Stars and bunches in planar graphs.
Part II : General planar graphs and colourings *

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CDAM Research Report LSE-CDAM-2002-05

May 2002

Abstract

Given a plane graph, a k-star at u is a set of k vertices with a common neighbour u; and a bunch is a maximal collection of paths of length at most two in the graph, such that all paths have the same end vertices and the edges of the paths form consecutive edges (in the natural order in the plane graph) around the two end vertices. We first prove a theorem on the structure of plane graphs in terms of stars and bunches. The result states that a plane graph contains a $(d-1)$-star centred at a vertex of degree $d \leq 5$ and the sum of the degrees of the vertices in the star is bounded, or there exists a large bunch.

This structural result is used to prove a best possible upper bound on the minimum degree of the square of a planar graph, and hence on a best possible bound for the number of colours needed in a greedy colouring of it. In particular, we prove that for a planar graph G with maximum degree $\Delta \geq 47$ the chromatic number of the square of G is at most $\lceil \frac{9}{5}\,\Delta \rceil + 1$. This improves existing bounds on the chromatic number of the square of a planar graph.

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*This is a translated and adapted version of a paper that appeared in Diskretn. Anal. Issled. Oper. Ser. 1 **8** (2001) no. 4, 9–33 (in Russian).

The first three authors are supported by NWO grant 047-008-006.

1 Introduction and main results

Throughout this paper, G is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set V and edge set E. The *distance* between two vertices u and v is the length of a shortest path joining them. We are mainly interested in pairs at distance one or two, for which we also can define : a pair of vertices u, v, $u \neq v$, have distance one if they are adjacent; and they have distance two if they are not adjacent but have a common neighbour.

A *distant-2-colouring* of G is a colouring of the vertices such that vertices at distance one or two have different colours. The least number for which a distant-2-colouring exists is called the *distant-2 chromatic number* of G, denoted by $\chi_2(G)$. Note that a distant-2-colouring of G is equivalent to an ordinary vertex colouring of the square G^2 of G. (The *square* of a graph G, denoted G^2, is the graph with the same vertex set and in which two vertices are joined by an edge if and only if they have distance one or two in G.) And hence the distant-2 chromatic number $\chi_2(G)$ equals the ordinary chromatic number $\chi(G^2)$.

The following conjecture was formulated in [11]. (See also Jensen & Toft [8, Section 2.18].)

Conjecture 1.1 (Wegner [11])
If G is a planar graph with maximum degree Δ, then

$$\chi_2(G) \le \begin{cases} \Delta + 5, & \text{if } 4 \le \Delta \le 7; \\ \lfloor \frac{3}{2}\,\Delta \rfloor + 1, & \text{if } \Delta \ge 8. \end{cases}$$

A first result towards a proof of this conjecture can be found in work of Jonas [9]. From one of the results in [9] it follows directly that $\chi_2(G) \le 8\,\Delta - 22$ for a planar graph G with maximum degree $\Delta \ge 7$. This bound was significantly improved in Van den Heuvel & McGuinness [7] to $\chi_2(G) \le 2\,\Delta + 25$. Independently, a result with a smaller factor in front of the Δ was proved by Agnarsson & Halldórsson [1] who showed that, provided $\Delta \ge 749$, for a planar graph G with maximum degree Δ we have $\chi_2(G) \le \lfloor \frac{9}{5}\,\Delta \rfloor + 2$.

The goal of this paper is to reduce the lower bound on Δ for this last bound.

Theorem 1.2
If G is a planar graph with maximum degree Δ, then

$$\chi_2(G) \le \begin{cases} 59, & \text{if } \Delta \le 20; \\ \max\{\, \Delta + 39,\ \lceil \frac{9}{5}\,\Delta \rceil + 1 \,\}, & \text{if } \Delta \ge 21. \end{cases}$$

In particular, if $\Delta \ge 47$, then $\chi_2(G) \le \lceil \frac{9}{5}\,\Delta \rceil + 1$.

The proof of Theorem 1.2 involves the establishment of the existence of certain *unavoidable configurations* in a planar graph. This approach goes back to Heawood's proof of the 5-Colour Theorem [6], the old and new proofs of the 4-Colour Theorem [2,3,10], and was also used in the proofs of the bounds mentioned above.

Our unavoidable configurations are defined in terms of "bunches" and "stars".

We say that G has a *bunch* of length $m \geq 3$ with *poles* the vertices p and q, where $p \neq q$, if G contains a sequence of paths $P_1, P_2, \ldots, P_m$ with the following properties. Each P_i has length 1 or 2 and joins p with q. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by P_i and P_{i+1} is not separating in G (i.e., has no vertex of G inside) (see Fig. 1.1). Moreover, this sequence

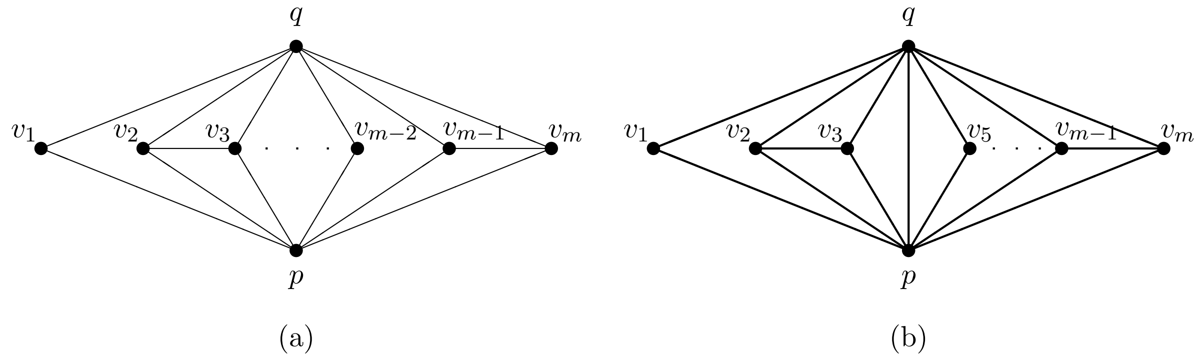


Fig. 1.1: A bunch without a parental edge (a) and with a parental edge (b)

of paths is maximal in the sense that there is no path P_0 (or P_{m+1}) that could be added to the bunch, preserving the above properties.

If a path P_i in the bunch has length 2, i.e., $P_i = pv_iq$, then the vertex v_i will be called a *brother* or a *bunch vertex*. A path $P_i = pq$ of length 1 in the bunch will be referred to as a *parental edge* (Fig. 1.1 (b)).

If the cycle bounded by P_1 and P_m separates G, then the edges in P_1 and P_m are called *boundary edges*, and the vertices v_1 and v_m (if they exist) are the *end vertices* (or *ends*) of the bunch. If $m \geq 3$, then the edges in P_2 and P_{m-1} are called *preboundary edges*. The vertex v_i in the bunch is *interior* if $2 \leq i \leq m-1$ and *strictly interior* if $3 \leq i \leq m-2$. Each edge v_iv_{i+1} joining consecutive bunch vertices is called *horizontal*, while the edges of the P_i's are called *vertical* in the bunch. Observe that each interior vertex has degree 2, 3 or 4 and is adjacent only to the poles and possibly to one or two brothers.

A *d-vertex* in G is a vertex of degree d. The *B-vertices* in G are those of degree at least 26, *L-vertices* have degree at most 25, and *minor vertices* at most 5.

Let u be a d-vertex, and let $v_1, \ldots, v_k$ be adjacent to u. We say that the vertices $u, v_1, \ldots, v_k$ and edges $uv_1, \ldots, uv_k$ form a *k-star at u*, defined by $v_1, \ldots, v_k$, of *weight* $\sum_{i=1}^{k} d(v_i)$. A $(d-1)$-star at a d-vertex is called *precomplete*, and a d-star at a d-vertex is *complete*.

The following result describes the unavoidable configurations used in our results on distant-2-colourings. The proof can be found in Section 3.

Theorem 1.3

For each plane graph G at least one of the following holds:

(A) G has a precomplete star of weight at most 38 that does not contain B-vertices and is centred at a minor vertex.

(B) G has a B-vertex b that satisfies at least one of the following conditions:

(i) b is a pole for a bunch of length greater than $d(b)/5$;

(ii) b is a pole for a bunch of length precisely $d(b)/5$ with a parental edge;

(iii) b is a pole for 5 bunches of length $d(b)/5$ without parental edges and with pairwise different end vertices. Moreover, among the end vertices there is a vertex v_0 of degree at most 11, and each other end vertex has degree at most 5 (see Fig. 1.2).

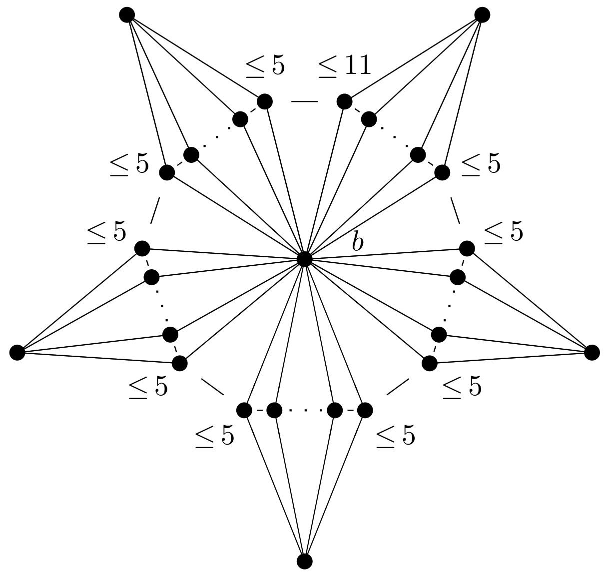


Fig. 1.2

Furthermore, if v_i and v_{i+1} are consecutive in the vicinity of b and are end vertices of two bunches such that $v_i \neq v_0$ and $d(v_i) = 5$, then v_i and v_{i+1} are adjacent in G.

As proved in Borodin & Woodall [5], each plane graph with minimum degree 5 has a precomplete star of weight at most 25 centred at a 5-vertex. On the other hand, planar graphs with vertices of degree less than 5 may have arbitrarily large weight of the precomplete stars at all minor vertices, as follows from the n-bipyramid. Theorem 1.3 shows that this is only possible if there are long enough bunches at big vertices. Moreover, Theorem 1.3 implies the following sufficient condition for the existence of an upper bound for the weight of precomplete stars at minor vertices.

Corollary 1.4
If G is a planar graph such that each B-vertex b is a pole only for bunches of length less than $d(b)/5$, then G has a precomplete star of weight at most 38 at a minor vertex. In particular, if the length of each bunch in G is at most 5, then G has a star of this kind.

In the next section we will discuss some corollaries of Theorem 1.3. We also discuss the sharpness of some of these corollaries. In particular we give the proof of Theorem 1.2. In Section 3 we will prove Theorem 1.3. This proof depends on a similar result for plane triangulations found in part I [4].

2 Distant-2 degrees and distant-2-colourings in planar graphs

The *distant-2 degree* $d_2(v)$ of a vertex v of a graph G is the number of vertices of G lying at distance 1 or 2 from v. Equivalently, the distant-2 degree of v in G is the ordinary degree of v in the square G^2 of G. We write $\delta_2(G)$ for the minimum distant-2 degree of vertices of G, and $\delta_2^*(G)$ for the minimum distant-2 degree of minor vertices of G. (Clearly, $\delta_2(G) \leq \delta_2^*(G)$ for every graph G.)

Another implication of Theorem 1.3 consists in obtaining the following upper bound for the minimum distant-2 degree $\delta_2(G)$ and $\delta_2^*(G)$ of plane graphs. These bounds are sharp whenever $\Delta \geq 47$.

Theorem 2.1
If G is a planar graph with maximum degree Δ, then
(a) $\delta_2(G) \leq \max\{\Delta + 38, \lceil \frac{9}{5}\Delta \rceil\}$;
(b) $\delta_2^*(G) \leq \max\{\Delta + 38, \lfloor \frac{9}{5}\Delta \rfloor + 1\}$.
In particular, if $\Delta \geq 47$, then $\delta_2(G) \leq \lceil \frac{9}{5}\Delta \rceil$ and $\delta_2^*(G) \leq \lfloor \frac{9}{5}\Delta \rfloor + 1$, and these bounds are best possible.

Proof Let G satisfy (A) in Theorem 1.3. Then there is a minor vertex u in G centred at a complete star of weight at most $\Delta + 38$. Since the distant-2 degree of u is not greater than the weight of the complete star at u, it follows that $\delta_2(G) \leq \delta_2^*(G) \leq \Delta + 38$.

Now let G satisfy (B) in Theorem 1.3. Then in each of the cases (i) – (iii), G has a bunch H of length k with poles b and t, which has a strictly interior vertex u. Let us bound $d_2(u)$ from above in terms of $d(b)$, $d(t)$ and k. First assume that H has no parental edge. By the definition of strictly interior vertex, each vertex of G that lies at distance 1 or 2 from u is adjacent to or coincides with one of the poles b and t of the bunch. Furthermore, each bunch vertex of H is adjacent to both poles. This yields

$$d_2(u) \leq d(b) + d(t) - k + 1. \tag{2.1}$$

When H has the parental edge bt, the only difference is that now one interior vertex "is missing" in the bunch, so that

$$d_2(u) \leq d(b) + d(t) - k. \tag{2.2}$$

For the cases (i) and (ii) of (B), using $k > d(b)/5$ and $k = d(b)/5$ in (2.1) and (2.2), respectively, we have

$$d_2(u) \;<\; \tfrac{4}{5}\, d(b) + d(t) + 1 \;\leq\; \tfrac{9}{5}\,\Delta + 1,$$

whence $\delta_2(G) \leq \delta_2^*(G) \leq \lceil \frac{9}{5}\,\Delta \rceil$.

Now suppose we are in case (iii) of (B). Then G has a B-vertex b which is a pole for five bunches $H_1, \ldots, H_5$ without parental edges and of length $d(b)/5$ each. Let the other poles of these bunches be $t_1, \ldots, t_5$. For each strictly interior vertex u of H_i it now follows from (2.1), where we can take $k = d(b)/5$, that

$$d_2(u) \;\leq\; \tfrac{4}{5}\, d(b) + d(t_i) + 1 \;\leq\; \tfrac{9}{5}\,\Delta + 1,$$

whence $\delta_2^*(G) \leq \lfloor \frac{9}{5}\,\Delta \rfloor + 1$.

Still for case (iii) of (B), let us estimate the distant-2 degree of b. Observe that apart from the vertices adjacent to b, the distant-2 vicinity of b also includes $t_1, \ldots, t_5$ and several vertices adjacent to the end of bunches H_i in G. These vertices will be called *exterior* for b. From the assumptions posed on the degrees of the end vertices and on their adjacency, it follows that each end vertex v_i other than v_0 (cf. the statement of Theorem 1.3), is adjacent to at most two exterior vertices of G, while v_0 is adjacent to at most nine exterior vertices. Hence we have

$$d_2(b) \;\leq\; d(b) + 5 + 9 \cdot 2 + 9 \;\leq\; \Delta + 32 \;<\; \Delta + 38,$$

whence $\delta_2(G) \leq \Delta + 38$.

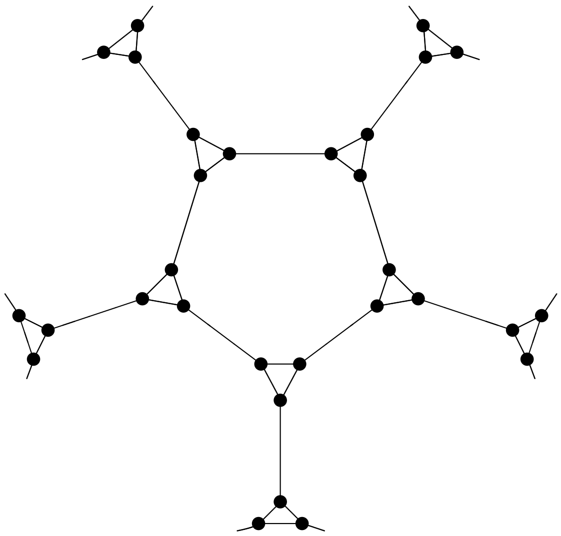
Thus we have proved the upper bounds (a) and (b) in Theorem 2.1. Since $\Delta \geq 47$ implies

$$\Delta + 38 \;\leq\; \lceil \tfrac{9}{5}\,\Delta \rceil \;\leq\; \lfloor \tfrac{9}{5}\,\Delta \rfloor + 1,$$

it follows that $\Delta \geq 47$ implies $\delta_2(G) \leq \lceil \frac{9}{5}\,\Delta \rceil$ and $\delta_2^*(G) \leq \lfloor \frac{9}{5}\,\Delta \rfloor + 1$.

To prove the sharpness of the last two bounds, we first consider the icosododecahedron graph J, partially shown in Fig. 2.1 (a). It is obtained by cutting off all the vertices of the dodecahedron, i.e., replacing each vertex by a 3-face incident with three new vertices of degree 3. As a result, each face of the initial dodecahedron gives rise to a face of size 10 in J, adjacent to five 10-faces and five 3-faces.

We replace each edge of J incident with two 10-faces by a path of length $k - 1$, where $k \geq 6$. The resulting graph J_k has 12 faces of size $5\,k$ and 20 triangles (Fig. 2.1 (b)). Next, we put a new vertex b_i into the centre of each $5k$-face f_i of J_k ($i = 1, \ldots, 12$) and join it with all the vertices in the boundary of f_i in J_k (Fig. 2.2 (a)). In the resulting triangulation T_k, each vertex b_i ($i = 1, \ldots, 12$) has degree $\Delta = 5\,k$ and is a pole for five bunches of length k which have no parental edges and whose end vertices have degree 5 each (i.e., T_k satisfies (iii) of (B) in Theorem 1.3). Now, counting the distant-2 degrees of minor vertices in T_k, we see that $\delta_2^*(T_k) = 9\,k + 1 = \lfloor \frac{9}{5}\,\Delta \rfloor + 1$, i.e., T_k attains the upper bound in (b). To extend this construction to Δ not divisible by 5, it suffices to increase the length of certain bunches in T_k from k to $k+1$ and leave the other bunches unchanged so that the degrees of all b_i's remain equal.



(a) (b)

Fig. 2.1 : The graphs J (a) and J_k (b)

Observe that the distant-2 degree of each b_i ($i = 1, \ldots, 12$) in T_k equals $\Delta + 5$, and therefore T_k fails to attain the upper bound in part (a) of Theorem 2.1. To attain (a), we replace one strictly interior vertex of each bunch in T_k by a parental edge $b_i b_j$ (Fig. 2.2 (b)). In the resulting triangulation T'_k, the minimum distant-2 degree δ_2 is attained on minor vertices and equals $9\,k = \lceil \frac{9}{5}\,\Delta \rceil$, so that we are done with (a) if Δ divides 5. The general case follows by replacing certain bunches in T'_k by bunches of length $k+1$ *without parental edges.* This completes the proof of Theorem 3. ■

The following result follows directly from Corollary 1.4.

Corollary 2.2
If G is a planar graph such that no B-vertex b is a pole for a bunch of length at least $d(b)/5$, then $\delta_2^*(G) \le \Delta + 38$. In particular, this inequality holds if G has no bunches of length at least 5.

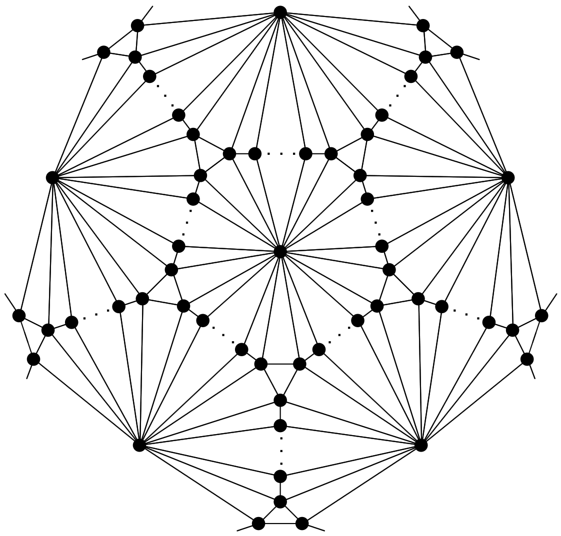
Theorem 1.3 and the upper bounds for $\delta_2^*(G)$ above can be used to prove the upper bounds in Theorem 1.2.

Theorem 2.3
Each planar graph G has

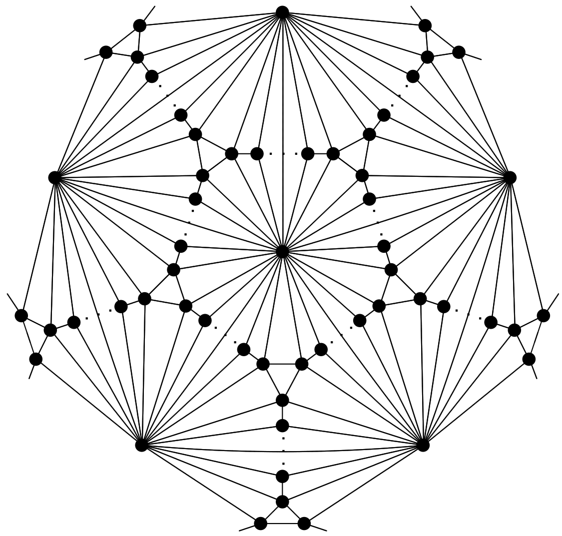
(a) $\chi_2(G) \le 59$ *whenever* $\Delta \le 20$, *and*

(b) $\chi_2(G) \le \max\{\, \Delta + 39,\ \lceil \frac{9}{5}\,\Delta \rceil + 1 \,\}$ *whenever* $\Delta > 20$.

In particular, if $\Delta \ge 47$, then $\chi_2(G) \le \lceil \frac{9}{5}\,\Delta \rceil + 1$.



(a)



(b)

Fig. 2.2 : The graphs T_k (a) and T'_k (b)

Proof Suppose that $\Delta \leq 20$ and that G is a minimal counterexample to the statement in part (a). Since for these Δ the graph G fails to satisfy condition (B) in Theorem 1.3, it follows that G has a minor vertex u that is a centre for a precomplete star of weight at most 38. Among the neighbours of u in G, choose a vertex v with smallest degree and denote by G_1 the plane graph obtained by contracting the edge uv into a new vertex $v_1 \in V(G_1)$.

We first prove that $\Delta(G_1) \leq 20$. It suffices to show that $d_{G_1}(v_1) \leq 20$. Observe that

$$d_{G_1}(v_1) \;\leq\; d(u) + d(v) - 2, \tag{2.3}$$

which readily implies that $d_{G_1}(v_1) \leq 20$ if $d(u) \leq 2$. If $d(u) = 3$ (or $d(u) \geq 4$), the choice of v and the bounds for the weight of a precomplete star at u together imply that $d(v) \leq 19$ ($d(v) \leq 12$, respectively). Hence, using (2.3), we again see that $d_{G_1}(v_1) \leq 20$.

So we have proved that $\Delta(G_1) \leq 20$. By the minimality of G, there exists a distant-2-colouring of G_1 with 59 colours. This colouring induces a distant-2-colouring at the vertex set $V(G) - u$ in G (since the distance between any two vertices from $V(G) - u$ in G is not greater than in G_1). Now from $d_2(u) \leq \Delta + 38 \leq 58$ we deduce that the distant-2-colouring obtained can be extended to u in G, which completes the proof of (a).

To prove (b), we again consider a minimal counterexample G (with $\Delta > 20$). If G satisfies statement (A) of Theorem 1.3, then we use the same argument as in proving (a). This leads to a plane graph G_1 with $\Delta(G_1) \leq \Delta$ that has a colouring with $\max\{\, \Delta + 39,\, \lceil \frac{9}{5}\,\Delta \rceil + 1 \,\}$ colours (by the minimality of G and the already proved statement (a)). Now, using the bound on the distant-2 degree of u in G, we deduce that the distant-2-colouring of G_1 yields a distant-2-

colouring of G with $\max\{\,\Delta+39,\ \lceil\frac{9}{5}\,\Delta\rceil+1\,\}$ colours.

Suppose G satisfies statement (B) of Theorem 1.3. Let u be a strictly interior vertex of a bunch centred at b (cf. the statement of Theorem 1.3). If we are in case (i) or (ii), then the same arguments as above combined with the fact that $d_2(u) \le \lceil\frac{9}{5}\,\Delta\rceil$ yield a distant-2-colouring of G with $\max\{\,\Delta+39,\ \lceil\frac{9}{5}\,\Delta\rceil+1\,\}$ colours. Suppose case (iii) in (B) holds. Then, as follows from the proof of Theorem 2.1, $d_2(u) \le \lfloor\frac{9}{5}\,\Delta\rfloor+1$ and $d_2(b) \le \Delta+38$. In this case, we first transfer a colouring of G_1 with $\max\{\,\Delta+39,\ \lceil\frac{9}{5}\,\Delta\rceil+1\,\}$ colours to the vertex set $V(G)-\{u,b\}$ in G (not giving a colour to b), and then colour consecutively u and b in G. This completes the proof of Theorem 2.3. ■

Remark.

Since the proof of Theorem 2.3 makes use only of the upper bound for the distant-2 degree of vertices in G, which gives the number of restrictions for the choice of colours for these vertices, it follows that the statement of Theorem 2.3 (along with the proof) is valid also for *list* distant-2-colourings and the *list* distant-2 chromatic number $\chi_{2(l)}(G)$.

Theorem 2.3 can also be generalised to so-called *$L(p,q)$-labellings* of planar graphs. For integers $p,q \ge 0$, this is any mapping $\varphi : V(G) \longrightarrow \{0,1,\ldots,k\}$ such that

(1) $|\varphi(u)-\varphi(v)| \ge p$ for all adjacent vertices p,q in G;

(2) $|\varphi(u)-\varphi(v)| \ge q$ for all vertices p and q in G at distance 2.

The *p,q-span* of a graph G, denoted $\lambda(G;p,q)$, is the minimum k for which an $L(p,q)$-labelling exists. Notice that this means that $\lambda(G;1,1) = \chi(G^2)-1$. An upper bound on $\lambda(G;p,q)$ for planar graph G can be proved similarly to Theorem 2.3. We obtain that for a planar graph G with maximum degree $\Delta \ge 47$, and for positive integers p,q with $p \ge q$, $\lambda(G;p,q) \le \lceil\frac{9}{5}\,\Delta\rceil\,(2\,q-1)+8\,p-8\,q+1$.

3 Proof of Theorem 1.3

The following result, which is essentially Theorem 1.3 for triangulations, is proved in [4].

Theorem 3.1

For each plane triangulation G at least one of the following holds:

(A) G has a precomplete star of weight at most 38 that does not contain B-vertices and is centred at a minor vertex.

(B) G has a B-vertex b that satisfies at least one of the following conditions:

- *(i) b is a pole for a bunch of length greater than $d(b)/5$;*
- *(ii) b is a pole for a bunch of length precisely $d(b)/5$ with a parental edge;*
- *(iii) b is a pole for 5 bunches of length $d(b)/5$ without parental edges and with pairwise different end vertices. Moreover, all but possibly one end vertices have degree 5, while the other end vertex has degree at most 11 (see Fig. 1.2 with all instances of "≤ 5" replaced by "$= 5$").*

For any plane graph G, we consider two numerical parameters $\tau(G)$ and $\beta(G)$. The first is defined as

$$\tau(G) \;=\; \sum_{f\in F(G)} (r(f)-3) \;=\; 2\,|E(G)| - 3\,|F(G)|$$

and characterizes the distance from G to a triangulation on the same vertices. Here $F(G)$ is the set of faces of the plane graph G, and $r(f)$ is the number of edges in the boundary of a face f. By $\beta(G)$ we denote the number of edges $e \in E(G)$ incident with two B-vertices in G. Such edges will hereafter be called *BB-edges*, and those incident with precisely one B-vertex, *BL-edges*. Finally, the vertices joining two L-vertices in G will be called *LL-edges*.

Now let G be a counterexample to Theorem 1.3. It follows from Theorem 3.1 that G is not a triangulation, whence $\tau(G) > 0$. Besides, G has no vertices of degree 1, and each vertex of degree 2 in G is adjacent to two B-vertices (due to (A)). We choose a counterexample G_0 with the minimal τ such that $\beta(G_0)$ is the least possible.

We define a *halfbunch* of length $m \geq 2$ with poles vertices p and q in a plane graph G as a sequence of paths $P_1, \ldots, P_m$ with the following properties. The path $P_m = pq$ has length 1, whereas all other paths have length 2. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by P_i and P_{i+1} is not separating in G.

Lemma 3.2
If G_0 has a bunch of length at least 6, or a bunch of length at least 5 without parental edge, or a halfbunch of length at least 4, then both poles of this bunch or halfbunch are B-vertices.

Proof Suppose H is a bunch in G_0 of length at least 6, or of length at least 5 and without parental edge. Then H contains a strictly interior vertex w. Let p and q be the poles of H. Vertex w has degree at most 4 and is adjacent to p, q and to at most two other interior vertices, also of degree at most 4. So there exist two precomplete stars centred at the minor vertex w of weight at most $d(p) + 8$ and $d(q) + 8$. Since G_0 does not satisfy Theorem 1.3 (A), we must have $d(p), d(q) \geq 30$, so p and q are certainly B-vertices.

We obtain the same result for a halfbunch of length at least 4 with poles p and q by considering the bunch vertex neighbouring the edge pq. □

A *step* remakes a current counterexample G to a counterexample G' such that $\tau(G') < \tau(G)$. A *substep* takes a counterexample G to a counterexample G' such that $\tau(G') = \tau(G)$ and $\beta(G') < \beta(G)$. By the definition of G_0, no step or substep can be applied to it. Thus, to prove Theorem 1.3 it suffices to make a step or substep with respect to G_0.

Note that if we can add to G_0 an edge such that the resulting graph G_1 is plane and simple, then $\tau(G_1) < \tau(G_0)$.

Lemma 3.3
Let a plane simple graph G_1 be obtained by adding to G_0 an edge. Then G_1 satisfies statement (B) of Theorem 1.3, but not statement (A).

Proof The graph G_1 must satisfy Theorem 1.3, otherwise we are able to make a step. Suppose G_1 satisfies statement (A), i.e., has a light precomplete star at the minor vertex u. Then this star induces a light precomplete star at u in G_0, a contradiction. □

Lemma 3.4
Let a plane simple graph G_1 be obtained by adding to G_0 an edge. If G_1 has a bunch of length at least 6, then both poles of that bunch are B-vertices in G_0.

Proof Suppose H is a bunch in G_1 of length at least 6, and let p and q be the poles of H. Then there exists a strictly interior vertex w in H. Since by Lemma 3.3, G_1 cannot contain a precomplete star of weight at most 38 centred at a minor vertex, we can follow the arguments in the proof of Lemma 3.2 to conclude that $d_{G_1}(p), d_{G_1}(q) \geq 30$. Hence $d_{G_0}(p), d_{G_0}(q) \geq 29$, and so p and q are B-vertices. □

Lemma 3.5
If there is a face in G_0 incident with two L-vertices, then these two L-vertices are adjacent.

Proof Suppose the lemma is false, and form the plane simple graph G_1 by adding an edge between the two L-vertices. From Lemma 3.3 it follows that G_1 must satisfy statement (B) in Theorem 1.3. If the new edge is a horizontal edge in one of the bunches involved, then (B) holds for G_0 too (since the length of a bunch or the degree of the poles does not depend on the presence of horizontal edges), a contradiction.

In all three cases in statement (B), G_1 has a B-vertex b that is a pole for one or three bunches of length at least $d_{G_1}(b)/5$. Since $d_{G_1}(b) \geq 26$, the length of these bunches is at least 6. From Lemma 3.4 it follows that all poles of these bunches must have been B-vertices in G_0, hence the new edge cannot be incident with any of them. It follows that (B) holds for G_0 too, again a contradiction. □

We now take a close look at the type of edges that can be added to G_0.

Lemma 3.6
Let a plane simple graph G_1 be obtained by adding to G_0 an edge e. Then e is vertical in a bunch H of length at least 6 in G_1 for which both poles are B-vertices in G_0. Moreover, e is either a boundary or a preboundary edge in H.

Proof From Lemma 3.3 it follows that G_1 must satisfy statement (B) in Theorem 1.3. Following the proof of Lemma 3.5, the new edge e must be incident with a pole of a bunch H in G_1 of length at least 6, and the poles of H are B-vertices in G_0. If e is not contained in the bunch, then, since the relevant poles are all B-vertices in G_0 as well, we find that G_0 satisfies (B) as well, a contradiction. (For case (iii) we use this argument for each of the five bunches.)

So we must have that e is contained in the bunch H and incident with at least one of its poles. Hence e is a vertical edge in H. Let the poles of H be p and q. First suppose e is a parental edge which is not a boundary or a preboundary edge. Then $e = pq$ is incident in G_1

with the edges in the paths $pv_{i-1}q$ and $pv_{i+1}q$, where v_{i-1} and v_{i+1} are two interior vertices in H. It follows that v_{i-1} and v_{i+1} are minor in G_0. These two vertices cannot be adjacent in G_0, because the length of H is at least 6 (Fig. 3.1). But then the vertices v_{i-1} and v_{i+1} in

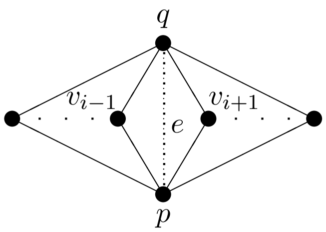


Fig. 3.1

the face $pv_{i-1}qv_{i+1}p$ violate Lemma 3.5.

So suppose e is a vertical edge pw or qw. If e is not a boundary or a preboundary edge, then w is a strictly interior vertex in H. This means that in G_0 w is adjacent to one of p, q and to at most two vertices of degree at most 4 (the two interior bunch vertices neighbouring w in H). Then G_0 has a precomplete star of weight at most 8, a contradiction. □

Lemma 3.7

Suppose G_0 has a halfbunch H of length at least 4 with poles p and q that are both B-vertices. Then the face incident with pq, but not contained in the halfbunch, contains B-vertices only.

Proof Following the terminology in the definition of a halfbunch, G_0 contains a 3-face $pv_{m-1}qp$ with $d_{G_0}(v_{m-1}) \leq 3$. In fact, because also $d_{G_0}(v_{m-2}) \leq 4$, it follows from Lemma 3.5 that v_{m-1} and v_{m-2} are adjacent. So v_{m-1} is adjacent to p, q and v_{m-2}, where p and q are B-vertices, while $d_{G_0}(v_{m-2}) \leq 4$.

Suppose G_0 contains an L-vertex v in the face incident with pq but not in the halfbunch (see left side of Fig. 3.2). Form the graph G_1 by putting a new vertex x on the BB-edge pq and add

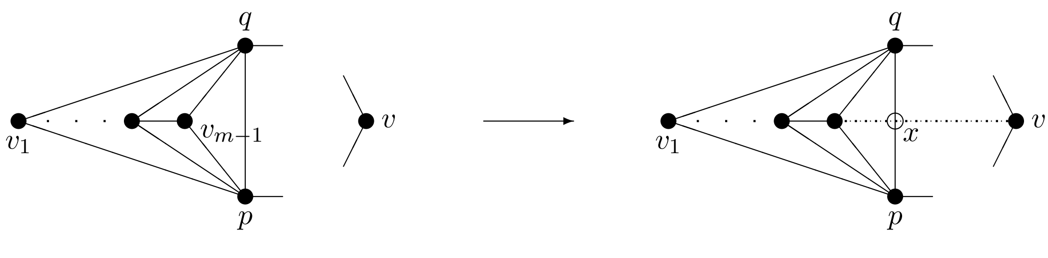


Fig. 3.2

the edges xv_{m-1} and xv (Fig. 3.2). Then we have $\tau(G_1) = \tau(G_0)$ and $\beta(G_1) < \beta(G_0)$, so G_1 cannot be a counterexample. First assume G_1 satisfies statement (A) of Theorem 1.3. Since G_0 does not satisfy (A), the only minor vertex in G_1 that can be the centre of a precomplete star of weight at most 38 is the new vertex x. But if that is the case, then also v_{m-1} is the centre of a precomplete star of weight at most 38, a contradiction.

So assume G_1 satisfies statement (B) of Theorem 1.3. If none of the new edges is contained in the bunch or bunches guaranteed by (B), then the same bunches exist in G_0. Since these bunches have length at least 6, Lemma 3.2 guarantees that their poles are B-vertices in G_0, and hence cannot be one of v_{m-1}, x, v. But then in fact a bunch or bunches with exactly the same properties exist in G_0, a contradiction.

Next assume that the new edges are horizontal in the bunch or bunches that exist in G_1 according to (B). If G_1 satisfies (i), then the poles of the special bunch are p and q. Removing the vertex x and edges xv_{m-1}, xv from G_1 gives a bunch of the same length in G_0. Since there is no edge pq in G_1, in fact G_1 cannot satisfy (ii) with poles p and q. And if G_1 satisfies (iii), then p and q are the poles of a bunch of length $d_{G_1}(p)/5$ or $d_{G_1}(q)/5$ without parental edge. Going back to G_0, noting that $d_{G_0}(p) = d_{G_1}(p)$ and $d_{G_0}(q) = d_{G_1}(q)$, we find that G_0 contains a bunch with poles p, q of length exactly $d_{G_0}(p)/5$ or $d_{G_0}(q)/5$ with parental edge, again a contradiction.

So we can conclude that at least one of the new edges must be vertical in a bunch H in G_1 according to (B). As before, we can determine that the length of H is at least 6, and hence the only candidate for a vertical new edge is xv, and v is one of the poles of H. But then the other pole must be p or q. Assume, without loss of generality that the poles of H are p and v. Then x is an end vertex of H, and we immediately find that G_0 has a bunch of length at least 5 without parental edge (if p and v are not adjacent) or G_0 has a halfbunch of length at least 5 (if p and v are not adjacent) with poles v and p. The fact that v is an L-vertex contradicts Lemma 3.2. $\square$

Now we use the observations above to make a step or substep in G_0. Since G_0 is not a triangulation, there are vertices u and v such that $G_1 = G_0 + e$, where $e = uv$, is a plane simple graph. As G_1 cannot be a counterexample, it follows from Lemma 3.6 that e is a vertical edge in a bunch H of length at least 6 in G_1 and the poles p and q of H are B-vertices in G_0. Moreover, by Lemma 3.6, we have the following alternatives:

(1) e is a preboundary parental edge in H;

(2) e is a preboundary non-parental edge in H;

(3) e is a boundary parental edge in H;

(4) e is a boundary non-parental edge in H.

In each of the cases (1) – (4), we show how to make a step or substep instead of unsuccessfully adding e to G_0.

Case 1. Here $e = pq$ is incident in G_1 with triangles pv_1q and pv_3q, where v_1 is an end vertex of H, and v_3 is its strictly interior vertex; in particular, $d(v_3) \leq 3$. Because the length of H is at least 6, v_1 and v_3 cannot be adjacent in G_0. So, by Lemma 3.5 v_1 is a B-vertex. Let G' be obtained by adding $e' = v_1v_3$ to G_0. Then by Lemma 3.6, the edge e' is vertical in a bunch H' of length at least 6 in G'. Since e' is incident with only one B-vertex v_1 in G', it follows that v_1 is one of the poles of H'. The second pole of H' is adjacent to v_3 and is a B-vertex. Since v_3 is adjacent with only two B-vertices p and q in G_0, the second pole coincides with one of these vertices.

By symmetry, we can assume that the poles of H' are v_1 and p, whence v_1p is parental for H'. It is not hard to see that v_3 is an end vertex for H', and that the vertex q which is the next neighbour to v_3 around a pole v_1 is not adjacent to the other pole p in G' (by the assumptions of Case 1) (Fig. 3.3). Let H'' be the halfbunch in G_0 with poles v_1 and p and length at least 5

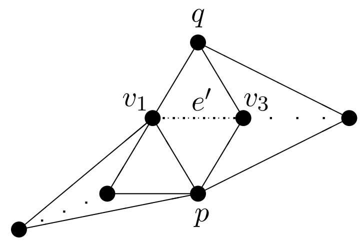


Fig. 3.3

formed by the vertices in H' minus v_3. Since v_3 is not a B-vertex, the existence of H'' violates Lemma 3.7.

CASE 2. Now $e = pv_2$ belongs to the path $P_2 = pv_2q$ in H, where the path P_1 next to P_2 in H is boundary, i.e., consists of boundary edges.

Observe that if P_1 a parental edge in H, then v_2 is adjacent in G_0 only with q and possibly with v_3, which is interior in H. Since v_3 is minor in G_0, the vertex v_2 is incident in G_0 with a light precomplete star; a contradiction. Hence, the path P_1 is not a parental edge of the bunch H in G_1 and v_2 is adjacent to v_1 in G_0.

We consider two subcases :

(2a) $P_1 = pv_1q$ and $P_3 = pq$, i.e., the path P_3 is a parental edge of H;

(2b) $P_1 = pv_1q$ and $P_3 = pv_3q$, i.e., the path P_3 is not a parental edge of H.

First we consider Case 2a. Since the length of H in G_1 is at least 6, it follows that the halfbunch in G_0 formed by removing v_1 and v_2 from H together with the L-vertex v_2 violates Lemma 3.7 (Fig. 3.4).

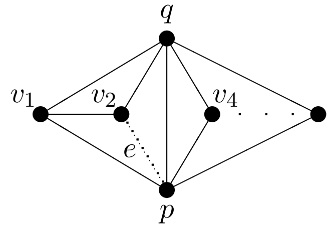


Fig. 3.4

If we are in Case 2b, then v_2 and v_3 must be adjacent, because of Lemma 3.5. So v_1 is a B-vertex because otherwise there is a precomplete star in G_0 of weight at most 29, centred at v_2 and consisting of v_1 and the minor vertex v_3. Also the path P_4 cannot be a parental edge pq, because otherwise the subgraph induced by p, q, v_1, v_2, v_3 together with the minor vertex v_5 would form a structure that cannot exist in G_0. This can be proved in exactly the same way as Lemma 3.7. Note that again we can conclude that v_3 and v_4 must be adjacent.

Form the graph G' by adding the edge $e' = v_1v_3$ to G_0 (Fig. 3.5). Due to the same argument

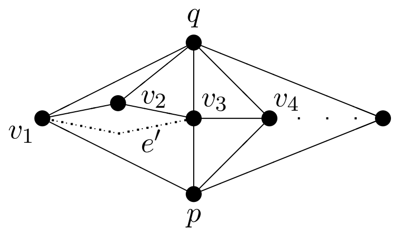


Fig. 3.5

as in Case 1, e' is a vertical edge in a bunch H' of length at least 6 in G'. Furthermore, the poles of H' are either v_1 and p, or v_1 and q.

First suppose that the poles of H' are v_1 and p, while v_1p is its parental edge. Observe that v_3 is an end vertex for H'. Since the length of H' is at least 6, we can find a halfbunch H'' having length at least 4 in G_0 formed by removing v_3 from H'. The existence of H'' together with the minor vertex v_3 violates Lemma 3.7.

Now suppose that the poles of H' are v_1 and q, while v_1q is its parental edge. This time we find that v_3 is an end vertex for H'. We can follow the reasoning from above by considering the halfbunch obtained by removing v_2 and v_3 from H'.

CASE 3. The edge $e = pq = P_1$ is incident in G_1 with a 3-face pv_2qp, where v_2 is interior in H. Also, e is incident in G_1 with a nontriangular face $ypqz \cdots y$ (otherwise e is not boundary in the bunch H). This implies that G_0 has a face $f_0 = ypv_2qz \cdots y$ of size at least 5. Furthermore, v_2 is not adjacent in G_0 to any vertex incident with f_0, except possibly p and q. The last claim follows from the fact that v_2, being an interior vertex of H in G_1, can be adjacent, except for p and q, only to an interior vertex of H. From Lemma 3.5 it follows that all vertices incident with the face f_0 in G_0 and different from v_2 are B-vertices.

Form G' by adding the edge $e' = yv_2$ to G_0 (Fig. 3.6). The argument used in Case 1 then

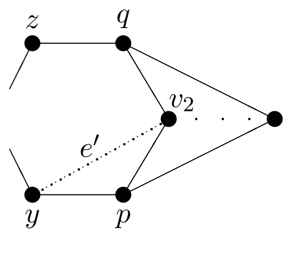


Fig. 3.6

shows that e' is a vertical edge in a bunch H' of length at least 6 in G'. Moreover, one of the poles in H' is y and the other coincides with one of the B-vertices p or q adjacent to v_2.

Let us prove that if the other pole of H' is q, then the length of H' in G' is at most 3. Indeed, both vertices adjacent to v_2 around y in G' must be B-vertices different from q. Hence, they must be end vertices for H', whence the length of H' is at most 3. This contradiction implies that y and p are the two poles of H' in G' and the edge yp is parental in H'. Furthermore, as in the cases above, v_2 is an end vertex for H', and we can form a halfbunch H'' by removing v_2 from H' which, together with the minor vertex v_2, violates Lemma 3.7.

Case 4. The edge $e = pv_1$ is a boundary in the bunch H, and e belongs to the path $P_1 = pv_1q$ in H. Two alternatives are possible:

(4a) $P_2 = pv_2q$, i.e. P_1 does not lie next to a parental edge of H;

(4b) $P_2 = pq$, i.e. P_1 lies next to a parental edge of H.

For (4a), we can assume that the bunch vertices v_1 and v_2 are adjacent in G_0 (otherwise our argument below works, even with obvious simplifications).

Suppose $f_0 = v_1v_2py\cdots v_1$ is a face in G_0. From Lemma 3.5 we conclude that all the vertices incident with f_0 and different from v_1 and v_2 are B-vertices in G_0. First suppose $r(f_0) \geq 5$. Then y and v_1 are not consecutive in the boundary cycle of f_0. Form the graph G' by adding $e' = yv_2$ to G_0 (Fig 3.7 (a)). An argument similar to that in Case 3 shows that the edge e' is

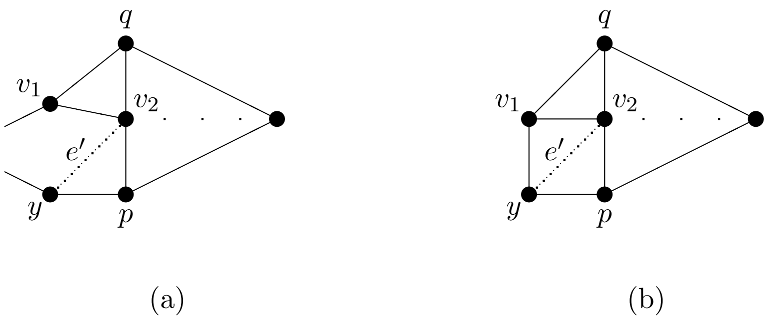


Fig. 3.7

vertical in a bunch H' of length at least 6 in G'. Furthermore, one of the poles of H' is y and the other coincides with p, q or v_1 (the latter is possible only if v_1 is a B-vertex in G_0).

The possibilities that the second pole of H' is v_1 or q are refuted in the same way as in Case 3. It remains to consider the case that the poles of H' are y and p, while the edge yp is parental. Then v_2 is an end vertex in H' and we can form a halfbunch with parental edge yp that violates Lemma 3.7 again.

Now suppose that f_0 is a quadrangular face, i.e., $f_0 = v_1v_2pyv_1$. Then after adding the edge $e' = yv_2$ to G_0 to get G' (Fig. 3.7 (b)), we may get a bunch H' with one of the following pairs of poles: (y, p), (y, v_1) (provided that v_1 is a B-vertex), or (y, q). Observe that in the first two cases, v_2 is an end vertex in H'. So in both cases we can form a halfbunch of length at least 5 with the outside minor vertex v_2, once again violating Lemma 3.7.

Next assume that the poles of H' are y and q. Then either v_2 or p is an end vertex of the bunch since p is a B-vertex. We deduce that v_1 is interior in the bunch H' and, in particular, v_1 is minor in G_0. Form the graph G'' by putting a vertex x on the BB-edge yp in G_0 and adding edges xv_1 and xv_2 (Fig. 3.8). From the fact that G'' cannot be a counterexample, we can follow

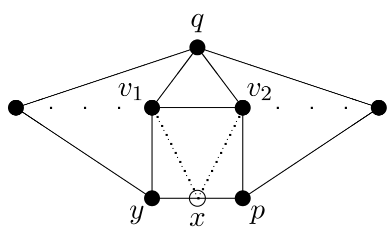


Fig. 3.8

the same reasoning as in the proof of Lemma 3.7 to obtain a contradiction (here we use that if there is a precomplete star of weight at most 38 centred at x, then there is one centered at v_1 or v_2 as well). This completes the proof of subcase (4a).

Next we consider subcase (4b). Then the edge pq is incident in G_0 with a nontriangular face $f_0 = v_1qpy \cdots v_1$ (since G_0 had no edge $e = pv_1$). Observe that the halfbunch obtained by removing v_1 from H has length at least 5. From Lemma 3.7 it follows that all vertices incident with f_0 are B-vertices.

Let G' be obtained by putting a vertex x on the BB-edge pq and adding the edges xv_3 and xy (Fig. 3.9 (a)). Following the proof of Lemma 3.2 we find that xy is a vertical edge in a bunch H'

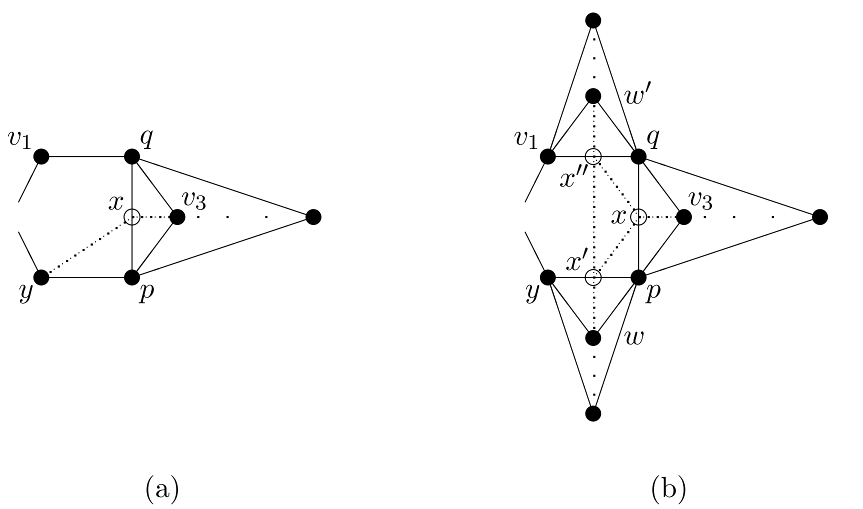


Fig. 3.9

in G' of length at least 6. Furthermore, one of the poles is y, while the other coincides with p or q. In the case q is the other pole, we find that x is an end vertex of H' and v_1 is an interior vertex. But this violates the observation that all vertices incident with f_0 are B-vertices.

Hence, the poles of H' are y and p, and the edge yp is parental in it. Since x is an end vertex in H', it follows that G_0 contains a halfbunch of length at least 5, with poles y and p. Let w be the interior vertex in that halfbunch neighbouring the edge yp. The graph obtained by putting a vertex x on the BB-edge pq and adding the edges xv_3 and v_1y fails, just as G', to be a counterexample. It follows by symmetry that G_0 contains a halfbunch of length at least 5, with poles v_1 and q. Let w' be the interior vertex in that halfbunch neighbouring the edge v_1q. Form the graph G_1^* by putting in G_0 vertices x, x' and x'' on the BB-edges pq, yp and v_1q, respectively, and adding the edges v_3x, xx', xx'', $x'x''$, $x'w$ and $x''w'$ (Fig. 3.9 (b)). The same sequence of arguments as in the proof of Lemma 3.7 will lead to a contradiction.

This completes the treatment of all the cases and subcases, and hence completes the proof of the theorem. ■

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