Properties and Classification of the Wheels of the OLS Polytope.

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Abstract

A wheel in a graph $G(V, E)$ is an induced subgraph consisting of an odd hole and an additional node connected to all nodes of the hole. In this paper, we study the wheels of the column intersection graph of the OLS polytope (P_I). These structures induce valid inequalities for this polytope, which are facet defining for its set packing relaxation. Our work builds on simple structural properties of wheels which are used to categorise them into a number of collectively exhaustive classes. Each such class gives rise to a set of valid inequalities for P_I. Moreover, this classification allows us to estimate the cardinality of the whole wheel class as well as to derive a recognition algorithm for the circulant matrices corresponding to wheels of a particular type.

In a forthcoming paper, we show for some of the wheel classes presented here that they give rise to facet-defining inequalities for P_I.

1 The problem

Consider four disjoint n-sets, namely I, J, K, L, and a mapping $w : I \times J \times K \times L \longrightarrow \mathbb{R}$. The *Orthogonal Latin Squares (OLS)* problem, formulated as a $0-1$ minimisation program, is the following:

$$\min \sum \{w_{ijkl} \cdot x_{ijkl} : i \in I, j \in J, k \in K, l \in L\}$$

s.t.

$$\sum \{x_{ijkl} : i \in I, j \in J\} = 1, \forall k \in K, l \in L, \tag{1a}$$

$$\sum \{x_{ijkl} : j \in J, k \in K\} = 1, \forall i \in I, l \in L, \tag{1b}$$

$$\sum \{x_{ijkl} : i \in I, k \in K\} = 1, \forall j \in J, l \in L, \tag{1c}$$

$$\sum \{x_{ijkl} : k \in K, l \in L\} = 1, \forall i \in I, j \in J, \tag{1d}$$

$$\sum \{x_{ijkl} : i \in I, l \in L\} = 1, \forall j \in J, k \in K, \tag{1e}$$

$$\sum \{x_{ijkl} : j \in J, l \in L\} = 1, \forall i \in I, k \in K, \tag{1f}$$

$$x_{ijkl} \in \{0, 1\}, \forall i \in I, j \in J, k \in K, l \in L \tag{1g}$$

This formulation is due to D. Gale (in [10]). Alternative formulations of the problem are presented in [1].

Let A denote the coefficient matrix of the constraint sets (1a), ..., (1f). Also let $P = \{x \in \mathbb{R}^{n^4} : Ax = e, x \geq 0\}$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^{n^4}$. Then, $P_I = \text{conv}\{x \in \{0,1\}^{n^4} : x \in P\}$ is the *Orthogonal Latin Squares* polytope of order n. We briefly introduce some definitions. A Latin square, of order n, is a $n \times n$ square matrix where each value $1, \ldots, n$ appears exactly once in each row and column. Given two Latin squares, let $\mathbb{L}$ denote the set of pairs, where each pair consists of the elements of the first and the second square lying in the same row and column. Then, the two Latin squares are called *orthogonal* (equivalently, form an OLS structure or a pair of OLS) if and only if $\mathbb{L} = \{(1,1), \ldots, (n,n)\}$. Sets of k Latin squares ($k > 2$) are called *mutually orthogonal (MOLS)* if and only if they are pairwise orthogonal. Thus, the OLS problem can be generalized to the k-*MOLS* problem. In this context, the *OLS* problem is the 2-*MOLS* problem.

The $1-1$ correspondence between Orthogonal Latin squares and points of P_I becomes apparent if, for the OLS structure, we consider the set I as the row set, J as the column set, K (L) as the set of elements of the first (second) square. Evidently the roles of the sets are conventional, therefore interchangeable. Polytope P_I is a special case of the set partitioning polytope defined as $P_{SPP} = \text{conv}\{x \in \{0,1\}^m : Dx = e\}$, where D is a $0-1$ matrix and e is a vector of ones of the appropriate size. If we substitute " $=$ " with " $\leq$ ", we obtain the *set packing* relaxation of P_{SPP}, denoted as $\tilde{P}_{SPP}$. Thus, the set packing relaxation of P_I is defined as $\tilde{P}_I = \text{conv}\{x \in \{0,1\}^{n^4} : Ax \leq e\}$. Observe that $P_I \subset \tilde{P}_I$.

Orthogonal Latin squares and related structures bear a long history in the literature of combinatorics. Their relation to finite algebra and to the theory of hypercubes, affine and projective planes along with the large number of applications in various fields, like (t,m,s)-nets, block designs and optimal codes, has motivated a vast amount of research. The reader may find an extensive discussion of the subject in [11, 15]. Nevertheless, only a limited part of the literature addresses the facial structure of P_I and its relaxations (see [2]). A recent classification ([16]), establishing that the Latin Square problem and the k-*MOLS* problem, where $k \leq n-1$ ([11, Theorem 5.1.5]), form the class of assignment problems of the *second* order (equivalently the class of *planar* assignment problems), provides further incentives for pursuing the study of P_I. This classification reveals the relation of P_I to other assignment polytopes. These are primarily the Latin square polytope and secondarily the polytopes of the assignment problems of the *first* order (equivalently the polytopes of *axial* assignment problems). For the Latin square polytope, a class of facets is presented in [12] and another in [3]. For P_I, all classes of clique facets are presented in [2] together with the corresponding separation procedures. The dimension of all the polytopes of the planar assignment problems is established in [16]. Among the axial assignment polytopes, the three-index assignment polytope has been substantially studied ([4, 5, 13, 18]). A non-trivial class of facets for all axial assignment polytopes is presented in [16].

In the current work, we study the subgraphs of the *column intersection* graph of the A matrix called *wheels*. These structures induce inequalities which are facet defining for $\tilde{P}_I$ and valid for P_I (see [14, p. 300]). Formal definitions are given in section 2. Properties leading to a concise characterisation of wheel structures are exhibited in sections 3 and 4. A classification scheme, encompassing all wheel structures, is presented in section 5. Finally, a recognition algorithm is given in section 6.

2 Basic definitions and conventions

We recall a number of definitions introduced in [2]. The columns of the A matrix are indexed by the tuples (i, j, k, l). Therefore, the index set of the columns of A is defined as $C = I \times J \times K \times L$. For the rows of A, observe that each of the constraint sets (1a), ..., (1g) is indexed by the product of two of the sets I, J, K, L. For example, the index set of (1a) is $K \times L$. Hence, the index set for the rows of A is $R = (K \times L) \cup (I \times L) \cup (J \times L) \cup (I \times J) \cup (J \times K) \cup (I \times K)$. For $c \in C$, let a^c denote a column of A. The *column intersection* graph of A, denoted as $G_A(C, E_C)$, has a node c for every column $a^c \in A$ and an edge $(c, d) \in E_C$ ($d \in C$) corresponding to columns a^c, a^d with $a^c \cdot a^d \geq 1$. Note that if c, d have two indices in common ($|c \cap d| = 2$), then $a^c \cdot a^d = 1$, whereas if c, d have three indices in common ($|c \cap d| = 3$), then $a^c \cdot a^d = 3$.

Let M_i ($i \in \{1, \ldots, 4\}$) denote any of the sets I, J, K, L. This definition allows us to refer to these sets in an abstract and general mode. For $c \in C$, let $m_i(c)$ denote the index of the set M_i at node c. The sets M_i, M_j ($i \neq j$, $i, j \in \{1, 2, 3, 4\}$) define the *double set* $M_i \otimes M_j = (M_i \times M_j) \cup (M_j \times M_i)$. By definition, $M_i \otimes M_j \equiv M_j \otimes M_i$. For $c, d \in C$ such that $|c \cap d| = 2$, the edge (c, d) is said to be *based* on a double set, for example $M_1 \otimes M_2$, if and only if $m_1(c) = m_1(d)$, $m_2(c) = m_2(d)$ and $m_t(c) \neq m_t(d), t = 3, 4$. Equivalently, the *ground* set of the edge (c, d) is $M_1 \otimes M_2$. In this case, we write $c \cap d \in M_1 \otimes M_2$.

Example 1 *Let* $c = (1,1,1,1), d = (1,1,2,2)$ *and* $M_1 = I, M_2 = J$. *Clearly,* $m_1(c) = m_1(d) = 1$ *and* $c \cap d \in M_1 \otimes M_2$.

All these conventions are extended in the case where nodes c and d have three indices in common. For example, assume that the common indices belong to the sets M_1, M_2, M_3. Then, the edge (c,d) is based on the *triple* set which is associated to these three single sets. To avoid a lengthy notation, we denote a triple set again with the use of operator $\otimes$. This operator is used in a slightly different context than in the double set case. Thus, if the triple set consists of sets M_1, M_2, M_3, it is denoted as $M_1 \otimes M_2 \otimes M_3 \equiv (M_1 \otimes M_2) \cup (M_2 \otimes M_3) \cup (M_1 \otimes M_3)$. Hence, each triple set is simply the union of three double sets. These double sets are called *components* of the triple set.

Occasionally, an edge based on a double (triple) set will be referred to as a double (triple) link. Equivalently, in the first case we have a *double-set* edge whereas in the second case a *triple-set* edge.

Remark 2 *There are six distinct double sets. Each of the four single sets appears in exactly three of the double sets. There are four distinct triple sets. Each of the double sets appears in exactly two triple sets. Two triple sets have exactly one common component, i.e. double set. Two triple sets differ in exactly one single set.*

For $H \subset C$, the subgraph of G_A, induced by H, is naturally defined as the graph formed by all nodes of subset H and all edges of E_C, which connect any two nodes in H. Let us define certain types of subgraphs of $G_A(C, E_C)$.

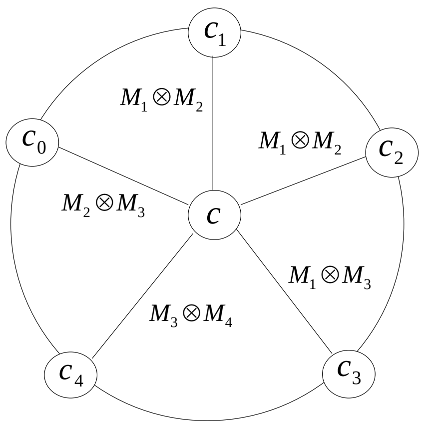
Definition 3 *A node set* $H \subset C$ *such that* $|H| = 2p+1$ *for some positive integer* $p \geq 2$, *induces an odd hole in* $G_A(C, E_C)$ *if and only if* H *can be ordered into a sequence* $\{c_0, \ldots, c_{2p}\}$ *such that for all* $c_s, c_t \in H$

$$|c_s \cap c_t| = \begin{cases} 2 \text{ or } 3 \text{ if } t = s \pm 1 \bmod 2p+1 \\ 0 \text{ or } 1 \text{ otherwise} \end{cases}$$

The above definition implies that two nodes of H are joined by an edge if and only if they are ordered consequently in the circular sequence $\{c_0, \ldots, c_{2p}\}$, i.e. the hole has no chords. The cardinality of the set H, denoted as q, is called the *size* of the odd hole ($q = 2p+1$). In general, inequalities arising from maximally lifted odd holes are known to be facet-inducing for $\tilde{P}_{SPP}$ ([17]). Wheels are essentially a special type of lifted odd holes. Specifically for $G_A(C, E_C)$, wheels are defined as follows.

Definition 4 *For* $c \in C$, *let* $H(c)$ *denote the node set of an odd hole of* $G_A(C, E_C)$, *such that* $c \notin H(c)$ *and* $|c \cap c_s| \geq 2$ *for every* $c_s \in H(c)$. *Then, the node set* $W_c = \{c\} \cup H(c)$ *induces a wheel in* $G_A(C, E_C)$.

Where appropriate, the node set of a wheel will be denoted as W_c^p, so that notation includes a reference to the size of the wheel. Node c is called the *hub* of the wheel. The node set $H(c)$ constitutes the *rim* of the wheel. Edges connecting the nodes of $H(c)$ are called *rim* edges, while edges (c, c_s), for $c_s \in H(c)$,



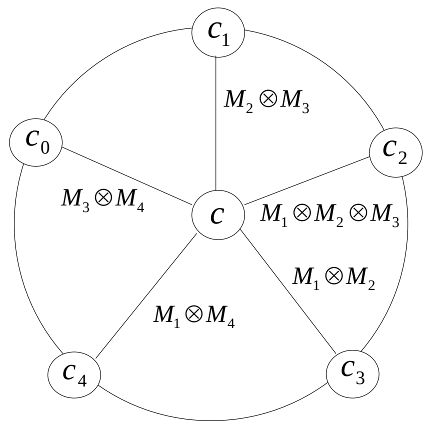


Figure 1a: Double-set spokes only

Figure 1b: Double- and triple-set spokes

are called *spokes*. Let $E_{S_{H(c)}}$, $E_{H(c)}$ denote the sets of spokes and rim edges, respectively. The set of the edges of the wheel is defined as $E_{W_c} = E_{S_{H(c)}} \cup E_{H(c)}$. The set $H(c)$ can be further partitioned into two subsets, namely $H^1(c), H^2(c)$. All the nodes incident to a double-set (triple-set) spoke belong to $H^1(c)$ ($H^2(c)$). Two wheels of size five ($p = 2$) are illustrated in Figures 1a, 1b, where the ground sets of the spokes are also depicted. Observe that, for the wheel of Figure 1a, $H^2(c) = \emptyset$.

We introduce a number of elaborate definitions, which will facilitate the discussion of the forthcoming section. Two rim nodes are called *adjacent* if they are incident to the same rim edge. Similarly, two spokes are adjacent if they are incident to adjacent rim nodes. A spoke is adjacent to a rim edge if they are both incident to the same rim node. Therefore, two adjacent spokes have a common adjacent rim edge. Let $c_1, c_2 \in H(c)$ be two adjacent nodes of the rim. If follows that nodes c_1, c_2 together with the hub (c) induce a 3-clique, normally denoted as K_3. To refer to the specific K_3, we write $K_3(c_1, c_2)$. The edges of this structure are the two spokes (c, c_1), (c, c_2) and the rim edge (c_1, c_2). Depending on the ground sets of the spokes, we distinguish 3-cliques of types 1, 2, 3 and 4, hereafter denoted as K_3^1, K_3^2, K_3^3 and K_3^4, respectively. The first two types denote K_3 structures where both spokes are based on double sets. If both spokes are based on the same double set, the structure is K_3^1; otherwise, it is K_3^2. Type K_3^3 implies that the one of the two spokes is based on a double set and the other on a triple set. Finally, K_3^4 consists of two spokes both based on triple sets. A double set appearing at the ground sets of both spokes of a K_3 (even as a component of a triple set), is called the *ground set* of the K_3. By definition, the ground set of K_3^1 is the double set of its spokes, whereas no ground set can be defined for K_3^2. In Section 4, we define more clearly the ground sets for K_3^3 and K_3^4.

In general, the ground set of a structure being an edge or a K_3 is denoted as $g(\textit{structure})$. Function g receives a structure as input and returns its ground set. In the case that the structure has no ground set, it returns $\emptyset$.

Example 5 *Let $c_1, c_2 \in C$ be two adjacent rim nodes such that $c \cap c_1, c \cap c_2 \in M_1 \otimes M_2$. The fact that $K_3^1(c_1, c_2)$ is based on $M_1 \otimes M_2$ is denoted as $g(K_3^1(c_1, c_2)) = M_1 \otimes M_2$. Notice that $g((c, c_2)) = M_1 \otimes M_2$ and $c \cap c_2 \in M_1 \otimes M_2$ are equivalent expressions, denoting that the spoke (c, c_2) is based on the double set $M_1 \otimes M_2$.*

The set of 3-cliques of type t $(1 \leq t \leq 4)$ for the wheel induced by W_c, is denoted as $W_c(K_3^t)$. Hence, the set of all K_3 in a wheel is defined as $W_c(K_3) = \bigcup_{t=1}^4 W_c(K_3^t)$. For the wheel of Figure 1b, $W_c(K_3^1) = \emptyset$, $W_c(K_3^2) = \{K_3(c_3, c_4),\ K_3(c_4, c_0),\ K_3(c_0, c_1)\}$, $W_c(K_3^3) = \{K_3(c_1, c_2),\ K_3(c_2, c_3)\}$, $W_c(K_3^4) = \emptyset$. The notion of adjacency can be extended to 3-cliques. Hence, two K_3 are called adjacent if they share a common spoke. In this case, we say that they are 0-*distant*. In general, two K_3 are t-*distant* if there exist t $(0 \leq t \leq 2p-1)$ K_3 structures between them. Note that the notion of distance assumes a direction according to which the wheel is examined (e.g. clockwise).

A wheel of OLS can be represented as sequence of the ground sets of its spokes. These sequences are cyclic, therefore it is of no importance which ground is first. To avoid repetition, a double set enclosed in square brackets represents two adjacent spokes, both based on it, i.e. a K_3^1 structure. According to this notation, the wheels of Figures 1a and 1b are presented as:

$$\text{(1a)} \qquad (M_2 \otimes M_3) - [M_1 \otimes M_2] - (M_1 \otimes M_3) - (M_3 \otimes M_4)$$

$$\text{(1b)} \qquad (M_3 \otimes M_4) - (M_2 \otimes M_3) - (M_1 \otimes M_2 \otimes M_3) - (M_1 \otimes M_2) - (M_1 \otimes M_4)$$

For the rest of the paper the hub of a wheel will be denoted as c $(c \in C)$.

3 Wheels with $\left|H^2(c)\right| = 0$

This section examines wheels of OLS whose spokes are based exclusively on double sets. Hence, $H(c) = H^1(c)$, i.e. wheels of this class include only K_3^1 and K_3^2 structures. Equivalently, $W_c(K_3) = W_c(K_3^1) \cup W_c(K_3^2)$ for all wheels examined in this section. We initiate the description of this class by presenting a number of elementary properties and their corollaries. The following discussion shows that the major aspects characterising a wheel are the number of distinct K_3^1 structures, the number of rim edges that are triple links and the number of distinct double sets appearing at its spokes. Moreover, this section establishes an upper bound on the wheel size and provides necessary and sufficient conditions for the existence of particular wheels.

Certain statements are presented without proofs, exactly because they constitute direct implications of previously exhibited results.

Lemma 6 *Let $c_s, c_t \in H(c)$ with $s \neq t$ and such that $c \cap c_s \in M_1 \otimes M_2$ and $c \cap c_t \in M_1 \otimes M_2$. Then $c \cap c_s = c \cap c_t$.*

Proof. By inspection. ■

Proposition 7 *Two spokes based on the same double set must be adjacent. At most two spokes can be based on the same double set.*

Proof. Let $c_s, c_t \in H(c)$. Assume that $c \cap c_s, c \cap c_t \in M_1 \otimes M_2$. According to Lem. 6, $|c_s \cap c_t| \geq 2$. Hence, if nodes c_s, c_t are not adjacent there exists a chord. Similarly, if there exists $c_r \in H(c)$, such that $c \cap c_r \in M_1 \otimes M_2$, at least one chord is bound to be formed for $p \geq 2$. ■

Two corollaries of Prop. 7 are the following.

Corollary 8 *Two K_3^1 are at least 1-distant. If they are exactly 1-distant then they are both adjacent to the same K_3^2.*

Corollary 9 *Wheels with $\left|H^2(c)\right| = 0$ include at least three consecutive spokes, each based on a distinct double set.*

Proof. Since the number of spokes is odd, at least one double set appears at a single spoke, whose two adjacent spokes must be based on different double sets. ■

An analogous argument shows the following.

Corollary 10 *For $\left|H^2(c)\right| = 0$, $p \leq 5$.*

Proof. There are six distinct double sets (Rem. 2) and at most two spokes can be based on each one (Prop. 7). By definition, the number of spokes is odd, implying $2p + 1 \leq 11$ or $p \leq 5$. ■

Proposition 11 *The double sets of two adjacent spokes have at least one common set.*

Proof. Let c_s, c_t be two rim adjacent nodes. Let, for example, $c \cap c_s \in M_1 \otimes M_2$, $c \cap c_t \in M_3 \otimes M_4$. Hence, $m_1(c_s) = m_1(c) \neq m_1(c_t)$, $m_2(c_s) = m_2(c) \neq m_2(c_t)$, $m_3(c_s) \neq m_3(c) = m_3(c_t)$ and $m_4(c_s) \neq m_4(c) = m_4(c_t)$. As a result, $|c_s \cap c_t| = 0$ and $(c_s, c_t) \notin E_C$, i.e. a contradiction. ■

The above proposition provides further implications regarding the structure of a wheel.

Corollary 12 *The two spokes of K_3^2 are based on double sets with one set common and one set different.*

Corollary 13 *The ground sets of two 1-distant K_3^1 have a common single set.*

Corollary 14 *There are 12 distinct K_3^2 configurations with respect to the double sets of the two spokes.*

Proof. Follows directly from Rem. 2 and Cor. 12. ■

Proposition 15 *The ground set of the rim edge of a K_3^1 is either the double set of the two spokes or a triple set which has the double set of the two spokes as component.*

Proof. Assume $K_3^1(c_s, c_t)$ such that $c \cap c_s, c \cap c_t \in M_1 \otimes M_2$. In other words, nodes c, c_s, c_t share common values for the indices of the sets M_1, M_2. If $m_3(c_s) = m_3(c_t)$ or $m_4(c_s) = m_4(c_t))$ then $c_s \cap c_t \in M_1 \otimes M_2 \otimes M_3$ or $M_1 \otimes M_2 \otimes M_4$, respectively. ■

Proposition 16 *The ground set of the rim edge of a K_3^2 is a double set consisting of the common set of the double sets of the two spokes and the set not appearing at the double sets of the spokes.*

Proof. Assume $K_3^2(c_s, c_t)$ such that $c \cap c_s \in M_1 \otimes M_2, c \cap c_t \in M_1 \otimes M_3$. This implies $m_1(c_s) = m_1(c) = m_1(c_t)$, $m_2(c_s) = m_2(c) \neq m_2(c_t)$, $m_3(c_s) \neq m_3(c) = m_3(c_t)$. Hence, nodes c_s, c_t have one index in common (for the set M_1) and two indices different (for the sets M_2, M_3). By assumption, $(c_s, c_t) \in E_C$, implying that $m_4(c_s) = m_4(c_t)$. Thus, $c_s \cap c_t \in M_1 \otimes M_4$. Observe that $m_4(c) \neq m_4(c_s), m_4(c_t)$. ■

The next statement is a direct consequence of Props. (15) and (16).

Corollary 17 *The double set of a rim edge has at least one set in common with each of its adjacent spokes.*

Proposition 18 *All sets appear in the distinct double sets of three consecutive spokes.*

Proof. Let c_s, c_t, c_r denote three consecutive rim nodes such that the double sets of $(c, c_s), (c, c_t), (c, c_r)$ are distinct. Assume the three double sets are formed by only three single sets. Observe that the three double sets cannot have a common single set. On the other hand, due to Prop. 11, the double sets of $(c, c_s), (c, c_t)$ must have a common single set. The same is true for $(c, c_t), (c, c_r)$. Without loss of generality, consider the configuration $c \cap c_s \in M_1 \otimes M_2$, $c \cap c_t \in M_2 \otimes M_3$, $c \cap c_r \in M_1 \otimes M_3$. By Prop. 16, $c_s \cap c_t \in M_2 \otimes M_4$, $c_t \cap c_r \in M_3 \otimes M_4$. Hence, $m_4(c_s) = m_4(c_r)$. In addition, $m_1(c_s) = m_1(c) = m_1(c_r)$. Therefore, $|c_s \cap c_r| = 2$, implying that a chord is formed between nodes c_s and c_r. ■

Proposition 19 *If the double sets of two adjacent rim edges have a common set then this set also appears at the double set of their adjacent spoke.*

Proof. Let c_s, c_t, c_r denote three consecutive rim nodes and assume that $c_s \cap c_t \in M_1 \otimes M_2$, $c_t \cap c_r \in M_2 \otimes M_3$. If set M_2 does not appear at the double set of (c, c_t), there are three choices for this double set viz., $M_3 \otimes M_4$, $M_1 \otimes M_4$, $M_1 \otimes M_3$. The case of $M_3 \otimes M_4$ ($M_1 \otimes M_4$) is excluded because, by Props. 15, 16, there exists no double set for the spoke (c, c_s) ((c, c_r)) such that the rim edge (c_s, c_t) ((c_s, c_t)) is based on $M_1 \otimes M_2$ ($M_2 \otimes M_3$). If (c, c_t) is based on $M_1 \otimes M_3$ then, according to Props. 15, 16, the double sets of edges (c, c_s) and (c, c_r) are $M_1 \otimes M_4$ and $M_3 \otimes M_4$. But then $m_4(c_s) = m_4(c) = m_4(c_r)$, which, in conjuction with $m_2(c_s) = m_2(c_r)$, implies the existence of a chord between nodes c_s and c_r. ■

Evidently, the number of K_3^1 structures is a central aspect of a wheel. The next proposition establishes tight bounds on this number.

Proposition 20 *For* $\left|H^2(c)\right| = 0$,

$$p - 1 \leq \left|W_c^p(K_3^1)\right| \leq p, \qquad \text{for } 2 \leq p \leq 4,$$
$$\left|W_c^5(K_3^1)\right| = 5$$

Proof. For $2 \leq p \leq 5$, the upper bound on $\left|W_c^p(K_3^1)\right|$ follows from the fact that a wheel consists of $2p+1$ K_3. According to Cor. 8, there are two K_3^2 adjacent to each K_3^1. Hence, there can be at most p K_3^1. The lower bound will be examined for each case individually.

Assume a wheel consisting of five spokes ($p = 2$), each based on a distinct double set ($\left|W_c^2(K_3^1)\right| = 0$). Without loss of generality, we can fix the double sets of two adjacent spokes by arbitrarily choosing two double sets with one set in common (Prop. 11). The choices for the remaining spokes are limited. Thus, it is easy to see that no combination of the remaining double sets can produce a valid configuration due to violation of Props. 11 and/or 18. Therefore, $\left|W_c^2(K_3^1)\right| \geq 1$.

For $p = 3$ observe that we cannot have $\left|W_c^p(K_3^1)\right| = 0$, since that would imply a wheel with more than seven spokes, each based on a distinct double set. Let $\left|W_c^3(K_3^1)\right| = 1$. In this case, assume that the double set of the spokes of the unique K_3^1 is $M_1 \otimes M_2$. The double sets of the two spokes adjacent to K_3^1 are formed by

(i) either one of the sets M_1, M_2 and the sets M_3, M_4,

(ii) or one of the sets M_3, M_4 and the sets M_1, M_2,

(iii) or the sets M_1, M_2 and the sets M_3, M_4 in such a way that the two double sets do not have a common set.

For case (i), assume that the double sets form the sequence $(M_1 \otimes M_3) - [M_1 \otimes M_2] - (M_1 \otimes M_4)$. According to Prop. 18, M_4 is one of the sets forming the double set of the second spoke adjacent to $(M_1 \otimes M_3)$. The only valid choice for the double set of this spoke is $M_3 \otimes M_4$ ($M_1 \otimes M_4$ and $M_2 \otimes M_4$ are excluded due to Props. 7 and 11 respectively). Similarly, M_3 is one of the sets forming the double set of the other adjacent spoke to $(M_1 \otimes M_4)$. Again the only choice for the double set of this choice is $M_3 \otimes M_4$. However, due to Prop. 7, these two spokes must be adjacent, thus contradicting the fact that $p = 3$.

For case (ii), assume that the double sets form the sequence $(M_1 \otimes M_3) - [M_1 \otimes M_2] - (M_2 \otimes M_3)$. Prop. 18 implies that the set M_4 must appear at the double sets of the two spokes adjacent to this structure. No matter which of the double sets, composed of M_4, will be used for these two spokes, the remaining double sets for the seventh spoke will always violate Prop. 18. Hence, this case is also infeasible.

For case (iii), assume that the double sets form the sequence $(M_1 \otimes M_3) - [M_1 \otimes M_2] - (M_2 \otimes M_4)$. By Prop. 18, the double set of the second spoke adjacent to $(M_1 \otimes M_3)$ must contain the set M_4. For the same reason, the second spoke adjacent to $(M_2 \otimes M_4)$ must contain the set M_3. Whatever the choice of these two double sets, the last spoke to be added for completing the wheel must be based on $M_1 \otimes M_2$, which contradicts Prop. 7. Thus, $\left|W_c^3(K_3^1)\right| \geq 2$.

For $p = 4$ ($p = 5$), if $\left|W_c^p(K_3^1)\right| \leq 2$ ($\left|W_c^5(K_3^1)\right| \leq 4$), we need more than six sets to cover all the spokes of a wheel of size 9 (11). Thus, $\left|W_c^4(K_3^1)\right| \geq 3$ and $\left|W_c^5(K_3^1)\right| \geq 5$. It follows that $\left|W_c^5(K_3^1)\right| = 5$. ■

Two important implications of the previous proposition as examined in the following lemmas.

Lemma 21 *Let D_p denote the number of distinct double sets of the spokes of a wheel. For $\left|H^2(c)\right| = 0$,*

$$p + 1 \leq D_p \leq \min\{6, p + 2\}, \quad 2 \leq p \leq 5$$

Proof. Observe that $D_p = 2p + 1 - \left|W_c^p(K_3^1)\right|$. Substituting from the inequalities of Prop. 20, we obtain the result. ■

Lemma 22 *Let U_p denote the number of spokes not belonging to a K_3^1. For $\left|H^2(c)\right| = 0$,*

$$U_p = \begin{cases} 1 \text{ or } 3, & \text{for } 2 \leq p \leq 4 \\ 1, & \text{for } p = 5 \end{cases}$$

Proof. Observe that $U_p = 2p + 1 - 2\left|W_c^p(K_3^1)\right|$. Substituting $\left|W_c^p(K_3^1)\right|$ by its bounds obtained from Prop. 20 yields the result. ■

Let us now focus on the conditions determining whether a rim edge can be a triple link.

Proposition 23 *Consider a configuration of four consecutive spokes such that the two middle spokes form a K_3^1. Then:*

(a) *the ground set of the rim edge of a K_3^1 is a double set if four sets appear in the double sets of the four consecutive spokes.*

(b) *the ground set of the rim edge of a K_3^1 can be a triple set only if three single sets appear in the double sets of the four consecutive spokes. The triple set consists of these three sets.*

Proof. Let c_s, c_t, c_r, c_u be four consecutive rim nodes such that $K_3^1(c_t, c_r)$. Consequently, we have $K_3^2(c_s, c_t)$, $K_3^2(c_r, c_u)$. Assume that $c \cap c_t, c \cap c_r \in M_1 \otimes M_2$.

(a) If (c_t, c_r) is based on the triple set then this is either $M_1 \otimes M_2 \otimes M_3$, or $M_1 \otimes M_2 \otimes M_4$. There are two cases if four single sets appear in the four double sets:

Case i: The same set appears in all double sets.

Assume that $c \cap c_s \in M_1 \otimes M_3$, $c \cap c_u \in M_1 \otimes M_4$. Then, $c_s \cap c_t \in M_1 \otimes M_4$, $c_r \cap c_u \in M_1 \otimes M_3$ (Prop. (16)). If $c_t \cap c_r \in M_1 \otimes M_2 \otimes M_3$ then $c_t \cap c_u \in M_1 \otimes M_3$. If $c_t \cap c_r \in M_1 \otimes M_2 \otimes M_4$ then $c_s \cap c_r \in M_1 \otimes M_4$.

Case ii: No single set appears in all double sets.

Assume that $c \cap c_s \in M_1 \otimes M_3$, $c \cap c_u \in M_2 \otimes M_4$. Then, $c_s \cap c_t \in M_1 \otimes M_4$, $c_r \cap c_u \in M_2 \otimes M_3$ (Prop. (16)). If $c_t \cap c_r \in M_1 \otimes M_2 \otimes M_3$ then $c_t \cap c_u \in M_2 \otimes M_3$. If $c_t \cap c_r \in M_1 \otimes M_2 \otimes M_4$ then $c_s \cap c_r \in M_1 \otimes M_4$

It follows that (c_t, c_r) cannot be based on a triple set, since this would imply the existence of a chord between either c_s, c_r or c_t, c_u.

(b) As shown in (a), if the double sets of the four spokes consist of all the four sets, the rim edge of the $K_3^1(c_t, c_r)$ is based on a double set. Consider, alternatively, that only three sets appear in the double sets of the four spokes. Without loss of generality, let $c \cap c_s \in M_1 \otimes M_3$, $c \cap c_u \in M_2 \otimes M_3$. According to Prop. (16), $c_s \cap c_t \in M_1 \otimes M_4$, $c_r \cap c_u \in M_2 \otimes M_4$. Hence, (c_t, c_r) can be based on the triple set $M_1 \otimes M_2 \otimes M_3$ without any of the chords (c_s, c_r), (c_t, c_u) being formed.

■

Proposition 24 *Consider two* 1*-distant* K_3^1*. The rim edge of only one of them can be based on a triple set.*

Proof. Assume that the first K_3^1 is based on $M_1 \otimes M_2$ and the second on $M_1 \otimes M_3$. The rim edge of the K_3^2 adjacent to these two K_3^1 is based on $M_1 \otimes M_4$ (Prop. 16). If the rim edges of the two K_3^1 were both based on a triple set, this set could only be $M_1 \otimes M_2 \otimes M_3$. If not, we would have a triple set consisting of M_4, in which case a chord would be formed. Then, the double set of the spoke adjacent to the first K_3^1, which does not belong to the second K_3^1, can only be $M_2 \otimes M_3$ (Prop. 23). By the same argument, the spoke which is adjacent to the second K_3^1, but does not belong to the first K_3^1, is based on $M_2 \otimes M_3$. By Prop. 7, the two spokes must be adjacent, therefore inducing a wheel of even size. ■

Before proceeding, let us summarise the last two propositions.

Remark 25 *Two necessary conditions for the rim edge of a* K_3^1 *to be based on a triple set are:*

(a) *The double sets of the adjacent (to the* K_3^1*) spokes and the double set of* K_3^1 *must be composed of three sets (Prop. 23).*

(b) *If there exists another* K_3^1*, 1-distant from the* K_3^1 *in question, then it must have a rim edge based on a double set (Prop. 24).*

The following result is the counterpart of Prop. 20, regarding triple links of the rim.

Proposition 26 *Let* T_p *denote the number of rim edges based on triple sets. For* $\left|H^2(c)\right| = 0$,

$$0 \leq T_p \leq \min\{3, p-1\}, \quad 2 \leq p \leq 5$$

Proof. The lower bound is trivial. Concerning the upper bound for $p \leq 4$, consider a wheel with $\left|W_c^p(K_3^1)\right| = p - 1$. This is the wheel with the maximum possible number of K_3^1 which are at least 2-distant from each other. For each of these wheels, the rim edges of their K_3^1 can be based on triple sets since both conditions of Props. 23 and 24 can be made satisfiable. For $p = 5$, the five K_3^1 are all adjacent, therefore at most three of them are 2-distant, implying $T_5 = 3$ (Prop. 24). ■

Observe that, for $2 \leq p \leq 4$, the upper bound on T_p is obtained for $|W_c^p| = p - 1$.

Corollary 27 *For* $\left|H^2(c)\right| = 0$*, if* $\left|W_c^p(K_3^1)\right| = p$ *then* $T_p \leq p - 2$.

Proof. If there are p K_3^1 in a wheel, they are bound to be consecutive, i.e. 1-distant. According to Prop. 24, for each pair of consecutive K_3^1 we can have only one rim edge based on a triple set. Hence, for $p \geq 4$, $T_p \leq p - 2$ and for $2 \leq p \leq 3$, $T_p \leq p - 1$. We will show that this bound cannot be obtained for the latter case.

For $p = 2$, assume a wheel having two K_3^1, one based on $M_1 \otimes M_2$ and the other on $M_2 \otimes M_3$. Then, the double set of the fifth spoke must contain M_4 (Prop. 18). This makes impossible the existence of a triple-set rim edge for either K_3^1 (Prop. 23).

Before examining the case for $p = 3$, we prove an intermediate lemma.

Lemma 28 *For* $\left|H^2(c)\right| = 0$*, each triple set appearing at a rim edge is distinct.*

Proof. For $p = 2$, we have already shown that we cannot have a wheel with two rim edges based on triple sets. For $p \geq 3$, suppose that there are two rim edges based on the same triple set. Assume that the triple set is $M_1 \otimes M_2 \otimes M_3$. According to Prop. 23, for this triple link, a series of four spokes is required, all of them based on the three double sets which are components of the above triple set. Hence, for the two triple-set rim edges we need two such four-spoke structures. It is easy to see that this cannot be done because then Prop. 7 would be violated for at least one double set. ■

(Back to the proof of Cor. 27) For $p = 3$, if $\left|W_c^3(K_3^1)\right| = 3$, then the wheel contains a spoke having a uniquely appearing double set. In order to have two rim edges, each based on a distinct triple set (Lem. 28), then these must appear at the two K_3^1 adjacent to that spoke (Prop. 24). Assume that the double set of this spoke is $M_1 \otimes M_2$. Due to Prop. 23 these sets must have $M_1 \otimes M_2$ as component.

According to Rem. 2, these triple sets are $M_1 \otimes M_2 \otimes M_3$ and $M_1 \otimes M_2 \otimes M_4$. This implies the sequence $(M_1 \otimes M_2) - [M_2 \otimes M_3] - (M_1 \otimes M_3) - (M_1 \otimes M_4) - [M_2 \otimes M_4]$, contradicting our assumption that $\left|W_c^3(K_3^1)\right| = 3$. ■

Theorem 29 *For $\left|H^2(c)\right| = 0$, a wheel of size $2p+1$ exists if and only if $n \geq \max\{3, p\}$.*

Proof. We show that there exists no wheel of size $2p+1$ with $n < \max\{3, p\}$ and afterwards we exhibit wheels having $n = \max\{3, p\}$. First, we need to prove two intermediate results.

Lemma 30 *Let $c_s, c_t \in H(c)$. Then $m_i(c_s) = m_i(c_t)$ (for some $i \in \{1, \ldots, 4\}$), if and only if either $m_i(c_s) = m_i(c)$ and $m_i(c_t) = m_i(c)$, or c_s, c_t are adjacent.*

Proof. The "if" part is trivial. For the "only if" part, assume that c_s, c_t are not adjacent and $m_i(c_s) = m_i(c_t)$, with $m_i(c_s), m_i(c_t) \neq m_i(c)$. By hypothesis, the double sets of (c, c_s), (c, c_t) have a single set in common. Hence, there exists M_j ($i \neq j$ and $i, j \in \{1, ..., 4\}$), such that $m_j(c_s), m_j(c_t) = m_j(c)$. Thus $(c_s \cap c_t) \in M_i \times M_j$, which implies the existence of chord (c_s, c_t). ■

Lemma 31 *Four consecutive spokes require $n \geq 3$.*

Proof. Let c_1, c_2, c_3, c_4 denote four consecutive rim nodes.

First assume that $K_3(c_2, c_3)$ is of type 1. We will show that the lower bound on n is not affected by whether (c_2, c_3) is based on a triple set. Assume that $|c_2 \cap c_3| = 3$. It follows that we need at least three distinct values for the index of the fourth set; one for each of the nodes c_2, c_3 and c. The same is true for the case of $|c_2 \cap c_3| = 2$, where we require three distinct values for the indices of at least two sets. In an analogous fashion, if $K_3(c_2, c_3)$ is of type 2, requires $n \geq 3$. ■

(Back to the proof of Thm. 29) For $p = 2, 3$ the result holds by Lem. 31.

Let $c_t, c_{t+1} \in H(c)$, such that $K_3^2(c_t, c_{t+1})$. Consider the set $\hat{H} \subset H(c)$ such that $\hat{H} = \{c_t\} \cup \{c_{t+1+2r \ (mod\ n)} \in H(c) : r = 0, \ldots, p-1\}$. Therefore, $\left|\hat{H}\right| = p+1$. Observe that none of the nodes of the set $\hat{H}$ is adjacent with another node in $\hat{H}$, except for c_t, c_{t+1}. Notice that nodes c_t and c_{t+1} are chosen in such a way that $c \cap c_t \neq c \cap c_{t+1}$. If $Q_{\hat{H}}$ denotes the set of the double sets of the spokes (c, q) for all $q \in \hat{H}$, then $\left|Q_{\hat{H}}\right| = p+1$. For the case of $p = 4$, $\left|Q_{\hat{H}}\right| = 5$. We observe that, for any collection of five distinct double sets, there are exactly two single sets each appearing at exactly two out of five double sets. Without loss of generality assume that M_4 is one of these two sets. As a result, there are exactly three spokes, whose rim nodes belong to $\hat{H}$, that are based on double sets not containing M_4. The rim nodes of these three spokes have distinct values for index $m_4 \in M_4$ since either

(a) no two of those are adjacent (Lem. 31), or

(b) two of them are adjacent, but with a different value for index m_4, while the third is not adjacent to any of these two; in this case, by Lem. 31, this third node requires a value for m_4 different from any of the other two.

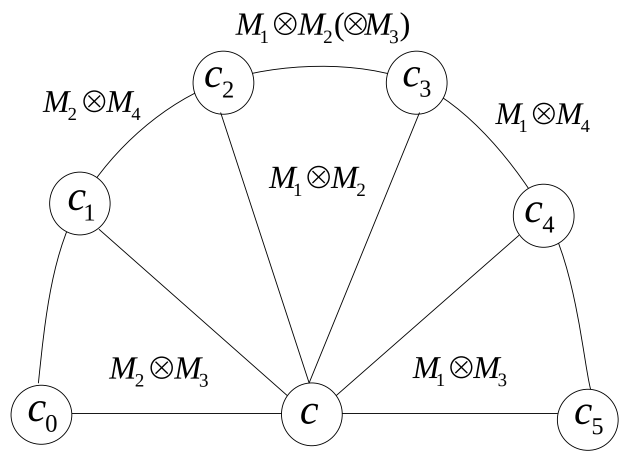


Figure 2: A structure of three K_3^1 consisting three sets.

On the other hand, since these three nodes are incident to spokes that are based on double sets not containing M_4, the value of m_4 at the hub is different from any of the values of m_4 at the three spokes. Thus, for $p = 4$ we need four distinct values for at least one of the indices, i.e. $n \geq 4$.

For $p = 5$, we consider two cases with respect to whether the double sets of the three consecutive K_3^1 structures consist of three or four single sets.

Case (i): Assume that the three single sets are M_1, M_2, M_3. The structure is shown in Figure 2 (the double set of a K_3^1 appears exactly once in the figure). Regardless of whether the edge (c_3, c_4) is based on a double or a triple set, we need at least five distinct values for the index of the set M_4, namely for nodes c, c_0, c_1, c_3, c_5.

Case (ii): If the wheel embeds no structure of the type described in Case (i), then the double sets of the distinct spokes in the two 0-distant K_3^2 do not have a set in common (spokes $(c, c_1), (c, c_{10})$ in Figure 3). The only wheel configuration for this case is illustrated in Figure 3, where the spoke with the unique double set $(M_2 \otimes M_3)$ is (c, c_0). Observe that even if we consider the two rim edges (c_1, c_2) and (c_9, c_{10}) based on triple sets, we need at least five distinct values for the indices of the sets M_2, M_3.

To show that the lower bound $\max\{3, p\}$ is attainable, we exhibit in Table 1, wheels for $p = 2, 3, 4, 5$. For all wheels, the hub is (n, n, n, n), whereas the rim nodes are illustrated as cyclic sequences of tuples. The proof is now complete. $\blacksquare$

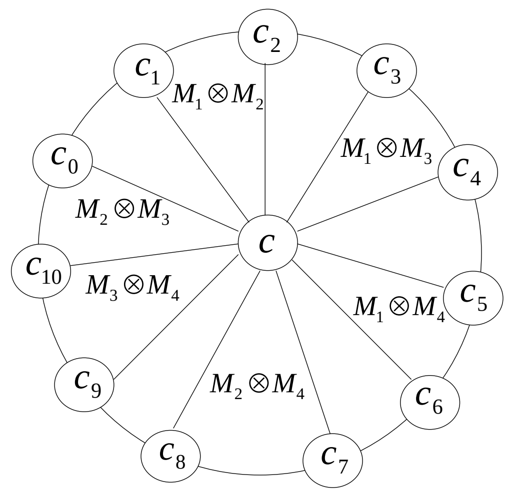


Figure 3: A wheel where any sequence of three K_3^1 consists of four sets.

Table 1: Wheels with $n = max\{3, p\}$

p	Rim nodes
2	$(i_0, n, n, l_1) - (n, n, k_0, l_1) - (n, n, k_1, l_0) - (n, j_0, n, l_0) - (i_0, j_0, n, n)$
3	$(i_0, n, k_0, n) - (i_0, n, n, l_1) - (n, n, k_1, l_1) - (n, n, k_1, l_0) -$ $(n, j_1, n, l_0) - (n; j_0, n, l_1) - (n, j_0, k_0, n)$
4	$(i_1, n, n, l_1) - (n, n, k_0, l_1) - (n, n, k_1, l_0) - (n, j_1, n, l_0) - (n, j_0, n, l_2) -$ $(i_2, j_0, n, n) - (i_0, j_2, n, n) - (i_0, n, k_2, n) - (i_1, n, k_2, n)$
5	$(i_1, n, n, l_1) - (n, n, k_2, l_1) - (n, n, k_2, l_0) - (n, j_2, n, l_0) -$ $(n, j_0, n, l_2) - (n, j_0, k_0, n) - (n, j_1, k_0, n) - (i_2, j_1, n, n) -$ $(i_0, j_3, n, n) - (i_0, n, k_1, n) - (i_1, n, k_1, n)$

Note that it is not necessary to have the maximum possible number of triple links in order to achieve the lower bound for n. For example, the wheel illustrated in Table 1, for $p = 4$, has exactly one triple link (edge $((i_0, n, k_2, n), (i_1, n, k_2, n))$), although there exist wheels of the same size with three triple links (Prop. 26). This completes the characterisation of wheels having no spokes based on a triple set. Next section discusses the properties of the remaining wheel classes.

4 Wheels with $\left|H^2(c)\right| \geq 1$

Wheels of this class have spokes based on double as well as on triple sets. Their main properties analysed are analogous to the properties discussed in the previous sections. We first characterise the spokes.

Lemma 32 *Two spokes, each based on a triple set, must be adjacent.*

Proof. Let $c_s, c_t \in H^2(c)$. Then the spokes $(c, c_s), (c, c_t)$ have at least one double set in common (Rem. 2). It follows that, unless nodes c_s and c_t are adjacent, a chord will be formed. ∎

Corollary 33 $\left|H^2(c)\right| \leq 2$.

Proof. For $p \geq 2$, if more than three spokes are based on triple sets, not all of them can be adjacent, as required by Lem. 32. ∎

The above corollary allows to distinguish between wheels with $\left|H^2(c)\right| = 1$ and those with $\left|H^2(c)\right| = 2$. If $\left|H^2(c)\right| = 1$, then $\left|W_c(K_3^3)\right| = 2$. The two K_3^3 are 0-distant and share the spoke based on the triple set. On the other hand, if $\left|H^2(c)\right| = 2$, then $\left|W_c(K_3^3)\right| = 2$, $\left|W_c(K_3^4)\right| = 1$. The two K_3^3 are 1-distant each including a different triple-set spoke.

Proposition 34 *The double set of a spoke participating on a K_3^3, is a component of the triple set of the other spoke of K_3^3.*

Proof. Let $c_s, c_t \in H^2(c)$ be two rim adjacent nodes such that spokes (c, c_s) and $(c, c_t))$ are based on a double and triple set, respectively. Clearly, c_s, c_t induce a K_3^3. Assume that $c \cap c_s \in M_1 \otimes M_4$ and $c \cap c_s \in M_1 \otimes M_2 \otimes M_3$, i.e. the double set of the one spokes is not part of the triple set of the other. Then, c_s, c_t have exactly one index in common, namely $m_1(c) = m_1(c_s) = m_1(c_t)$. Hence, the two nodes are not connected, which is a contradiction. ∎

We say that K_3^3 is based on the double set, which is the ground set of the one spoke and the component of the triple set of the other. Similarly, K_3^4 is based on the double set, which is the common component of the two triple sets (Rem. 2).

Proposition 35 *The triple set of a spoke is unique with respect to other triple sets of the spokes of the same wheel. Equivalently, the triple sets of the two spokes of a K_3^4 are distinct.*

Proof. Let $c_s, c_t \in H^2(c)$ such that $(c \cap c_s), (c \cap c_t) \in M_1 \otimes M_2 \otimes M_3$. By Lem. 32, the two spokes are adjacent. Let c_r denote the second rim node adjacent to c_t. The spoke (c, c_r) is based on a double set (Cor. 33). Thus, the nodes c_t, c_r induce a K_3^3. The double set of (c, c_r) must be one of the three double sets composing $M_1 \otimes M_2 \otimes M_3$. Hence, c_r has two indices in common to c_s implying the existence of the chord (c_s, c_r). ■

The following proposition unifies results accomplished for all K_3 structures. Its implications allow us to better characterise the adjacency of different K_3 in a wheel.

Proposition 36 *The double set of a K_3^1, (K_3^3), (K_3^4) does not appear at any other spoke not belonging to the same K_3^1, (K_3^3), (K_3^4).*

Corollary 37 *For $\left|H^2(c)\right| = 1$, a K_3^3 is adjacent to the second K_3^3 and one K_3^2. For $\left|H^2(c)\right| = 2$, a K_3^3 is adjacent to the K_3^4 and a K_3^2.*

Proposition 38 *For each triple-set spoke, one of the double sets forming it does not appear at any other spoke of the wheel. This is the double set not appearing at the other spokes of the two adjacent K_3 containing it.*

Proof. Let $c_s \in H^2(c)$. If $\left|H^2(c)\right| = 1$ then the triple-set spoke (c, c_s) belongs to two 0-distant K_3^3, each based on a distinct double set. These double sets are among the components of the triple set of the spoke (c, c_s) (Props. 34, 36). The third double set cannot appear at any other spoke of the wheel because there will be a chord connecting the node incident to it and c_s. In a similar manner, we prove the case of $\left|H^2(c)\right| = 2$. ■

The first significant property to be established is the maximum size of wheels with triple-set spokes.

Proposition 39

$$\begin{aligned} p &\leq 4, \qquad \textit{for} \quad \left|H^2(c)\right| = 1 \\ p &= 2, \qquad \textit{for} \quad \left|H^2(c)\right| = 2 \end{aligned}$$

Proof. For $\left|H^2(c)\right| = 1$, there are two 0-distant K_3^3. These require exactly three double sets, none of which appears at any other spoke of the wheel (Props. 36, 38). Hence, only three double sets can appear at the remaining spokes. Given that a double set appears at no more than two spokes (Prop. 7), there can exist at most six spokes in addition to the three forming the two 0-distant K_3^3. Hence, $p \leq 4$.

For $\left|H^2(c)\right| = 2$, there are four spokes forming two 1-distant K_3^3 and one K_3^4. We need two double sets for the two K_3^3 and one double set for the K_3^4. All these double sets cannot appear at any other spoke of the wheel (Prop. 36). For each of the two triple sets, there is a double set that cannot be used at any other spoke due to Prop. 38. Hence, only one double set remains available for any additional spoke. Because the wheel must be of odd size, $p = 2$. ■

Next, we exhibit properties of rim edges.

Lemma 40 *The rim edge of a K_3^3 is based either*

(a) *on the ground set of the K_3^3, or*

(b) *on a triple set consisting of the two sets of the ground set of the K_3^3 and the set not appearing at the ground set of any of the spokes of K_3^3.*

Proof. Let $c_s \in H^1(c)$, $c_t \in H^2(c)$, such that $c \cap c_s \in M_1 \otimes M_2$, $c \cap c_t \in M_1 \otimes M_2 \otimes M_3$. It follows that $m_1(c_s) = m_1(c) = m_1(c_t)$, $m_2(c_s) = m_2(c) = m_2(c_t)$ and $m_3(c_s) \neq m_3(c) = m_3(c_t)$. If $m_4(c_s) \neq m_4(c_t)$ we obtain case (a); otherwise, we have case (b). ■

Lemma 41 *The rim edge of a K_3^4 is based on the ground set of K_3^4.*

Proof. Let $c_s, c_t \in H^2(c)$. Obviously, the two indices of the double set of the K_3^3 have identical values at nodes c, c_s, c_t. The third index appears in one of c_s, c_t and also in c. Similarly, the fourth index is the second of nodes c_s, c_t and again in c. Thus, (c_1, c_2) is based on the double set of the K_3^3. ■

Lemma 42 *For $\left|H^2(c)\right| = 1$, the rim edge of only one of the two 0-distant K_3^3 can be based on a triple set.*

Proof. The rim edges of the two 0-distant K_3^3 are adjacent edges of an odd hole. It is easy to see that any two adjacent edges of an odd hole being based on triple sets implies the existence of a chord. ■

Proposition 43 *Consider the K_3^2 adjacent to a K_3^3 (Cor. 37). Then, the spoke of K_3^2 not belonging to K_3^3, is based on a double set consisting of one of the sets forming the ground set of K_3^3 and the set not present at the triple set of the corresponding spoke of K_3^3.*

Proof. Let c_s, c_t, c_r be three consecutive nodes of the rim such that $c_s, c_t \in H^1(c)$ and $c_r \in H^2(c)$. Assume that $c \cap c_t \in M_1 \otimes M_2$ and $c \cap c_r \in M_1 \otimes M_2 \otimes M_3$. Hence, nodes c_s, c_t induce a K_3^2 (Cor. 37) The double set of (c, c_s) must have one set in common with $M_1 \otimes M_2$ (Prop. 11). Assume that this set is M_1. The second set of the double set cannot be M_3 because then a chord (c_s, c_r) would be formed. The only remaining option is M_4. ■

The following corollaries explore further structural issues.

Corollary 44 *If there exists a K_3^1 which is 1-distant from a K_3^3, then its double set consists of one of the sets forming the ground set of K_3^3 and the set not present at the triple set of the corresponding spoke of K_3^3.*

Proof. The 1-distant K_3^1 is adjacent to a K_3^2 which is adjacent to the K_3^3. By Prop. 43, the double set of the spoke of K_3^2, not belonging to K_3^3 has this property. This spoke belongs to K_3^1 as well, therefore the second spoke of K_3^1 also has this property. ■

Corollary 45 *If there exists a K_3^1, which is 1-distant from a K_3^3 whose rim edge is based on a triple set, then its rim edge can be based only on the same triple set as the rim edge of K_3^3.*

Proof. Let c_0, c_1, c_2, c_3 be four consecutive rim nodes such that $c_0, c_1, c_2 \in H^1(c)$, $c_3 \in H^2(c)$ and $K_3^1(c_0, c_1)$. Suppose that $c \cap c_3 \in M_1 \otimes M_2 \otimes M_3$, $c \cap c_2 \in M_1 \otimes M_2$. Then, $c \cap c_1 \in M_1 \otimes M_4$ (Cor. 37) and $c \cap c_0 \in M_1 \otimes M_4$ (Cor. 44). If (c_2, c_3) is based on a triple set, then this set is $M_1 \otimes M_2 \otimes M_4$ (Prop. 40). The ground set of (c_1, c_2) is $M_1 \otimes M_3$. The ground set of (c_0, c_1) consists of the sets M_1, M_4 and, if it is a triple set, M_2. Observe that the third set cannot be M_3 because then we would have two adjacent edges of an odd hole based on a double and a triple set having this double set as component. It is not difficult to show that in such a case a chord is formed. ■

At this point, we can provide characterisation results analogous to those of the previous section. The number of K_3^1 structures is evidently a significant factor in our classification. The same holds for the number of triple-set rim edges.

Proposition 46 *For $\left|H^2(c)\right| = 1$,*

$$\begin{aligned} p-2 \leq \left|W_c^p(K_3^1)\right| &\leq p-1, \qquad \textit{for } p = \{2,3\}, \\ \left|W_c^4(K_3^1)\right| &= 3 \end{aligned}$$

Proof. $\left|H^2(c)\right| = 1$ implies two 0-distant K_3^3. As observed in the proof of Prop. 39, the three sets appearing in this configuration cannot appear at any other spoke of the wheel (Props. 36, 38). For each of the remaining double sets we can have two spokes, yielding $\left|W_c^p(K_3^1)\right| \leq p-1$, for $p \in \{2,3,4\}$. For $p = 4$, this inequality is satisfied as equality. This is because the only way to have a wheel of size 9, for $\left|H^2(c)\right| = 1$, is to have two spokes based on each of the three remaining double sets. For $p = 3$, we have three double sets to cover four spokes, thus $\left|W_c^3(K_3^1)\right| \geq 1$. For $p = 2$, we can have a wheel with $\left|W_c^2(K_3^1)\right| = 0$, for example:

$$(M_1 \otimes M_2) - (M_1 \otimes M_2 \otimes M_3) - (M_2 \otimes M_3) - (M_3 \otimes M_4) - (M_1 \otimes M_4)$$

■

Proposition 47 *For $\left|H^2(c)\right| = 1$, $0 \leq T_p \leq p-1$.*

Proof. The lower bound is trivial. For the upper bound, we consider wheels with $\left|W_c^p(K_3^1)\right| = p-1$. Given that the rim edge of only one of the two K_3^3 can be based on a triple edge, it remains to show that the rim edges of at most $p-2$ K_3^1 can be based on a triple set.

For $p = 2$, assume that the two 0-distant K_3^3 consists of the sequence $(M_1 \otimes M_2) - (M_1 \otimes M_2 \otimes M_3) - (M_2 \otimes M_3)$. By Prop. 43, set M_4 must appear at the double sets of the two remaining spokes. In the case that these two spokes form a K_3^1, its rim edge cannot be based on a triple set because then there will be four consecutive spokes with all four sets appearing in their double sets (Prop. 23).

For $p = 3$, we can have at most two K_3^1, which are 1-distant. According to Prop. 24, only the rim edge of one of the two K_3^1 can be based on a triple set. For $p = 4$, we can have at most three consecutive K_3^1. Via the same argument, the rim edge of only two of these three K_3^1 can be based on a triple set. ■

Lemma 48 *Let D_p be defined as in Lem. 21. By convention, the triple set of the spoke incident to a node belonging to $H^2(c)$ increases D_p by 3. For $\left|H^2(c)\right| = 1$:*

$$p + 2 \leq D_p \leq min\{6, p + 3\}, \quad 2 \leq p \leq 4$$

Proof. There are three distinct double sets to be used for the spokes of the two 0-distant K_3^3. Additionally for

$p = 2:$ we can have one or two distinct double sets for the two remaining spokes. In total, $4 \leq D_2 \leq 5$.

$p = 3:$ we can have two or three distinct double sets for the four remaining spokes. In total, $5 \leq D_3 \leq 6$.

$p = 4:$ all three sets are used for the six remaining spokes. Thus, $D_4 = 6$.

■

Lemma 49 *For $\left|H^2(c)\right| = 2$, we have $\left|W_c^2(K_3^1)\right| = 0$ and $0 \leq T_2 \leq 2$, where T_2 is defined as in Prop. 26.*

Proof. For $\left|H^2(c)\right| = 2$, there are only wheels of size 5 (Prop. 39). Each of these wheels includes two 1-distant K_3^3 and one K_3^4 adjacent to the two K_3^3. Thus, $\left|W_c^2(K_3^1)\right| = 0$. It follows that only the rim edges of the two K_3^3 structures can be based on triple sets. Therefore, $0 \leq T_2 \leq 2$. ■

The final aspect to be addressed is the minimum value of n required for wheels with $\left|H^2(c)\right| = 1$.

Theorem 50 *For $\left|H^2(c)\right| = 1$, a wheel of size $2p + 1$ exists if and only if $n \geq \max\{3, p\}$.*

Proof. To show that no wheel exists for $n < \max\{3, p\}$, we first prove the following lemma.

Lemma 51 *Two 0-distant K_3^3 imply $n \geq 3$.*

Proof. Let c_s, c_t, c_r denote three consecutive rim nodes such that $c_s, c_r \in H^1(c)$ and $c_t \in H^2(c)$. The edge (c, c_t) is based on a triple set. Thus, the value of the index of the fourth set in c_t is different from the value of the same index in c. This index value is also different in the nodes c_s, c_r and the hub because the double sets of the spokes (c, c_s) and (c, c_r) are components of the triple set of (c, c_t). On the other hand, observe that this index can have the same value as in c_t in only one of nodes c_s, c_r. Thus, we need at least three distinct values for this index, i.e. one for c, one for c_s, c_t (c_t, c_r) and one for c_r (c_s). ■

Table 2: Wheels with $n = max\{3, p\}$ for $H^2(c) = 1$

p	Rim nodes
2	$(i_0, n, n, l_0) - (n, n, n, l_0) - (n, n, k_0, l_1) - (i_1, n, k_0, n) - (i_0, n, k_1, n)$
3	$(n, j_0, n, l_0) - (n, n, n, l_0) - (n, n, k_0, l_1) - (i_0, n, k_0, n)-$ $(i_0, j_1, n, n) - (n; j_1, k_1, n) - (n, j_0, k_1, n)$
4	$(n, n, k_0, l_0) - (n, n, n, l_0) - (i_2, n, n, l_1) - (i_2, j_1, n, n)-$ $(i_0, j_1, n, n) - (i_0, n, k_2, n) - (i_1, n, k_1, n) - (n, j_0, k_1, n) - (n, j_0, k_0, n)$

(Back to the proof of Thm. 50) Lem. 51 proves the result for $p = 2, 3$. For $p = 4$, we use exactly the same method as in the corresponding case in the proof of Thm. 29, with the additional restriction that t should be chosen in a way that the node belonging to $H^2(c)$ is not included in $\hat{H}$.

In Table 2 we illustrate wheels for $p = 2, 3, 4$ with $n = 3, 3, 4$, respectively. ■

Lemma 52 *For $\left|H^2(c)\right| = 2$, a wheel of size 5 exists if and only if $n \geq 2$.*

Proof. We need at least $n > 1$ in order for a wheel to exist. For $\left|H^2(c)\right| = 2$ we have $p = 2$ (Prop. 39). The following wheel has $\left|H^2(c)\right| = 2$, $p = 2$ and $n = 2$.

$$(n, n, k_0, l_0) - (n, n, k_0, n) - (i_0, n, n, n) - (i_0, j_0, n, n) - (n, j_0, n, l_0)$$

■

This completes the structural properties of wheels. Apart from providing a concise description, these results are critical for the classification scheme and the recognition algorithm presented next.

5 Classification

In Sections 3, 4, we have explicitly categorised wheels into three main families, one for each of the values of $\left|H^2(c)\right|$ ($\left|H^2(c)\right| \in \{0, 1, 2\}$). However, the recognition of classes of wheel-induced inequalities requires a more elaborate classification scheme. This scheme, apart from the values of $\left|H^2(c)\right|$ and p, takes also into account structures based on ground sets. These are, primarily, the 3-cliques of the four types as well as their compositions and, secondarily, the rim edges. In general, for each $t(t \in \{1, 2, 3, 4\})$, the cardinality of the set $W_c(K_3^t)$ as well as the relative placement of its members within the wheel yields the specific characteristics of each class. For each such class, we can further derive subclasses depending on the set of values of parameter T_p.

As shown previously, the value of $\left|H^2(c)\right|$ determines the range of values for p (Cor. 10, Prop. 39), $\left|W_c(K_3^t)\right|, t \in \{1, \ldots, 4\}$, (Props. 20, 46 and Lem. 49) and, consequently, the relative positions of K_3^3, K_3^4 (Cor. 37) if they exist. Hence, given the values of $\left|H^2(c)\right|$ and p, we are interested in compositions of K_3^1, K_3^2 that essentially define a wheel class. Observe that the roles of K_3^1, K_3^2 are complementary, implying that it is adequate for our classification scheme to be solely based on K_3^1. Therefore, apart of the value of $\left|W_c(K_3^1)\right|$, we are interested on the positions that K_3^1 occupy within the wheel. This is quantified by the parameter $\mathrm{seq}(K_3^1)$, denoting the largest number of consecutive K_3^1. Clearly, $\mathrm{seq}(K_3^1) \leq \left|W_c(K_3^1)\right|$, equality holding only if all K_3^1 are consecutive.

The compositions of K_3^1 and the way they interrelate to K_3 of other types, within a wheel structure, are explored through the notion of a 3-*sequence*. In general terms, a 3-sequence is a composite structure consisting of a consecutive occurrence of three structures based on ground sets. We are specifically interested on 3-sequences involving K_3^1. Our classification is derived from two 3-sequence configurations. The first configuration consists of three consecutive K_3^1, to be denoted as $K_3^1 - K_3^1 - K_3^1$. Observe that, if $\left|H^2(c)\right| = 0$, this structure requires $p \geq 3$, since every two consecutive K_3^1 are 1-distant, while the structure intervening between the two K_3^1 is a K_3^2 (Cor. 8). The second configuration consists of a K_3^1 with its two adjacent spokes both based on double sets. Two K_3^2 are formed, each sharing a spoke with the particular K_3^1. Hence, this structure is denoted as a $K_3^2 - K_3^1 - K_3^2$.

Both 3-sequences are related to three double sets. For $K_3^1 - K_3^1 - K_3^1$, we have one double set for each of K_3^1. For $K_3^2 - K_3^1 - K_3^2$, the three double sets are the double set of the first spoke not belonging to K_3^1, the double set of K_3^1 and the double set of the second spoke not belonging to K_3^1. Thus, both configurations result in a 3-sequence of double sets. We distinguish three distinct types called 3, $4a$ and $4b$. Type 3 implies that the three double sets are composed of three single sets (e.g. $M_1 \otimes M_2 - M_1 \otimes M_3 - M_2 \otimes M_3$). Type 4 implies that the three double sets are composed of four sets. $4a$ implies that one of the single sets appear in every double set (e.g. $M_1 \otimes M_2 - M_1 \otimes M_3 - M_1 \otimes M_4$). Exactly the opposite is implied by $4b$ (e.g. $M_1 \otimes M_2 - M_2 \otimes M_3 - M_3 \otimes M_4$). Observe that each configuration $[K_3^1 - K_3^1 - K_3^1, t]$ embeds a configuration $[K_3^2 - K_3^1 - K_3^2, t]$, $t \in \{3, 4a, 4b\}$.

The concepts discussed above are encoded into the *class identity* string, uniquely associated with each class. This string has the format

$$\left|H^2(c)\right| - p - \left|W_c(K_3^1)\right| - \mathrm{seq}(K_3^1) - (3, 4a, 4b)^{K_3^1 - K_3^1 - K_3^1} - (3, 4a, 4b)^{K_3^2 - K_3^1 - K_3^2}$$

where $(3, 4a, 4b)$ denotes the number of 3-sequences of type 3, $4a$, $4b$, respectively, for $K_3^1 - K_3^1 - K_3^1$ and $K_3^2 - K_3^1 - K_3^2$. In this string, the parameters appear in decreasing order with respect to their importance concerning class categorisation.

The values of the parameters composing the class identity string, for all distinct wheel classes, are illustrated in Table 3. For reference, each class is assigned a number. The following theorem establishes that our classification is complete.

Theorem 53 *All wheel classes of the $G_A(C, E_C)$ are illustrated in Table 3.*

Proof. The distinct wheel classes are revealed through considering the different values of the parameters embedded in the class identity string, in the order from left to right. All assumptions on the double

Table 3: The Wheel classes of $G_A(C, E_C)$.

Num.	$\lvert H^2(c)\rvert$	p	$\lvert W_c^p(K_3^1)\rvert$	seq(K_3^1)	$K_3^1 - K_3^1 - K_3^1$			$K_3^2 - K_3^1 - K_3^2$		
					3	$4a$	$4b$	3	$4a$	$4b$
1	0	2	2	2	0	0	0	0	2	0
2	0	2	1	1	0	0	0	1	0	0
3	0	2	1	1	0	0	0	0	0	1
4	0	3	3	3	0	0	1	1	1	1
5	0	3	3	3	0	0	1	0	0	3
6	0	3	3	3	1	0	0	1	0	2
7	0	3	2	2	0	0	0	1	1	0
8	0	3	2	1	0	0	0	2	0	0
9	0	3	2	1	0	0	0	1	0	1
10	0	4	4	4	0	0	2	2	0	2
11	0	4	4	4	0	1	1	1	1	2
12	0	4	4	4	1	1	0	2	1	1
13	0	4	4	4	1	1	0	1	2	1
14	0	4	3	3	0	0	1	2	0	1
15	0	4	3	2	0	0	0	2	1	0
16	0	4	3	1	0	0	0	2	0	1
17	0	4	3	1	0	0	0	3	0	0
18	0	5	5	5	1	1	1	2	1	2
19	0	5	5	5	0	2	1	2	2	1
20	0	5	5	5	1	2	0	3	2	0
21	0	5	5	5	2	0	1	2	2	1
22	1	2	1	1	0	0	0	0	1	0
23	1	2	0	0	0	0	0	0	0	0
24	1	3	2	2	0	0	0	0	0	2
25	1	3	2	2	0	0	0	1	0	1
26	1	3	1	1	0	0	0	1	0	0
27	1	4	3	3	0	1	0	1	1	1
28	1	4	3	3	0	1	0	2	1	0
29	2	2	0	0	0	0	0	0	0	0

sets of the spokes are done without loss of generality.

$\underline{H^2(c) = 0}$.

$\underline{p = 2}$.

For all subcases, assume $K_3^1(c_0, c_1)$ and $g(K_3^1(c_0, c_1)) = M_1 \otimes M_2$.

$\underline{\left|W_c^2(K_3^1)\right| = 2}$.

The two K_3^1 are consecutive. Assume $K_3^1(c_2, c_3)$. Let M_1 be the common set among the double sets of $K_3^1(c_0, c_1)$, $K_3^1(c_2, c_3)$ (Cor. 13) by assuming $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$. Then, the double set of (c, c_4) must also contain the set M_1. In the opposite case, $c \cap c_4 \in M_3 \otimes M_4$ (Prop. 11) which then causes violation of Prop. 18 for $(c, c_3), (c, c_4), (c, c_0)$. Hence if one of the sets of $g((c, c_4))$ is M_1, then the other must be M_4 (Prop. 18).

$\underline{\left|W_c^2(K_3^1)\right| = 1}$.

Observe that a configuration $[K_3^2 - K_3^1 - K_3^2, 4a]$ is impossible. To prove that, consider $c \cap c_4 \in M_1 \otimes M_3$, $c \cap c_2 \in M_1 \otimes M_4$. None of the double sets containing M_1 can be used for the spoke (c, c_3); double sets $M_1 \otimes M_3$, $M_1 \otimes M_4$ are excluded because we require one K_3^1 and $M_1 \otimes M_2$ is excluded due to Prop. 7. The only possible double set for this spoke is $M_3 \otimes M_4$ (Prop. 11). But then Prop. 18 is violated for $(c, c_2), (c, c_3), (c, c_4)$. Configurations $[K_3^2 - K_3^1 - K_3^2, 3]$, $[K_3^2 - K_3^1 - K_3^2, 4b]$ yield classes num. 2,3.

$\underline{p = 3}$.

For all subcases, assume $K_3^1(c_0, c_1)$ with ground set $M_1 \otimes M_3$.

$\underline{\left|W_c^3(K_3^1)\right| = 3}$.

Evidently all K_3^1 must be consecutive. Thus, assume $K_3^1(c_2, c_3)$, $K_3^1(c_4, c_5)$ with $g(K_3^1(c_2, c_3)) = M_1 \otimes M_2$ ($g(K_3^1(c_0, c_1)), g(K_3^1(c_2, c_3))$ must have one set in common (Cor. 13)). A configuration $[K_3^1 - K_3^1 - K_3^1, 4a]$ is impossible. To see that, assume such a configuration, i.e. $g(K_3^1(c_4, c_5)) = M_1 \otimes M_4$. The set M_1 cannot appear at $g((c, c_6))$. This is because all the double sets involving M_1 have been used for two spokes each (Prop. 7). This implies that $g((c, c_6)) = M_3 \otimes M_4$ (Prop. 11), which contradicts Prop. 18 for $(c, c_5), (c, c_6), (c, c_0)$.

A configuration $[K_3^1 - K_3^1 - K_3^1, 4b]$ is formed if $g(K_3^1(c_4, c_5)) = M_2 \otimes M_4$. This structure embeds $[K_3^2(c_1, c_2) - K_3^1(c_2, c_3) - K_3^2(c_3, c_4), 4b]$.Two more $[K_3^2 - K_3^1 - K_3^2]$ are formed, i.e. $K_3^2(c_3, c_4) - K_3^1(c_4, c_5) - K_3^2(c_5, c_6)$, $K_3^2(c_6, c_0) - K_3^1(c_0, c_1) - K_3^2(c_1, c_2)$. If $c \cap c_6 \in M_1 \otimes M_4 (M_2 \otimes M_3)$ then the first is of type $3(4a)$ and the second is of type $4a(3)$. In both cases we have class num. 4. If $c \cap c_6 \in M_3 \otimes M_4$ then the two $[K_3^2 - K_3^1 - K_3^2]$ are both of type 3 (class num. 5).

If $g(K_3^1(c_4, c_5)) = M_2 \otimes M_3$ then $[K_3^1(c_0, c_1),\ K_3^1(c_2, c_3),\ K_3^1(c_4, c_5), 3]$. Observe that spoke (c, c_6) must be based on the double set $M_3 \otimes M_4$, because the use of any other available double set will contradict Prop. 18. Thus, $c \cap c_6 \in M_3 \otimes M_4$ implies $[K_3^2(c_6, c_1) - K_3^1(c_0, c_1) - K_3^2(c_1, c_2), 4b]$ and $[K_3^2(c_3, c_4) - K_3^1(c_4, c_5) - K_3^2(c_5, c_6), 4b]$ (class num. 6).

$\underline{\left|W_c^3(K_3^1)\right| = 2}$.

Either the two K_3^1 are 1-distant (Case 1), or 2-distant (Case 2).

Case 1:

Assume, again, $g(K_3^1(c_2,c_3)) = M_1 \otimes M_2$. The wheel cannot have a configuration $[K_3^2 - K_3^1 - K_3^2, 4b]$. To see that, assume such a configuration by considering $c \cap c_6 \in M_3 \otimes M_4$ which yields $[K_3^2(c_6,c_0) - K_3^1(c_0,c_1) - K_3^2(c_1,c_2), 4b]$. The remaining double sets, to be used for the two spokes $(c,c_4),(c,c_5)$, are $M_1 \otimes M_4$, $M_2 \otimes M_4$, $M_2 \otimes M_3$. All combinations yield the contradiction of either Prop. 11 or Prop. 18 involving one of (or both) the above spokes and its (their) other adjacent spokes.

If $c \cap c_6 \in M_1 \otimes M_4$ then $[K_3^2(c_6,c_0) - K_3^1(c_0,c_1) - K_3^2(c_1,c_2), 4a]$. Three double sets are candidates for spokes (c,c_4), (c,c_5), i.e. $M_2 \otimes M_4$, $M_2 \otimes M_3$, $M_3 \otimes M_4$. Observe that $c \cap c_4 \notin M_3 \otimes M_4$ (the opposite contradicts Prop. 11 $(c,c_3),(c,c_4)$) and $c \cap c_4 \notin M_2 \otimes M_4$ (the opposite yields a $[K_3^2 - K_3^1 - K_3^2, 4b]$, which has been proven infeasible). Thus, $c \cap c_4 \in M_2 \otimes M_3$. This implies $c \cap c_5 \in M_2 \otimes M_4$ yielding the class num. 7. The same class is derived if we assume $c \cap c_6 \in M_2 \otimes M_3$ (symmetric case).

Case 2:

Assume $K_3^1(c_3,c_4)$. First, we will prove that $[K_3^1(c_0,c_1) - (c,c_2) - K_3^1(c_3,c_4), 4a]$. Evidently $K_3^1(c_0,c_1) - (c,c_2) - K_3^1(c_3,c_4)$ cannot be of type 3, because that would contradict Prop. 18 for $(c,c_1),(c,c_2),(c,c_3)$. To see that type $4b$ is also excluded, assume $c \cap c_2 \in M_2 \otimes M_3$ and $g(K_3^1(c_3,c_4)) = M_2 \otimes M_4$. The double sets for the remaining two spokes $(c,c_5),(c,c_6)$ are $M_1 \otimes M_2, M_1 \otimes M_4, M_3 \otimes M_4$. These two spokes are adjacent, hence the only valid combinations of double sets are either $M_1 \otimes M_2, M_1 \otimes M_4$ or $M_1 \otimes M_4, M_3 \otimes M_4$. For each pair, no matter which double set is assigned to which spoke, a contradiction for Prop. 18 occurs, involving either the spokes $(c,c_4),(c,c_5),(c,c_6)$ or the spokes $(c,c_5),(c,c_6),(c,c_0)$.

To implement $[K_3^1(c_0,c_1) - (c,c_2) - K_3^1(c_3,c_4), 4a]$, let $c \cap c_2 \in M_1 \otimes M_2$ and $g(K_3^1(c_3,c_4)) = M_1 \otimes M_4$. Prop. 11, for spokes $(c,c_4),(c,c_5)$, implies $c \cap c_5 \notin M_2 \otimes M_3$. Prop. 11, for spokes $(c,c_5),(c,c_6)$, implies $c \cap c_6 \notin M_2 \otimes M_4$. If $c \cap c_5 \in M_2 \otimes M_4$ and $c \cap c_6 \in M_2 \otimes M_3$ then $[K_3^2(c_6,c_0) - K_3^1(c_0,c_1) - K_3^2(c_1,c_2), 3]$, $[K_3^2(c_2,c_3) - K_3^1(c_3,c_4) - K_3^2(c_4,c_5), 3]$ (class num. 8). If $c \cap c_5 \in M_2 \otimes M_4$ and $c \cap c_6 \in M_3 \otimes M_4$ then $[K_3^2(c_6,c_0) - K_3^1(c_0,c_1) - K_3^2(c_1,c_2), 4b]$, $[K_3^2(c_2,c_3) - K_3^1(c_3,c_4) - K_3^2(c_4,c_5), 3]$ (class num. 9).

<u>$p = 4$</u>.

For all subcases assume $K_3^1(c_0,c_1)$ with $g(K_3^1(c_0,c_1)) = M_1 \otimes M_2$.

<u>$\left|W_c^4(K_3^1)\right| = 4$</u>.

All K_3^1 are consecutive. Thus, assume $K_3^1(c_2,c_3)$, $K_3^1(c_4,c_5)$, $K_3^1(c_6,c_7)$. There are two $K_3^1 - K_3^1 - K_3^1$ structures having two K_3^1 in common. It is trivial to show that the two 3-sequences cannot be both of type 3 or $4a$. Also it is impossible to have one structure of type $4b$ and the other of type 3. To see that this is so, create such a configuration by assuming, for example, $g(K_3^1(c_2,c_3)) = M_2 \otimes M_3$, $g(K_3^1(c_4,c_5)) = M_3 \otimes M_4$, $g(K_3^1(c_6,c_7)) = M_2 \otimes M_4$. This is infeasible because a) if $c \cap c_8 \in M_1 \otimes M_3$ yields a contradiction of Prop. 11 for $(c,c_7),(c,c_8)$ and b) if $c \cap c_8 \in M_1 \otimes M_4$ yields a contradiction of Prop. 18 for $(c,c_7),(c,c_8),(c,c_0)$.

The remaining cases are: (a) one of the 3-sequences is of type $4b$ and the other can either be of type $4b$ or $4a$, (b) one of the 3-sequences is of type 3 and the other is of type $4a$.

Case (a) is implemented if we assume $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$, $g(K_3^1(c_4, c_5)) = M_3 \otimes M_4$. Then, $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 4b]$. Further, if $g(K_3^1(c_6, c_7)) = M_2 \otimes M_4$, then $[K_3^1(c_2, c_3) - K_3^1(c_4, c_5) - K_3^1(c_6, c_7), 4b]$. Observe that $c \cap c_8 \in M_2 \otimes M_3$ because the alternative choice ($c \cap c_8 \in M_1 \otimes M_4$) yields a contradiction of Prop. 18 for $(c, c_7), (c, c_8)(c, c_0)$. The resulting class is num. 10. Alternatively, if $g(K_3^1(c_6, c_7)) = M_2 \otimes M_3$, then $[K_3^1(c_2, c_3) - K_3^1(c_4, c_5) - K_3^1(c_6, c_7), 4a]$. Observe that $c \cap c_8 \in M_2 \otimes M_4$ because the alternative choice ($c \cap c_8 \in M_1 \otimes M_4$) yields a contradiction of Prop. 11 for spokes $(c, c_7), (c, c_8)$. The resulting class is num. 11.

Case (b) is implemented if we assume $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$, $g(K_3^1(c_4, c_5)) = M_2 \otimes M_3$, $g(K_3^1(c_6, c_7)) = M_3 \otimes M_4$. Then, $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 3]$, $[K_3^1(c_2, c_3) - K_3^1(c_4, c_5) - K_3^1(c_6, c_7), 4a]$. Observe that $c \cap c_8 \in M_2 \otimes M_4$ since $c \cap c_8 \in M_1 \otimes M_4$ yields a contradiction of Prop. 18 for $(c, c_7), (c, c_8), (c, c_0)$. The resulting class is num. 12.

Class num. 13 is produced if $K_3^1(c_2, c_3), K_3^1(c_4, c_5)$ exchange double sets, i.e. $g(K_3^1(c_2, c_3)) = M_2 \otimes M_3$ and $g(K_3^1(c_4, c_5)) = M_1 \otimes M_3$. Again, $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 3]$, $[K_3^1(c_2, c_3) - K_3^1(c_4, c_5) - K_3^1(c_6, c_7), 4a]$.

$\underline{\left|W_c^4(K_3^1)\right| = 3}$.

The three K_3^1 are either all consecutive ($\text{seq}(K_3^1) = 3$), or two of them are 1-distant from each other and the third is more than 1-distant from any of the two ($\text{seq}(K_3^1) = 2$), or all of them are more than 1-distant from each other ($\text{seq}(K_3^1) = 1$). We consider each case separately.

$\underline{\text{seq}(K_3^1) = 3}$.

Assume $K_3^1(c_2, c_3)$, $K_3^1(c_4, c_5)$. Two double sets will be used for these K_3^1. Three double sets remain each to be assigned the spokes $(c, c_6), (c, c_7), (c, c_8)$.

We claim that $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 3]$ is impossible. This is implemented if $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$ and $g(K_3^1(c_4, c_5)) = M_2 \otimes M_3$. The remaining double sets are $M_1 \otimes M_4, M_2 \otimes M_4, M_3 \otimes M_4$. Observe that $c \cap c_6 \notin M_1 \otimes M_4$ (Prop. 11 for $(c, c_5), (c, c_6)$) and $c \cap c_8 \notin M_3 \otimes M_4$ (Prop. 11 for $(c, c_8), (c, c_0)$). All other possible assignments of double sets to spokes yield a contradiction of Prop. 18 either for spokes $(c, c_6), (c, c_7), (c, c_8)$ or $(c, c_7), (c, c_8), (c, c_0)$.

It is also impossible to have $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 4a]$. To see that, create a 3-sequence of this type by assuming that $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$ and $g(K_3^1(c_4, c_5)) = M_1 \otimes M_4$. The remaining double sets are $M_2 \otimes M_3, M_2 \otimes M_4, M_3 \otimes M_4$, to be assigned each to one of three consecutive spokes. This contradicts Prop. 18.

Next, assume $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$, $g(K_3^1(c_4, c_5)) = M_3 \otimes M_4$. This produces $[K_3^1(c_0, c_1) - K_3^1(c_2, c_3) - K_3^1(c_4, c_5), 4b]$. The remaining double sets are $M_1 \otimes M_4, M_2 \otimes M_3, M_2 \otimes M_4$. Observe that the first two sets do not have a common single set, hence they cannot be assigned to adjacent spokes. Thus, $c \cap c_7 \in M_2 \otimes M_4$. Also, $c \cap c_6 \in M_2 \otimes M_3$ contradicts Prop. 18 for $(c, c_5), (c, c_6), (c, c_7)$. Similarly, $c \cap c_8 \in M_1 \otimes M_4$ contradicts Prop. 18 for $(c, c_7), (c, c_8), (c, c_0)$. Thus, we can only have $c \cap c_6 \in M_1 \otimes M_4$, $c \cap c_8 \in M_2 \otimes M_3$ resulting in class num. 14.

$\underline{\text{seq}(K_3^1) = 2}$.

Such a configuration is formed if we assume $K_3^1(c_2, c_3)$ and $K_3^1(c_5, c_6)$. Observe that the double sets of $K_3^1(c_0, c_1)$, $K_3^1(c_2, c_3)$ must have one set in common (Cor. 13). So, let $g(K_3^1(c_2, c_3)) = M_1 \otimes M_3$. The 3-sequence $K_3^1(c_2, c_3) - (c, c_4) - K_3^1(c_5, c_6)$ cannot be of type 3 (contradiction of Prop. 18 for spokes $(c, c_3), (c, c_4), (c, c_5)$). It cannot be of type $4a$ either. To see that, create such a configuration, i.e. assume $c \cap c_4 \in M_2 \otimes M_3$ $(M_3 \otimes M_4)$ and $g(K_3^1(c_5, c_6)) = M_3 \otimes M_4$ $(M_2 \otimes M_3)$. Then, the remaining double sets for the two last spokes are $M_1 \otimes M_4$, $M_2 \otimes M_4$. This yields a contradiction of Prop. 18 for spokes $(c, c_7), (c, c_8), (c, c_0)$.

To find out if $[K_3^1(c_2, c_3) - (c, c_4) - K_3^1(c_5, c_6), 4b]$, we examine the following alternatives, each implementing such a configuration.

(a) If $c \cap c_4 \in M_1 \otimes M_4$, then we must have $g(K_3^1(c_5, c_6)) = M_2 \otimes M_4$. The remaining double sets for the adjacent spokes $(c, c_7), (c, c_8)$ are $M_2 \otimes M_3, M_3 \otimes M_4$. This yields a contradiction of Prop. 18 for $(c, c_6), (c, c_7), (c, c_8)$.

(b) If $c \cap c_4 \in M_3 \otimes M_4$, then we must have $g(K_3^1(c_5, c_6)) = M_2 \otimes M_4$. The remaining double sets for the two adjacent spokes $(c, c_7), (c, c_8)$ are $M_1 \otimes M_4$ and $M_2 \otimes M_3$. This contradicts Prop. 11.

(c)If $c \cap c_4 \in M_2 \otimes M_3$, then we must have $g(K_3^1(c_5, c_6)) = M_2 \otimes M_4$. Consequently, $c \cap c_7 \in M_3 \otimes M_4$, $c \cap c_8 \in M_1 \otimes M_4$. Assigning the two last double sets to the spokes in reverse order, yields a contradiction of Prop. 11 for spokes $(c, c_8)(c, c_0)$. The wheel produced belongs to class num.15.

$\underline{\text{seq}(K_3^1) = 1}$.

Each K_3^1 is exactly 2-distant from the other two. Thus, assume $K_3^1(c_3, c_4)$ and $K_3^1(c_6, c_7)$. To see that it is impossible to have $[K_3^1(c_0, c_1) - K_3^1(c_3, c_4) - K_3^1(c_6, c_7), 4a]$, assume $g(K_3^1(c_3, c_4)) = M_1 \otimes M_3$, $g(K_3^1(c_6, c_7)) = M_1 \otimes M_4$, which implements a configuration of such a type. Then, any assignment of the remaining double sets $M_2 \otimes M_3$, $M_2 \otimes M_4$, $M_3 \otimes M_4$ to the spokes $(c, c_2), (c, c_5), (c, c_8)$ contradicts either Prop. 11 or 18.

Next, assume $g(K_3^1(c_3, c_4)) = M_2 \otimes M_4$, $g(K_3^1(c_6, c_7)) = M_3 \otimes M_4$, yielding $[K_3^1(c_0, c_1) - K_3^1(c_3, c_4) - K_3^1(c_6, c_7), 4b]$. There are three double sets for the remaining spokes, namely $M_1 \otimes M_3$, $M_1 \otimes M_4$, $M_2 \otimes M_3$. $c \cap c_5$ must be based on $M_1 \otimes M_4$ because if $c \cap c_5 \in M_2 \otimes M_3$ spokes $(c, c_4), (c, c_5), (c, c_6)$ contradict Prop. 18. Alternatively, if $c \cap c_5 \in M_1 \otimes M_3$ then the double sets of adjacent spokes $(c, c_4), (c, c_5)$ contradict Prop. 11. For analogous reasons, $c \cap c_2 \in M_2 \otimes M_3$, implying $c \cap c_8 \in M_1 \otimes M_3$. The resulting class is num. 16.

Lastly, assume $g(K_3^1(c_3, c_4)) = M_2 \otimes M_3$, $g(K_3^1(c_6, c_7)) = M_1 \otimes M_3$, yielding $[K_3^1(c_0, c_1) - K_3^1(c_3, c_4) - K_3^1(c_6, c_7), 3]$. The double sets of the spokes $(c, c_2), (c, c_5), (c, c_8)$ are uniquely defined as $M_2 \otimes M_4$, $M_3 \otimes M_4$, $M_1 \otimes M_4$, respectively (class num. 17). Observe that any other assignment will cause a violation of either Prop. 11, or Prop. 18.

$\underline{p = 5}$.

All five K_3^1 are consecutive. Thus, assume $K_3^1(c_0, c_1)$, $K_3^1(c_2, c_3)$, $K_3^1(c_4, c_5)$, $K_3^1(c_6, c_7)$, $K_3^1(c_8, c_9)$, form three distinct 3-sequences, each having two K_3^1 in common from the next.

First, we prove that these 3-sequences cannot all be of the same type (Claim 1). This is obvious for type 3. Type $4a$ is also excluded because each 3-sequence of this type presupposes the appearance of a single set in three double sets. Consequently, for three such sequences we need three single sets, each to appear on a triple of double sets. Observe that, for any 5 -set of double sets, there are two single sets appearing in three double sets.

Next, assume that one of the 3-sequences is of type $4b$. For example, let $K_3^1(c_0, c_1)$, $K_3^1(c_2, c_3)$, $K_3^1(c_4, c_5)$ be based on $M_1 \otimes M_2$, $M_2 \otimes M_3$, $M_3 \otimes M_4$, respectively. To form a second 3-sequence of the same type $K_3^1(c_6, c_7)$ should be based either on $M_1 \otimes M_4$ or $M_1 \otimes M_3$. In the first case, the remaining double sets are $M_2 \otimes M_4$ and $M_1 \otimes M_3$. One of these sets must be assigned to (c, c_9) and the other to (c, c_{10}), yielding a contradiction of Prop. 11. An analogous situation occurs in the second case. Therefore, because a configuration of two 3-sequences of type $4b$ with two common K_3^1 is infeasible, it is impossible to have three 3-sequences, with two K_3^1 in common, of type $4b$. The proof of Claim 1 is now complete.

Further, we will show that among the three 3-sequences, there cannot be two of the type $4b$ (Claim 2). In the proof of Claim 1, we have shown that among five consecutive double sets there cannot be two 3-sequences of type $4b$ with two K_3^1 in common. To see that we can neither have two 3-sequences of type $4b$ with one K_3^1 in common, assume $g(K_3^1(c_0, c_1)) = M_1 \otimes M_2$, $g(K_3^1(c_2, c_3)) = M_2 \otimes M_3$, $g(K_3^1(c_4, c_5)) = M_3 \otimes M_4$. Hence, $[K_3^1(c_0, c_1), K_3^1(c_2, c_3), K_3^1(c_4, c_5), 4b]$. Then, $K_3^1(c_6, c_7)$ is based either on $M_1 \otimes M_3$, or on $M_1 \otimes M_4$, or on $M_2 \otimes M_4$. In the first case, we have $[K_3^1(c_2, c_3), K_3^1(c_4, c_5), K_3^1(c_6, c_7), 4a]$, so that $[K_3^1(c_4, c_5), K_3^1(c_6, c_7), K_3^1(c_8, c_9), 4b]$ occurs only if $g(K_3^1(c_8, c_9)) = M_1 \otimes M_2$ or $g(K_3^1(c_8, c_9)) = M_2 \otimes M_3$. In both cases, these double sets constitute the ground sets of other K_3^1, which contradicts Prop. 7. In the second case, we have $[K_3^1(c_2, c_3), K_3^1(c_4, c_5), K_3^1(c_6, c_7), 4b]$, which was proven infeasible above. In the third case, $[K_3^1(c_2, c_3), K_3^1(c_4, c_5), K_3^1(c_6, c_7), 3]$. Then, for $K_3^1(c_4, c_5), K_3^1(c_6, c_7), K_3^1(c_8, c_9)$ to be of type $4a$ or $4b$, M_1 must appear in $g(K_3^1(c_8, c_9))$. If $g(K_3^1(c_8, c_9)) = M_1 \otimes M_2$ then there is a contradiction of Prop. 7, since this double set has also been used for $K_3^1(c_0, c_1)$. If $g(K_3^1(c_8, c_9)) = M_1 \otimes M_3$ then there is a contradiction of Prop. 11 for spokes (c, c_7), (c, c_8). Finally, if $g(K_3^1(c_8, c_9)) = M_1 \otimes M_4$, then $[K_3^1(c_4, c_5), K_3^1(c_6, c_7), K_3^1(c_8, c_9), 4a]$. The proof of Claim 2 is now complete.

Our last claim is that if there are two $[K_3^1 - K_3^1 - K_3^1, 3]$, then the third 3-sequence cannot be of type $4a$ (Claim 3). It is easy to see that we cannot have two $[K_3^1 - K_3^1 - K_3^1, 3]$, with two K_3^1 in common. Thus, consider a configuration, of five consecutive K_3^1, embedding two $[K_3^1 - K_3^1 - K_3^1, 3]$ with one K_3^1 in common and a $[K_3^1 - K_3^1 - K_3^1, 4a]$. Such a configuration is implemented if we assume $g(K_3^1(c_0, c_1)) = M_1 \otimes M_2$, $g(K_3^1(c_2, c_3)) = M_2 \otimes M_3$, $g(K_3^1(c_4, c_5)) = M_1 \otimes M_3$, $g(K_3^1(c_6, c_7)) = M_3 \otimes M_4$, $g(K_3^1(c_8, c_9)) = M_1 \otimes M_4$. This implies that $c \cap c_{10} \in M_3 \otimes M_4$, which contradicts Prop. 18 for spokes $(c, c_9), (c, c_{10}), (c, c_0)$.

There are four distinct configurations of the five K_3^1 not contradicting Claims 1-3. Each of these forms a different wheel class (class nums. 18-21). A representative from each class is presented below.

$$[M_1 \otimes M_2] - [M_1 \otimes M_3] - [M_1 \otimes M_4] - [M_3 \otimes M_4] - [M_2 \otimes M_3] - (M_2 \otimes M_4)$$
$$[M_1 \otimes M_2] - [M_1 \otimes M_3] - [M_1 \otimes M_4] - [M_2 \otimes M_4] - [M_3 \otimes M_4] - (M_2 \otimes M_3)$$
$$[M_1 \otimes M_2] - [M_1 \otimes M_3] - [M_1 \otimes M_4] - [M_3 \otimes M_4] - [M_2 \otimes M_4] - (M_2 \otimes M_3)$$
$$[M_1 \otimes M_2] - [M_2 \otimes M_3] - [M_1 \otimes M_4] - [M_2 \otimes M_4] - [M_3 \otimes M_4] - (M_1 \otimes M_4)$$

$\underline{\left|H^2(c)\right| = 1}$.

For all subcases of this case, assume that $c \cap c_0 \in M_1 \otimes M_2$, $c \cap c_1 \in M_1 \otimes M_2 \otimes M_3$, $c \cap c_2 \in M_2 \otimes M_3$. Thus, $g(K_3^3(c_0, c_1)) = M_1 \otimes M_2$, $g(K_3^3(c_1, c_2)) = M_2 \otimes M_3$. The double sets $M_1 \otimes M_4$, $M_2 \otimes M_4$, $M_3 \otimes M_4$ are left for the remaining spokes (Props. 36, 38).

$\underline{p = 2}$.

There can be either one or none K_3^1 (Prop. 46).

$\underline{\left|W_c^2(K_3^1)\right| = 1}$

We must have $K_3^1(c_3, c_4)$ (Cor. 37). It is impossible to have $[K_3^2(c_2, c_3) - K_3^1(c_3, c_4) - K_3^2(c_4, c_0), 3]$. This is because the double sets of $(c, c_0), (c, c_2)$ are two of the components of the triple set (c, c_1) and the third component cannot appear as a double set of any other spoke (Prop. 38). In order for the 3-sequence above to be either of type $4a$ or $4b$, the set M_4 must appear at $g(K_3^1(c_3, c_4))$. Only $M_2 \otimes M_4$ is a valid choice, because all alternatives lead to a contradiction of Prop. 11 either for spokes $(c, c_0), (c, c_4)$ or for $(c, c_2), (c, c_3)$. The resulting class is num. 22.

$\underline{\left|W_c^2(K_3^1)\right| = 0}$

The available double sets for spokes $(c, c_3), (c, c_4)$ are $M_1 \otimes M_4$, $M_2 \otimes M_4$, $M_3 \otimes M_4$. Observe that if either $c \cap c_3 \in M_1 \otimes M_4$, or $c \cap c_4 \in M_3 \otimes M_4$, there is a contradiction of Prop. 11 for spokes $(c, c_2), (c, c_3)$ and $(c, c_4), (c, c_0)$, respectively. Also observe that if $c \cap c_3 \in M_2 \otimes M_4$, and $c \cap c_4 \in M_1 \otimes M_4$, there is a contradiction of Prop. 18 for spokes (c, c_3),(c, c_4), (c, c_0), whereas if $c \cap c_3 \in M_3 \otimes M_4$, and $c \cap c_4 \in M_2 \otimes M_4$, there is a contradiction of Prop. 18 for spokes (c, c_2),(c, c_3),(c, c_4). Hence, $c \cap c_3 \in M_3 \otimes M_4$ and $c \cap c_4 \in M_1 \otimes M_4$ (class num. 23).

$\underline{p = 3}$.

$\underline{\left|W_c^3(K_3^1)\right| = 2}$

We must have $K_3^1(c_3, c_4)$, $K_3^1(c_5, c_6)$. Observe that $g(K_3^1(c_3, c_4)) = M_1 \otimes M_4$ leads to a contradiction of Prop. 11 for spokes (c, c_2), (c, c_3). For the same reason, $g(K_3^1(c_5, c_6)) \neq M_3 \otimes M_4$. Thus, the possible double sets for $K_3^1(c_3, c_4)$, $K_3^1(c_5, c_6)$ are either $M_3 \otimes M_4$, $M_1 \otimes M_4$, or $M_2 \otimes M_4$, $M_1 \otimes M_4$, or $M_3 \otimes M_4$, $M_2 \otimes M_4$, respectively. In the first case, two $[K_3^2 - K_3^1 - K_3^2, 4b]$ are formed (class num.24). The second and the third case are symmetric, both yielding the class num 25.

$\underline{\left|W_c^3(K_3^1)\right| = 1}$

First we claim that, the one K_3^1 must be 1-distant from one of the two K_3^3. To do this, we consider the alternative which is that the K_3^1 is exactly 2-distant from each of the K_3^3. Thus, assume $K_3^1(c_4, c_5)$. Because the available double sets all involve the set M_4, we have $[K_3^2(c_3, c_4) - K_3^1(c_4, c_5) - K_3^1(c_5, c_6), 4a]$. Then, $c \cap c_3 \notin M_1 \otimes M_4$ because the opposite contradicts Prop. 11. For analogous reasons, $c \cap c_6 \notin M_3 \otimes M_4$. If $c \cap c_3 \in M_3 \otimes M_4$, then the double sets of $K_3^1(c_4, c_5)$, and (c, c_6) are $M_1 \otimes M_4$, $M_2 \otimes M_4$, not necessarily in this order. This yields a contradiction of Prop. 18 for spokes $(c, c_5), (c, c_6), (c, c_0)$. If $c \cap c_3 \in M_2 \otimes M_4$ then $K_3^1(c_4, c_5)$ must be based on $M_3 \otimes M_4$ (remember that $c \cap c_6 \notin M_3 \otimes M_4$), yielding a contradiction of Prop. 18 for spokes $(c, c_2), (c, c_3), (c, c_4)$. The proof of our claim is complete.

Consider that K_3^1 is 1-distant from one K_3^3. Thus, assume $K_3^1(c_3, c_4)$. Then, $g(K_3^1(c_3, c_4)) \neq M_1 \otimes M_4$ because the opposite contradicts Prop. 11 for (c, c_2), (c, c_3). Also, $g(K_3^1(c_3, c_4)) \neq M_3 \otimes M_4$. The opposite implies that $M_1 \otimes M_4$, $M_2 \otimes M_4$ are left to be assigned to spokes (c, c_5), (c, c_6). Irrespectively of which double set is assigned to which spoke, there is a contradiction of Prop. 18 for (c, c_5), (c, c_6), (c, c_0). The only feasible assignment is $g(K_3^1(c_3, c_4)) = M_2 \otimes M_4$, $g((c, c_5)) = M_3 \otimes M_4$ and $g((c, c_6)) = M_1 \otimes M_4$, yielding class num. 26.

$\underline{p = 4}$.

The three consecutive K_3^1, are of type $4a$ (the set M_4 is common to all three double sets), that is $[K_3^1(c_3, c_4) - K_3^1(c_5, c_6) - K_3^1(c_7, c_8), 4a]$, $g(K_3^1(c_3, c_4)) \neq M_1 \otimes M_4$. The opposite contradicts Prop. 11 for (c, c_2), (c, c_3). For analogous reasons, $g(K_3^1(c_7, c_8)) \neq M_3 \otimes M_4$. If $g(K_3^1(c_3, c_4)) = M_2 \otimes M_4$, then $g(K_3^1(c_5, c_6)) = M_3 \otimes M_4$ and $g(K_3^1(c_7, c_8)) = M_1 \otimes M_4$, yielding class num. 27. The same class is produced if $g(K_3^1(c_3, c_4)) = M_3 \otimes M_4$, $g(K_3^1(c_5, c_6)) = M_1 \otimes M_4$ and $g(K_3^1(c_7, c_8)) = M_2 \otimes M_4$. However, if $g(K_3^1(c_3, c_4)) = M_3 \otimes M_4$, $g(K_3^1(c_5, c_6)) = M_2 \otimes M_4$ and $g(K_3^1(c_7, c_8)) = M_1 \otimes M_4$ then we end up with class num. 28.

$\underline{\left|H^2(c)\right| = 2}$.

By Prop. 39, $p = 2$. Assume $K_3^3(c_0, c_1)$, $K_3^4(c_1, c_2)$, $K_3^3(c_2, c_3)$ and that $c \cap c_1 \in M_1 \otimes M_2 \otimes M_3$, $c \cap c_1 \in M_1 \otimes M_2 \otimes M_4$. Observe that $c \cap c_0, c \cap c_3 \notin M_1 \otimes M_2$ (Prop. 36). We claim that (a) the double sets of (c, c_0) (c, c_3) should have no set in common, and (b) the double set of the spoke (c, c_4) is the double set having no set in common with the double set of K_3^4, which in our case is $M_3 \otimes M_4$.

To show (a) assume that $c \cap c_0 \in M_1 \otimes M_3$ and $c \cap c_3 \in M_1 \otimes M_4$. Then, due to Prop. 18, the double set of (c, c_4) must include M_2. However, if $c \cap c_4 \in M_1 \otimes M_2$, there is a contradiction of Prop. 36, else if $c \cap c_4 \in M_2 \otimes M_3$, we have a contradiction of Prop. 11 for spokes (c, c_3) and (c, c_4), else if $c \cap c_4 \in M_2 \otimes M_4$ yields a contradiction of Prop. 11 for spokes (c, c_0) and (c, c_4). This completes the proof of (a). Assume $c \cap c_0 \in M_1 \otimes M_3$ and $c \cap c_3 \in M_2 \otimes M_4$. Then, due to Prop. 18 for (c, c_0), (c, c_4), (c, c_3), the spoke (c, c_4) must be based on $M_3 \otimes M_4$. This proves (b) and yields class num. 29. ■

Note that some of the classes illustrated in Table 3 can be further divided into subclasses with respect to the number of triple-set rim edges. For each wheel class with $\left|H^2(c)\right| = 0$, the number of possible triple-set rim edges can be calculated, mainly, with the use of Rem. 25, Props. 23, 24, 26, Cor. 27, Lem. 28. The corresponding information for wheel classes having $\left|H^2(c)\right| \geq 1$ is included, mainly, in Lems. 40, 41, 42, 49, Prop. 47, and Cor. 45.

Another important issue is that of the cardinality of the set of wheels of $G_A(C, E_C)$. Obviously, the cardinality of the largest class determines the order of the size. It is easy to see that the classes of wheels of the largest size, i.e. $\left|H^2(c)\right| = 0$, $p = 5$, contain most members. This is formally described in the proof of the following theorem.

Theorem 54 *The cardinality of the set of wheels of OLS is of $O(n^{20})$.*

Proof. There are $|C| = n^4$ options for the hub of a wheel. For a certain hub, at least two indices for each rim node are determined. Observe that for wheels having $\left|H^2(c)\right| = 0$, exactly two indices of each node are not determined by the hub. On the other hand, for $\left|H^2(c)\right| \geq 1$, there exists at least one node having one index not determined by the hub. For each p, this observation implies that the number of wheels having $\left|H^2(c)\right| = 0$ is larger than the one of wheels with $\left|H^2(c)\right| \geq 1$. Recall also the fact that the wheels of the largest size belong exclusively to this family ($\left|H^2(c)\right| = 0$). Hence, we can only consider the cardinality of the family having $\left|H^2(c)\right| = 0$, in order to establish the size of the whole set of wheels in $G_A(C, E_C)$.

Hence, for wheels having $\left|H^2(c)\right| = 0$, the options for the two indices of a rim node depend on whether the node belongs to a K_3^1. Observe that up to 7 indices are required from each set. To see that, consider set M_1, and a wheel of size 11. All three double sets, namely $M_1 \otimes M_2$, $M_1 \otimes M_3$, $M_1 \otimes M_4$ appear at the spokes and at least two of them appear twice. Therefore, at least 5 nodes of the rim have value n for index m_1. Hence, there are at most 6 nodes, where index m_1 has value different from n. As a result, at most 7 values of index m_1 are required for a wheel of size 11. The argument can be repeated for each of the indices m_2, m_3, m_4 and also for wheels of smaller size. There are n options for selecting the value of index m_1 for the hub, $n-1$ options for selecting the value of index m_1 for the first node encountered in the rim, which has index m_1 different from n, and so on. It follows that there are $O(n)$ options for selecting each value of index m_1, the same being true for all other indices.

Two nodes inducing a K_3^2, are bound to have one index in common with value different from n (Prop. 16). Assuming a direction for examining the nodes of the rim, each node has $O(n)$ options for one of its indices, if it is part of a K_3^2 structure. However, if it induces a K_3^1, it has $O(n)$ options for each of its two indices, which is not equal to n, provided that the rim edge connecting it to its successor is not based on a triple set. If this rim edge is a triple link, the node has $O(n)$ options for only one of its indices. By restricting our analysis to wheels with no triple links, such a node has $O(n) \cdot O(n)$ options for its indices. Its successor has again $O(n)$ options for one of its indices, since it is the first of two nodes inducing a K_3^2 (Cor. 8). It follows that there are $\lfloor O(n) \rfloor^3$ options for each pair of nodes inducing a K_3^1. There are $\left|W_c^p(K_3^1)\right|$ such pairs and $U_p = 2p + 1 - 2\left|W_c^p(K_3^1)\right|$ nodes not belonging to such a pair (Lem. 22). Therefore, the number of rims of size $2p+1$, which include $|W_c^p|$ K_3^1 structures, is:

$$\lfloor O(n) \rfloor^{3\left|W_c^p(K_3^1)\right|} \cdot \lfloor O(n) \rfloor^{2p+1-2\left|W_c^p(K_3^1)\right|} = \lfloor O(n) \rfloor^{2p+1+\left|W_c^p(K_3^1)\right|} = O(n^{2p+1+\left|W_c^p(K_3^1)\right|})$$

Given that there exist n^4 options for the hub, the number of wheels of size $2p+1$ is: $O(n^{2p+1+\left|W_c^p(K_3^1)\right|})$. $n^4 = O(n^{2p+5+\left|W_c^p(K_3^1)\right|})$. Since $\left|W_c^p(K_3^1)\right| \leq p$ (Prop. 20), the function $2p+5+\left|W_c^p(K_3^1)\right|$ is strictly increasing with respect to p. Both p and $\left|W_c^p(K_3^1)\right|$ achieve their maximum value for 11-wheels, namely $p=5$, $\left|W_c^p(K_3^1)\right| = 5$ (Prop. 20). Therefore, the number of wheels of size 11 dominates the number of wheels of smaller size. Substituting the values of p, $\left|W_c^p(K_3^1)\right|$ in the above expression yields the result. ∎

6 Circulants and Recognition

Having classified the wheels of $G_A(C, E_C)$, we focus on the recognition issue. Since wheels can be regarded as special cases of lifted odd holes, it is important to devise a mechanism for distinguishing the two. To achieve this, we establish an association of odd-holes and rims of wheels with row sets of matrix A. Based on this association, we propose an algorithm that determines whether a particular submatrix of A gives rise to an odd-hole, which can constitute the rim of a wheel.

For any $Q \subseteq R$ and $T \subseteq C$, let A_Q^T denote the submatrix of A with rows and columns indexed by Q and T, respectively. Let $Q_{M_i \times M_j} = Q \cap (M_i \times M_j)$, for $(i \neq j,\ i,j \in \{1,\ldots,4\})$. Finally, let C_k^2 denote the circulant matrix of order k with two ones in each row and column and zero everywhere else. For any node set H $(H \subset C)$ inducing an odd hole.

Proposition 55 *Let H denote the node set of odd hole of $G_A(C, E_C)$, where $|H| = 2p+1$. If the number of triple-set edges of the odd hole is T_P, then A^H has 3^{T_p} distinct row sets Q, $|Q| = 2p+1$, such that*

1. $A_Q^H = C_{2p+1}^2$ *up to row and column permutations,*
2. *for* M_i, M_j, $(i \neq j,\, i,j \in \{1,2,3,4\})$, $0 \leq \left|Q_{M_i \times M_j}\right| \leq p$
3. *the rows of $A_{R \setminus Q}^H$ either are distinct copies of rows of C_{2p+1}^2 corresponding to edges based on triple sets, or else contain at most one 1.*

Proof. The rows of A^H correspond to the edges connecting each node of H with other nodes of H and nodes of $C \setminus H$. The nodes of H correspond to the columns of A^H, whereas there are no columns in A^H for the nodes of $C \setminus H$. Thus, the rows of A^H have either two or one 1. It is not difficult to see that when two nodes of H are connected by an edge based on a double set, this edge is represented by a single row in A^H. In contrast, when based on a triple set, the edge is represented by three rows of A^H. Therefore, the set of rows of A^H that contain two $1s$ corresponding to the edges connecting c_0 and c_1, c_1 and c_2, $\ldots$, c_{2p} and c_0, form a set of cardinality $2p+1+2T_p$, where T_p is the number of edges based on a triple set. This set contains exactly 3^{T_p} further subsets Q of cardinality $2p+1$ obtained by selecting a single row (out of the three) for each edge based on a triple set and one row for each edge based on a double set. Each such subset forms a square submatrix A_Q^H of order $2p+1$ that has exactly two 1s in every row and column, hence becoming C^2_{2p+1} after the necessary row and column permutations. This proves (1).

For (2), it is trivial that $\left|Q_{M_i\times M_j}\right| \geq 0$. To show that $\left|Q_{M_i\times M_j}\right| \leq p$, assume $\left|Q_{M_i\times M_j}\right| = p+1$. This means that there are $p+1$ edges of the odd hole based on $M_i \otimes M_j$. Consequently, two adjacent edges are based on the double set $M_i \otimes M_j$. This contradicts to the hypothesis, since it implies a chord connecting the two nodes that are incident to these edges.

To prove (3), consider that each row of A^H has either one or two 1s. Since A^H_Q for a given set $Q \subset R$ contains only rows that have exactly two 1s, all rows containing a single 1 belong to $A^H_{R\setminus Q}$. Further, for every edge based on a triple set, there are three rows in A^H containing two 1s in the same columns. In other words, for each triple-set edge there are three similar rows, only one of which is included in A^H_Q. The remaining two are included in $A^H_{R\setminus Q}$. ■

Obviously, Prop. 55 is true when $H = H(c)$, i.e. H is the node set of a rim of a wheel. However, we can achieve a slightly better upper bound on the value of $\left|Q_{M_i\times M_j}\right|$, for the case of a wheel with $\left|H^2(c)\right| = 0$.

Lemma 56 *Let H be the node set of a rim ($H = H(c)$). If $\left|H^2(c)\right| = 0$ then $\left|Q_{M_i\times M_j}\right| \leq \min\{3, p\}$.*

Proof. To show that, for wheels with $\left|H^2(c)\right| = 0$, 3 is an upper bound on $\left|Q_{M_i\times M_j}\right|$, assume $i = 1$ and $j = 2$. In the case that the rim edges are based on double sets, it is easy to show that there can be at most three rim edges based on $M_1 \otimes M_2$. This holds because there are exactly three K_3 having a rim edge based on $M_1 \otimes M_2$. These are: a K^1_3 such that $g(K^1_3) = M_1 \otimes M_2$, a K^2_3 with spokes based on the double sets $(M_1 \otimes M_3), (M_1 \otimes M_4)$ and a K^2_3 with spokes based on the double sets $(M_2 \otimes M_3), (M_2 \otimes M_4)$ (Prop. 16).

Suppose now that there is a rim edge based on a triple set, which has $M_1 \otimes M_2$ as one of its components. Clearly, if this is the rim edge of a K^1_3 such that $g(K^1_3) = M_1 \otimes M_2$, the previous case appears. In the opposite case, we claim that we cannot have two K^2_3, one with spokes based on the double sets $(M_1 \otimes M_3), (M_1 \otimes M_4)$ and the other based on $(M_2 \otimes M_3), (M_2 \otimes M_4)$. Without loss of generality, assume the sequence $(M_1 \otimes M_4) - [M_2 \otimes M_4] - (M_1 \otimes M_2)$. It is easy to see that the rim edge of the K^1_3 can be based on a triple set induced by the single sets M_1, M_2, M_4. We can see that at most two other rim edges can be based on $M_1 \otimes M_2$, since we can never have a K^2_3 with spokes based on $(M_2 \otimes M_3), (M_2 \otimes M_4)$ because $M_2 \otimes M_4$ cannot be used for any other spoke (Prop. 36). It is easy to see that the same is true in the case that $M_1 \otimes M_2$ is a component of two triple sets, each appearing at a rim edge of a K^1_3 not based on $M_1 \otimes M_2$. As an example, consider the wheel:

$$(M_1 \otimes M_4) - [M_2 \otimes M_4] - [M_1 \otimes M_2] - [M_2 \otimes M_3] - (M_1 \otimes M_3)$$

Note that no other case exists, since a double set appears as a component (Rem. 2) at no more than two triple sets. In addition, for $\left|H^2(c)\right| = 0$, all triple sets appearing at the rim edges of a wheel are distinct

(Lem. 28). ■

Table 4 illustrates wheels for all distinct values of $|H^2(c)|$ and p. Notice that the upper bound on $|Q_{I\times J}|$ is attained in all these wheels. Again, the hub is node (n,n,n,n). In the case that $H = H(c)$,

Table 4: Wheels for which $|Q_{I\times J}|$ attains the upper bound illustrated by (2) of prop. 55

$\lvert H^2(c)\rvert$	p	Wheel	$Q_{I\times J}$
0	2	$(n,n,k_0,l_1)-(n,n,k_1,l_0)-(n,j_1,n,l_0)-(n,j_0,n,l_2)$ $-(n,j_0,k_0,n)$	$\{(n,n),(n,j_0)\}$
0	3	$(i_1,n,n,l_1)-(i_0,n,n,l_3)-(i_0,n,k_0,n)-(n,n,k_0,l_2)$ $-(n,n,k_1,l_0)-(n,j_0,n,l_0)-(n,j_0,n,l_1)$	$\{(i_0,n),(n,n),$ $(n,j_0)\}$
0	4	$(i_0,n,k_4,n)-(i_0,n,n,l_1)-(n,n,k_1,l_1)-(n,n,k_1,l_0)$ $-(n,j_2,n,l_0)-(n,j_0,n,l_2)-(n,j_0,k_3,n)-(n,j_3,k_0,n)$ $-(i_1,n,k_0,n)$	$\{(i_0,n),(n,n),$ $(n,j_0)\}$
0	5	$(i_0,n,k_4,n)-(i_0,n,n,l_1)-(n,n,k_2,l_1)-(n,n,k_3,l_0)$ $-(n,j_2,n,l_0)-(n,j_0,n,l_2)-(n,j_0,k_0,n)-(n,j_1,k_0,n)$ $-(i_2,j_1,n,n)-(i_1,j_3,n,n)-(i_1,n,k_1,n)$	$\{(i_0,n),(n,n),$ $(n,j_0)\}$
1	2	$(n,n,n,l_0)-(n,n,k_0,l_1)-(i_1,n,k_0,n)-(i_0,n,k_1,n)$ $-(i_0,n,n,l_0)$	$\{(i_0,n),(n,n)\}$
1	3	$(n,n,k_1,l_1)-(n,n,k_1,l_0)-(n,j_0,n,l_0)-(n,j_0,n,n)$ $-(n,j_1,k_0,n)-(i_0,n,k_0,n)-(i_0,n,n,l_1)$	$\{(i_0,n),(n,n),$ $(n,j_0)\}$
1	4	$(i_0,n,k_0,n)-(i_0,n,n,n)-(i_1,n,n,l_2)-(n,j_1,n,l_2)$ $-(n,j_1,n,l_0)-(n,n,k_2,l_0)-(n,n,k_1,l_1)-(n,j_0,k_1,n)$ $-(n,j_0,k_0,n)$	$\{(i_0,n),(n,n),$ $(n,j_0),(n,j_1)\}$
2	2	$(n,j_0,n,n)-(n,j_0,n,l_0)-(n,n,k_0,l_0)-(i_0,n,k_0,n)$ $-(i_0,n,n,n)$	$\{(i_0,n),(n,j_0)\}$

the bounds on the parameter T_p (see Props. 26, 47 and Lem. 49) determine a range of values regarding the number of distinct circulants. One of these circulants is the *primary* circulant of the wheel.

Definition 57 *A circulant associated with the node set $H(c)$ of a wheel, will be called primary if for every rim edge based on a triple set, the corresponding row of the circulant is indexed by the double set constituting the ground set of the K_3 this rim edge belongs to.*

Recognising whether an odd hole is also the rim of a wheel of $G_A(C,E_C)$ constitutes the *recognition* problem.

Problem 58 *Given a circulant of an odd hole of the appropriate size, is it associated to a wheel?*

The question posed by Prob. 58 is that of the existence of a hub, for the circulant at hand. The reference on the size is required because circulants of size greater than eleven are not associated to wheels of $G_A(C,E_C)$ (see Cor. 10, Prop. 39). The hub of the wheel can be explicitly identified by the ground sets of the spokes. In fact, the identification of the hub requires a number of spokes sufficient for each single set to appear at least once. Then, the double sets of the remaining spokes need to be identified in order to produce a certificate that the odd hole is indeed the rim of a wheel with the given hub.

The complication of the recognition problem is that the circulant at hand might include, for some of the triple-set rim edges, rows of A indexed by any of the double sets forming the corresponding triple sets. The problem becomes easier if we consider primary circulants. Hence, this section addresses the following relaxed version of Prob. 58.

Problem 59 *Given a circulant of an odd hole of the appropriate size, is it a primary circulant of a wheel having* $\left|H^2(c)\right| = 0$*?*

To answer this question, certain auxiliary results are necessary.

Lemma 60 *Let* c_1, c_2 *be two adjacent rim nodes such that either* $K_3^1(c_1, c_2)$ *or* $K_3^2(c_1, c_2)$*. Given* $g((c, c_1))$ *and* $g((c_1, c_2))$*,* $g((c, c_2))$ *can be identified.*

Proof. Notice first that $g((c, c_1))$, $g((c, c_2))$, $g((c_1, c_2))$ must have one set in common (Prop. 11, Cor. 17). If $g((c_1, c_2)) = g((c, c_1))$, then the structure is $K_3^1(c_1, c_2)$, implying $g((c, c_2)) = g((c, c_1))$. Alternatively, suppose that $g((c_1, c_2)) \neq g((c, c_1))$. Without loss of generality, assume that $g((c, c_1)) = M_1 \otimes M_3$ and $g((c_1, c_2)) = M_1 \otimes M_2$. It follows that the underlying structure is $K_3^2(c_1, c_2)$ and $g((c, c_2)) = M_1 \otimes M_4$ (Prop. 16). ∎

Corollary 61 *For a wheel having* $\left|H^2(c)\right| = 0$*, given the double set of one of its spokes and the primary circulant, the double sets of the remaining spokes can be identified.*

Proof. Wheels having $\left|H^2(c)\right| = 0$, include only K_3^1 and K_3^2. Hence, starting from a spoke adjacent to the spoke, whose double set is known, and iteratively applying Lem. 60 yields the result. ∎

Observe that, given the primary circulant and having recognized the double set of at least one spoke, the above corollary solves Prob. 59. This is because this process reveals the double sets of all spokes. In this sense, the hub is denitrified and a certificate is produced by verifying that the double set of the initial spoke is the one recognized. In the case that the double set of the initial spoke is not verified, the answer to Prob. 59 is negative. The recognition of the ground set of at least one spoke, given the primary circulant of a wheel having $\left|H^2(c)\right| = 0$, is based on the following proposition.

Proposition 62 *Consider three consecutive rim edges whose double sets are formed by three sets. Then, for wheels having* $\left|H^2(c)\right| = 0$*, this structure implies* $[K_3^2 - K_3^1 - K_3^2, 3]$ *and vice versa.*

Proof. Assume a wheel having $\left|H^2(c)\right| = 0$ and let c_1, c_2, c_3, c_4 denote four consecutive rim nodes. Without loss of generality, assume that $g((c_1, c_2)) = M_1 \otimes M_3$, $g((c_2, c_3)) = M_1 \otimes M_2$, $g((c_3, c_4)) = M_2 \otimes M_3$. Then, the set M_1 must appear at $g((c, c_2))$ and M_2 at $g((c, c_3))$ (Prop. 19).Also, $g((c, c_2))$, $g((c, c_3))$ must have one set in common because the spokes (c, c_2), (c, c_3) are adjacent (Prop. 11). If this

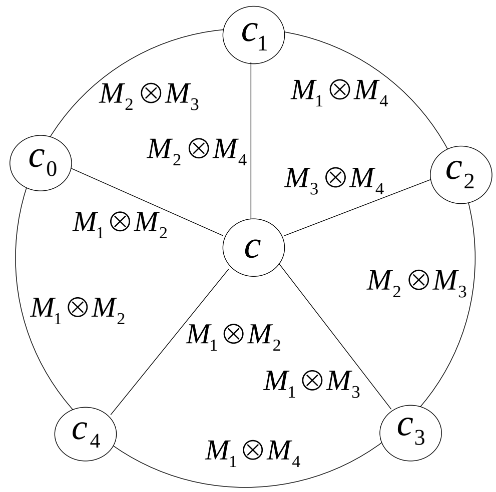


Figure 4: A wheel of class num. 3.

set is M_3 (M_4), then we have the structure $K_3^2(c_2, c_3)$ and M_3 (M_4) must appear at $g((c_2, c_3))$ (Prop. 16). In both cases, our initial assumption that $g((c_2, c_3)) = M_1 \otimes M_2$ is contradicted. Thus, $K_3^1(c_2, c_3)$ exists and $g((c, c_2)) = g((c, c_3)) = M_1 \otimes M_2$. By applying Lem. 60, we obtain that $g((c, c_1)) = M_1 \otimes M_4$ and $g((c, c_4)) = M_2 \otimes M_4$.

The inverse part of the proof is trivial. ■

Hence, if a structure like the one described in Prop. 62 is identified, in the circulant at hand, we can immediately guess the double sets of four spokes. A look at Table 3, for wheels having $\left|H^2(c)\right| = 0$, asserts that at least one $[K_3^2 - K_3^1 - K_3^2, 3]$ appears in all wheels except the ones belonging to classes num. 1, 3, 5. However, the wheels of these classes exhibit specific characteristics, which can be used to identify the ground sets of the spokes. In particular, wheels of classes 3 and 5 have at least one pair of adjacent rim edges based on ground sets with no set in common (see Figures 4 and 5, respectively). Observe that each of the edges of such a pair, belongs to a K_3^2 and the two K_3^2 are 0-distant. Moreover, each such pair forms a $[K_3^2 - K_3^2, 4b]$, i.e. the double sets of the three spokes forming the two K_3^2 constitute a 3-sequence of type $4b$. For the wheels of class 3, there exist three distinct $[K_3^2 - K_3^2, 4b]$, each two having with one K_3^2 in common, whereas for those of class 5 there exists one such configuration. In the first case, there exists a K_3^1 formed by the first spoke of the first K_3^2 and the second spoke of the fourth K_3^2 ($K_3^1(c_4, c_0)$ in Figure 4). In the second case, one K_3^1 is formed, which is exactly 2-distant from the $[K_3^2 - K_3^2, 4b]$ ($K_3^1(c_4, c_5)$ in Figure 5). Therefore, in both cases, the primary circulant can be used to identify pairs of adjacent rim edges with ground sets having no set in common. Afterwards, depending on the size of the circulant, it is possible to identify the rim edge belonging to a K_3^1. Evidently, the double set of that edge

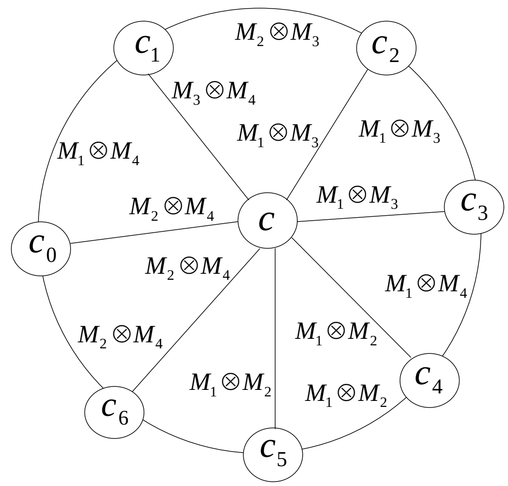


Figure 5: A wheel of class num. 5.

is the ground set of the two spokes forming the K_3^1.

Finally, the wheels belonging to class num. 1 have the property that the same single set appears in every double set of the rim edges (wheel of Figure 6). Again this characteristic can be traced from the primary circulant. In this class, there exist two K_3^1. Observe that the double set of the rim edge of the K_3^2 adjacent to the two K_3^1 is the same with the ground set of the spoke adjacent to the pair of spokes, each belonging to a different K_3^1.

Now we are ready to describe an algorithm for Prob. 59. The input of the algorithm is a circulant (of odd size less than or equal to 11). The output is the answer to the question posed by the problem. The algorithm consists of two steps. In the first step an attempt is made to guess the double set of at least one spoke. This is done initially with the use of Prop. 62. If this fails, we examine the circulant with respect to the particular characteristics of one of the classes 1, 3, 5. If this fails too, we can assert that this is not the primary circulant of a wheel with $\left|H^2(c)\right| = 0$ and skip the second step. If at Step 1 a double set is assigned to a spoke, then we proceed to the second step, where we attempt to evaluate all the double sets of the spokes based on this information (Cor. 61). Observe that we also need to re-evaluate the double set(s) of the spoke(s) guessed at Step 1. There are two cases where the algorithm returns a negative answer. The first case occurs during the evaluation of the double sets of the spokes. At this stage, if we find a spoke and its adjacent rim edge with double sets with no set in common, we cannot find the double set of the next spoke. This cannot happen for the primary circulant of a wheel with $\left|H^2(c)\right| = 0$ (Lem. 60). The second case occurs if the re-evaluated double sets do not match the guesses made at the first step. Again this shows that we are not dealing with the primary circulant of a wheel

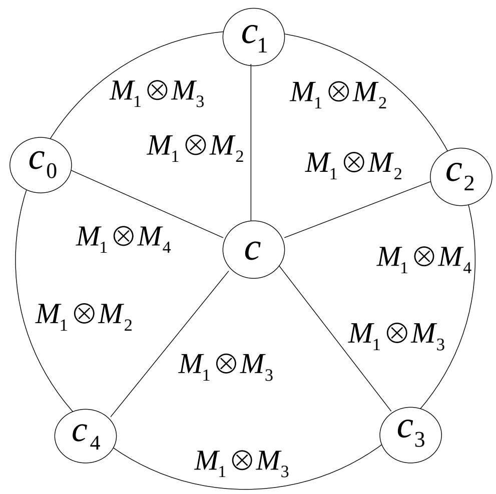


Figure 6: A wheel of class num. 1.

with $\left|H^2(c)\right| = 0$. The algorithm, in pseudocode, is illustrated below. Comments are included in /* */.

Algorithm 63 *"Solving Problem 59"*

    /*STEP 1*/

    IF *(*$[K_3^2 - K_3^1 - K_3^2, 3]$ *is detected)*

        */*Look for structure of Prop. 62*/*

    {

        Label the nodes as $c_0, c_1, c_2, c_3, \cdots, c_{2p}$ *such that* $[K_3^2(c_0, c_1) - K_3^1(c_1, c_2) - K_3^2(c_2, c_3), 3]$*;*

        Let $g((c, c_1)) = g((c, c_2)) = g((c_1, c_2))$*;*

    }

    ELSE IF *(*$(p = 3)$ ***AND*** *(*$[K_3^2 - K_3^2, 4b]$ *is detected))*

        */*Look for structure of class* 5**/*

    {

        Label the nodes as $c_0, c_1, c_2, \cdots, c_6$ *such that* $[K_3^2(c_0, c_1) - K_3^2(c_1, c_2), 4b]$*;*

        Let $g((c, c_4)) = g((c, c_5)) = g((c_4, c_5))$*;*

    }

    ELSE IF *(*$(p = 2)$ ***AND*** *(*$[K_3^2 - K_3^2, 4b]$ *is detected))*

        */*Look for structure of class* 3**/*

    {

        Label the nodes as c_0, c_1, c_2, c_3, c_4 *such that* $[K_3^2(c_0, c_1) - K_3^2(c_1, c_2), 4b]$,
            $[K_3^2(c_1, c_2) - K_3^2(c_2, c_3), 4b]$, $[K_3^2(c_2, c_3) - K_3^2(c_3, c_4), 4b]$*;*

Let $g((c, c_4)) = g((c, c_0)) = g((c_4, c_0))$;

}

ELSE IF *(($p = 2$)* ***AND*** *(the same set appears in the ground sets of all rim edges))*

*/*Look for structure of class 1*/*

{

Label the nodes as c_0, c_1, c_2, c_3, c_4 such that

$g((c_0, c_1)) = g((c_3, c_4))$ and $g((c_4, c_0)) = g((c_1, c_2))$;

Let $g((c, c_0)) = g((c_2, c_3))$;

}

ELSE *"This is not a primary circulant of a wheel with $H^2(c) = 0$";*

/*STEP 2*/

IF *(there is at least one assignment of a double set to a spoke)*

{

Attempt to find all double sets of the spokes by iteratively applying Lem. 60;

IF *((this is impossible)* ***OR*** *(the assignment(s) made at Step 1 are not valid))*

"This is not a primary circulant of a wheel with $H^2(c) = 0$";

ELSE

"This is a primary circulant of a wheel with $H^2(c) = 0$";

}

Lemma 64 *Algorithm 63 is of $O(p)$.*

Proof. At Step 1, each of the cases, excluding the first, is checked only if the circulant is of the appropriate size. This yields a constant number of operations for each such case. For the first alternative we must consider, in the worst case, $2p+1$ triplets of consecutive rim edges. For each such triplet, depending on the implementation, we need a constant number of operations to decide if we have a $[K_3^2 - K_3^1 - K_3^2, 3]$. Hence, Step 1 requires $O(p)$ operations in the worst case.

At Step 2, the algorithm evaluates the ground sets of $2p+1$ spokes. For each ground set to be evaluated, we need at most four comparisons to detect the common set of two adjacent spokes or assert that no such set exists. In the case of a common set, we need one more comparison to evaluate the second single set which appears in the ground set. Hence, the second step is also performed in $O(p)$ operations. ■

Notice that, since $p \leq 5$, Algorithm 63 essentially runs in a constant number of steps.

7 Conclusions

This paper has presented all classes of wheels of the OLS polytope. The structural properties of the wheels have been extracted through the study of their components (spokes, rim edges, 3-cliques) based on ground sets. Through the relations revealed, we were able to devise a classification useful for deriving families of valid inequalities. In a forthcoming paper we examine some of these families, showing that they are facet-defining for P_I. The same methodology can be used for the study of more involved wheel structures like the ones described in [6, 7, 8, 9]. It can also be used for examining wheel-induced subgraphs of other polytopes.

References

[1] G. Appa, Mathematical programming formulations of the Orthogonal Latin squares problem, LSEOR01 37.

[2] G. Appa, D. Magos, I. Mourtos, J. C. M. Janssen, On the Orthogonal Latin Squares polytope, Under revision for Discrete Math.

[3] G. Appa, D. Magos, I. Mourtos, A new class of facets for the Latin square polytope. Submitted to SIAM J. Discrete Math.

[4] E. Balas, M. J. Saltzman, Facets of the three-index assignment polytope, Discrete Applied Math. 23 (1989) 201-229.

[5] E. Balas, L. Qi, Linear time separation algorithms for the three index assignment polytope, Discrete Applied Math. 43 (1993) 201 209.

[6] F. Barahona, A. R. Mahjoub, Compositions of graphs and polyhedra II: Stable sets, SIAM J. on Disc. Math. 7 (1994) 359-371.

[7] E. Cheng, W. H. Cunningham, Wheel inequalities for stable set polytopes, Math. Program. **77** (1997) 389-421.

[8] E.Cheng, S. de Vries, Antiweb-wheel inequalities and their separation problems over the stable set polytopes, Math. Program. 92 (2002) 153-175.

[9] E.Cheng, S. de Vries, On the facet-inducing antiweb-wheel inequalities for stable set polytopes, SIAM J. Disc. Math. 15 4 (2002) 470-487.

[10] G. B. Dantzig, Linear Programming and Extensions (Princeton University Press, 1963).

[11] J. Dénes, A. D. Keedwell, Latin Squares and Their Applications (Academic Press, New York, 1974).

[12] R. Euler, R. E. Burkard, R. Grommes, On latin squares and the facial structure of related polytopes, Discrete Math. 62 (1986) 155-181.

[13] R. Euler, Odd cycle and a class of facets of the axial 3-index assignment polytope, Zastosowania Mathematyki XIX 3-4 (1987) 375-386.

[14] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combinatorial Optimization, 2nd corr. ed. (Springer, 1993).

[15] C. F. Laywine, G. L. Mullen, Discrete Mathematics Using Latin Squares (John Wiley & Sons, 1998).

[16] D. Magos, I. Mourtos, G. Appa, Polyhedral results for assignment porblems, LSE-CDAM-2002-01, http://www.cdam.lse.ac.uk/Reports.

[17] M. W. Padberg, On the facial structure of the set packing polyhedra, Math. Program. 5 (1973) 199-215.

[18] L. Qi, E. Balas, G. Gwan, A new class of facet defining inequalities for the three index assignment problem, in Dung-Zu Du and Jie Sun, eds., Advances in Optimization and Approximation (Kluwer Academics Pub., 1993) 256-274.