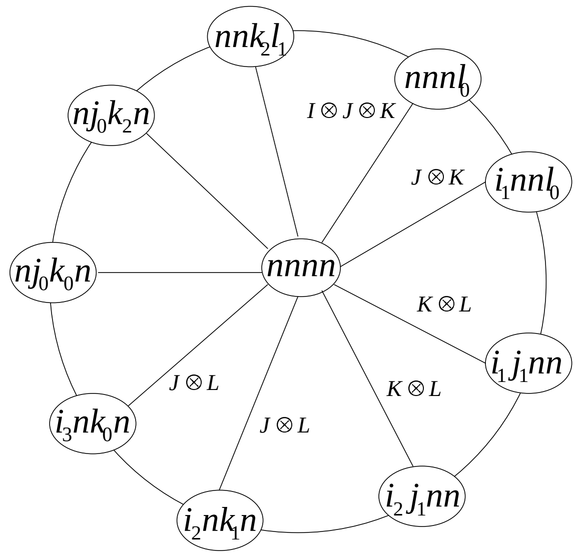

Figure 1: A wheel having a triple-set spoke and six nodes connected to (i_0, n, n, n)

indices. In all cases, node s is connected to all nodes of the rim whose spokes are based on one of the double sets $J \otimes K$, $J \otimes L$, $K \otimes L$. There can be at most six such nodes, two for each double set ([2, prop. 3.2]). Setting $x_s = 1$, implies that the variables indexed by the hub and these six nodes will be set to 0. This leaves the $2p + 1 - 6 = 2 \cdot (p - 2) - 1$ variables, indexed by the remaining rim nodes, free. At least $p - 2$ of these variables can be set to one simultaneously. Hence, the lifted wheel inequality yields $a_s + p - 2 \leq p$ or $a_s \leq 2$. Observe that if among the six nodes there exist nodes having i_0 among their indices, then the edge connecting each such node to s is based on a triple set formed by set I and two of J, K, L.

The situation is the same in the case of wheels having one triple-set spoke. In this case observe that the maximum number of pairs of spokes, each spoke of the pair being based on the same double set, is three ([2, Table 3]). This occurs for wheels belonging to classes nums. 27, 28 (column $\left|W_c^p(K_3^1)\right|$). In both classes the three pairs are consecutive (column seq(K_3^1)). However, there is no sequence of three such spoke pairs with the property that the three double sets of the pairs are formed by three single sets (column $K_3^1 - K_3^1 - K_3^1$, subcolumn 3). Therefore, one of the six spokes of the sequence must be based on a triple set having as a component one of $J \otimes K$, $J \otimes L$, $K \otimes L$. One such wheel is illustrated in Figure 1. s is connected to at most six such nodes implying $a_s \leq 2$.

Finally, if the wheel has two triple-set spokes we can only have $p = 2$ ([2, Table 3]), implying that $a_s \leq 1$.

Case 2.3.2 $|c \cap s| = 2$.

Without loss of generality assume $s = (i_0, j_0, n, n)$, $i_0 \in I \setminus \{n\}$, $j_0 \in J \setminus \{n\}$. In the case of wheels having only double-set spokes, node s can be connected to at most six nodes, i.e.

(a) two nodes incident to spokes based on $K \otimes L$,

(b) two nodes having i_0 and n for one of the indices of the sets K, L,

(c) two nodes having j_0 and n for one of the indices of the sets K, L.

The nodes of (b), (c) will have one more index equal to n such that the double sets of their incident spokes are different from $K \otimes L$. One such collection of six nodes is:

$$\{(\cdot, \cdot, n, n), (\cdot, \cdot, n, n), (i_0, n, n, \cdot), (i_0, n, n, \cdot), (n, j_0, \cdot, n), (n, j_0, \cdot, n)\}$$

Observe that it is not necessary for the nodes of (b) to have the same pair of indices equal to n.The same is true for the nodes of (c). In all cases, we cannot have another node connected to s because then we would have a chord between this node and one of the six nodes of the collection. Hence $a_s \leq 2$.

For wheels with a triple-set spoke, it is also easy to see that we cannot have more than six nodes connected to s.

■

The wheel classes presented in ([2, Table 3]) give rise to families of valid inequalities for P_I. Each such inequality is produced by lifting an inequality of the type (2.2) obtained from a wheel belonging to a particular class. The lifted inequality depends on the lifting sequence; it is easy to see that a wheel class can give rise to more than one family of lifted inequalities. For example, the wheel illustrated in Figure 2a, belonging to class num. 1, induces the inequalities

$$\begin{aligned} 2x_{nnnn} + \sum_{i \in I \setminus \{n\}} x_{innn} + \sum_{j \in J \setminus \{n\}} x_{njnn} + \sum_{l \in L \setminus \{n\}} x_{nnnl} & \\ + \sum_{l \in L \setminus \{l_0, n\}} x_{nj_0nl} + \sum_{j \in J \setminus \{j_0, n\}} x_{njnl_0} & \\ + \sum_{l \in L \setminus \{l_0, n\}} x_{i_0nnl} + \sum_{i \in I \setminus \{i_0, n\}} x_{innl_0} & \\ x_{i_0nnl_0} + x_{nj_0nl_0} + x_{i_0j_0nn} & \leq 2 \end{aligned} \tag{2.5}$$

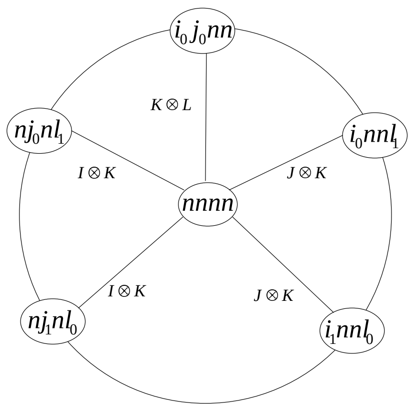

Figure 2a: A wheel of class num. 1

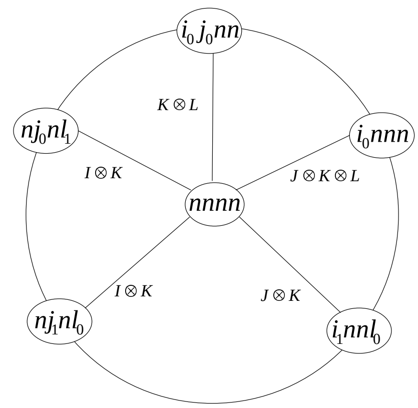

Figure 2b: A wheel of class num. 22

$$
\begin{aligned}
2x_{nnnn} + \sum_{i\in I\setminus\{n]} x_{innn} + \sum_{j\in J\setminus\{n\}} x_{njnn} + \sum_{l\in L\setminus\{n\}} x_{nnnl} & \\
+ \sum_{l\in L\setminus\{l_0,n\}} x_{i_0nnl} + \sum_{i\in I\setminus\{i_0,n\}} x_{innl_0} & \\
+ \sum_{l\in L\setminus\{l_0,n\}} x_{nj_0nl} + \sum_{i\in I\setminus\{i_0,n\}} x_{ij_0nn} & \\
x_{i_0nnl_0} + x_{nj_0nl_0} + x_{i_0j_0nn} + x_{nj_1nl_0} & \leq 2
\end{aligned} \tag{2.6}
$$

On the other hand, the same family of lifted inequalities can be obtained from wheels belonging to two distinct classes. For example, the above inequalities can also be obtained from the wheel illustrated in Figure 2b which belongs to the wheel class num. 22.

Another important observation is that the number of triple-set rim edges of a wheel is related to the induced lifted wheel inequality. The bigger this number is, the larger the set of the common indices of the nodes of the wheel. This implies a larger set of variables to be considered for the lifted wheel inequality. Therefore, for a given wheel class, the wheels having the maximum number of rim edges based on triple sets are likely to produce more families of valid inequalities than wheels without this property. Results regarding triple-set rim edges are presented in [2].

3 Wheel and clique inequalities

The definition of a wheel assumes $p \geq 2$ If we extend it to include wheels having $p = 1$, we can derive all classes of clique-induced inequalities ([1]) as wheel inequalities. This is illustrated in Figures 3a, 3b, 3c. For each of these wheels, the resulting lifted inequality belongs to a distinct class of clique inequalities,

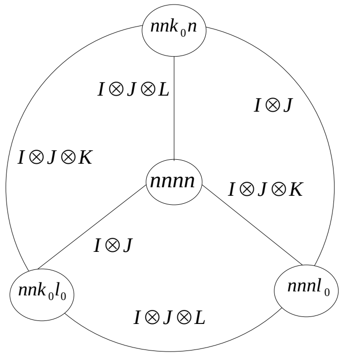

Figure 3a

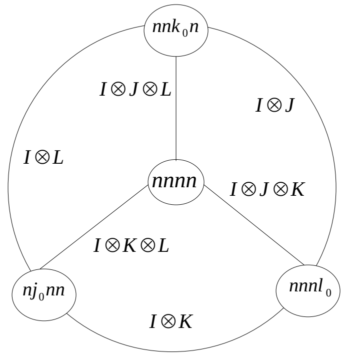

Figure 3b

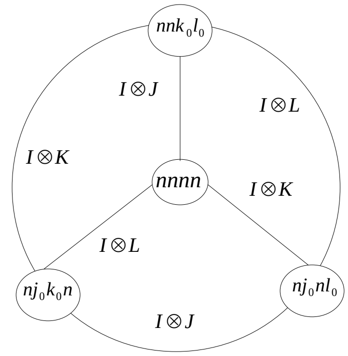

Figure 3c

Table 1: Clique inequalities induced by wheels

Figure	Clique class	Inequality
3a	I	$\sum_{k\in K}\sum_{l\in L} x_{nnkl} \leq 1$
3b	II	$x_{nnnn} + \sum_{i\in I\setminus\{n\}} x_{innn} + \sum_{j\in J\setminus\{n\}} x_{njnn} + \sum_{k\in K\setminus\{n\}} x_{nnkn} + \sum_{l\in L\setminus\{n\}} x_{nnnl} \leq 1$
3c	III	$x_{nnnn} + x_{nnk_0l_0} + x_{nj_0k_0n} + x_{nj_0nl_0} \leq 1$

as illustrated in Table 1.

Obviously, all the wheels of size three are in fact K_4 graphs. However, considering the specific structures illustrated in Figures 3a, 3b, 3c as wheels allows as to derive three additional wheel classes to be added to the ones described in [2, Table 3]. For the wheel of Figure 3a, we have two spokes based on triple sets ($\left|H^2(c)\right| = 2$). This is the second wheel class with this property. The wheel of Figure 3b has the unique property that it has three spokes based on triple sets, i.e. $\left|H^2(c)\right| = 3$. This cannot happen for $p \geq 2$ ([2]). Finally, the wheel of Figure 3c does not include two spokes based on the same double set although it consists exclusively of double-set spokes. Again this property is unique ([2]). Observe that the inequality induced by the last class of wheels is facet-defining for P_I as is (no lifting is required).

4 Wheel facets

The general form of the inequalities considered hereafter is

$$px_c + \sum\{x_q : q \in Q\} \leq p \tag{4.1}$$

where $c \in C$, $Q \subset C$ such that there exists $H(c) \subset Q$ with $|c \cap h| \geq 2$, for all $h \in H(c)$ and $|H(c)| = 2p+1$. Hence, (4.1) describes a different family of lifted wheel inequalities for each distinct set Q. Consequently, each family of inequalities of the type (4.1) is induced by a wheel class described in [2]. For each such family presented in one of the following subsections, we show that the face $P_I(Q) = \{x \in P_I : px_c + \sum\{x_q : q \in Q\} = p\}$ is a facet of P_I. For every family examined, the associated set Q is defined with respect to two elements of C, namely c, s, such that $|c \cap s| = 0$. It follows easily that the cardinality of each family is of $O(n^8)$, since for any given c there are exactly $(n-1)^4$ elements of C available for the the role of s.

The procedure for showing that $P_I(Q)$ is a facet of P_I includes the following steps. We give specific values to p, c, s and, if necessary, to other parameters, so as to define a particular inequality belonging to the family examined. We show that this inequality is maximal with respect to set inclusion. This proves that the inequality is facet-defining for $\tilde{P}_I$, and because $P_I \subset \tilde{P}_I$, the inequality is valid for P_I. Then we show that $P_I(Q) \neq \emptyset$ and $P_I(Q) \neq P_I$. Finally, let (4.1) be written as $dx \leq d_0$. We show that if any other inequality $ax \leq a_0$ is satisfied as equality by all points of $P_I(Q)$, then there exist $\lambda \in \mathbb{R}^m$ and $\pi > 0$ such that $a = \lambda A + \pi d$ and $a_0 = \lambda e + \pi d_0$ (see also [11]). For showing this last step we use exclusively points of $P_I(Q)$. An observation used throughout the facet proofs is that for any two points $x, y \in P_I(Q)$ the equation $ax = ay$ is valid.

Another relation mostly exploited in the proofs is the $1-1$ correspondence between Orthogonal Latin squares and integer points of P_I. This relation is more evident if we consider two of the sets I, J, K, L as the row and column set of the OLS structure, and the remaining two sets as the sets of values included in the cells of the first and the second Latin square,respectively. Conventionally, I is considered to be the row set, J the column set and K (L) the set of elements of the first (second) Latin square. As in ([1]), we denote $k(i,j)$ (respectively $l(i,j)$) the value of the cell in row i, column j of the first (second) Latin square. Thus, $k(i,j) \in K$ and $l(i,j) \in L$. Occasionally, in order to emphasize the value of $k(i,j)$ ($l(i,j)$) at a given point x of P_I, we use the notation $k(x; i, j)$ ($l(x; i, j)$). For the rest of the paper we illustrate a pair of OLS as a Latin square containing in each cell a pair of indices, each belonging to a different set. The above convention implies that this pair belongs to $K \times L$. In the case that a different convention is used with respect to the sets I, J, K, L and the rows, columns and elements of the cells of the two squares, the above notation as well as the illustration of the OLS structure are adjusted accordingly.

The inherent symmetry of the integer points of P_I is exploited through the notion of *interchange*. Given a pair of OLS and $m_1, m_2 \in M$ where M is any of the disjoint sets I, J, K, L (inter)changing all m_1 values m_2 and all m_2 values to m_1 yields another pair of OLS ([1, rem. 13]). The two OLS structures are called *equivalent* ([4, p. 168]) and the corresponding points of P_I *isotopic*. The *interchange* operator ($\leftrightarrow$), introduced in [1], is used to denote such an operation. Thus, setting $x^* = x(i_1 \leftrightarrow i_2)_1$ implies that at a given integer point x, we set $x_{i_1jk(i_1,j)l(i_1,j)} = x_{i_2jk(i_2,j)l(i_2,j)} = 0$ and $x_{i_1jk(i_2,j)l(i_2,j)} = x_{i_2jk(i_1,j)l(i_1,j)} = 1$, for all $j \in J$, obtaining point x^*. The expression is subscripted by a number denoting the set to which the elements participating in the interchange belong (1 for the set I, 2 for J, 3 for K, 4 for L). The *conditional* interchange is defined as the interchange to be executed only if a certain condition is met. Thus, $x^* = x(i_1 = n?n \leftrightarrow i_2)_1$ implies that the interchange is carried out only if $i_1 = n$. Otherwise $x^* = x$.

Another type of interchange involves the permutation of the roles of the four sets. This form of symmetry is known as *conjugacy* or *parastrophy* ([4, Section 2.1]). Hence, $x^* = x(I \leftrightarrow J)$ denotes the swap of values of the indices i, j applied to point x resulting in point $x^*(x^* \in P_I)$. The points x, x^* are called *conjugates*.

The following proposition, introduced in [1, prop. 16], establishes an equation involving elements of the vector a defined previously.

Proposition 4.1 *[1, Prop. 16]For $n \geq 3$ and $n \neq 6$*

$$\begin{aligned} & a_{i_1 j_1 k(i_1,j_1) l(i_1,j_1)} + a_{i_1 j_2 k(i_1,j_2) l(i_1,j_2)} + a_{i_2 j_1 k(i_2,j_1) l(i_2,j_1)} + a_{i_2 j_2 k(i_2,j_2) l(i_2,j_2)} \\ & + a_{i_1 j_1 k(i_2,j_2) l(i_2,j_2)} + a_{i_1 j_2 k(i_2,j_1) l(i_2,j_1)} + a_{i_2 j_1 k(i_1,j_2) l(i_1,j_2)} + a_{i_2 j_2 k(i_1,j_1) l(i_1,j_1)} \\ = \; & a_{i_1 j_1 k(i_2,j_1) l(i_2,j_1)} + a_{i_1 j_2 k(i_2,j_2) l(i_2,j_2)} + a_{i_2 j_1 k(i_1,j_1) l(i_1,j_1)} + a_{i_2 j_2 k(i_1,j_2) l(i_1,j_2)} \\ & + a_{i_1 j_1 k(i_1,j_2) l(i_1,j_2)} + a_{i_1 j_2 k(i_1,j_1) l(i_1,j_1)} + a_{i_2 j_1 k(i_2,j_2) l(i_2,j_2)} + a_{i_2 j_2 k(i_2,j_1) l(i_2,j_1)} \end{aligned}$$

for $i_1, i_2 \in I, i_1 \neq i_2$, $j_1, j_2 \in J, j_1 \neq j_2$.

To derive this equation we use an integer point $x \in P_I$ and points $x', \bar{x}, \bar{x}'$ derived from x as $x' = x(i_1 \leftrightarrow i_2)_1$, $\bar{x} = x(j_1 \leftrightarrow j_2)_2$, $\bar{x}' = \bar{x}(i_1 \leftrightarrow i_2)_1$. Subtracting $a\bar{x} = a\bar{x}'$ from $ax = ax'$ and cancelling out equivalent terms yields the equation. Hence, the point x and the two pairs of indices, each pair belonging to one of the sets I, J, K, L, are enough for defining such an equation. Consequently, the equation is referred to as $x((i_1, i_2)_1; (j_1, j_2)_2)$ and the collection $(x, x', \bar{x}, \bar{x}')$ as $X((i_1, i_2)_1; (j_1, j_2)_2)$. As usual, the set to which the indices of the pair belong is indicated by the subscript of the pair. Observe that $x((i_1, i_2)_1; (j_1, j_2)_2)$ can be used in the proofs that follow, only if $x, x', \bar{x}, \bar{x}' \in P_I(Q)$, in which case we use the notation $X \in P_I(Q)$.

To facilitate our presentation, we introduce some additional definitions and conventions. Hence, for $c \in C$, $Q^m(c) = \{q \in C : |q \cap c| = m\}$. We refer to a point x illustrated in Table numbered by (#) as $x^{\mathrm{Tbl}(\#)}$. As in [1], an equation which is numbered by (#) and whose terms are indexed by i, j, k, l, is referred to as (#)$[a, b, c, d]$ implying that the indices i, j, k, l, take the specific values a, b, c, d, respectively.

$\mathring{x}$ denotes a point of P_I induced by the OLS structure which has the elements of the first columns of the two squares in standard order as well as the first row of the first square. A pair of OLS of this form is called *standardised* or *reduced* ([4, p. 159]) and can be derived, provided that $P_I \neq \emptyset$. Thus, $\mathring{x}_{m1mm} = 1, \mathring{x}_{1mk(1,m)l(1,m)} = 1, \forall m \in \{1, \ldots, n\}$. The following Lemma establishes another point to be used in the proofs that follow.

Lemma 4.2 *For $n \geq 7$, $i_q, i_1 \in I \setminus \{1, n\}$, $j_q, j_1 \in J \setminus \{1, n\}$, $k_2, k_3 \in K \setminus \{1, k_0, n\}$ where $k_0 \in K \setminus \{1, n\}$, $l_1 \in L \setminus \{1, l_0, n\}$, $l_2 \in L \setminus \{1, l_0\}, l_3 \in L \setminus \{1\}$ where $l_0 \in L \setminus \{1, n\}$, there exists the point x illustrated in Table 2*

Table 2: Point x (Lemma 4.2)

	1	$\cdots$	j_q	$\cdots$	j_1	$\cdots$	n
1	$(1,1)$		(k_2,l_2)				
$\vdots$							
i_q	(n,l_1)		(k_3,l_3)				
$\vdots$							
i_1					$(1,n)$		
$\vdots$							
n							(n,n)

Proof. At point $\mathring{x}$, let $i_1 \in I \setminus \{1,n\}$, $j_1 \in J \setminus \{1\}$ be such that $k(i_1,j_1) = 1$, $l(i_1,j_1) = n$. There also exist i_2, i_q such that $i_q \in I \setminus \{1,i_1,n\}$ with $k(i_q,1) = k_1$ and $i_2 \in I \setminus \{i_1,n\}$ with $\mathring{x}_{i_2 j_2 k_1 n} = 1$, where $j_2 \in I \setminus \{j_1,n\}$. For $n \geq 5$, it is safe to assume that $i_q \neq i_1$ because if $i_q = i_1$ then there exist $k_2 \in K \setminus \{1,k_1,n\}$ and (another) i_q (this time different from i_1) such that $k(i_q,1) = k_2$. By denoting i_3, j_3 the row and column at which the pair (k_2,n) appears ($\mathring{x}_{i_3 j_3 k_2 n} = 1, i_3 \in I \setminus \{i_1,i_2,n\}, j_3 \in I \setminus \{j_1,j_2,n\}$) and by performing the interchanges $(k_2 \leftrightarrow k_1)_3 (i_3 \leftrightarrow i_2)_1 (j_3 \leftrightarrow j_2)_2$, we derive point $\mathring{x}$ with $i_q \neq i_1$.

We denote $l(i_q,1)$ as l_1. Therefore, $\mathring{x}_{i_q 1 k_1 l_1} = \mathring{x}_{i_1 j_1 1 n} = \mathring{x}_{i_2 j_2 k_1 n} 1$. Let $\tilde{x} = \mathring{x}(k_1 \leftrightarrow n)_3 (i_2 \leftrightarrow n)_1 (j_2 \leftrightarrow n)_2$. As shown in Table 3, for $n \geq 7$ there exists $j_q \in J \setminus \{1,n\}$ such that $k(i_q,j_q), k(1,j_q) \in K \setminus \{1,n\}$, $l(i_q,j_q), l(1,j_q) \in L \setminus \{1\}$. Observe that we can safely assume that $k(i_q,j_q), k(1,j_q) \neq k_0$. To show

Table 3: Point $\tilde{x}$ (Lemma 4.2)

	1	$\cdots$	j_q	$\cdots$	j_1	$\cdots$	j_3	$\cdots$	j_4	$\cdots$	j_5	$\cdots$	n
1	$(1,1)$		(k_2,l_2)								$(n,\cdot)$		
$\vdots$													
i_q	(n,l_1)		(k_3,l_3)				$(1,\cdot)$		$(\cdot,1)$				
$\vdots$													
i_1					$(1,n)$								
$\vdots$													
n													(n,n)

that this is so, consider $k(i_q,j_q) = k_0$. Then there exists $k_t \in K \setminus \{1,k_0,n\}$ such that the interchange $(k_t \leftrightarrow k_0)_3$ will set $k(i_q,j_q) = k_t \neq k_0$. By the same argument, $l(i_q,j_q), l(1,j_q), l_1 \neq l_0$. By denoting $k(1,j_q)$ as k_2, $k(i_q,j_q)$ as k_3, $l(1,j_q)$ as l_2 and $l(i_q,j_q)$ as l_3, we obtain the point $x^{\mathrm{Tbl}(2)}$ of Table 2. $\blacksquare$

Each of the following subsections is entitled with respect to the wheel class from which the inequality at hand is induced (see [2]).

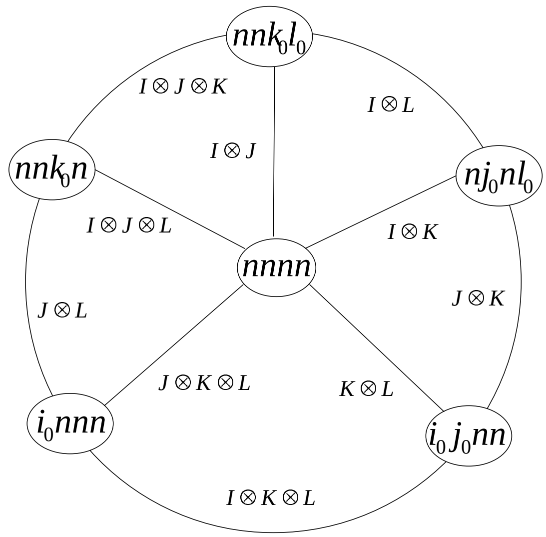

Figure 4: A wheel of class num. 29

4.1 Wheel class num. 29

Let $p = 2$ and $Q = (Q^2(c) \cap Q^2(s)) \cup Q^3(c)$. For $c = (n, n, n, n)$ and $s = (i_0, j_0, k_0, l_0)$, (4.1) becomes

$$\begin{aligned} 2x_{nnnn} + \sum_{i \in I \setminus \{n\}} x_{innn} + \sum_{j \in J \setminus \{n\}} x_{njnn} + \sum_{k \in K \setminus \{n\}} x_{nnkn} + \sum_{l \in L \setminus \{n\}} x_{nnnl} \\ + x_{i_0j_0nn} + x_{i_0nk_0n} + x_{nj_0k_0n} + x_{i_0nnl_0} + x_{nj_0nl_0} + x_{nnk_0l_0} \quad \leq \quad 2 \end{aligned} \tag{4.2}$$

This inequality is induced by the wheel illustrated in Figure 4, which belongs to wheel class num. 29.

Lemma 4.3 *(4.2) is maximal.*

Proof. Suppose that there exists a variable x_q ($q = (i_q, j_q, k_q, l_q) \in C$) which can be added to the left-hand side of (4.2) without increasing its right-hand side. Clearly x_q must have either three or two indices in common with (n, n, n, n), since otherwise it can be set to one together with x_{nnnn}. In the former case, x_q is already included in (4.2), i.e. x_q is one of $x_{innn}, x_{njnn}, x_{nnkn}, x_{nnnl}, i \in I \setminus \{n\}, j \in J \setminus \{n\}, k \in K \setminus \{n\}, l \in L \setminus \{n\}$. In the latter case, assume $i_q = n, j_q = n$. Then we can simultaneously set $x_{nnk_ql_q}$ and $x_{nj_0nl_0}, x_{i_0nnn}$ to one. Observe that this is valid even if $k_q = k_0$, $l_q \neq l_0$ or $l_q = l_0$, $k_q \neq k_0$. ■

Theorem 4.4 *For $n \geq 7$, (4.2) defines a facet of P_I.*

Proof. Let $x = \mathring{x}(1 \leftrightarrow n)_2$. Then $x \in P_I(Q)$ since $x_{nnnn} = 1$. Hence, $P_I(Q) \neq \emptyset$.

At point $\mathring{x}$ either $\mathring{x}_{nnk_0l_0} = 1$ or $\mathring{x}_{nnk_0l_0} \neq 1$. In the former case, there exists $j_1 \in J \setminus \{1, n\}$ such that $(k(i_0, j_1), l(i_0, j_1)) \neq (k_0, n), (n, l_0)$. We need $n \geq 5$, because we might have $\mathring{x}_{i_0j_2k_0n} = \mathring{x}_{i_0j_3nl_0} = 1$. Let $x = \mathring{x}(j_1 \leftrightarrow n)_2$. Clearly $x \in P_I \setminus P_I(Q)$. In the latter case, if $\mathring{x}_{i_0nk_0n} = 0$ and $\mathring{x}_{i_0nnl_0} = 0$ we are done, i.e. $\mathring{x} \in P_I \setminus P_I(Q)$. Otherwise, assume without loss of generality that $\mathring{x}_{i_0nk_0n} = 1$. Then there exists $j_1 \in J \setminus \{1, n\}$ such that $(k(i_0, j_1), l(i_0, j_1)) \neq (n, l_0)$ and $(k(n, j_1), l(n, j_1)) \neq (k_0, l_0)$. Let $x = \mathring{x}(j_1 \leftrightarrow n)_2$. Clearly $x \in P_I \setminus P_I(Q)$. Again we need $n \geq 5$.

Suppose that there exist $a \in \mathbb{R}^{n^4}$, $a_0 \in \mathbb{R}$ such that $ax = a_0$, for every $x \in P_I(Q)$. We will show that (a, a_0) is a linear combination of the rows of A and inequality (4.2). Equivalently, we will exhibit scalars λ^1_{kl}, λ^2_{il}, λ^3_{jl}, λ^4_{ij}, λ^5_{jk}, λ^6_{ik}, $\pi \in \mathbb{R}$, $\forall i \in I, j \in J, k \in K, l \in L$, satisfying

$$a_{ijkl} = \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik}, \qquad \forall (i,j,k,l) \in C \setminus (Q \cup \{(n,n,n,n)\}), \tag{4.3}$$

$$a_{ijkl} = \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik} + \pi, \quad \forall (i,j,k,l) \in Q, \tag{4.4}$$

$$a_{ijkl} = \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik} + 2\pi, \quad (i,j,k,l) = (n,n,n,n) \tag{4.5}$$

$$\begin{aligned} a_0 = {} & \sum_{k \in K, l \in L} \lambda^1_{kl} + \sum_{i \in I, l \in L} \lambda^2_{il} + \sum_{j \in J, l \in L} \lambda^3_{jl} \\ & + \sum_{i \in I, j \in J} \lambda^4_{ij} + \sum_{j \in J, k \in K} \lambda^5_{jk} + \sum_{i \in I, k \in K} \lambda^6_{ik} + 2\pi \end{aligned} \tag{4.6}$$

We denote (4.4) and (4.5) as $(4.3)_\pi$ and $(4.3)_{2\pi}$, respectively. We define:

$$\begin{array}{llll} \lambda^1_{kl} = a_{11kl}, & \forall k \in K, l \in L, & \lambda^4_{ij} = a_{ij11} - a_{i111} - a_{1j11} + a_{1111}, & \forall i \in I, j \in J, \\ \lambda^2_{il} = a_{i11l} - a_{111l}, & \forall i \in I, l \in L, & \lambda^5_{jk} = a_{1jk1} - a_{1j11} - a_{11k1} + a_{1111}, & \forall j \in J, k \in K, \\ \lambda^3_{jl} = a_{1j1l} - a_{111l}, & \forall j \in J, l \in L, & \lambda^6_{ik} = a_{i1k1} - a_{i111} - a_{11k1} + a_{1111}, & \forall i \in I, k \in K \end{array}$$

If we substitute λs in (4.3), we obtain

$$\begin{aligned} a_{ijkl} = {} & a_{ij11} + a_{i1k1} + a_{i11l} + a_{1jk1} + a_{1j1l} + a_{11kl} \\ & -2a_{i111} - 2a_{1j11} - 2a_{11k1} - 2a_{111l} + 3a_{1111} \end{aligned} \tag{4.7}$$

Observe that (4.7) is true for all cases where at least two of the indices are equal to one. All other cases of $(i,j,k,l) \in C \setminus (Q \cup \{(n,n,n,n)\})$ are grouped with respect to the number of indices equal to n. There are three such groups defined when none, one, or two of the indices are equal to n, respectively. For the first two we show (4.3) by proving (4.7), whereas for the third one (4.3) is proven directly.

Case 4.4.1 None of i, j, k, l is equal to n.

Consider $(i,j,k,l) = (i_q, j_q, k_q, l_q)$ where $i_q \neq n, j_q \neq n, k_q \neq n, l_q \neq n$. Let $x^1 = x^{\mathrm{Tbl}(2)}$ and $x^2 = x^1(1 \leftrightarrow k_q)_3$ where $k_q \in K \setminus \{1, k_2, k_3, n\}$. Further, for $n \geq 6$, there exists $l_q \in L \setminus \{1, l_1, l_2, l_3, n\}$ such that $x^3 = x^2(1 \leftrightarrow l_q)_4$, $x^4 = x^1(1 \leftrightarrow l_q)_4$. Then, $X^m((1,i_q)_1;(1,j_q)_2) \in P_I(Q)$ since $x_{nnnn} = 1$, for all $x \in X^m$, for all $m = 1,\ldots,4$. $x^1((1,i_q)_1;(1,j_q)_2) - x^2((1,i_q)_1;(1,j_q)_2)$ yields (4.7)$[i_q, j_q, k_q, 1]$. By symmetry, we obtain (4.7)$[i_q, j_q, 1, l_q]$, (4.7)$[i_q, 1, k_q, l_q]$, (4.7)$[1, j_q, k_q, l_q]$.

$x^3((1,i_q)_1;(1,j_q)_2) - x^4((1,i_q)_1;(1,j_q)_2)$ yields

$$a_{i_q j_q k_q l_q} = (a_{i_q j_q 1 l_q} + a_{i_q 1 k_q l_q} + a_{1 j_q k_q l_q}) - a_{i_q 1 1 l_q} - a_{1 j_q 1 l_q} - a_{1 1 k_q l_q} + a_{1 1 1 l_q}$$

Substituting terms in brackets from (4.7)$[i_q, j_q, 1, l_q]$, (4.7)$[i_q, 1, k_q, l_q]$, (4.7)$[1, j_q, k_q, l_q]$ and cancelling out identical terms, we obtain (4.7)$[i_q, j_q, k_q, l_q]$

Case 4.4.2 One of i, j, k, l is equal to n.

Consider $(i,j,k,l) = (i_q, j_q, n, l_q)$ where $i_q \neq n, j_q \neq n, l_q \neq n$. Let $l_q \in L \setminus \{1, l_1, l_2, l_3, n\}$. Then, $x^1 = x^{\mathrm{Tbl}(2)}(l_1 \leftrightarrow l_q)_4$ $x^2 = x^1(1 \leftrightarrow n)_3(i_1 \leftrightarrow n)_1(j_1 \leftrightarrow n)_2$. Also let $\hat{x}^1 = x^1(l_1 \leftrightarrow l_q)_4$ and $\hat{x}^2 = x^2(l_1 \leftrightarrow l_q)_4$. Then, $X^m((1,i_q)_1;(1,j_q)_2), \hat{X}^m((1,i_q)_1;(1,j_q)_2) \in P_I(Q)$ since $x_{nnnn} = 1$, for all $x \in X^m, \hat{X}^m$, for all $m = 1,\ldots,4$.

$x^1((1,i_q)_1;(1,j_q)_2) - x^2((1,i_q)_1;(1,j_q)_2)$ yields

$$\begin{aligned} & a_{1111} + a_{1 j_q n 1} + a_{i_q 1 n 1} + a_{i_q j_q 1 1} - (a_{i_q j_q n 1} + a_{i_q 1 1 1} + a_{1 j_q 1 1} + a_{1 1 n 1}) \\ = \quad & a_{i_q j_q n l_q} + a_{i_q 1 1 l_q} + a_{1 j_q 1 l_q} + a_{1 1 n l_q} - (a_{i_q j_q 1 l_q} + a_{i_q 1 n l_q} + a_{1 j_q n l_q} + a_{1 1 1 l_q}) \end{aligned} \tag{4.8}$$

Adding $\hat{x}^1((1,i_q)_1;(1,j_q)_2) - \hat{x}^2((1,i_q)_1;(1,j_q)_2)$ to (4.8) yields

$$\begin{aligned} & 2(a_{1111} + a_{1 j_q n 1} + a_{i_q 1 n 1} + a_{i_q j_q 1 1} - (a_{i_q j_q n 1} + a_{i_q 1 1 1} + a_{1 j_q 1 1} + a_{1 1 n 1})) \\ = \quad & \sum_{l \in \{l_1, l_q\}} \{a_{i_q j_q n l} + a_{i_q 1 1 l} + a_{1 j_q 1 l} + a_{1 1 n l} - (a_{i_q j_q 1 l} + a_{i_q 1 n l} + a_{1 j_q n l} + a_{1 1 1 l})\} \end{aligned} \tag{4.9}$$

Let $x^3 = x^2(1 \leftrightarrow l_q)_4(1 \leftrightarrow l_1)_4$, $x^4 = x(1 \leftrightarrow l_q)_4$. Again $X^m((1,i_q)_1;(1,j_q)_2) \in P_I(Q)$ since $x_{nnnn} = 1$, for all $x \in X^m$, for $m = 3, 4$. $x^3((1,i_q)_1;(1,j_q)_2) - x^4((1,i_q)_1;(1,j_q)_2)$ leads the right-hand side of (4.8) to zero, thus proving (4.7)$[i_q, j_q, n, 1]$. Observe that this result is valid independently of whether $l_1 = l_3$ or $l_1 \neq l_3$ at point $x^{\mathrm{Tbl}(2)}$. For each case we obtain different points for the collections X^1, X^2, X^3, X^4, but the operations performed yield the same result. (4.7)$[i_q, 1, n, l_q]$, (4.7)$[1, j_q, n, l_q]$ follow by symmetry.

By virtue of (4.7)$[i_q, j_q, n, 1]$, the right-hand side of (4.8) is equal to zero. Thus, by substituting term $a_{i_q j_q 1 l_q}$ from (4.7)$[i_q, j_q, 1, l_q]$, $a_{i_q 1 n l_q}$ from (4.7)$[i_q, 1, n, l_q]$ and $a_{1 j_q n l_q}$ from (4.7)$[1, j_q, n, l_q]$, we obtain (4.7)$[i_q, j_q, n, l_q]$.

Taking the conjugates of the above used points with respect to the interchange $(K \leftrightarrow L)$, we obtain

(4.7)$[i_q, j_q, k_q, n,]$. (4.7)$[i_q, n, k_q, l_q]$, (4.7)$[n, , j_q, k_q, l_q]$ are obtained by considering the conjugate points derived from the interchanges $(J \leftrightarrow K)$ and $(I \leftrightarrow K)$, respectively.

Case 4.4.3 Two of i, j, k, l are equal to n.

Observe that the remaining two indices cannot simultaneously take values from the tuple (i_0, j_0, k_0, l_0), because then the tuple under consideration would belong to Q. Hence, assume $(i, j, k, l) = (n, n, k_q, l_q)$ with $k_q \neq n, l_q \neq l_0, n$.

Consider the point x illustrated in the Table 4, where $k_1 \in K \setminus \{1, k_0, n\}$. It is easy to establish

Table 4: Point x (Theorem 4.4, Case 4.4.3)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0				(n, l_0)
$\vdots$				
i_1				(k_1, n)
$\vdots$				
n		(n, n)		$(1, 1)$

existence of this point, for $n \geq 4$. Let $\tilde{x} = \mathring{x}(l(1, n) \neq l_0 ? l(1, n) \leftrightarrow l_0)_4 (1 \leftrightarrow n)_2 (1 \leftrightarrow n)_1 (n \leftrightarrow l_0)_4$. Therefore, $x_{i_1 n k_{(i_1, n)} n} = 1$. If $k(i_1, n) \neq k_0$ then we denote it k_1 and obtain point x as illustrated in Table 4. Otherwise, for $n \geq 4$ there exists $k_1 \in K \setminus \{1, k_0, n\}$ such that $x = \tilde{x}(k_1 \leftrightarrow k_0)_3$.

Let $x^1 = x^{\text{Tbl}(4)}$, $x^2 = x^1(1 \leftrightarrow k_q)_3$ where $k_q \in K \setminus \{1, k_1, n\}$. Observe that $x^m \in P_I(Q)$, because $x^m_{n j_0 n n} = x^m_{i_0 n n l_0} = 1$, for $m = 1, 2$. Expressing indices of the sets J, L in terms of indices of the sets I, K in $ax^1 = ax^2$ yields

$$\begin{aligned} & a_{nn11} + \sum_{i \in I \setminus \{n\}} a_{ij(i,1)1l(i,1)} + \sum_{i \in I} a_{ij(i,k_q)k_q l(i,k_q)} \\ = \quad & a_{nnk_q1} + \sum_{i \in I \setminus \{n\}} a_{ij(i,1)k_q l(i,1)} + \sum_{i \in I} a_{ij(i,k_q)1l(i,k_q)} \end{aligned} \tag{4.10}$$

All terms in (4.10), except a_{nnk_q1}, a_{nn11}, have at most one index equal to n. For these terms, (4.3) is proven in one of the previous cases through (4.7). Therefore, solving (4.10) with respect to term a_{nnk_q1}, substituting terms in the summands from (4.3), adding and subtracting λs so that the summation index

i runs for all values of the set I and cancelling out identical terms, (4.10) yields

$$\begin{aligned} a_{nnk_q1} &= a_{nn11} \\ &+\sum_{i\in I}\{\lambda^1_{1l(i,1)}+\lambda^1_{k_ql(i,k_q)}-\lambda^1_{k_ql(i,1)}-\lambda^1_{1l(i,k_q)}\} \\ &+\sum_{i\in I}\{\lambda^5_{j(i,1)1}+\lambda^5_{j(i,k_q)k_q}-\lambda^5_{j(i,1)k_q}-\lambda^5_{j(i,k_q)1}\} \\ &+\lambda^1_{k_ql(n,1)}+\lambda^5_{j(n,1)k_q}+\lambda^6_{nk_q} \\ &-\lambda^1_{1l(n,1)}-\lambda^5_{j(n,1)1}-\lambda^6_{n1} \end{aligned} \tag{4.11}$$

It is easy to see that summands cancel out, i.e. $\sum_{i\in I}\lambda^1_{1l(i,1)}=\sum_{i\in I}\lambda^1_{1l(i,k_q)}$, $\sum_{i\in I}\lambda^1_{k_ql(i,k_q)}=\sum_{i\in I}\lambda^1_{k_ql(i,1)}$, $\sum_{i\in I}\lambda^5_{j(i,1)1}=\sum_{i\in I}\lambda^5_{j(i,k_q)1}$, $\sum_{i\in I}\lambda^5_{j(i,1)k_q}=\sum_{i\in I}\lambda^5_{j(i,k_q)k_q}$. This is because the n-tuple $(j(i,1)_{i\in I})$ $((l(i,1)_{i\in I}))$ is an array containing the elements $1,\ldots,n$ in some order. The same is true for $(j(i,k_q)_{i\in I})$ $((l(i,k_q)_{i\in I}))$. If we consider the n-tuples as unordered, we have $\bigcup_{i\in I}\{j(i,1)\}=\bigcup_{i\in I}\{j(i,k_q)\}\equiv J$ and $\bigcup_{i\in I}\{l(i,1)\}=\bigcup_{i\in I}\{l(i,k_q)\}\equiv L$.

Also, observe that (4.7)$[n,n,1,1]$ is valid because tuple $(n,n,1,1)$ has two indices equal to one. Substituting term a_{nn11} from (4.7), (4.11) becomes

$$\begin{aligned} a_{nnk_q1} &= \lambda^1_{11}+\lambda^2_{n1}+\lambda^3_{n1}+\lambda^4_{nn}+\lambda^5_{n1}+\lambda^6_{n1} \\ &+\lambda^1_{k_ql(n,1)}+\lambda^5_{j(n,1)k_q}+\lambda^6_{nk_q} \\ &-\lambda^1_{1l(n,1)}-\lambda^5_{j(n,1)1}-\lambda^6_{n1} \end{aligned}$$

Since $l(n,1)=1, j(n,1)=n$ (Table 4), the above equation becomes

$$a_{nnk_q1}=\lambda^1_{k_q1}+\lambda^2_{n1}+\lambda^3_{n1}+\lambda^4_{nn}+\lambda^5_{nk_q}+\lambda^6_{nk_q}$$

which is (4.7)$[n,n,k_q,1]$.

Let $x^3=x^2(1\leftrightarrow l_q)_4$ where $l_q\in L\setminus\{1,l_0,l_1,n\}$. $x^3\in P_I(Q)$. Expressing indices of the sets J,K in respect to the indices of sets I,L, $ax^2=ax^3$ yields

$$\begin{aligned} & a_{nnk_ql_q}+\sum_{i\in I\setminus\{n\}}a_{ij(i,1)k(i,1)l_q}+\sum_{i\in I}a_{ij(i,k_q)k(i,l_q)1} \\ = \;& a_{nnk_q1}+\sum_{i\in I\setminus\{n\}}a_{ij(i,1)k(i,1)1}+\sum_{i\in I}a_{ij(i,k_q)k(i,l_q)l_q} \end{aligned} \tag{4.12}$$

Equation (4.12), dealt with in a manner analogous to that of (4.11), yields (4.7)$[n,n,k_q,l_q]$. By symmetry we obtain the same equation but with $k_q\neq k_0,n$, $l_q\neq n$.

Applying the same procedure to the points $x^m(J\leftrightarrow K)$, $x^m(I\leftrightarrow K)$, $x^m(J\leftrightarrow L)$, $x^m(I\leftrightarrow L)$, for $m = 1, 2, 3$, yields (4.7)$[n, j_q, n, l_q]$, (4.7)$[i_q, n, n, l_q]$, (4.7)$[n, j_q, k_q, n]$, (4.7)$[i_q, n, k_q, n]$, respectively. Finally, the same procedure applied to points $x^m(I \leftrightarrow L)(J \leftrightarrow K)$, for $m = 1, 2, 3$, yields (4.7)$[i_q, j_q, n, n]$. Observe that all points used belong to $P_I(Q)$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we define

$$\pi_{ijkl} = a_{ijkl} - (\lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik}), \forall (i, j, k, l) \in Q \tag{4.13}$$

We must show that all π_{ijkl} are equal. We do this in a series of steps.

Step1: Consider the point x illustrated in Table 5. It is easy to establish existence for this point. At

Table 5: Point x (Theorem 4.4, Step 1)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0						(n, n)
$\vdots$						
n		(n, l_0)		(k_0, n)		(k_q, l_q)

point $\mathring{x}$ let $i_1 \in I \setminus \{1, n\}$ be such that $k(i_1, 1) = k_0$, $l(i_1, 1) = l_1$. Then, $\hat{x} = \mathring{x}(l_1 \leftrightarrow n)_4(l_1 \neq l_0 ? l_1 \leftrightarrow l_0)_4(1 \leftrightarrow n)_2(n \leftrightarrow i_0)_1$. Observe that pair $(n, n) \in K \times L$ does not lie at rows i_0, i_1 and column n. Thus, we can place it to row n and column j_0 by performing the necessary row and/or column interchange, without affecting the positions of pairs (n, l_0) (row i_0, column n) and pairs (k_0, n) (row i_1, column n). Thus, we derive point $\bar{x}$. We denote $k(\bar{x}; n, n)$ as k_q, $l(\bar{x}; n, n)$ as l_q. Then, $x = \bar{x}(I \leftrightarrow J)$. Observe that $k(x; n, n) = k_q$, $l(x; n, n) = l_q$.

Let $x^1 = x^{\mathrm{Tbl}(5)}$ and $x^2 = x^1(j_1 \leftrightarrow j_0)_2$. $x^1, x^2 \in P_I(Q)$ because $x^1_{i_0nnn} = x^1_{nj_0nl_0} = 1$ and $x^2_{i_0nnn} = x^2_{nj_0k_0n} = 1$. $ax^1 = ax^2$ yields

$$\begin{aligned} & a_{nj_0nl_0} + \sum_{i \in I \setminus \{n\}} a_{ij_0k(i,j_0)l(i,j_0)} + a_{nj_1k_0n} + \sum_{i \in I \setminus \{n\}} a_{ij_1k(i,j_1)l(i,j_1)} \\ = \; & a_{nj_0k_0n} + \sum_{i \in I \setminus \{n\}} a_{ij_0k(i,j_1)l(i,j_1)} + a_{nj_1nl_0} + \sum_{i \in I \setminus \{n\}} a_{ij_1k(i,j_0)l(i,j_0)} \end{aligned}$$

If we substitute the first terms of both sides from (4.13), the rest of the terms from (4.7), and cancel out identical terms, we obtain $\pi_{nj_0k_0n} = \pi_{nj_0nl_0} = \pi$.

Next we consider point $x^3 = x^1(I \leftrightarrow J)$ and let $x^4 = x^3(i_1 \leftrightarrow i_0)_1$. Then $x^3, x^4 \in P_I(Q)$ because $x^3_{nj_0nn} = x^3_{i_0nnl_0} = 1$ and $x^4_{nj_0nn} = x^4_{i_0nk_0n} = 1$. Then $ax^3 = ax^4$ yields $\pi_{i_0nnl_0} = \pi_{i_0nk_0n}$.

Let $x^5 = x^1(k_0 \leftrightarrow k_q)_3$ and $x^6 = x^5(l_q \leftrightarrow l_0)_4$. Then $x^5, x^6 \in P_I(Q)$ because $x^5_{i_0nnn} = x^5_{nj_0nl_0} = 1$ and $x^6_{i_0nnn} = x^6_{nnk_0l_0} = 1$. Then $ax^5 = ax^6$ yields $\pi_{nj_0nl_0} = \pi_{nnk_0l_0}$. Let $x^7 = x^3(k_0 \leftrightarrow k_q)_3$ and $x^8 = x^7(l_q \leftrightarrow l_0)_4$. Then $x^7, x^8 \in P_I(Q)$ because $x^7_{nj_0nn} = x^7_{i_0nnl_0} = 1$ and $x^8_{nj_0nn} = x^8_{nnk_0l_0} = 1$. $ax^7 = ax^8$ yields $\pi_{i_0nnl_0} = \pi_{nnk_0l_0}$.

Hence,

$$\pi_{i_0nk_0n} = \pi_{nj_0k_0n} = \pi_{i_0nnl_0} = \pi_{nj_0nl_0} = \pi_{nnk_0l_0} = \pi$$

Step 2: Consider the point x illustrated in Table 6. It is easy to establish existence for this point. At

Table 6: Point x (Theorem 4.4, Step 2)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0						
$\vdots$						
n		(n,n)		(k_1,l_1)		(k_0,l_0)

point $\mathring{x}$ let $i_1 \in I \setminus \{1,n\}$ be such that $k(i_1,1) = k_0$. We denote $l(i_1,1)$ as l_2. Then $\hat{x} = \mathring{x}(l_2 \neq l_0?l_2 \leftrightarrow l_0)_4(i_0 \leftrightarrow n)_1$. Let $i_1 \in I \setminus \{1,n\}$ be such that $k(i_1,1) = k_1$. We denote $l(\hat{x}; i_1, 1)$ as l_1. Let $\bar{x} = \hat{x}(1 \leftrightarrow n)_2$. Then point $x = \bar{x}(I \leftrightarrow J)$.

Let $x^1 = x^{\text{Tbl}(6)}$ and $x^2 = x^1(j_0 \leftrightarrow j_1)_2$. Observe that $x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nn} = x^1_{nnk_0l_0} = 1$ and $x^2_{nj_1nn} = x^2_{nnk_0l_0} = 1$. Thus, $ax^1 = ax^2$ implies

$$\begin{aligned} & a_{nj_0nn} + \sum_{i \in I\setminus\{n\}} a_{ij_0k(i,j_0)l(i,j_0)} + a_{nj_1k_1l_1} + \sum_{i \in I\setminus\{n\}} a_{ij_1k(i,j_1)l(i,j_1)} \\ = \quad & a_{nj_1nn} + \sum_{i \in I\setminus\{n\}} a_{ij_1k(i,j_0)l(i,j_0)} + a_{nj_0k_1l_1} + \sum_{i \in I\setminus\{n\}} a_{ij_0k(i,j_1)l(i,j_1)} \end{aligned}$$

If we substitute the first terms of both sides from (4.13), the rest of the terms from (4.7), and cancel out identical terms, we obtain $\pi_{nj_0nn} = \pi_{nj_1nn} = \pi^2$.

Next consider the point $x^3 = x^1(I \leftrightarrow J)$ and let $x^4 = x^3(i_0 \leftrightarrow i_1)_1$. $x^3, x^4 \in P_I(Q)$ because $x^3_{i_0nnn} = x^3_{nnk_0l_0} = 1$, $x^4_{i_1nnn} = x^4_{nnk_0l_0} = 1$. Then $ax^3 = ax^4$ yields $\pi_{i_0nnn} = \pi_{i_1nnn} = \pi^1$.

Let $x^5 = x^1(j_1 \neq j_0?j_1 \leftrightarrow j_0)_2(l_0 \leftrightarrow n)_4$ and $x^6 = x^5(k_1 \leftrightarrow k_0)_3$ where $k_1 \in K \setminus \{k_0, n\}$. $x^5, x^6 \in P_I(Q)$ because $x^5_{nj_0nl_0} = x^5_{nnk_0n} = 1$, $x^6_{nj_0nl_0} = x^6_{nnk_1n} = 1$. Thus, $ax^5 = ax^6$ yields $\pi_{nnk_0n} = \pi_{nnk_1n} = \pi^3$.

In a similar manner, let $x^7 = x^1(j_1 \neq j_0?j_1 \leftrightarrow j_0)_2(k_0 \leftrightarrow n)_3$ and $x^8 = x^7(l_1 \leftrightarrow l_0)_4$ where $l_1 \in L \setminus \{l_0, n\}$. $x^7, x^8 \in P_I(Q)$ because $x^7_{nj_0k_0n} = x^7_{nnnl_0} = 1$, $x^8_{nj_0k_0n} = x^8_{nnnl_1} = 1$. Thus, $ax^7 = ax^8$ yields $\pi_{nnnl_0} = \pi_{nnnl_1} = \pi^4$.

Hence,

$$\begin{aligned}
\pi_{i_0nnn} &= \pi_{i_1nnn} = \pi^1, \forall i_1 \in I \setminus \{i_0, n\}, \\
\pi_{nj_0nn} &= \pi_{nj_1nn} = \pi^2, \forall j_1 \in J \setminus \{j_0, n\}, \\
\pi_{nnk_0n} &= \pi_{nnk_1n} = \pi^3, \forall k_1 \in K \setminus \{k_0, n\}, \\
\pi_{nnnl_0} &= \pi_{nnnl_1} = \pi^4, \forall l_1 \in L \setminus \{l_0, n\}
\end{aligned}$$

Step 3: Let $x^1 = x^{\mathrm{Tbl}(5)}(j_1 \leftrightarrow n)_2$ and $x^2 = x^1(i_0 \leftrightarrow n)_1$. $x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nl_0} = x^1_{nnk_0n} = 1$ and $x^2_{nj_1nn} = x^2_{i_0nk_0n} = 1$. Thus, $ax^1 = ax^2$ implies

$$\begin{aligned}
& a_{nnk_0n} + a_{nj_0nl_0} + \sum_{j \in J \setminus \{j_0, n\}} a_{njk(n,j)l(n,j)} + a_{i_0j_1nn} + \sum_{j \in J \setminus \{j_1\}} a_{i_0jk(i_0,j)l(i_0,j)} \\
= \quad & a_{nj_1nn} + a_{i_0nk_0n} + \sum_{j \in J \setminus \{j_1\}} a_{njk(i_0,j)l(i_0,j)} + a_{i_0j_0nl_0} + \sum_{j \in J \setminus \{j_0, n\}} a_{i_0jk(n,j)l(n,j)}
\end{aligned}$$

Substituting the first two terms of both sides from (4.13) and the rest of the terms from (4.7), and taking into account the results obtained in the previous two steps, we obtain $\pi_{nnk_0n} = \pi_{nj_1nn} \Rightarrow \pi^3 = \pi^2$.

Let $x^3 = x^{\mathrm{Tbl}(5)}, x^4 = x^3(n \leftrightarrow l_0)_4$. $x^3, x^4 \in P_I(Q)$ because $x^3_{nj_0nl_0} = x^3_{i_0nnn} = 1$ and $x^4_{i_0nnl_0} = x^4_{nj_0nnn} = 1$. $ax^3 = ax^4$ yields $\pi_{nj_0nl_0} + \pi_{i_0nnn} = \pi_{i_0nnl_0} + \pi_{nj_0nnn}$ which by Step 1 results in $\pi_{i_0nnn} = \pi_{nj_0nnn} \Rightarrow \pi^1 = \pi^2$.

Finally, let $x^5 = x^3(j_0 \leftrightarrow n)_2$ and $x^6 = x^5(j_1 \leftrightarrow n)_2$. $x^5, x^6 \in P_I(Q)$ because $x^5_{i_0j_0nn} = x^5_{nnnl_0} = 1$, $x^6_{i_0j_0nn} = x^6_{nnk_0n} = 1$. $ax^5 = ax^6$ yields $\pi^3 = \pi^4$.

Hence,

$$\pi^1 = \pi^2 = \pi^3 = \pi^4 = \kappa$$

Step 4: Consider the point x illustrated in Table 7. It is easy to see that this point exists. At an

Table 7: Point x (Theorem 4.4, Step 4)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0				(k_0, n)		
$\vdots$						
i_1						(n, n)
$\vdots$						
n		(n, l_0)				

arbitrary point $x \in P_I$ pairs $(n, l_0), (n, n)$ cannot lie at the same row or column. Thus we can easily derive a point $\hat{x}$ such that $\hat{x}_{nj_0nl_0} = \hat{x}_{i_1nnn} = 1$. Also let $\hat{x}_{i_pj_pk_0n} = 1$. Clearly $j_p \neq n, i_p \neq i_1$. If $i_p \neq n$ and $j_p \neq j_0$ then $x = \hat{x}(i_p \neq i_0?i_p \leftrightarrow i_0)_1(j_p \neq j_1?j_p \leftrightarrow j_1)_2$. Otherwise assume that $i_p = n$. Then there exist $i_2 \in I \setminus \{i_1, n\}$, $j_2 \in J \setminus \{j_0, j_p, n\}$, $k_1 \in K \setminus \{k_0, n\}$ such that $\hat{x}_{i_2j_2k_1n} = 1$. Then $x = \hat{x}(i_2 \neq i_0?i_2 \leftrightarrow i_0)_1(j_2 \neq j_1?j_2 \leftrightarrow j_1)_2(k_1 \leftrightarrow k_0)_3$.

Let $x^1 = x^{\text{Tbl}(7)}$ and $x^2 = x^1(j_1 \leftrightarrow n)_2$. Note that $x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nl_0} = x^1_{i_1nnn} = 1$ and $x^2_{nj_0nl_0} = x^2_{i_0nk_0n} = 1$. $ax^1 = ax^2$ yields

$$\begin{aligned} & a_{i_1nnn} + \sum_{i \in I \setminus \{i_1\}} a_{ink(i,n)l(i,n)} + a_{i_0j_1k_0n} + \sum_{i \in I \setminus \{i_0\}} a_{ij_1k(i,j_1)l(i,j_1)} \\ = \quad & a_{i_0nk_0n} + \sum_{i \in I \setminus \{i_1\}} a_{ij_1k(i,n)l(i,n)} + a_{i_1j_1nn} + \sum_{i \in I \setminus \{i_0\}} a_{ink(i,j_1)l(i,j_1)} \end{aligned}$$

Substituting the first terms of both sides from (4.13), the rest of the terms from (4.7), and cancelling equivalent terms, we obtain $\pi_{i_1nnn} = \pi_{i_0nk_0n} \Rightarrow .\pi^1 = \pi \Rightarrow \kappa = \pi$.

Step 5: Let $x^1 = x^{\text{Tbl}(5)}$ and $x^2 = x^1(j_0 \leftrightarrow n)_2$. $x^2 \in P_I(Q)$ because $x^2_{i_0j_0nn} = x^2_{nnnl_0} = 1$. $ax^1 = ax^2$ yields $\pi_{i_0nnn} + \pi_{nj_0nl_0} = \pi_{i_0j_0nn} + \pi_{nnnl_0}$ which due to the previous steps implies $\pi_{i_0j_0nn} = \pi_{nj_0nl_0} = \pi$ resulting in $\kappa = \pi$.

The proof of $(4.3)_\pi$ is complete.

To show $(4.3)_{2\pi}$, we define

$$\pi_{nnnn} = a_{nnnn} - (\lambda^1_{nn} + \lambda^2_{nn} + \lambda^3_{nn} + \lambda^4_{nn} + \lambda^5_{nn} + \lambda^6_{nn}) \tag{4.14}$$

It remains to show that $\pi_{nnnn} = 2\pi$. Consider the point x illustrated in Table 8. Let $x^1 = x^{\text{Tbl}(8)}$

Table 8: Point x (Theorem 4.4, $(i, j, k, l) = (n, n, n, n)$)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0		(n, l_0)		
$\vdots$				
n				(n, n)

$x^2 = x^1(j_0 \leftrightarrow n)_2$. Note that $x^1, x^2 \in P_I(Q)$ because $x^1_{nnnn} = 1$ and $x^2_{nj_0nn} = x^2_{i_0nnl_0} = 1$. Thus, $ax^1 = ax^2$, after substituting terms from (4.7), (4.13), (4.14) and taking into account the results of steps 1-4, yields $\pi_{nnnn} = \pi_{nj_0nn} + \pi_{i_0nnl_0} = 2\pi$.

Finally, we have established that $P_I(Q) \neq \emptyset$ if $P_I \neq \emptyset$ which is true for $n \geq 7$ ([7, Thm. 2.9]). Therefore, for every $x \in P_I(Q)$, the constraints defining the A matrix of the OLS problem are satisfied.

Thus, by multiplying each row of A by the corresponding λ and (4.2) by π and summing over all rows we obtain

$$\begin{aligned} ax &= \sum_{k\in K, l\in L} \lambda^1_{kl} + \sum_{i\in I, l\in L} \lambda^2_{il} + \sum_{j\in J, l\in L} \lambda^3_{jl} \\ &+ \sum_{i\in I, j\in J} \lambda^4_{ij} + \sum_{j\in J, k\in K} \lambda^5_{jk} + \sum_{i\in I, k\in K} \lambda^6_{ik} + 2\pi \end{aligned}$$

which proves (4.6). ■

In the proofs that follow, we present only the parts in which they differ from the proof of Thm. 4.4. Specifically for the part of the proof regarding $(i,j,k,l) \in C \setminus (Q \cup \{(n,n,n,n)\})$ when at least two of the indices are equal to n, we establish the existence of two points, namely $x, y \in P_I(Q)$, such that (4.3) is valid for all terms, but one, of the equation $ax = ay$. To show (4.3) for this term, we follow a procedure similar to that of Case 4.4.3. Hereafter, we will briefly refer to this procedure as '$ax = ay$ yields (4.3)'. The same expression will be used when proving $(4.3)_\pi$ implying a procedure analogous to that followed in Steps 1-5. In this case, the terms of $ax = ay$ indexed by tuples $(i,j,k,l) \in Q$ are substituted from (4.13), whereas the rest from (4.3).

4.2 Wheel class num. 3

Let $p = 2$ and $t \in C$ such that $|c \cap t| = 2$, $|s \cap t| = 2$. We define $Q = (C^2(t) \cap C^3(c)) \cup (C^3(t) \cap C^3(c)) \cup (C^2(c) \cap C^3(t)) \cup (C^2(c) \cap C^2(s))$. For $c = (n,n,n,n), s = (i_0, j_0, k_0, l_0), t = (i_0, j_0, n, n)$, (4.1) becomes

$$\begin{aligned} 2x_{nnnn} + \sum_{i\in I\setminus\{n\}} x_{innn} + \sum_{j\in J\setminus\{n\}} x_{njnn} + \sum_{i\in I\setminus\{i_0,n\}} x_{ij_0nn} + \sum_{j\in J\setminus\{j_0,n\}} x_{i_0jnn} \\ + x_{i_0j_0nn} + x_{i_0nk_0n} + x_{nj_0k_0n} + x_{i_0nnl_0} + x_{nj_0nl_0} + x_{nnk_0l_0} \;\leq\; 2 \qquad (4.15) \end{aligned}$$

This inequality is induced by the wheel illustrated in Figure 5, which belongs to wheel class num. 3. Observe that there are n^4 choices for c, $(n-1)^4$ choices for s and $\binom{4}{2}$ choices for t.

Lemma 4.5 *(4.15) is maximal.*

Proof. Suppose that there exists a variable x_q ($q = (i_q, j_q, k_q, l_q) \in C$) which can be added to the left-hand side of (4.15) without increasing its right-hand side. At least two indices of x_q must be equal to n, since otherwise it can be set to one together with x_{nnnn}. Then $|q \cap (n,n,n,n)| = 3$ or 2. In the former case, $q \neq (i_q, n, n, n), (n, j_q, n, n)$, for all $i_q \in I, j_q \in J$, because in these cases x_q is already included in (4.15). Assume that $q = (n,n,n,l_q), l_q \in L$. Then we can simultaneously have $x_{nnnl_q} = x_{nj_0k_0n} = x_{i_0j_1nn} = 1$. In the latter case, we consider two further subcases with respect to the two indices not equal to n. Hence, values from the collection (i_0, j_0, k_0, l_0) are (a) forbidden, or (b) allowed but the two indices cannot obtain values from the collection simultaneously. In particular,

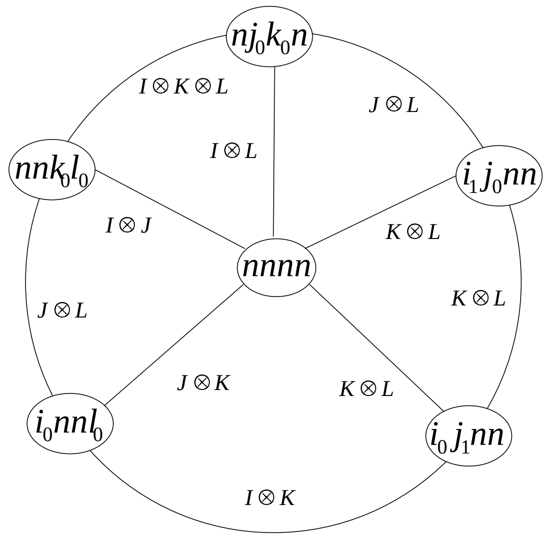

Figure 5: A wheel of class num. 3

(a) Assume $q = (i_q, j_q, n, n)$ with $i_q \in I \setminus \{i_0, n\}$ and $j_q \in J \setminus \{j_0, n\}$, because otherwise x_q is already included in (4.15). Then $x_{i_q j_q nn} = x_{i_0 nnl_0} = x_{nj_0k_0n} = 1$.

(b) Assume $q = (i_q, n, n, l_q)$. Observe that $i_q \in I \setminus \{n\}$, $l_q \in L \setminus \{n\}$, $(i_q, l_q) \neq (i_0, l_0)$, because otherwise x_q is already included in (4.15). Then, if $l_q \neq l_0$ consider $x_{i_q nnl_q} = x_{nj_0nn} = x_{nnk_0l_0} = 1$. In this case we can have $i_q = i_0$. If $i_q \neq i_0$ consider $x_{i_q nnl_q} = x_{nj_0nn} = x_{i_0nk_0n} = 1$, in which case we can have $l_q = l_0$.

■

Theorem 4.6 *For $n \geq 7$, (4.15) defines a facet of P_I.*

Proof. For tuples $(i,j,k,l) \in C \setminus (Q \cup \{(n,n,n,n)\})$ there is an additional case to these examined in the proof of Thm. 4.4, regarding the number of indices being equal to n. This case involves three indices equal to n. (4.3) for the cases where none or one of i,j,k,l, is equal to n, is shown in exactly the same way as in the cases 4.4.1, 4.4.2 of Thm. 4.4.

If two of the indices are equal to n, we consider three further cases, viz., $|(i,j,k,l) \cap (i_0, j_0, n, n)| = 0$ or 1 or 2. For the first case, $(i,j,k,l) = (n,n,k_q,l_q)$ and the proof of (4.3) is exactly the same as in case 4.4.3 of Thm. 4.4. The remaining two cases are considered analytically.

Case 4.6.1 $|(i,j,k,l) \cap (i_0, j_0, n, n)| = 1$.

Table 9: Point x (Theorem 4.6, Case 4.6.1)

	$\cdots$	j_0	$\cdots$	n
1				$(n,1)$
$\vdots$				
i_0				(k_0,n)
$\vdots$				
i_q				
$\vdots$				
n		(n,n)		$(1,l_1)$

Let $\hat{x} = x^{\mathrm{Tbl}(4)}(K \leftrightarrow L)$ and $x = \hat{x}(1 \leftrightarrow i_1)_1(1 \leftrightarrow l_1)_4$ (Table 9). Let $x^1 = x$ and $x^2 = x^1(1 \leftrightarrow i_q)_1$, $x^3 = x^2(1 \leftrightarrow l_q)_4$ where $i_q \in I \setminus \{1, i_0, n\}, l_q \in L \setminus \{1, l_1, n\}$. $x^m \in P_I(Q)$ because $x^m_{nj_0nn} = x^m_{i_0nk_0n} = 1$, for $m = 1,2,3$. Hence, $ax^1 = ax^2$ yields

$$
\begin{aligned}
& a_{1nn1} + \sum_{j\in J\setminus\{n\}} a_{1jk(1,j)l(1,j)} + \sum_{j\in J} a_{i_qjk(i_q,j)l(i_q,j)} \\
= \; & a_{i_qnn1} + \sum_{j\in J\setminus\{n\}} a_{i_qjk(1,j)l(1,j)} + \sum_{j\in J} a_{1jk(i_q,j)l(i_q,j)}
\end{aligned}
$$

All terms of the above equation, except a_{i_qnn1}, a_{1nn1}, have at most one index equal to n. For these terms, (4.3) has been proven previously. (4.3) is also valid for a_{1nn1}, because this term satisfies (4.7). Substituting these terms from (4.3) and following the same procedure as in Case 4.4.3 of Thm. 4.4, we obtain (4.3)$[i_q, n, n, 1]$. Substituting in $ax^2 = ax^3$ all terms including a_{i_qnn1} from (4.3) results (4.3)$[i_q, n, n, l_q]$. By symmetry, this equation is valid for $i_q \in I \setminus \{n\}, l_q \in L \setminus \{l_0, n\}$.

By applying the interchange $(K \leftrightarrow L)$ to the points defined above and following the same procedure, we obtain (4.3)$[i_q, n, k_q, n]$. In a similar manner, the interchange $(I \leftrightarrow J)$ yields (4.3)$[n, j_q, n, l_q]$. Finally, the interchanges $(I \leftrightarrow J)(K \leftrightarrow L)$ yield (4.3)$[n, j_q, k_q, n]$.

Case 4.6.2 $|(i,j,k,l) \cap (i_0, j_0, n, n)| = 2$.

Consider the point x illustrated in Table 10 It is easy to establish existence of point x. At any arbitrary integer point $\hat{x} \in P_I$, $(n,n) \in K \times L$ cannot lie in the same row or column with either of (n, l_0), (k_0, n). Thus, we can place (n,n) in the first row and column without affecting the positions of these two pairs. Let us denote this point as $\hat{x}$. If the two pairs do not lie in the same row or column it is trivial to obtain point x from $\hat{x}$. Therefore, assume that the two pairs lie in the same column, denoted by j_1. Additionally, suppose that (n, l_0) lies in row i_2 and let j_2 be such that $k(i_2, j_2) = k_2$, $l(i_2, j_2) = n$. Then for $n \geq 5$, there exists $k_1 \in K \setminus \{1, k_0, k_2, n\}$. Then, at point $\tilde{x} = \hat{x}(k_1 \leftrightarrow k_0)_3$ pairs (n, l_0), (k_0, n) lie in different rows and columns.

Table 10: Point x (Theorem 4.6, Case 4.6.2)

	1	$\cdots$	j_0	$\cdots$	n
1	(n,n)				
$\vdots$					
i_0					(n,l_0)
$\vdots$					
i_q					
$\vdots$					
n			(k_0,n)		

Let $x^1 = x^{\mathrm{Tbl}(10)}$ and let $x^2 = x^1(1 \leftrightarrow i_q)_1$, where $i_q \in I \setminus \{1, i_0, n\}$. $x^m \in P_I(Q)$ because $x^m_{nj_0k_0n} = x^m_{i_0nnl_0} = 1$, for $m = 1, 2$. Observe that all terms in equation $ax^1 = ax^2$, except a_{i_q1nn}, can be substituted from (4.3) yielding (4.3)$[i_q, 1, n, n]$. The substitution is valid even if $l(x^1; i_q, n) = n$. In this case, apart from a_{i_q1nn}, a_{11nn}, there are exactly two terms, each with two indices equal to n. For each of these terms (4.3) is shown to be valid in one of the previous cases.

Let $x^3 = x^2(1 \leftrightarrow j_q)_2$where $j_q \in J \setminus \{1, j_0, n\}$. $x^3 \in P_I(Q)$. Then, $ax^2 = ax^3$ yields (4.3)$[i_q, j_q, n, n]$.

The proof of (4.3) for the case where two of i, j, k, l are equal to n is complete.

For the case where three of the indices equal n, consider $x = x^{\mathrm{Tbl}(4)}(i_1 \leftrightarrow n)_1(k_q \leftrightarrow k_1)_3$ (Table 11).

Table 11: Point x (Theorem 4.6, three indices equal to n)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0				(n,l_0)
$\vdots$				
i_1		(n,n)		
$\vdots$				
i_q				(k_2,l_2)
$\vdots$				
n				(k_q,n)

Let $k(x; i_q, n) = k_2$ and $l(x; i_q, n) = l_2$ where $i_q \in I \setminus \{i_0, i_1, n\}, k_2 \in K \setminus \{k_q, n\}$, $l_2 \in L \setminus \{l_q, n\}$. Let $x^1 = x^{\mathrm{Tbl}(11)}$ and $x^2 = x^1(i_q \leftrightarrow n)_1$. $x^m \in P_I(Q)$ because $x^m_{i_1j_0nn} = x^m_{i_0nnl_0} = 1$, for $m = 1, 2$. $ax^1 = ax^2$

yields

$$\begin{aligned} & a_{nnk_qn} + \sum_{j\in J\setminus\{n\}} a_{njk(n,j)l(n,j)} + a_{i_qnk_2l_2} + \sum_{j\in J\setminus\{n\}} a_{i_qjk(i_q,j)l(i_q,j)} \\ = \quad & a_{nnk_2l_2} + \sum_{j\in J\setminus\{n\}} a_{njk(i_q,j)l(i_q,j)} + a_{i_qnk_qn} + \sum_{j\in J\setminus\{n\}} a_{i_qjk(n,j)l(n,j)} \end{aligned}$$

Observe that all terms, except a_{nnk_qn}, have at most two indices equal to n. Also observe that the tuples, indexing terms of the above equation, with two indices equal to n, belong to $C\setminus(Q\cup\{(n,n,n,n)\})$. Thus, substituting all terms, except a_{nnk_qn}, from (4.3), we obtain (4.3)$[n,n,k_q,n]$. Let $\hat{x}^m = x^m(K \leftrightarrow L)$, for $m=1,2$. $x^m \in P_I(Q)$ because $\hat{x}^m_{i_1j_0nn} = \hat{x}^m_{i_0nk_0n} = 1$, for $m=1,2$. $a\hat{x}^1 = a\hat{x}^2$ yields (4.3)$[n,n,n,l_q]$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we consider (4.13) and must show that all π_{ijkl} are equal.

Step 1: Exactly as in the corresponding step of the proof of Thm. 4.4.

Step 2: Exactly as in the corresponding step of the proof of Thm. 4.4 but limited for the cases $\pi_{i_0nnn} = \pi_{i_1nnn} = \pi^1, \forall i_1 \in I\setminus\{i_0,n\}$ and $\pi_{nj_0nn} = \pi_{nj_1nn} = \pi^2, \forall j_1 \in J\setminus\{j_0,n\}$.

Step 3: Exactly as in the corresponding step of the proof of Thm. 4.4 but limited for the cases $\pi_{i_0nnn} = \pi_{nj_0nn} \Leftrightarrow \pi^1 = \pi^2 = \kappa$.

Step 4: Exactly as in the corresponding step of the proof of Thm. 4.4.

Step 5: Let $x^1 = x^{\mathrm{Tbl}(5)}(j_1 \leftrightarrow n)_2$ where $j_1 \in J\setminus\{j_0,n\}$ and $x^2 = x^1(i_0 \leftrightarrow n)_1$. $x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nl_0} = x^1_{i_0j_1nn} = 1$ and $x^2_{i_0nk_0n} = x^2_{nj_1nn} = 1$. $ax^1 = ax^2$ yields $\pi_{nj_1nn} = \pi_{i_0j_1nn}, j_1 \in J\setminus\{j_0,n\}$. By symmetry, $\pi_{i_1nnn} = \pi_{i_1j_0nn}, i_1 \in I\setminus\{i_0,n\}$. Hence, $\pi_{i_0j_1nn} = \pi_{i_1j_0nn}, \forall i_1 \in I\setminus\{i_0,n\}, j_1 \in J\setminus\{j_0,n\}$, since by step 3 we have $\pi_{i_1nnn} = \pi_{nj_1nn}$.

Step 6: Let $i_2 \in I\setminus\{i_0,i_1,n\}$ be such that $k(x^{\mathrm{Tbl}(4)}; i_2, n) = k_0$. We denote $l(i_2,n)$ as l_2. Let $x^1 = x(l_2 \leftrightarrow l_0)_4(i_0 \leftrightarrow n)_1(i_2 \leftrightarrow n)_1$ and $x^2 = x^1(j_0 \leftrightarrow j_1)_2$, $j_1 \in J\setminus\{j_0,n\}$. $x^1, x^2 \in P_I(Q)$, because $x^1_{nnk_0l_0} = x^1_{i_0j_0nn} = 1$ and $x^2_{i_0j_1nn} = x^2_{nnk_0l_0} = 1$. $ax^1 = ax^2$ yields $\pi_{i_0j_0nn} = \pi_{i_0j_1nn}, j_1 \in J\setminus\{j_0,n\}$. If we consider $x^2 = x^1(i_0 \leftrightarrow i_1)_1$, $i_1 \in I\setminus\{i_0,n\}$, we get $\pi_{i_0j_0nn} = \pi_{i_1j_0nn}, i_1 \in I\setminus\{i_0,n\}$.

The proof of $(4.3)_\pi$ is complete. The rest of the proof is exactly as in Thm. 4.4. ∎

4.3 Wheel class num. 1 (22)

Let $p=2$ and consider $t_m \in C$, such that $|c\cap t_m| = |s\cap t_m| = 2$, for $m=1,2,3$, and $t_1\cap t_2\cap t_3 \subset c$. We define $Q = \cup_{m=1}^{m=2}((C^3(c)\cap C^3(t_m)) \cup (C^3(c)\cap C^2(t_m)) \cup (C^3(t_m)\cap C^2(c)) \cup \{t_m\}) \cup t_3$.

For $c=(n,n,n,n)$, $s=(i_0,j_0,k_0,l_0)$, $t_1=(i_0,n,n,l_0)$, $t_2=(n,j_0,n,l_0)$, $t_3=(i_0,j_0,n,n)$, (4.1) becomes (2.5). Obviously, the roles of the tuples t_1,t_2,t_3 are interchangeable. Hence, we can derive three inequalities for each such collection of three tuples. The inequality (2.5) is induced by either of the wheels illustrated in Figures 2a and 2b. There are n^4 choices for c, $(n-1)^4$ choices for s, and 4 different ways for defining t_1, t_2, t_3. Then every collection of t_1, t_2, t_3 yields three distinct inequalities.

Lemma 4.7 *(2.5) is maximal.*

Proof. Suppose that there exists a variable x_q $(q = (i_q, j_q, k_q, l_q) \in C)$ which can be added to the left-hand side of (2.5) without increasing its right-hand side. At least two indices of x_q must be equal to n, since otherwise it can be set to one together with x_{nnnn}. If $|q \cap (n,n,n,n)| = 3$ then for $q \notin Q$, we must have $q = (n, n, k_q, n), k_q \in K \setminus \{n\}$ in which case, x_{nnk_qn}, $x_{i_0nnl_1}$, $x_{nj_0nl_0}$ can all be set to one for $l_1 \in L \setminus \{l_0, n\}$. If $|q \cap (n,n,n,n)| = 2$ then the two indices of q having values different than n can either (a) obtain all other values in their domains, or (b) obtain all other values in their domains but the values i_0, j_0, k_0, l_0, simultaneously, or (c) obtain all other values in their domains but the values i_0, j_0, k_0, l_0, independently from each other.

(a) Let $q = (n, n, k_q, l_q), k_q \in K\setminus\{n\}$, $l_q \in L\setminus\{n\}$. Consider $x_{nnk_ql_q} = x_{nj_0nl_1} = x_{i_1nnn} = 1, i_1 \in I\setminus\{n\}$, $l_1 \in L \setminus \{l_q, n\}$.

(b) Let $q = (i_q, j_q, n, n), i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}, (i_q, j_q) \neq (i_0, j_0)$. If $j_q \neq j_0$ then $x_{i_qj_qnn} = x_{i_1nnl_0} = x_{nj_0nl_1} = 1, i_1 \in I \setminus \{i_q, n\}$. If $i_q \neq i_0$ then $x_{i_qj_qnn} = x_{i_0nnl_0} = x_{nj_1nl_1} = 1, j_1 \in J \setminus \{j_q, n\}$. In both cases $l_1 \in L \setminus \{l_0, n\}$.

(c) Let $q = (n, j_q, n, l_q,), j_q \in J \setminus \{j_0, n\}, l_q \in L \setminus \{l_0, n\}$. Consider $x_{nj_qnl_q} = x_{i_1nnl_0} = x_{i_0j_0nn} = 1, i_1 \in I \setminus \{j_0, n\}$.

■

Theorem 4.8 *For $n \geq 7$, (2.5) defines a facet of P_I.*

Proof. Let $\mathring{x}_{i_qj_qnl_0} = 1$. If $j_q \in J \setminus \{j_0, n\}$ then $\mathring{x} \in P_I \setminus P_I(Q)$. If $j_q = j_0$ or $j_q = n$ then there exists $j_1 \in J \setminus \{j_0, n\}$ such that $x = \mathring{x}(j_1 \leftrightarrow n)_2$. Then $x \in P_I \setminus P_I(Q)$ yielding $P_I \neq P_I(Q)$.

For $(i, j, k, l) \in C \setminus (Q \cup \{(n, n, n, n)\})$ such that $|(i, j, k, l) \cap (n, n, n, n)| = 2$, we consider cases analogous to (a), (b), (c) of Lem. 4.7.

Case 4.8.1 *(Analogous to (a)).*

For $(i, j, k, l) = (n, n, k_q, l_q), k_q \in K \setminus \{n\}, l_q \in L \setminus \{n\}$, if $k_q \neq k_0$ or $l_q \neq l_0$ then the proof is exactly the same as in the case 4.4.3 of Thm. 4.4. Observe that $x^1, x^2 \in P_I(Q)$.

In the case that $(k_q, l_q) = (k_0, l_0)$ let $x^3 = x^{\mathrm{Tbl}(4)}(l_1 \leftrightarrow l_0)_4(1 \leftrightarrow k_0)_3$, $x^4 = x^3(1 \leftrightarrow l_0)_4$. Then, $x^m \in P_I(Q)$, because $x^m_{nj_0nn} = x^m_{i_0nnl_1} = 1$, for $m = 3, 4$. Thus, $ax^3 = ax^4$ after substituting all terms except $a_{nnk_0l_0}$ from (4.3) yields (4.3)$[n, n, k_0, l_0]$.

By considering the conjugate points of x^m $(m = 1, \ldots, 4)$ with respect to the interchanges $(I \leftrightarrow L)$ and $(J \leftrightarrow L)$, we obtain (4.3)$[i_q, n, k_q, n]$ and (4.3)$[n, j_q, k_q, n]$, respectively.

Case 4.8.2 *(Analogous to (b)).*

Consider the point x illustrated in Table 12. To establish existence of $x^{\text{Tbl}(12)}$, let $i_q \in I \setminus \{i_0, i_1, n\}$

Table 12: Point x (Theorem 4.6, Case 4.8.2)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0						(n, l_1)
$\vdots$						
i_1						(k_1, n)
$\vdots$						
i_q		(n, n)				
$\vdots$						
n				(n, l_0)		$(\cdot, l_2)$

such that $\hat{x} = x^{\text{Tbl}(4)}(i_q \leftrightarrow n)_1$. Obviously there exists $j_1 \in J \setminus \{j_0, n\}$ such that $k(\hat{x}; n, j_1) = n$. We denote $l(\hat{x}; n, j_1)$ as l_1 and $l(\hat{x}; n, n)$ as l_2. Then $x^{\text{Tbl}(12)} = \hat{x}(l_1 \leftrightarrow l_0)_4$

Let $x^1 = x^{\text{Tbl}(12)}$ and $x^2 = x^1(l_q \leftrightarrow n)_4$ with $l_q \in L \setminus \{l_0, l_1, l_2, n\}$. $x^m \in P_I(Q)$, because $x^m_{nj_1nl_0} = x^m_{i_0nnl_1} = 1$, for $m = 1, 2$. By construction there is no term with three indices equal to n in the equation $ax^1 = ax^2$. Thus, by substituting all terms, except $a_{i_qj_0nn}$, from (4.3) we obtain (4.3)$[i_q, j_0, n, n]$. Let $x^3 = x^1(j_q \leftrightarrow j_0)_2$, $x^4 = x^2(j_q \leftrightarrow j_0)_2$ with $j_q \in J \setminus \{j_0, n\}$. $x^3, x^4 \in P_I(Q)$. $ax^3 = ax^4$ yields (4.3)$[i_q, j_q, n, n]$ for $j_q \neq j_0, n$. By symmetry, we obtain (4.3)$[i_q, j_q, n, n]$ for $i_q \in I \setminus \{n\}, j_q \in J \setminus \{j_0, n\}$.

Case 4.8.3 *(Analogous to (c)).*

Let $x^1 = x^{\text{Tbl}(12)}(i_0 \leftrightarrow i_q)_1(l_1 \leftrightarrow l_0)_4(l_1 \leftrightarrow l_q)_4(j_1 \leftrightarrow j_q)_2$, where $j_q \in J \setminus \{j_0, n\}, l_q \in L \setminus \{l_0, n\}$. Also let $x^2 = x^1(1 \leftrightarrow n)_1$. Then $x^m \in P_I(Q)$, because $x^m_{i_0j_0nn} = x^m_{i_qnnl_0} = 1$, for $m = 1, 2$. By construction $ax^1 = ax^2$ contains only one term with two of the indices equal to n (term $a_{nj_qnl_q}$), whereas the rest of the indices have at most one of the indices equal to n. Thus, by substituting all terms, except $a_{nj_qnl_q}$, from (4.3) we obtain (4.3)$[n, j_q, n, l_q], j_q \in J \setminus \{j_0, n\}, l_q \in L \setminus \{l_0, n\}$.

By considering the conjugates of x^1, x^2, with respect to the interchanges $(I \leftrightarrow J)$ and following the same procedure, we obtain (4.3)$[i_q, n, n, l_q]$.

The proof of (4.3) for the case where $|(i, j, k, l) \cap (n, n, n, n)| = 2$ is complete.

For the case where $|(i, j, k, l) \cap (n, n, n, n)| = 3$, consider $(i, j, k, l) = (n, n, k_q, n)$, $k_q \in K \setminus \{n\}$. Let $x^1 = x^{\text{Tbl}(4)}(i_0 \leftrightarrow n)_1(i_1 \leftrightarrow n)_1(k_1 \leftrightarrow k_q)_3$ and $x^2 = x^1(i_2 \leftrightarrow n)_1$. Then $x^m \in P_I(Q)$, because $x^m_{i_0j_0nn} = x^m_{i_1nnl_0} = 1$, for $m = 1, 2$. Substituting all terms in $ax^1 = ax^2$, except a_{nnk_qn}, from (4.3) yields (4.3)$[n, n, k_q, n]$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we consider (4.13) and must show that all π_{ijkl} are equal.

Step1: Let $i_1 \in I \setminus \{n\}$ be such that $k(x^{\text{Tbl}(6)}; i_1, n) = n$. We denote $l(i_1, n)$ as l_1. Then, let $x = x^{\text{Tbl}(6)}(l_0 \leftrightarrow l_1)_4(i_1 \neq i_0 ? i_1 \leftrightarrow i_0)_1$ (Table 13). Let $x^1 = x$, $x^2 = x^1(j_0 \leftrightarrow j_q)_2$ where $j_q \in J \setminus \{j_0, n\}$.

Table 13: Point x (Theorem 4.8, Step 1)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0						(n, l_0)
$\vdots$						
n		(n, n)		(k_1, l_1)		

$x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nn} = x^1_{i_0nnl_0} = 1$ and $x^2_{nj_qnn} = x^2_{i_0nnl_0} = 1$. $ax^1 = ax^2$ after substituting terms from (4.3) and (4.13), we obtain $\pi_{nj_0nn} = \pi_{nj_qnn} = \pi^2, j_q \in J \setminus \{j_0, n\}$.

By imposing the interchange $(I \leftrightarrow J)$ to both x^1, x^2, we obtain $\pi_{i_0nnn} = \pi_{i_qnnn} = \pi^1, i_q \in I \setminus \{i_0, n\}$.

In a analogous manner, let $x^3 = x^1(J \leftrightarrow L)$. Let $x^4 = x^3(l_q \leftrightarrow l_0)_4$ where $l_q \in L \setminus \{l_0, n\}$. $x^3, x^4 \in P_I(Q)$ because $x^3_{i_0j_0nn} = x^3_{nnnl_0} = 1$ and $x^4_{i_0j_0nn} = x^4_{nnnl_q} = 1$. $ax^3 = ax^4$ yields $\pi_{nnnl_0} = \pi_{nnnl_q} = \pi^4, l_q \in L \setminus \{l_0, n\}$.

Hence,

$$\begin{aligned}
\pi_{i_0nnn} &= \pi_{i_qnnn} = \pi^1, \forall i_q \in I \setminus \{i_0, n\}, \\
\pi_{nj_0nn} &= \pi_{nj_qnn} = \pi^2, \forall j_q \in J \setminus \{j_0, n\}, \\
\pi_{nnnl_0} &= \pi_{nnnl_q} = \pi^4, \forall l_q \in L \setminus \{l_0, n\}
\end{aligned}$$

Step 2: Consider point x^1 of Step 1. Let $x^2 = x^1(i_q \leftrightarrow i_0)_1, i_q \in I \setminus \{i_0, n\}$. $x^2 \in P_I(Q)$ because $x^2_{i_qnnl_0} = x^2_{nj_0nn} = 1$. Performing the usual substitutions in $ax^1 = ax^2$, we derive $\pi_{i_0nnl_0} = \pi_{i_qnnl_0}$. Then $a(x^1(I \leftrightarrow L)) = a(x^2(I \leftrightarrow L))$ yields $\pi_{i_0nnl_0} = \pi_{i_0nnl_q}$, $a(x^1(I \leftrightarrow J)) = a(x^2(I \leftrightarrow J))$ yields $\pi_{nj_0nl_0} = \pi_{nj_qnl_0}$. In an analogous manner we obtain $\pi_{nj_0nl_0} = \pi_{nj_0nl_q}$. Observe that all points used belong to $P_I(Q)$.

Hence,

$$\begin{aligned}
\pi_{i_0nnl_0} &= \pi_{i_qnnl_0} = \pi_{i_0nnl_q}, \forall i_q \in I \setminus \{i_0, n\}, l_q \in L \setminus \{l_0, n\}, \\
\pi_{nj_0nl_0} &= \pi_{nj_qnl_0} = \pi_{nj_0nl_q}, \forall j_q \in J \setminus \{j_0, n\}, l_q \in L \setminus \{l_0, n\}
\end{aligned}$$

Step 3: Consider point x^1 of Step 1. Let $x^3 = x^1(i_0 \leftrightarrow n)_1$ and $x^4 = x^3(j_q \leftrightarrow n)_2$, where $j_q \in J \setminus \{j_0, n\}$. Then $x^3, x^4 \in P_I(Q)$ because $x^3_{i_0j_0nn} = x^3_{nnnl_0} = 1$ and $x^4_{i_0j_0nn} = x^4_{nj_qnl_0} = 1$. Then $ax^3 = ax^4$ yields $\pi_{nnnl_0} = \pi_{nj_qnl_0}$. Then $a(x^3(J \leftrightarrow L)) = a(x^4(J \leftrightarrow L))$ yields $\pi_{nj_0nn} = \pi_{nj_0nl_q}$. Because $\pi_{nj_qnl_0} = \pi_{nj_0nl_q}$ (Step 2), we get $\pi_{nnnl_0} = \pi_{nj_0nn}$ which by Step 1 yields $\pi^2 = \pi^4$.

Consider the interchange $(I \leftrightarrow J)$ applied to all points of Step 3. Then applying the same procedure to the corresponding points, we get $\pi_{nnnl_0} = \pi_{i_q nnl_0}$ and $\pi_{i_0 nnn} = \pi_{i_0 nnl_q}$. These two equalities result in $\pi^1 = \pi^4$. Hence, in this step we have shown

$$\pi_{i_0 nnl_0} = \pi_{nj_0 nl_0} = \pi^1 = \pi^2 = \pi^4 = \pi$$

Step 4: Consider point x^1 of Step 1. Let $x^2 = x^1(i_0 \leftrightarrow n)_1$. $x^2 \in P_I(Q)$ because $x^2_{nj_0 nn} = x^2_{i_0 nnl_0} = 1$. Then $ax^1 = ax^2$ yields $\pi_{i_0 j_0 nn} + \pi_{nnnl_0} = \pi_{nj_0 nn} + \pi_{i_0 nnl_0}$ implying $\pi_{i_0 j_0 nn} = \pi$.

The proof of $(4.3)_\pi$ is complete. The rest of the proof is exactly as in Thm. 4.4. ■

4.4 Wheel class num. 23

Let $p = 2$ and $t, r \in C$ such that $|c \cap t| = 2$, $|s \cap t| = 2$ and $|c \cap r| = 1, |s \cap r| = 3$ and $|t \cap r| = 3$. We define $Q(c,s,t) = \{q \in (C^2(c) \cap C^2(s)) : |q \cap t| \geq 1\}$ and $Q(c,s,t,r) = \{q \in (C^2(c) \cap C^1(s)) : q \cap t = q \cap r\}$. Then, $Q = (C^3(c) \cap C^1(r)) \cup (C^3(c) \cap C^2(r)) \cup Q(c,s,t) \cup Q(c,s,t,r) \cup \{t\}$.

For $c = (n,n,n,n)$, $s = (i_0, j_0, k_0, l_0)$, $t = (i_0, j_0, n, n)$, $r = (i_0, j_0, n, l_0)$, (4.1) becomes

$$\begin{aligned} 2x_{nnnn} + \sum_{i \in I \setminus \{n\}} x_{innn} + \sum_{j \in J \setminus \{n\}} x_{njnn} + \sum_{l \in L \setminus \{n\}} x_{nnnl} + \sum_{l \in L \setminus \{l_0, n\}} x_{i_0 nnl} & \\ + \sum_{l \in L \setminus \{l_0, n\}} x_{nj_0 nl} + x_{i_0 j_0 nn} + x_{i_0 nk_0 n} + x_{nj_0 k_0 n} + x_{i_0 nnl_0} + x_{nj_0 nl_0} & \leq 2 \end{aligned} \tag{4.16}$$

The inequality 4.16 is induced by the wheel illustrated in Figure 6, which belongs to wheel class num. 23. There are n^4 choices for c, $(n-1)^4$ choices for 6, 6 choices for t and 2 for r.

Lemma 4.9 *(4.16) is maximal.*

Proof. Suppose that there exists a variable x_q ($q = (i_q, j_q, k_q, l_q) \in C$) which can be added to the left-hand side of (4.16) without increasing its right-hand side. At least two indices of x_q must be equal to n, since otherwise it can be set to one together with x_{nnnn}. If q has three indices equal to n then the proof is exactly the same as in the corresponding case of Lem. 4.7. If q has two indices equal to n then we consider cases analogous to (a), (b) of Lem. 4.7. Specifically for case (b), we consider two further subcases, viz. $q \notin Q(c,s,t,r)$ and $q \in Q(c,s,t,r)$.

(a) Exactly as in the corresponding case of Lem. 4.7.

(b.1) Let $q = (i_q, j_q, n, n), i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}, (i_q, j_q) \neq (i_0, j_0)$. If $j_q \neq j_0$ then $x_{i_q j_q nn} = x_{nj_0 k_0 n} = x_{nnnl_1} = 1, l_1 \in L \setminus \{n\}$. If $i_q \neq i_0$ then $x_{i_q j_q nn} = x_{i_0 nk_0 n} = x_{nnnl_1} = 1$.

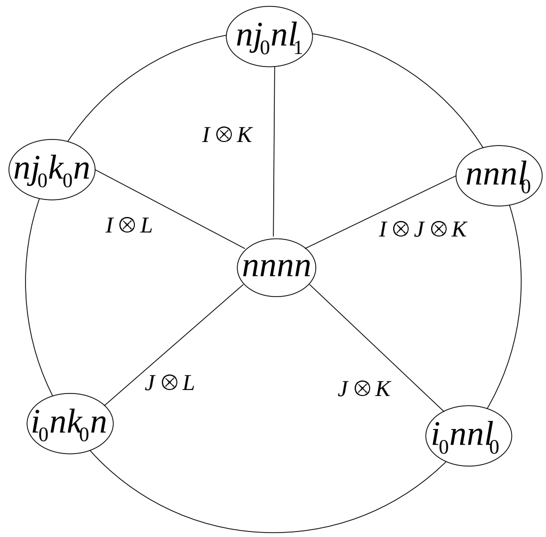

Figure 6: A wheel of class num. 23

(b.2) Let $q = (i_q, n, n, l_q), i_q \in I \setminus \{i_0, n\},\ l_q \in L \setminus \{n\}$. Consider $x_{i_q nnl_q} = x_{i_0nk_0n} = x_{nj_0nn} = 1$. Observe that we can have $l_q = l_0$.

■

Theorem 4.10 *For $n \geq 7$, (4.16) defines a facet of P_I.*

Proof. We denote $l(\mathring{x}; 1, n), k(\mathring{x}; n, n), l(\mathring{x}; n, n)$ as l_2, k_1, l_1, respectively. Let $x = \mathring{x}(k_1 \neq k_0 ? k_1 \leftrightarrow k_0)_3(l_1 \neq l_0 ? l_1 \leftrightarrow l_0)_4$. At point x among the variables participating in (4.16) only $\mathring{x}_{n1nn}$ is set to one. Therefore $x \in P_I \setminus P_I(Q)$ implying $P_I \neq P_I(Q)$.

For $(i, j, k, l) \in C \setminus (Q \cup \{(n, n, n, n)\})$ such that $|(i, j, k, l) \cap (n, n, n, n)| = 2$, we consider cases analogous to (a), (b.1), (b.2) of Lem. 4.9.

Case 4.10.1 *(Analogous to (a)).*

Consider $(i, j, k, l) = (n, n, k_q, l_q), k_q \in K \setminus \{n\}, l_q \in L \setminus \{n\}$. The proof is exactly the same as in case 4.8.1.

Case 4.10.2 *(Analogous to (b.1)).*

Let $\hat{x} = x^{\mathrm{Tbl}(12)}(k_1 \leftrightarrow k_0)_3(i_1 \leftrightarrow i_0)_1(i_1 \leftrightarrow n)_1(1 \leftrightarrow j_0)_2$. Let $j_1 \in J \setminus \{j_0, n\}$ be such that $l(\hat{x}; n, j_2) = n$. Let $x = \hat{x}(k(\hat{x}; n, j_1) \neq 1 ? 1 \leftrightarrow k(\hat{x}; n, j_1))_3$ (Table 14). Let $x^1 = x^{\mathrm{Tbl}(14)}$, $x^2 = x(1 \leftrightarrow i_q)_1, x^3 = x(1 \leftrightarrow j_q)_2, j_q \in J \setminus \{n\}$. $x^m \in P_I(Q)$ because $x^m_{i_0nk_0n} = x^m_{nnnl_1} = 1, m = 1, 2, 3$. By

Table 14: Point x (Theorem 4.6, Case 4.10.2)

	1	$\cdots$	j_1	$\cdots$	n
$\vdots$					
i_0					(k_0, n)
$\vdots$					
i_q	(n, n)				
$\vdots$					
n			$(1, n)$		(n, l_1)

construction $ax^1 = ax^2$ has only two terms indexed by two indices each equal to n, i.e. a_{11nn}, a_{i_q1nn}. The rest of the indices have at most one index equal to n. Thus, we can substitute all terms in this equation, except a_{i_q1nn}, from (4.3), yielding (4.3)$[i_q, 1, n, n]$. Following the same procedure in $ax^1 = ax^3$, we obtain (4.3)$[i_q, j_q, n, n]$. Applying to all points defined above the interchange $(I \leftrightarrow J)$ yields the same results for $i_q \in I \setminus \{n\}, j_q \in J \setminus \{j_0, n\}$.

In a similar manner, by considering the conjugate points of x^1, x^2, x^3, with respect to the interchanges $(J \leftrightarrow K)$ and $(I \leftrightarrow K)$, we prove (4.3)$[i_q, n, k_q, n]$, (4.3)$[n, j_q, k_q, n]$, respectively. Observe that all points used belong to $P_I(Q)$.

Case 4.10.3 *(Analogous to (b.2)).*

Let x^1, x^2, x^3 be defined as above. Then $a(x^1(J \leftrightarrow L)) = a(x^2(J \leftrightarrow L))$ yields (4.3)$[i_q, n, n, 1]$. $a(x^1(J \leftrightarrow L)) = a(x^3(J \leftrightarrow L))$ yields (4.3)$[i_q, n, n, l_q], i_q \in I \setminus \{i_0, n\}, j_q \in J \setminus \{n\}$.

Let $x = x^{\mathrm{Tbl}(5)}(j_0 \leftrightarrow j_q)_2(j_1 \leftrightarrow j_0)_2$, where $j_q \in J \setminus \{j_0, n\}$. Let $x^1 = x(1 \leftrightarrow l_0)_4$ and $x^2 = x^1(1 \leftrightarrow j_q)_2$. $x^m \in P_I(Q)$ because $x^m_{nj_0k_0n} = x^m_{i_0nnn} = 1$. All terms of $ax^1 = ax^2$ have at most two indices equal to n. As in the previous cases, substituting terms, except a_{nj_qn1}, from (4.3) and cancelling out identical terms yields (4.3)$[n, j_q, n, 1]$. Let $x^3 = x^1(1 \leftrightarrow l_q)_4$, where $l_q \in L \setminus \{n\}$. Observe that $x^3 \in P_I(Q)$. $ax^1 = ax^3$ yields (4.3)$[n, j_q, n, l_q]$.

The proof of (4.3) for the case where $|(i, j, k, l) \cap (n, n, n, n)| = 2$ is complete.

For the case where three of the indices are equal to n, we can only have $(i, j, k, l) = (n, n, k_q, n), k_q \in K \setminus \{n\}$. Let $x^1 = x^{\mathrm{Tbl}(12)}$ and $x^2 = x^1(j_1 \leftrightarrow j_0)_2(l_0 \leftrightarrow l_1)_4(l_2 \leftrightarrow n)_4$. We denote $k(x^2; n, n)$ as k_q. Let $x^3 = x^2(n \leftrightarrow l_q)_4$. $x^m \in P_I(Q)$ because $x^m_{nj_0nl_1} = x^m_{i_0nnl_0} = 1, m = 2, 3$. Then all terms in $ax^2 = ax^3$ have at most two indices equal to n, except a_{nnk_qn}. Substituting all these terms from (4.3) we obtain (4.3)$[n, n, k_q, n]$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we consider (4.13) and must show that all π_{ijkl} are equal. For all of the following cases where the steps are similar to the corresponding proofs of previous theorems, the points used belong to $P_I(Q)$.

Step 1: Exactly as the corresponding step of the proof of Thm. 4.8 for (i_q, n, n, n) and (n, j_q, n, n) where $i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}$. Hence,

$$\begin{aligned}
\pi_{i_0nnn} &= \pi_{i_qnnn} = \pi^1, \forall i_q \in I \setminus \{i_0, n\}, \\
\pi_{nj_0nn} &= \pi_{nj_qnn} = \pi^2, \forall j_q \in J \setminus \{j_0, n\}
\end{aligned}$$

Step 2: Exactly as the corresponding step of the proof of Thm. 4.8 for $\pi_{i_0nnl_0}, \pi_{i_0nnl_q}$ and $\pi_{nj_0nl_0}, \pi_{nj_0nl_q}$. Hence,

$$\begin{aligned}
\pi_{i_0nnl_0} &= \pi_{i_0nnl_q}, \forall l_q \in L \setminus \{l_0, n\}, \\
\pi_{nj_0nl_0} &= \pi_{nj_0nl_q}, \forall l_q \in L \setminus \{l_0, n\}
\end{aligned}$$

Step 3: We show that $\pi^1 = \pi^2$, exactly as in the corresponding step of the proof of Thm. 4.4. Thus,

$$\pi_{i_qnnn} = \pi_{nj_qnn} = \pi, \forall i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}$$

Step 4: We show that $\pi_{i_0j_0nn} = \pi_{nj_0nl_0}$, exactly as in Step 5 of the proof of Thm. 4.4. Applying the interchange $(I \leftrightarrow J)$ to all points of this step and following the same procedure, we obtain $\pi_{i_0j_0nn} = \pi_{i_0nnl_0}$. Then we show that $\pi_{nj_0nl_0} = \pi_{nj_0k_0n}$ and $\pi_{i_0nnl_0} = \pi_{i_0nk_0n}$, exactly as in Step 1 of the proof of Thm. 4.4. Hence,

$$\pi_{i_0nk_0n} = \pi_{nj_0k_0n} = \pi_{i_0nnl_0} = \pi_{nj_0nl_0} = \pi_{i_0j_0nn}$$

Step 5: Let $x^1 = x^{\mathrm{Tbl}(4)}$ and $x^2 = x^1(l_q \leftrightarrow n)_4, l_q \in L \setminus \{l_0, n\}$. $x^1, x^2 \in P_I(Q)$ because $x^1_{nj_0nn} = x^1_{i_0nnl_0} = 1$, $x^2_{nj_0nl_q} = x^2_{i_0nnl_0} = 1$. Thus, after substituting in $ax^1 = ax^2$ terms $a_{nj_0nn}, a_{nj_0nl_q}$ from (4.13) and the remaining terms from (4.3) and cancelling out identical terms, we obtain $.\pi_{nj_0nn} = \pi_{nj_0nl_q}$.

The proof of $(4.3)_\pi$ is complete. The rest of the proof is exactly as in Thm. 4.4. ■

4.5 Wheel class num. 5

Let $p = 3$ and $t_1, t_2, t_3 \in C$ such that $|c \cap t_m| = 2, |s \cap t_m| = 2$, for $m = 1, \ldots, 3$, and $|t_2 \cap t_1| = 2, |t_2 \cap t_3| = 2, |t_1 \cap t_3| = 0$. We define $Q = C^3(c) \cup (C^2(c) \cap C^2(s)) \cup_{m=1}^{m=3} (C^2(c) \cap C^3(t_m))$. For

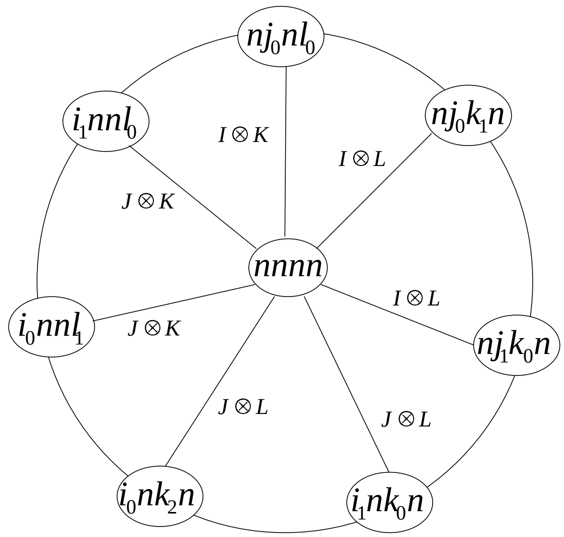

Figure 7: A wheel of class num. 5

$c = (n, n, n, n), s = (i_0, j_0, k_0, l_0), t_1 = (i_0, n, n, l_0), t_2 = (i_0, n, k_0, n), t_3 = (n, j_0, k_0, n)$, (4.1) becomes

$$\begin{aligned} 3x_{nnnn} + \sum_{i\in I\setminus\{n\}} x_{innn} + \sum_{j\in J\setminus\{n\}} x_{njnn} + \sum_{k\in K\setminus\{n\}} x_{nnkn} + \sum_{l\in L\setminus\{n\}} x_{nnnl} & \\ + \sum_{i\in I\setminus\{i_0,n\}} x_{innl_0} + \sum_{l\in L\setminus\{l_0,n\}} x_{i_0nnl} + \sum_{i\in I\setminus\{i_0,n\}} x_{ink_0n} & \\ + \sum_{j\in J\setminus\{j_0,n\}} x_{njk_0n} + \sum_{k\in K\setminus\{k_0,n\}} x_{nj_0kn} + \sum_{k\in K\setminus\{k_0,n\}} x_{i_0nkn} & \\ + x_{i_0j_0nn} + x_{i_0nk_0n} + x_{nj_0k_0n} + x_{i_0nnl_0} + x_{nj_0nl_0} + x_{nnk_0l_0} & \leq 3 \end{aligned} \tag{4.17}$$

This inequality is induced by the wheel illustrated in Figure 7, which belongs to wheel class num. 5.

Lemma 4.11 *(4.17) is maximal.*

Proof. Suppose that there exists a variable x_q ($q = (i_q, j_q, k_q, l_q) \in C$) which can be added to the left-hand side of (4.17) without increasing its right-hand side. Exactly two indices of x_q must be equal to n. We consider two cases for the indices not equal to n: (a) they can both obtain any value in their domains, but if the value of one of the indices is taken from (i_0, j_0, k_0, l_0), the other should obtain a value not belonging to this tuple, or (b) same as (a) but all indices are restricted from taking the value from (i_0, j_0, k_0, l_0).

(a) $q = (n, n, k_q, l_q), k_q \in K \setminus \{n\}, l_q \in L \setminus \{n\}, (k_q, l_q) \neq (k_0, l_0)$.

Consider $x_{nnk_ql_q} = x_{nj_0k_1n} = x_{i_0nk_2n} = x_{i_1nnl_0} = 1$, where $i_1 \in I \setminus \{i_0, n\}, k_1, k_2 \in K \setminus \{k_0, n\}$ with $k_1 \neq k_2, l_q \in L \setminus \{l_0, n\}$. Observe that k_q can be equal to k_0. In an analogous way, we can set variables $x_{nnk_ql_q}, x_{nj_0k_1n}, x_{i_1nk_0n}, x_{i_0nnl_1}$ to one. In this case observe that l_q can be equal to l_0 but $k_q \neq k_0$.

(b) $q = (i_q, n, n, l_q), i_q \in I \setminus \{i_0, n\}, l_q \in L \setminus \{l_0, n\}$.

Consider $x_{i_qnnl_q} = x_{nj_1k_0n} = x_{i_0nk_1n} = x_{nj_0nl_0} = 1$, where $j_1 \in J \setminus \{j_0, n\}, k_1 \in K \setminus \{k_0, n\}$.

■

Theorem 4.12 *For $n \geq 7$, (4.17) defines a facet of P_I.*

Proof. Let $x = x^{\mathrm{Tbl}(4)}$. Then $x \in P_I \setminus P_I(Q)$ because only two variables $(x_{i_0nnl_0}, x_{nj_0nn})$ appearing in (4.17) are set to one, since x_{ink_0n} is set to zero for every $i \in I$, as a consequence of setting $x_{i_1nk_1n}$ to one $(k_1 \neq k_0)$.

Instead of $(4.3)_{2\pi}$, we must prove $(4.3)_{3\pi} : a_{nnnn} = \lambda^1_{nn} + \lambda^2_{nn} + \lambda^3_{nn} + \lambda^4_{nn} + \lambda^5_{nn} + \lambda^6_{nn} + 3\pi$, and instead of (4.6), we must show

$$\begin{aligned} a_0 \quad = \quad & \sum_{k\in K, l\in L} \lambda^1_{kl} + \sum_{i\in I, l\in L} \lambda^2_{il} + \sum_{j\in J, l\in L} \lambda^3_{jl} \\ & + \sum_{i\in I, j\in J} \lambda^4_{ij} + \sum_{j\in J, k\in K} \lambda^5_{jk} + \sum_{i\in I, k\in K} \lambda^6_{ik} + 3\pi \end{aligned} \tag{4.18}$$

Observe that (4.3) is valid for all cases where at least two of the indices are equal to one. For the remaining $(i, j, k, l) \in C \setminus (Q \cup \{(n, n, n, n)\})$, we consider three cases, viz. none, one, two of the indices are equal to n. The first two cases are shown in exactly the same way as in Thm. 4.4. Observe that all points used belong to $P_I(Q)$. For the case where two of the indices are equal to n, we consider cases analogous to these of Lem. 4.11.

Case 4.12.1 *(Analogous to(a)).*

Let $x^1 = x^{\mathrm{Tbl}(4)}(i_1 \leftrightarrow i_0)_1$ and $x^2 = x^1(1 \leftrightarrow k_q)_3, k_q \in K \setminus \{1, k_1, n\}$. Clearly $x^m \in P_I(Q)$ since $x^m_{nj_0nn} = x^m_{i_0nk_1n} = x^m_{i_1nnl_0} = 1$, for $m = 1, 2$. Hence, $ax^1 = ax^2$ is valid. Observe that all terms of the equation $ax^1 = ax^2$ are indexed by tuples having at most one index equal to n, except a_{nn11} and a_{nnk_q1}. Therefore, substituting all terms, except a_{nnk_q1}, from (4.3) we obtain $(4.3)[n, n, k_q, 1]$. Let $x^3 = x^2(1 \leftrightarrow l_q)_4$, $l_q \in L \setminus \{1, l_0, l_1, n\}$. $ax^2 = ax^3$ yields $(4.3)[n, n, k_q, l_q]$, $k_q \in K \setminus \{n\}, l_q \in L \setminus \{l_0, n\}$. Applying the interchange $(K \leftrightarrow L)$ to x^1, x^2, x^3 we obtain $(4.3)[n, n, k_q, l_q]$ for $k_q \in K \setminus \{k_0, n\}, l_q \in L \setminus \{n\}$.

Let $x^4 = x^1(J \leftrightarrow K)(j_1 \leftrightarrow j_0)_2$, $x^5 = x^4(j_q \leftrightarrow j_1)_2$, $j_q \in J \setminus \{1, j_0, n\}$, $x^6 = x^5(1 \leftrightarrow l_q)_4, l_q \in L \setminus \{1, l_0, n\}$. $x^m \in P_I(Q)$, since $x^m_{i_0j_0nn} = x^m_{nnk_0n} = x^m_{i_1nnl_0} = 1$, for $m = 4, 5, 6$. Then, $ax^4 = ax^5$ yields $(4.3)[n, j_q, n, 1]$ and $ax^5 = ax^6$ yields $(4.3)[n, j_q, n, l_q]$ where $j_q \neq j_0, n$, $l_q \neq l_0, n$.

Let $\hat{x} = x^{\mathrm{Tbl}(12)}(k_1 \leftrightarrow k_0)_3(j_1 \leftrightarrow j_q)_2$, where $j_q \in J \setminus \{j_0, n\}$. Let $j_2 \in J \setminus \{j_0, j_q, n\}$ such that $l(\hat{x}; n, j_2) = n$, and denote $k(\hat{x}; n, j_2)$ as k_2. Let $x^6 = \hat{x}(j_0 \leftrightarrow j_2)_2$ and $x^7 = x^6(1 \leftrightarrow l_0)_4$. $x^6, x^7 \in P_I(Q)$ because $x^m_{nj_0k_2n} = x^m_{i_0nnl_1} = x^m_{i_1nk_0n} = 1$. $ax^6 = ax^7$ yields (4.3)$[n, j_q, n, l_0]$.where $j_q \neq j_0$.

Let $\hat{x} = \mathring{x}(k_0 \leftrightarrow n)_3$. It is valid to assume that $k(\hat{x}; n, n) \neq n$, because if $k(\hat{x}; n, n) = n$ then we apply the interchange $(k_2 \leftrightarrow n)_3, k_2 \in K \setminus \{k_0, n\}$. Hence, let k_2 denote $k(\hat{x}; n, n)$. Again it is safe to assume that the pair (n, n) does not lie at the column n. If this is the case, apply the interchange $(k_1 \leftrightarrow n)_3, k_1 \in K \setminus \{k_0, k_2, n\}$. Hence, let k_1 denote $k(\hat{x}; n, n)$. Let $i_1, i_2 \in I \setminus \{n\}, i_1 \neq i_2$, be such that $k(\hat{x}; i_1, n) = n$ and $l(\hat{x}; i_2, n) = n$. Let $x = \hat{x}(l(\hat{x}; i_1, n) \neq l_0 ? l(\hat{x}; i_1, n) \leftrightarrow l_0)_4(i_2 \neq i_0 ? i_2 \leftrightarrow i_0)_1(1 \leftrightarrow j_1)_2$, $j_1 \in J \setminus \{1, j_0, n\}$ (Table 15). Let $j_q \in J \setminus \{1, j_1, n\}$ be such that $k(x^{\mathrm{Tbl}(15)}; n, j_q) = n$. The assumption

Table 15: Point x (Theorem 4.12, Case 4.12.1)

	$\cdots$	j_q	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0						(k_1, n)
$\vdots$						
i_1						(n, l_0)
$\vdots$						
n		(n, l_q)		(k_0, n)		$(k_2, \cdot)$

that $j_q \neq 1$ is valid because if $k(x^{\mathrm{Tbl}(15)}; n, 1) = n$, for $n \geq 4$, there exists a $j_q \in J \setminus \{1, j_1, n\}$ such that we can apply the interchange $(1 \leftrightarrow j_q)_2$. We denote $l(x^{\mathrm{Tbl}(15)}; n, j_q)$ as l_q. Observe that $l_q \in L \setminus \{l_0, n\}$. Let $x^8 = x(l_q \neq 1 ? 1 \leftrightarrow l_q)_4$, $x^9 = x^8(1 \leftrightarrow j_q)_2$ and $x^{10} = x^9(1 \leftrightarrow l_q)_4$. $x^m \in P_I(Q)$ because $x^m_{nj_1k_0n} = x^m_{i_0nk_1n} = x^m_{i_1nnl_0} = 1$, for $m = 8, 9, 10$. Thus, $ax^8 = ax^9$ yields (4.3)$[n, j_q, n, 1]$ and $ax^9 = ax^{10}$ yields (4.3)$[n, j_q, n, 1]$, $j_q \in J \setminus \{n\}, l_q \in L \setminus \{l_0, n\}$.

Let $\hat{x} = x^{\mathrm{Tbl}(15)}(i_0 \leftrightarrow i_2)_1(k_0 \leftrightarrow k_1)_3(j_1 \leftrightarrow j_0)_2$, where $i_2 \in I \setminus \{i_0, i_1, n\}$. Then $\hat{x}_{i_3j_3nn} = 1$, for $i_3 \in I \setminus \{i_1, i_2, n\}, j_3 \in J \setminus \{j_0, n\}$. Let $x^{12} = \hat{x}(i_3 \neq 1 ? 1 \leftrightarrow i_3)_1(j_3 \neq 1 ? 1 \leftrightarrow j_3)_2$. Then $x^{12}_{11nn} = 1$. Also, let $x^{13} = x^{12}(1 \leftrightarrow i_q)_1$ and $x^{14} = x^{13}(1 \leftrightarrow j_q)_2$, with $i_q \in I \setminus \{i_1, i_2, n\}$, $j_q \in J \setminus \{j_0, n\}$. $x^m \in P_I(Q)$ because $x^m_{nj_0k_1n} = x^m_{i_1nnl_0} = x^m_{i_2nk_0n} = 1$, for $m = 12, 13, 14$. Thus, $ax^{12} = ax^{13}$ yields (4.3)$[i_q, 1, n, n]$ and $ax^{13} = ax^{14}$ (4.3)$[i_q, j_q, n, n]$ where $i_q \in I \setminus \{n\}, j_q \in J \setminus \{j_0, n\}$.

Finally, $x^{\mathrm{Tbl}(15)}_{i_3j_3nn} = 1$ for $i_3 \in I \setminus \{i_0, i_1, n\}, j_3 \in J \setminus \{j_1, n\}$. As in the previous case, let $x^{15} = x^{\mathrm{Tbl}(15)}(i_3 \neq 1 ? 1 \leftrightarrow i_3)_1(j_3 \neq 1 ? 1 \leftrightarrow j_3)_2$. Then $x^{15}_{11nn} = 1$. Also, let $x^{16} = x^{15}(1 \leftrightarrow j_q)_2$ and $x^{17} = x^{16}(1 \leftrightarrow i_q)_1$, with $i_q \in I \setminus \{i_0, i_1, n\}$, $j_q \in J \setminus \{j_1, n\}$. $x^m \in P_I(Q)$, because $x^m_{nj_1k_0n} = x^m_{i_0nk_1n} = x^m_{i_1nnl_0} = 1$, for $m = 15, 16, 17$ Thus, $ax^{15} = ax^{16}$ yields (4.3)$[1, j_q, n, n]$ and $ax^{16} = ax^{17}$ (4.3)$[i_q, j_q, n, n]$ where $i_q \in I \setminus \{i_0, n\}, j_q \in J \setminus \{n\}$.

Case 4.12.2 *(Analogous to (b)).*

Let $x = x^{\mathrm{Tbl}(4)}(1 \leftrightarrow i_0)_1(i_0 \leftrightarrow i_1)_1(1 \leftrightarrow l_0)_4(1 \leftrightarrow k_0)_3$ (Table 16). Let $x^1 = x$ and $x^2 = x^1(1 \leftrightarrow i_q)_1$,

Table 16: Point x (Theorem 4.12, Case 4.12.2)

	$\cdots$	j_0	$\cdots$	n
1				$(n,1)$
$\vdots$				
i_0				(k_1,n)
$\vdots$				
n		(n,n)		(k_0,l_0)

$i_q \in I \setminus \{1, i_0, i_1, n\}$. $x^m \in P_I(Q)$ because $x^m_{nj_0nn} = x^m_{i_0nk_1n} = x^m_{nnk_0l_0} = 1$, for $m = 1, 2$. Hence, $ax^1 = ax^2$ is valid. Observe that all terms of the equation $ax^1 = ax^2$ are indexed by tuples having at most one index equal to n except a_{1nn1} and a_{i_qnn1}. Therefore, substituting all terms but a_{i_qnn1} from (4.3) and cancelling out identical terms, we obtain (4.3)$[i_q, n, n, 1]$. Let $x^3 = x^2(1 \leftrightarrow l_q)_4$, $l_q \in L \setminus \{1, l_0, l_1, n\}$. $ax^2 = ax^3$ yields (4.3)$[i_q, n, n, l_q]$, where $i_q \neq i_0, l_q \neq l_0$.

Considering the conjugates of the above points with respect to the interchange $(K \leftrightarrow L)$, we obtain (4.3)$[i_q, n, k_q, n]$. Similarly the conjugates with respect to the interchanges $(J \leftrightarrow L)(I \leftrightarrow K)$ yield (4.3)$[n, j_q, k_q, n]$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we consider (4.13) and must show that all π_{ijkl} are equal.

Step 1: Let $x^1 = x^{\text{Tbl}(16)}$ and $x^2 = x^1(j_0 \leftrightarrow j_q)_2$ where $j_q \in J \setminus \{j_0, n\}$. $x^2 \in P_I(Q)$ since $x^2_{nj_qnn} = x^2_{i_0nk_1n} = x^2_{nnk_0l_0} = 1$. Hence, $ax^1 = ax^2$ is valid. Substituting a_{nj_0nn}, a_{nj_qnn} from (4.13) and the remaining terms from (4.3), and cancelling out identical terms, we obtain $\pi_{nj_0nn} = \pi_{nj_qnn}$. In an analogous way, $a\bar{x}^1 = a\bar{x}^2$, where $\bar{x}^m = x^m(J \leftrightarrow L)$, for $m = 1, 2$, yields $\pi_{nnnl_0} = \pi_{nnnl_q}, l_q \in L \setminus \{n\}$. Similarly, $a\hat{x}^1 = a\hat{x}^2$, where $\hat{x}^m = x^m(I \leftrightarrow J)$, for $m = 1, 2$, yields $\pi_{i_0nnn} = \pi_{i_qnnn}, i_q \in I \setminus \{n\}$.

Let $\tilde{x} = x^1(l_0 \leftrightarrow n)_4(k_1 \leftrightarrow n)_3(i_0 \leftrightarrow i_1)_1$, where $i_1 \in I \setminus \{i_0, n\}$. Then, let $\tilde{x}_{i_2j_2nn} = 1, i_2 \in I \setminus \{i_0, n\}, j_2 \in J \setminus \{j_0, n\}$. Let $x^3 = \tilde{x}(i_0 \leftrightarrow i_2)_1(j_0 \leftrightarrow j_2)_2$ and $x^4 = x^3(k_0 \leftrightarrow k_q)_3, k_q \in K \setminus \{n\}$. $x^m \in P_I(Q)$ since $x^m_{i_0j_0nn} = x^m_{i_1nnl_0} = 1$, for $m = 3, 4$ and $x^3_{nnk_0n} = 1, x^4_{nnk_qn}$. Hence, $ax^3 = ax^4$ is valid, yielding $\pi_{nnk_0n} = \pi_{nnk_qn}, k_q \in K \setminus \{n\}$.

Therefore,

$$\begin{aligned}
\pi_{i_0nnn} &= \pi_{i_qnnn} = \pi^1, \forall i_q \in I \setminus \{n\}, \\
\pi_{nj_0nn} &= \pi_{nj_qnn} = \pi^2, \forall j_q \in J \setminus \{n\}, \\
\pi_{nnk_0n} &= \pi_{nnk_qn} = \pi^3, \forall k_q \in K \setminus \{n\}, \\
\pi_{nnnl_0} &= \pi_{nnnl_q} = \pi^4, \forall l_q \in L \setminus \{n\}
\end{aligned}$$

Step 2: Let $x = x^{\mathrm{Tbl}(16)}(1 \leftrightarrow l_0)_4(k_0 \leftrightarrow k_1)_3(i_0 \leftrightarrow i_1)_1(1 \leftrightarrow i_0)_1$ (Table 17). Let $x^1 = x^{\mathrm{Tbl}(17)}$,

Table 17: Point x (Theorem 4.12, Step 2)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0				(n, l_0)
$\vdots$				
i_1				(k_0, n)
$\vdots$				
n		(n, n)		

$x^2 = x^1(i_0 \leftrightarrow i_q)_1$, where $i_q \in I \setminus \{i_0, i_1, n\}$, and $x^3 = x^1(l_0 \leftrightarrow l_q)_4$, where $l_q \in L \setminus \{l_0, n\}$. $x^m \in P_I(Q)$ since $x^m_{nj_0nn} = x^m_{i_1nk_0n} = 1$, for $m = 1, 2, 3$ and $x^1_{i_0nnl_0} = 1, x^2_{i_qnnl_0} = 1, x^3_{i_0nnl_q} = 1$. Hence, $ax^1 = ax^2$ ($ax^1 = ax^3$) is valid, yielding $\pi_{i_0nnl_0} = \pi_{i_qnnl_0}$ ($\pi_{i_0nnl_0} = \pi_{i_0nnl_q}$). The interchange $(K \leftrightarrow L)$ applied to x^1, x^2, x^3 yields points belonging to $P_I(Q)$. The corresponding equations lead to $\pi_{i_0nk_0n} = \pi_{i_qnk_0n}$ and $\pi_{i_0nk_0n} = \pi_{i_0nk_qn}$.

At point x^1, there exist $i_2 \in I \setminus \{i_0, i_1, n\}, j_2 \in J \setminus \{j_0, n\}, k_q \in K \setminus \{k_0, n\}$ such that $x^1_{i_2j_2k_qn} = 1$. Let $x^5 = x^1(i_0 \leftrightarrow i_1)_1(i_2 \leftrightarrow n)_1(j_2 \leftrightarrow j_0)_2$ and $x^6 = x^5(k_0 \leftrightarrow k_q)_3$. The point x^6 is illustrated in Table 18. Let . Then, $x^5, x^6 \in P_I(Q)$ since $x^5_{i_0nk_0n} = x^5_{i_1nnl_0} = x^5_{nj_0k_qn} = 1$ and $x^6_{i_0nk_qn} = x^6_{i_1nnl_0} =$

Table 18: Point x^6 (Theorem 4.12, Step 2)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0				(k_q, n)
$\vdots$				
i_1				(n, l_0)
$\vdots$				
n		(k_0, n)		

$x^6_{nj_0k_0n} = 1$. Thus, $ax^5 = ax^6$ is valid, yielding $\pi_{i_0nk_0n} + \pi_{nj_0k_qn} = \pi_{i_0nk_qn} + \pi_{nj_0k_0n} \Rightarrow \pi_{nj_0k_qn} = \pi_{nj_0k_0n}$. Let $x^7 = x^6(j_q \leftrightarrow j_0)_2$, $j_q \in J \setminus \{j_0, n\}$. Observe that $x^7 \in P_I(Q)$. Hence, $ax^7 = ax^6$ is valid yielding $\pi_{nj_0k_0n} = \pi_{nj_qk_0n}$.

At this step, we have shown that for every $i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}, k_q \in K \setminus \{n\}, l_q \in L \setminus \{n\}$

$$\pi_{i_0nnl_0} = \pi_{i_qnnl_0} = \pi_{i_0nnl_q}, \pi_{i_0nk_0n} = \pi_{i_qnk_0n} = \pi_{i_0nk_qn}, \pi_{nj_0k_0n} = \pi_{nj_0k_qn} = \pi_{nj_qk_0n}$$

Step 3: Let $\hat{x} = x^{\mathrm{Tbl}(16)}(i_0 \leftrightarrow i_q)_1(i_0 \leftrightarrow n)_1$ and $x = \hat{x}(i_q \leftrightarrow n)_1, i_q \in I \setminus \{i_0, i_1, n\}$ (Table 19). Let $x^1 =$

Table 19: Point x (Theorem 4.12, Step 3)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0		(n,n)		
$\vdots$				
i_1				(k_0,n)
$\vdots$				
n				(n,l_0)

$\hat{x}$, $x^2 = x$. $x^1, x^2 \in P_I(Q)$ since $x^m_{i_0 j_0 nn} = x^m_{i_1 n k_0 n} = 1$, for $m = 1,2$, and $x^1_{i_q nnl_0} = 1, x^2_{nnnl_0} = 1$. $ax^1 = ax^2$ yields $\pi_{i_q nnl_0} = \pi_{nnnl_0}$.

Applying the interchange $(K \leftrightarrow L)$ to x^1, x^2, we obtain points belonging to $P_I(Q)$. The corresponding equation yields $\pi_{i_q nk_0 n} = \pi_{nnk_0 n}$

At point x^6, illustrated in Table 18, let $(n,n) \in K \times L$ lie in cell (i_q, j_q). Obviously $i_q \in I \setminus \{i_0, i_1, n\}, j_q \in J \setminus \{j_0, n\}$. Let $x^7 = x^6(i_q \leftrightarrow n)_1$. $x^7 \in P_I(Q)$ since $x^7_{i_0 nk_q n} = x^7_{i_1 nnl_0} = x^7_{nj_q nn} = 1$ Therefore, $ax^7 = ax^6$ yields $\pi_{nj_0 k_0 n} = \pi_{nj_q nn}$.

Up to this point, we have shown

$$
\begin{aligned}
\pi^4 &= \pi_{nnnl_q} = \pi_{nnnl_0} = \pi_{i_q nnl_0} = \pi_{i_0 nnl_0} = \pi_{i_0 nnl_q}, \forall i_q \in I \setminus \{n\}, l_q \in L \setminus \{n\}, \\
\pi^3 &= \pi_{nnk_q n} = \pi_{nnk_0 n} = \pi_{i_q nk_0 n} = \pi_{i_0 nk_0 n} = \pi_{i_0 nk_q n}, \forall i_q \in I \setminus \{n\}, k_q \in K \setminus \{n\}, \\
\pi^2 &= \pi_{nj_q nn} = \pi_{nj_0 nn} = \pi_{nj_q k_0 n} = \pi_{nj_0 k_0 n} = \pi_{nj_0 k_q n}, \forall j_q \in J \setminus \{n\}, k_q \in K \setminus \{n\}
\end{aligned}
$$

Step 4: We denote $k(x^{\mathrm{Tbl}(17)}; n,n), (l(x^{\mathrm{Tbl}(17)}; n,n))$ as $k_q, (l_q)$. Let $x = x^{\mathrm{Tbl}(17)}(k_0 \leftrightarrow k_q)_3(l_0 \leftrightarrow l_q)_4(i_0 \leftrightarrow i_1)_1$ (Table 20), Let $x^1 = x^{\mathrm{Tbl}(20}$ and $x^2 = x^1(i_0 \leftrightarrow i_1)_1$. $x^m \in P_I(Q)$ since $x^m_{nj_0 nn} =$

Table 20: Point x (Theorem 4.12, Step 4)

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_0				(k_q,n)
$\vdots$				
i_1				(n,l_q)
$\vdots$				
n		(n,n)		(k_0,l_0)

$x^m_{nnk_0 l_0} = 1$, for $m = 1,2$, and $x^1_{i_0 nk_q n} = 1, x^2_{i_0 nnl_q} = 1$. $ax^1 = ax^2$ yields $\pi_{i_0 nnl_q} = \pi_{i_0 nk_q n}$, implying

$\pi^3 = \pi^4$.

Let $x^3 = x^{\text{Tbl}(19)}$ and $x^4 = x^3(I \leftrightarrow J)$. $x^4 \in P_I(Q)$ since $x^4_{i_0j_0nn} = x^4_{nj_1k_0n} = x^4_{nnnl_0} = 1$. $ax^3 = ax^4$ yields $\pi_{i_1nk_0n} = \pi_{nj_1k_0n}$, implying $\pi^3 = \pi^2$.

Let $x^5 = x^1(I \leftrightarrow J)$. $x^5 \in P_I(Q)$ since $x^5_{i_0nnn} = x^5_{nnk_0l_0} = x^5_{nj_0k_qn} = 1$. $ax^1 = ax^5$ yields $\pi_{nj_0nn} + \pi_{nnk_0l_0} + \pi_{i_0nk_qn} = \pi_{i_0nnn} + \pi_{nnk_0l_0} + \pi_{nj_0k_qn}$, implying $\pi^1 = \pi^2$.

Hence,

$$\pi^1 = \pi^2 = \pi^3 = \pi^4 = \pi$$

Step 5: Let $x^1 = x^{\text{Tbl}(19)}$ and $x^2 = x^1(i_0 \leftrightarrow n)_1$. $ax^1 = ax^2$ yields $\pi_{i_0j_0nn} + \pi_{nnnl_0} = \pi_{nj_0nn} + \pi_{i_0nnl_0}$ implying $\pi_{i_0j_0nn} = \pi$. Let $x^3 = x^{\text{Tbl}(17)}$ and $x^4 = x^3(I \leftrightarrow J)$. $x^6 \in P_I(Q)$ since $x^4_{i_0nnn} = x^4_{nj_1k_0n} = x^4_{nj_0nl_0} = 1$. $ax^3 = ax^4$ yields $\pi_{nj_0nl_0} = \pi$.

The proof of $(4.3)_\pi$ is complete. To show $(4.3)_{3\pi}$, we define

$$\pi_{nnnn} = a_{nnnn} - (\lambda^1_{nn} + \lambda^2_{nn} + \lambda^3_{nn} + \lambda^4_{nn} + \lambda^5_{nn} + \lambda^6_{nn}) \tag{4.19}$$

We will show that $\pi_{nnnn} = 3\pi$.

Let $x^1 = x^{\text{Tbl}(17)}$ and $x^2 = x^1(j_0 \leftrightarrow n)_2$. $x^2 \in P_I(Q)$ since $x^2_{nnnn} = 1$. Thus, $ax^1 = ax^2$ yields the desired result.

Finally, because $P_I(Q) \neq \emptyset$, for $n \geq 7$, there exists at least one solution to the system defined by the constraints of the problem. Hence, multiplying each row of A with the corresponding λ and (4.17) with π and summing over all rows we obtain

$$\begin{aligned} ax \quad = \quad & \sum_{k\in K,l\in L} \lambda^1_{kl} + \sum_{i\in I,l\in L} \lambda^2_{il} + \sum_{j\in J,l\in L} \lambda^3_{jl} \\ & + \sum_{i\in I,j\in J} \lambda^4_{ij} + \sum_{j\in J,k\in K} \lambda^5_{jk} + \sum_{i\in I,k\in K} \lambda^6_{ik} + 3\pi \end{aligned}$$

which proves (4.18). ■

4.6 Wheel class num. 24

Let $p = 3$ and $v, u \in C$ such that $|v \cap u| = 0, |v \cap c| = 1, |v \cap s| = 3, |u \cap s| = 1, |u \cap c| = 3$. We define $Q = C^3(c) \cup (C^2(c) \cap C^2(s)) \cup (C^2(v) \cap C^2(c)) \cup (C^1(v) \cap C^2(c) \cap C^2(u))$. For $c = (n, n, n, n), s =$

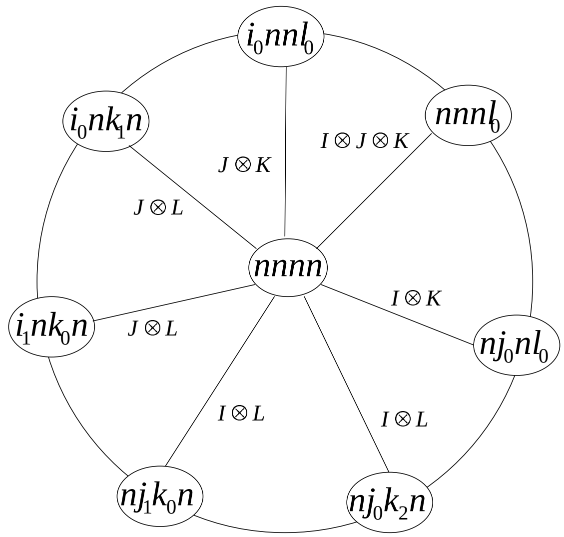

Figure 8: A wheel of class num. 24

$(i_0, j_0, k_0, l_0), v = (i_0, j_0, k_0, n), u = (n, n, n, l_0)$, (4.1) becomes

$$\begin{aligned} 3x_{nnnn} + \sum_{i \in I\setminus\{n\}} x_{innn} + \sum_{j \in J\setminus\{n\}} x_{njnn} + \sum_{k \in K\setminus\{n\}} x_{nnkn} + \sum_{l \in L\setminus\{n\}} x_{nnnl} & \\ + \sum_{i \in I\setminus\{i_0,n\}} x_{ink_0n} + \sum_{k \in K\setminus\{k_0,n\}} x_{i_0nkn} & \\ + \sum_{j \in J\setminus\{j_0,n\}} x_{njk_0n} + \sum_{k \in K\setminus\{k_0,n\}} x_{nj_0kn} & \\ + \sum_{l \in L\setminus\{l_0,n\}} x_{i_0nnl} + \sum_{l \in L\setminus\{l_0,n\}} x_{nj_0nl} + \sum_{l \in L\setminus\{l_0,n\}} x_{nnk_0l} & \\ +x_{i_0j_0nn} + x_{i_0nk_0n} + x_{nj_0k_0n} + x_{i_0nnl_0} + x_{nj_0nl_0} + x_{nnk_0l_0} & \leq 3 \end{aligned} \tag{4.20}$$

This inequality is induced by the wheel illustrated in Figure 8, which belongs to wheel class num. 24.

Lemma 4.13 *(4.20) is maximal.*

Proof. Suppose that there exists a variable x_q ($q = (i_q, j_q, k_q, l_q) \in C$) which can be added to the left-hand side of (4.20) without increasing its right-hand side. Exactly two indices of x_q must be equal to n. We consider two cases for the indices not equal to n: (a) they can both obtain any value in their domains but if the value of one of the indices is taken from (i_0, j_0, k_0, l_0), the other should obtain a value not belonging to this tuple, or (b) same as (a) but all indices are restricted from taking any value from (i_0, j_0, k_0, l_0). Then for case (a), we consider two further subcases, viz. $q \in C^1(v) \cap C^2(c) \cap C^2(u)$ (case a.1) and $q \notin C^1(v) \cap C^2(c) \cap C^2(u)$ (case a.2).

(a.1) $q = (i_q, n, n, l_q), i_q \in I \setminus \{i_0, n\}, l_q \in L \setminus \{n\}$.

Consider $x_{i_q n n l_q} = x_{n j_0 n l_1} = x_{n j_1 k_0 n} = x_{i_0 n k_1 n} = 1$. Observe that since $l_1 \neq l_0$, we can have $l_q = l_0$.

(a.2) $q = (i_q, j_q, n, n), i_q \in I \setminus \{n\}, j_q \in J \setminus \{n\}, (i_q, j_q) \neq (i_0, j_0)$. If $i_q \neq i_0$ then consider $x_{i_q, j_q, n, n} = x_{i_0 n k_1 n} = x_{n j_1 k_0 n} = x_{n n n l_0} = 1$. Observe that we can have $j_q = j_0$. If $j_q \neq j_0$ then consider $x_{i_q, j_q, n, n} = x_{n j_0 k_1 n} = x_{i_1 n k_0 n} = x_{n n n l_0} = 1$. In this case we can have $i_q = i_0$.

(b) $q = (i_q, n, k_q, n), i_q \in I \setminus \{i_0, n\}, k_q \in k \setminus \{k_0, n\}$.

Consider $x_{i_q n k_q n} = x_{i_0 n n l_0} = x_{n j_0 n l_1} = x_{n n k_0 l_2} = 1$.

■

Theorem 4.14 *For $n \geq 7$, (4.20) defines a facet of P_I.*

Proof. $P_I(Q) \neq \emptyset$ and $P_I(Q) \neq P_I$ are both shown in exactly the same way as in the proof of Thm. 4.12.

(4.3) is valid for all cases where at least two of the indices are equal to one. (4.3) for all cases where $|(i, j, k, l) \cap (n, n, n, n)| = 0$ or 1 is shown in exactly the same way as in Thm. 4.4. Observe that all points used belong to $P_I(Q)$. For the case where two of the indices are equal to n, we consider cases analogous to these of Lem. 4.13.

Case 4.14.1 (Analogous to case a.1) $(i, j, k, l) \in C^1(v) \cap C^2(c) \cap C^2(u)$.

Let $x^1 = x^{\mathrm{Tbl}(4)}(k_0 \leftrightarrow k_1)_3(l_0 \leftrightarrow l_1)_4$, where $l_1 \in L \setminus \{1, l_0, n\}$. Also, let $x^2 = x^1(1 \leftrightarrow k_q)_3, x^3 = x^2(1 \leftrightarrow l_q)_4$, where $k_q \in K \setminus \{k_0, n\}, l_q \in L \setminus \{1, l_1, n\}$. $x^m \in P_I(Q)$ since $x^m_{n j_0 n n} = x^m_{i_0 n n l_1} = x^m_{i_1 n k_0 n} = 1$, for $m = 1, 2, 3$. Thus, $ax^1 = ax^2$ yields (4.3)$[n, n, k_q, 1]$ and $ax^2 = ax^3$ yields (4.3)$[n, n, k_q, l_q]$ where .$k_q \neq k_0, n$, $l_q \neq n$.

Let $\bar{x}^m = x^m(I \leftrightarrow K)$. $\bar{x}^m \in P_I(Q)$ since $\bar{x}^m_{n j_0 n n} = \bar{x}^m_{n n k_0 l_1} = \bar{x}^m_{i_0 n k_1 n} = 1$, for $m = 1, 2, 3$. Thus, $a\bar{x}^1 = a\bar{x}^2$ yields (4.3)$[i_q, n, n, 1]$ and $a\bar{x}^2 = a\bar{x}^3$ (4.3)$[i_q, n, n, l_q]$, $i_q \neq i_0, n$, $l_q \neq n$.

Let $\hat{x}^1 = x(1 \leftrightarrow j_0)_2(k_0 \leftrightarrow k_1)_3(1 \leftrightarrow n)_4(l_0 \leftrightarrow l_1)_4$, where $l_1 \in L \setminus \{1, l_0, n\}$. Also, let $\hat{x}^2 = \hat{x}^1(1 \leftrightarrow j_q)_3, \hat{x}^3 = \hat{x}^2(1 \leftrightarrow l_q)_4$, where $j_q \in J \setminus \{j_0, n\}, l_q \in L \setminus \{1, l_1, n\}$. $\hat{x}^m \in P_I(Q)$ since $\hat{x}^m_{i_0 n n l_1} = \hat{x}^m_{i_1 n k_0 1} = \hat{x}^m_{n n 1 n} = 1$, for $m = 1, 2, 3$. Thus, $a\hat{x}^1 = a\hat{x}^2$ yields (4.3)$[n, j_q, n, 1]$ and $a\hat{x}^2 = a\hat{x}^3$ yields (4.3)$[n, j_q, n, l_q]$, $j_q \neq j_0, n$, $l_q \neq n$.

Case 4.14.2 (Analogous to case a.2)

Consider points $\bar{x}^m$ as defined in the previous case and let $x^m = \bar{x}^m(J \leftrightarrow L)$. $x^m \in P_I(Q)$ since $x^m_{nnnl_0} = x^m_{nj_1k_0n} = x^m_{i_0nk_1n} = 1$, for $m = 1,2,3$. As in the previous cases, $ax^1 = ax^2$ yields (4.3)$[i_q, j_q, n, n]$ and $ax^2 = ax^3$ yields (4.3)$[i_q, j_q, n, n]$. The same result is obtained for $i_q \in I \setminus \{n\}, j_q \in J \setminus \{j_0, n\}$ if we use the points resulting from the interchange $(I \leftrightarrow J)$ to x^m, for $m = 1,2,3$.

Case 4.14.3 (Analogous to case b).

Let $x^1 = x^{\mathrm{Tbl}(4)}(1 \leftrightarrow i_1)_1(1 \leftrightarrow k_1)_3(k_0 \leftrightarrow k_1)_3(1 \leftrightarrow l_0)_4$. Also, let $x^2 = x^1(1 \leftrightarrow i_q)_1, x^3 = x^2(1 \leftrightarrow k_q)_3$, where $i_q \in I \setminus \{i_0, n\}, k_q \in K \setminus \{k_0, n\}$. $x^m \in P_I(Q)$ because $x^m_{nj_0nn} = x^m_{i_0nn1} = x^m_{nnk_0l_0} = 1$, for $m = 1,2,3$. As in the previous cases, $ax^1 = ax^2$ yields (4.3)$[i_q, n, 1, n]$ and $ax^2 = ax^3$ yields (4.3)$[i_q, n, k_q, n]$. In exactly the same way, we obtain (4.3)$[n, j_q, k_q, n], j_q \in J \setminus \{j_0, n\}, k_q \in K \setminus \{k_0, n\}$, if we use points $\bar{x}^m = x^m(I \leftrightarrow J)$. Observe that $\bar{x}^m \in P_I(Q)$.

The proof of (4.3) is complete.

To prove $(4.3)_\pi$, we consider (4.13) and must show that all π_{ijkl} are equal.

Step1: Consider the points x^3, x^4 of step 2 of Thm. 4.12. Let $\hat{x}^m = x^m(I \leftrightarrow L)$, for $m = 3,4$. $\hat{x}^m \in P_I(Q)$, since $\hat{x}^m_{nj_0nl_0} = \hat{x}^m_{i_0nnl_1} = 1$, for $m = 3,4$, and $\hat{x}^3_{nnk_0n} = 1, \hat{x}^4_{nnk_qn} = 1$. $a\hat{x}^3 = a\hat{x}^4$ is valid, yielding $\pi_{nnk_0n} = \pi_{nnk_qn}$. The proof for tuples $(i,n,n,n), (n,j,n,n), (n,n,n,l), .i,j,l \neq n$, is exactly the same as in Step 1 of Thm. 4.12. Hence,

$$\begin{aligned}
\pi_{i_0nnn} &= \pi_{i_qnnn} = \pi^1, \forall i_q \in I \setminus \{i_0, n\}, \\
\pi_{nj_0nn} &= \pi_{nj_qnn} = \pi^2, \forall j_q \in J \setminus \{j_0, n\}, \\
\pi_{nnk_0n} &= \pi_{nnk_qn} = \pi^3, \forall k_q \in K \setminus \{k_0, n\}, \\
\pi_{nnnl_0} &= \pi_{nnnl_q} = \pi^4, \forall l_q \in L \setminus \{l_0, n\}
\end{aligned}$$

Step 2: Consider points x^1, x^3 of step 2 of Thm. 4.12. $ax^1 = ax^3$ yields $\pi_{i_0nnl_0} = \pi_{i_0nnl_q}$. Let $\hat{x}^m = , x^m(I \leftrightarrow J)$, $\bar{x}^m =, x^m(I \leftrightarrow K)$, for $m = 1,3$. Then $\hat{x}^m, \bar{x}^m \in P_I(Q)$ since $\hat{x}^m_{i_0nnn} = \hat{x}^m_{nj_1k_0n} = 1, \bar{x}^m_{nj_0nn} = \bar{x}^m_{i_1nk_0n} = 1$, for $m = 1,3$, and $\hat{x}^1_{nj_0nl_0} = \hat{x}^3_{nj_0nl_q} = 1, \bar{x}^1_{nnk_0l_0} = \bar{x}^3_{nnk_0l_q} = 1$. Thus, $a\hat{x}^1 = a\hat{x}^3$ yields $\pi_{nj_0nl_0} = \pi_{nj_0nl_q}$ and $a\bar{x}^1 = a\bar{x}^3$ yields $\pi_{nnk_0l_0} = \pi_{nnk_0l_q}$.

Let $x' = x^1(i_0 \leftrightarrow n)_1(i_0 \leftrightarrow i_1)_1$. At x', let $j_1 \in J \setminus \{j_0, n\}$ be such that $l(n, j_1) = n$. We denote $k(n, j_1)$ as k_1. Let $\tilde{x}^1 = x'(j_1 \leftrightarrow j_0)_2$. Also, let $\tilde{x}^2 = \tilde{x}^1(i_0 \leftrightarrow i_q)_1, \tilde{x}^3 = \tilde{x}^1(k_0 \leftrightarrow k_q)_3$, $i_q \in I \setminus \{i_0, n\}, k_q \in K \setminus \{k_0, n\}$. $\tilde{x}^m \in P_I(Q)$ since $\tilde{x}^m_{nj_0k_1n} = \tilde{x}^m_{nnnl_0} = 1$, for $m = 1,2,3$, and $\tilde{x}^1_{i_0nk_0n} = 1$, $\tilde{x}^2_{i_qnk_0n} = 1$, $\tilde{x}^3_{i_0nk_qn} = 1$. $a\tilde{x}^1 = a\tilde{x}^2$ and $a\tilde{x}^2 = a\tilde{x}^3$ yield $\pi_{i_0nk_0n} = \pi_{i_qnk_0n} = \pi_{i_0nk_qn}$. Following exactly the same procedure for points $\check{x}^m = \tilde{x}^m(I \leftrightarrow J)$ we obtain $\pi_{nj_0k_0n} = \pi_{nj_qk_0n} = \pi_{nj_0k_qn}$.

Hence, for $i_q \in I \setminus \{i_0, n\}, j_q \in J \setminus \{j_0, n\}, k_q \in K \setminus \{k_0, n\}, l_q \in L \setminus \{l_0, n\}$, we have shown that

$$\begin{aligned} \pi_{i_0 n n l_0} &= \pi_{i_0 n n l_q}, \\ \pi_{n n k_0 l_0} &= \pi_{n n k_0 l_q}, \\ \pi_{n j_0 n l_0} &= \pi_{n j_0 n l_q}, \\ \pi_{i_0 n k_0 n} &= \pi_{i_0 n k_q n} = \pi_{i_q n k_0 n}, \\ \pi_{n j_0 k_0 n} &= \pi_{n j_0 k_q n} = \pi_{n j_q k_0 n} \end{aligned}$$

Step 3: Consider points x^1, x^2 of step 3 of Thm. 4.12. Let $\hat{x}^m = x^m(I \leftrightarrow L)$, for $m = 1, 2$. $\hat{x}^m \in P_I(Q)$ since $\hat{x}^m_{n j_0 n l_0} = \hat{x}^m_{n n k_0 l_1} = 1$, for $m = 1, 2$, and $\hat{x}^1_{i_0 n n l_q} = 1$, $\hat{x}^2_{i_0 n n n} = 1$. Thus, $a\hat{x}^1 = a\hat{x}^2$ yields $\pi_{i_0 n n l_q} = \pi_{i_0 n n n}$. Let $\bar{x}^m = x^m(K \leftrightarrow L)(I \leftrightarrow L)$, for $m = 1, 2$. $\bar{x}^m \in P_I(Q)$ since $\bar{x}^m_{n j_0 n l_0} = \bar{x}^m_{i_0 n n l_1} = 1$, for $m = 1, 2$, and $\bar{x}^1_{n n k_0 l_q} = 1$, $\bar{x}^2_{n n k_0 n} = 1$. Thus, $a\bar{x}^1 = a\bar{x}^2$ yields $\pi_{n n k_0 l_q} = \pi_{n n k_0 n}$. Let $\tilde{x}^m = x^m(I \leftrightarrow L)$, for $m = 1, 2$. $\tilde{x}^m \in P_I(Q)$ since $\tilde{x}^m_{n n k_0 l_0} = \tilde{x}^m_{i_0 n n l_1} = 1$, for $m = 1, 2$, and $\tilde{x}^1_{n j_0 n l_q} = 1$, $\tilde{x}^2_{n j_0 n n} = 1$. Thus, $a\tilde{x}^1 = a\tilde{x}^2$ is valid, yielding $\pi_{n j_0 n l_q} = \pi_{n j_0 n n}$.

Hence,

$$\pi_{i_0 n n l_0} = \pi^1, \pi_{n j_0 n l_0} = \pi^2, \pi_{n n k_0 l_0} = \pi^3$$

Step 4: Consider points x^1, x^2, x^3, x^4, x^5 established at step 4 of Thm. 4.12. Observe that $x^1, x^2, x^4, x^5 \in P_I(Q)$. $ax^1 = ax^2$ yields $\pi_{i_0 n n l_q} = \pi_{i_0 n k_q n}$, $ax^3 = ax^4$ yields $\pi_{i_1 n k_0 n} = \pi_{n j_1 k_0 n}$. These two equations imply

$$\pi^1 = \pi_{i_0 n n l_0} = \pi_{i_0 n n l_q} = \pi_{i_0 n k_q n} = \pi_{i_0 n k_0 n} = \pi_{i_q n k_0 n} = \pi_{n j_q n k_0 n} = \pi_{n j_0 k_q n} = \pi_{n j_0 k_0 n}$$

$ax^1 = ax^5$ yields $\pi_{n j_0 n n} + \pi_{n n k_0 l_0} + \pi_{i_0 n k_q n} = \pi_{i_0 n n n} + \pi_{n n k_0 l_0} + \pi_{n j_0 k_q n}$, which by the above equation becomes $\pi_{n j_0 n n} = \pi_{i_0 n n n}$, resulting in $\pi^1 = \pi^2$.

Let $x^6 = x^{\mathrm{Tbl}(16)}$. Let $x^7 = x^6(k_0 \leftrightarrow n)_3$. $x^6, x^7 \in P_I(Q)$ since $x^6_{i_0 j_0 n n} = x^6_{n n k_0 l_0} = x^6_{i_0 n k_1 n} = 1$, $x^7_{n j_0 k_0 n} = x^7_{n n n l_0} = x^7_{i_0 n k_1 n} = 1$. Hence, $ax^6 = ax^7$ yields $\pi_{n n k_0 l_0} = \pi_{n n n l_0}$, implying $\pi^3 = \pi^4$.

Let $x^8 = x^3(I \leftrightarrow L)$. $x^8 \in P_I(Q)$ since $x^8_{n j_0 n l_0} = x^8_{i_0 n n n} = x^8_{n n k_0 l_1} = 1$. Hence, $ax^3 = ax^8$ yields $\pi_{i_0 j_0 n n} + \pi_{n n n l_0} + \pi_{i_1 n k_0 n} = \pi_{n j_0 n l_0} + \pi_{i_0 n n n} + \pi_{n n k_0 l_1}$, implying $x_{i_0 j_0 n n} = \pi^2$.

Finally, let $x^9 = x^3(i_0 \leftrightarrow n)_1$. $x^9 \in P_I(Q)$ since $x^9_{n j_0 n n} = x^9_{i_0 n n l_0} = x^9_{i_1 n k_0 n} = 1$. Hence, $ax^3 = ax^9$ yields $\pi_{i_0 j_0 n n} + \pi_{n n n l_0} = \pi_{n j_0 n n} + \pi_{i_0 n n l_0}$, which leads to $\pi_{n n n l_0} = \pi^1$, implying $\pi^1 = \pi^4$.

The proof of $(4.3)_\pi$ is complete. The rest of the proof is exactly as in Thm. 4.12. ■

5 Separation

Facet-defining inequalities are of great importance since they describe the convex hull of integer solutions for a problem. Therefore, if we knew all facets of an integer polytope, we would be able to solve the integer problem by incorporating them into the constraint matrix and then solving the linear programming (LP) relaxation. In practice, however, this is not easy, since for most problems a) not all the facets of the underlying convex hull of integer points are known, and b) the number of facets is not polynomially bounded on the size of the problem, therefore resulting in a constraint matrix of exponential size. For these reasons, most algorithms consider the known facet defining inequalities only when they are violated by the solution of the LP-relaxation. To identify the facets violated by such a solution constitutes the *separation* problem. In general, this problem is $\mathcal{NP}$-hard. However, for some families of facet-defining inequalities this problem can be solved in polynomial time.

For the OLS problem, a polynomial time separation algorithm for each of the two classes of clique facets is described in [1]. Motivated by that work, we present two polynomial separation algorithms for wheel facets; one for inequalities induced by the wheel class 29 and the other for inequalities induced by the wheel class 3.

The following conventions are used. P_L denotes the linear relaxation of P_I. For $u \in U$ $(U \subseteq C)$ we denote $\bar{u}$ any element of U such that $|u \cap \bar{u}| = 0$. In the algorithms that follow, U is defined in such a way that for every $u \in U$ there exists exactly one $\bar{u} \in U$. For $U \subseteq C, x(U) = \sum\{x_u : u \in U\}$.

5.1 The inequalities of wheel class 29

Observe that (4.1), for $p = 2$, $Q = (Q^2(c) \cap Q^2(s)) \cup Q^3(c)$, can be written as

$$d(c) + d(c, s) \leq 2 \tag{5.1}$$

where $d(c) = x_s + x(Q^3(c)), d(c, s) = x_c + x(Q^2(c) \cap Q^2(s))$. Observe that if no clique of type II is violated then $d(c) \leq 1, \forall c \in C$. For specific values of c and s the inequality (5.1) is denoted as (5.1)$[c, s]$.

To establish the complexity of the algorithm 5.1, we make use of the following lemma.

Lemma 5.1 ([1]) *For a point $x \in P_L$ and $z > 0$, the number of components of x with value $\geq z$ is $\leq \frac{n^2}{z}$.*

Proposition 5.2 *Algorithm 5.1 determines in $O(n^6)$ steps whether a given $x \in P_L \setminus P_I$ which does not violate a clique facet of type II violates a wheel inequality (5.1).*

Proof. (Correctness) Suppose that no clique inequality of type II is violated. Then, (5.1) is violated only if

$$x_{i_c j_c k_c l_c} + x_{i_c j_c k_s l_s} + x_{i_c j_s k_c l_s} + x_{i_s j_c k_c l_s} + x_{i_s j_c k_s l_c} + x_{i_c j_s k_s l_c} + x_{i_s j_s k_c l_c} > 1 \tag{5.2}$$

Algorithm 5.1. Separation of inequalities induced by wheel class 29.
Let $x \in P_L \setminus P_I$ be such that no clique inequality of type II is violated.

STEP 1 FOR ALL $c \in C$ let $d(c) = x_c + \sum\{x_q : q \in Q^3(c)\}$;
STEP 2 FOR ALL $c \in C$ IF $1 > x_c > \frac{1}{7}$ THEN
{
STEP 3 FOR ALL $t \in C$ such that $|c \cap t| = 2$ IF $x_t > \frac{1-x_c}{6}$ THEN
STEP 4 FOR ALL $s \in C$ such that $|c \cap s| = 0, |s \cap t| = 2$,
STEP 5 {
$U = (Q^2(c) \cap Q^2(s)) \cup \{c\} \setminus \{\bar{t}\}$;
FOR ALL $u \in U$ IF $d(u) + x(U) - x_{\bar{u}} > 2$ THEN (5.1)$[u, \bar{u}]$ is violated;
};
STEP 6 FOR ALL $s \in C$ such that $|c \cap s| = 0$, IF $x_s > \frac{1-x_c}{6}$ THEN
STEP 7 {
$U = (Q^2(c) \cap Q^2(s))$
FOR ALL $u \in U$ IF $d(u) + x(U) - x_{\bar{u}} > 2$ THEN (5.1)$[u, \bar{u}]$ is violated;
};
};

This implies that at least one of the variables of (5.2) has a value greater than $\frac{1}{7}$. Let this variable be denoted as $x_c (c \in C)$. Among the remaining variables of (5.2) at least one must be greater than $\frac{1-x_c}{6}$. Let this variable be denoted as $x_v (v \in C)$. There are two cases; either $|c \cap v| = 2$ or $|c \cap v| = 0$. In both cases we calculate the left-hand side for every inequality (5.1) containing both x_c, x_v.

(Complexity) In Step 1 for every $c \in C$, we perform $4(n-1)$ additions. Hence, in total, we perform $4n^4(n-1)$ additions and n^4 assignments. In Step 2, the block of code containing all other steps is executed at most $7n^2$ times (Lem. 5.1). For each c of Step 3 we scan 6 rows of the A matrix. For each row we consider $(n-1)^2$ elements to play the role of t. Hence, Step 3 (i.e. the comparison $\frac{1-x_s}{6}$) is executed at most $42n^2(n-1)^2$. Observe that at each row we can have at most 6 variables with value $> \frac{1-x_c}{6}$. Thus Step 4 is executed at most $36 \cdot 7n^2$ times. At each such iteration $(n-1)^2$ elements are considered for the role of s. Hence, Step 5 is executed at most $36 \cdot 7n^2 \cdot (n-1)^2$ times.

For each of the $c \in C$, such that $1 > x_c > \frac{1}{7}$, Step 6 is executed $(n-1)^4$ times. In total, the comparison $x_s > \frac{1-x_c}{6}$ of Step 6 is executed at most $7n^2(n-1)^4$ times. Observe that the number of operations of steps 5 and 7 is constant. This is because $|U| = 6$. Hence, the overall complexity of the algorithm is $O(n^6)$. ■

5.2 The inequalities of wheel class 3

The set Q is defined with respect to $c, s, t \in C$ ($|c \cap s| = 0, |c \cap t| = 2 = |s \cap t|$). Observe that $Q = Q(c,t) \cup (Q^2(c) \cap Q^2(s))$, where $Q(c,t) = (Q^2(t) \cap Q^3(c)) \cup (Q^3(t) \cap Q^3(c)) \cup (Q^2(c) \cap Q^3(t))$. Then, the inequalities of this class can be written as

$$x_c + x_t + x(Q(c,t)) + x_c + x((Q^2(c) \cap Q^2(s)) \setminus \{t\}) \leq 2 \tag{5.3}$$

Observe that for any $x \in P_L \setminus P_I$, $x_c + x_t + x(Q(c,t)) \leq 1$, because all these terms appear in the same row of the A matrix. Hence, a necessary condition for (5.3) to be violated is $x_s + x((Q^2(c) \cap Q^2(s)) \setminus \{t\}) > 1$ (by definition $t \in Q^2(c) \cap Q^2(s)$). This observation is used by the following algorithm. Again (5.3)$[c, s, t]$ denotes (5.3) for specific values of c, s, t.

Proposition 5.3 *Algorithm 5.2 determines in $O(n^6)$ steps whether a given $x \in P_L \setminus P_I$ violates a wheel inequality (5.3).*

Proof. (Correctness) As stated previously, (5.3) is violated only if

$$x_c + x((Q^2(c) \cap Q^2(s)) \setminus \{t\}) > 1 \tag{5.4}$$

There are six variables in (5.4), therefore at least one of them must be greater than $\frac{1}{6}$. Let this variable be denoted as $x_c (c \in C)$. Among the remaining variables of (5.2) at least one must be greater than $\frac{1-x_c}{5}$. Let this variable be denoted as $x_v (v \in C)$. There are two cases; either $|c \cap v| = 2$ or $|c \cap v| = 0$. In both cases we calculate the left hand side for every inequality (5.3) containing both x_c,x_v.

(Complexity) In Step 1 for every $c \in C$, we perform $6 \cdot 2(n-1)$ additions. In total, $12 \cdot n^4(n-1)$ additions and n^4 assignments are executed. The boolean expression in Step 2 evaluates true at most $6n^2$ times (Lem. 5.1). For each c of Step 3 we scan 6 rows of the A matrix. For each row we consider $(n-1)^2$ elements to play the role of t. Hence Step 3 (i.e. the comparison $\frac{1-x_c}{5}$) is executed at most $42n^2(n-1)^2$. Observe that at each row we can have at most 5 variables with value $> \frac{1-x_c}{5}$. Thus, Step 4 is executed at most $30 \cdot 6n^2$ times. At each such iteration $(n-1)^2$ elements are considered for the role of s. Hence, Step 5 is executed at most $30 \cdot 6n^2 \cdot (n-1)^2$ times. For each of the $c \in C, 1 > x_c > \frac{1}{6}$ Step 6 is executed $(n-1)^4$ times. In total, the comparison $x_s > \frac{1-x_c}{5}$ of Step 6 is executed at most $6n^2(n-1)^4$ times.

The number of additions performed in Steps 5 and 7 is constant. First observe that for given $u_1, u_2 \in C$ such that $|u_1 \cap u_2| = 2$, $\left|Q^3(u_1) \cap Q^3(u_2)\right| = 2$. For Step 5, if $u_1 = c$ or $u_1 = t$ then $|U \setminus \{u_1, \bar{u}_1, c, t\}| = 5$ else if $u_1 \neq c, t$ then $|U \setminus \{u_1, \bar{u}_1, c, t\}| = 4$. Hence, 26 comparisons will be performed in Step 6, each of which requires a constant number of additions. Due to the same reasoning $6 \cdot 4$ comparisons will be executed in Step 7.

Hence, the overall complexity of the algorithm is $O(n^6)$. ∎

Algorithm 5.2. Separation of inequalities induced by wheel class 3.
Let $x \in P_L \setminus P_I$.

STEP 1 FOR ALL $c \in C$

{

$d^{K\otimes L}(c) = x_c + \sum_{i\neq i_c} x_{ij_ck_cl_c} + \sum_{j\neq j_c} x_{i_cjk_cl_c}$;

$d^{J\otimes L}(c) = x_c + \sum_{i\neq i_c} x_{ij_ck_cl_c} + \sum_{k\neq k_c} x_{i_cj_ckl_c}$;

$d^{I\otimes L}(c) = x_c + \sum_{j\neq j_c} x_{i_cjk_cl_c} + \sum_{k\neq k_c} x_{i_cj_ckl_c}$;

$d^{J\otimes K}(c) = x_c + \sum_{i\neq i_c} x_{ij_ck_cl_c} + \sum_{l\neq l_c} x_{i_cj_ck_cl}$;

$d^{I\otimes K}(c) = x_c + \sum_{j\neq j_c} x_{i_cjk_cl_c} + \sum_{l\neq l_c} x_{i_cj_ck_cl}$;

$d^{I\otimes J}(c) = x_c + \sum_{k\neq k_c} x_{i_cj_ckl_c} + \sum_{l\neq l_c} x_{i_cj_ck_cl}$;

};

STEP 2 FOR ALL $c \in C$ IF $1 > x_c > \frac{1}{6}$ THEN

{

STEP 3 FOR ALL $t \in C$ such that $|c \cap t| = 2$ if $x_t > \frac{1-x_c}{5}$ THEN

STEP 4 FOR ALL $s \in C$ such that $|s \cap c| = 0, |t \cap s| = 2$,

STEP 5 {

$U = (Q^2(c) \cap Q^2(s)) \cup \{c, s\}$;

FOR ALL $u_1 \in U \setminus \{s, t\}$

FOR ALL $u_2 \in U \setminus \{u_1, \bar{u}_1, c, t\}$

{

Let M, M' $(M \neq M')$ be any of I, J, K, L such that $(u_1 \cap u_2) \in M \otimes M'$;

IF $d^{M\otimes M'}(u_1) + d^{M\otimes M'}(u_2) + x(U) - x_{\bar{u}_1} - x_{u_2} + x(Q^3(u_1) \cap Q^3(u_2)) > 2$ THEN (5.3)$[u_1, \bar{u}_1, u_2]$ is violated;

}

};

STEP 6 FOR ALL $s \in C$ such that $|c \cap s| = 0$, IF $x_s > \frac{1-x_c}{5}$ THEN

STEP 7 {

$U = (Q^2(c) \cap Q^2(s)) \cup \{c, s\}$;

FOR ALL $u_1 \in U \setminus \{c, s\}$

FOR ALL $u_2 \in U \setminus \{u_1, \bar{u}_1, c, s\}$

{

Let M, M' $(M \neq M')$ be any of I, J, K, L such that $(u_1 \cap u_2) \in M \otimes M'$;

IF $d^{M\otimes M'}(u_1) + d^{M\otimes M'}(u_2) + x(U) - x_{\bar{u}_1} - x_{u_2} + x(Q^3(u_1) \cap Q^3(u_2)) > 2$ THEN (5.3)$[u_1, \bar{u}_1, u_2]$ is violated;

}

}

};

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