Stars and bunches in planar graphs.

Part I : Triangulations *

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Abstract

Given a plane graph, a k-star at u is a set of k vertices with a common neighbour u; and a bunch is a maximal collection of paths of length at most two in the graph, such that all paths have the same end vertices and the edges of the paths form consecutive edges (in the natural order in the plane graph) around the two end vertices. We prove a theorem on the structure of plane triangulations in terms of stars and bunches. The result states that a plane triangulation contains a $(d-1)$-star centred at a vertex of degree $d \leq 5$ and the sum of the degrees of the vertices in the star is bounded, or there exists a large bunch.

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1 Introduction

Throughout this paper, G is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set V and edge set E. A *k-star at u* is a set of k vertices with a common neighbour u. A *bunch* is a maximal collection of paths of length at most two in the graph, such that all paths have the same end vertices and the edges of the paths form consecutive edges (in the natural order in the plane graph) around the two end vertices. The *weight* of a subgraph is the sum of the degrees of the vertices in that subgraph.

A significant amount of research has been done on the structure of plane triangulations, especially concerning bounds on the weights of small subgraphs. See for instance [1,2,4,5] and references in those. In [4] the conjecture of Kotzig (1978) that a plane triangulation with minimum degree 5 contains a cycle of length 4 of weight at most 25 is proved. Another result, more directly related to the main theorem in this paper, can be found in [2] and gives a best possible upper bound on the minimum weight of a face in a plane triangulation depending on the maximum length of a path of vertices of degree 4 in the graph.

The proof of Kotzig's Conjecture in [4] is based on the existence in plane triangulations with minimum degree 5 of a 4-star of weight at most 25 centred at some vertex of degree 5. On the other hand, for triangulations that contain vertices with degree less than 5 it is impossible to give a maximum value for the weight of a $(d-1)$-star centred at a vertex of degree $d \leq 5$ (a so-called *minor vertex*). For instance, the n-bipyramid shows that every minor vertex in a plane triangulation can be adjacent to at least two vertices of arbitrarily high degree.

In this paper we prove a theorem on the structure of plane triangulations in terms of stars centred at minor vertices and bunches. (Note that a large bunch in a plane triangulation always contains a long path of vertices of degree 4.) We prove that a plane triangulation contains a $(d-1)$-star centred at a vertex of degree $d \leq 5$ of bounded weight, if and only if there is no large bunch. The bound of the size of the bunch in the main theorem is best possible. In a sequel paper [3], this result is generalised to general planar graphs. That generalisation is used to prove a best possible upper bound on the minimum degree and on the minimum number of colours needed in a greedy colouring of the square of a planar graph.

2 Definitions and result

Throughout this paper, G is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set V and edge set E. The *distance* between two vertices u and v is the length of a shortest path joining them. We are mainly interested in pairs at distance one or two, for which we also can define : a pair of vertices u, v, $u \neq v$, have distance one if they are adjacent; and they have distance two if they are not adjacent but have a common neighbour.

In this paper we will prove a result on so-called unavoidable configurations in plane triangulations. In the sequel [3] this is generalised to plane graphs in general, and used to prove an upper bound on the number of colours needed for a planar graph in which vertices at distance one or two have different colours (a so-called *distant-2-colouring*). Some background and earlier work on distant-2-colourings can also be found in [3].

Before we can state our main result, we need some more definitions.

We say that G has a *bunch* of length $m \geq 3$ with *poles* at vertices p and q, where $p \neq q$, if G contains a sequence of paths $P_1, P_2, \ldots, P_m$ with the following properties. Each P_i has length 1 or 2 and joins p with q. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by P_i and P_{i+1} is not separating in G (i.e., has no vertex of G inside) (see Fig. 2.1). Moreover, this sequence of paths is maximal in the sense that there is no path P_0 (or P_{m+1}) that could be added to the bunch, preserving the above properties.

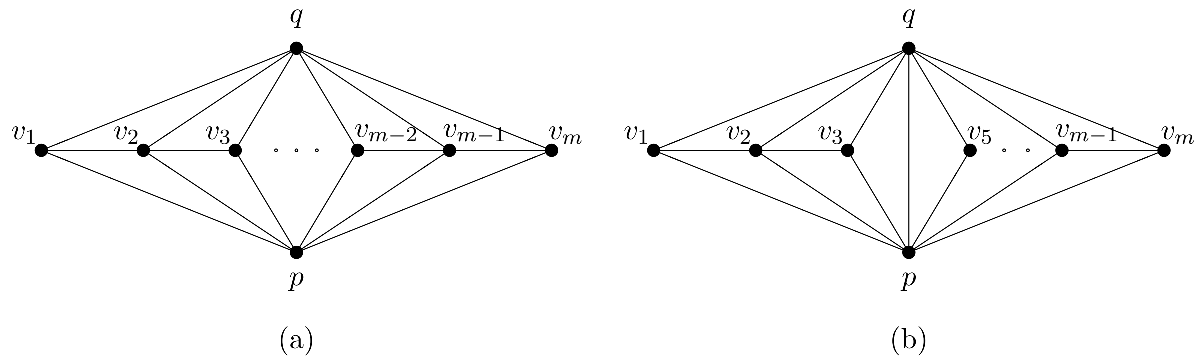


Fig. 2.1 : A bunch without a parental edge (a) and with a parental edge (b)

If a path P_i in the bunch has length 2, i.e., $P_i = pv_iq$, then the vertex v_i will be called a *brother* or a *bunch vertex.* A path $P_i = pq$ of length 1 in the bunch will be referred to as a *parental edge* (Fig. 2.1 (b)).

If the cycle bounded by P_1 and P_m separates G, then the edges in P_1 and P_m are called *boundary edges*, and the vertices v_1 and v_m (if they exist) are the *end vertices* (or *ends*) of the bunch. The vertex v_i in the bunch is *interior* if $2 \leq i \leq m-1$ and *strictly interior* if $3 \leq i \leq m-2$. Each edge v_iv_{i+1} joining consecutive bunch vertices is called *horizontal*, while the edges of the P_i's are called *vertical* in the bunch. Observe that each interior vertex has degree 2, 3 or 4 and is adjacent only to the poles and possibly to one or two brothers.

A *d-vertex* in G is a vertex of degree d. The *B-vertices* in G are those of degree at least 26, *L-vertices* have degree at most 25, and *minor vertices* at most 5.

Let u be a d-vertex, and let $v_1, \ldots, v_k$ be adjacent to u. We say that the vertices $v_1, \ldots, v_k$ form a *k-star at u*, of *weight* $\sum_{i=1}^{k} d(v_i)$. Each $(d-1)$-star at u is called *precomplete*, and each d-star *complete.*

The following is the main result in this paper.

Theorem 2.1
For each plane triangulation G at least one of the following holds:

(A) G has a precomplete star of weight at most 38 that does not contain B-vertices and is centred at a minor vertex.

(B) G has a B-vertex b that satisfies at least one of the following conditions:

(i) b is a pole for a bunch of length greater than $d(b)/5$;

(ii) b is a pole for a bunch of length precisely $d(b)/5$ with a parental edge;

(iii) b is a pole for 5 bunches of length $d(b)/5$ without parental edges and with pairwise different end vertices. Moreover, all but possibly one end vertices have degree 5, while the other end vertex has degree at most 11 (see Fig. 2.2).

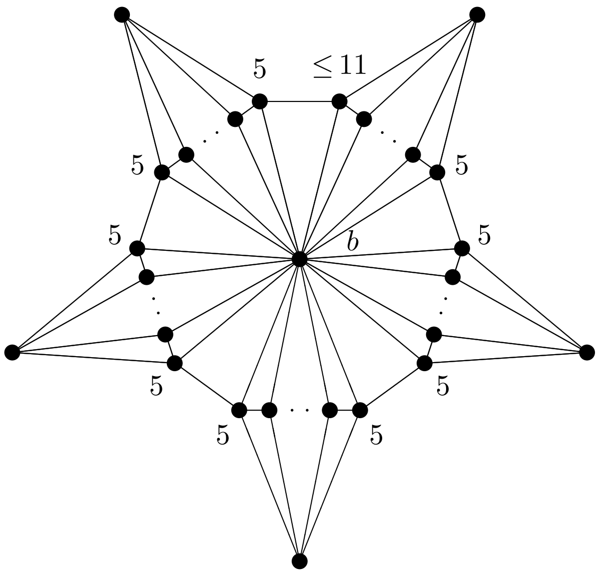


Fig. 2.2

3 Proof of Theorem 2.1

Let G be a counterexample, then the following facts are obvious:

(A') Each precomplete star in G at a minor vertex either contains a B-vertex or has weight at least 39.

(B') Each B-vertex b in G can be a pole for bunches of length at most $d(b)/5$.

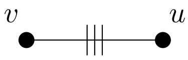
Euler's formula for G can be written as

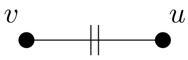
$$\sum_{v\in V}(d(v)-6) \;=\; \sum_{v\in V}\mu(v) \;=\; -12. \tag{3.1}$$

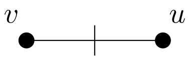
Here, $\mu(v) = d(v) - 6$ is called a *charge* of $v \in V$. Observe that only minor vertices in G have a negative charge. We redistribute the charges among the vertices of G so that each $v \in V$ gets a nonnegative *new charge* $\mu^*(v)$, while the sum of all charges in G remains the same. This will contradict (3.1):

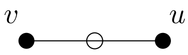
$$0 \leq \sum_{v \in V} \mu^*(v) = \sum_{v \in V} \mu(v) = -12. \tag{3.2}$$

We will say that v *gives* u *charge* c if an amount c is subtracted from $\mu(v)$, transferred and added to $\mu(u)$. We use the following notation:


if v gives u charge 3/2;


if v gives u charge 1;


if v gives u charge 1/2;


if v gives u no charge.

The edge vu will be called *sesquialteral* in the first case, *unitary* in the second, *half* in the third, and *zero* in the last case. The direction of transferring charge will always be clear from the context.

Next, we define the rules R1 − R3 of transferring charge from vertices of degree at least 9 to minor vertices so that the new charge of each minor vertex becomes nonnegative.

R1: Let u be a 4-vertex adjacent to vertices v_1, v_2, v_3, v_4 in a cyclic order. By (A'), at least two of the v_i's have degree at least 12.

(a) If $d(v_i) \geq 12$ for all $i = 1, \dots, 4$, then each v_i gives u charge 1 (Fig. 3.1 (a)).

(b) If $d(v_1) \leq 11$ and $d(v_i) \geq 12$ for $i = 2, 3, 4$, then v_3 gives u charge 1, while each of v_2 and v_4 gives 1/2 (Fig. 3.1 (b)).

(c) If $d(v_1) \leq 11$, $d(v_3) \leq 11$, $d(v_2) \geq 12$, and $d(v_4) \geq 12$, then each of v_2 and v_4 gives u charge 1 (Fig. 3.1 (c)).

(d) If $9 \leq d(v_1) \leq 11$, $9 \leq d(v_2) \leq 11$, $d(v_3) \geq 12$, and $d(v_4) \geq 12$, then each v_i gives u charge 1/2 (Fig. 3.1 (d)).

(e) If $d(v_1) \leq 8$, $9 \leq d(v_2) \leq 11$, $d(v_3) \geq 12$, and $d(v_4) \geq 12$, then v_3 gives u charge 1, while each of v_2 and v_4 gives 1/2 (Fig. 3.1 (e)).

(f) Finally, if $d(v_1) \leq 8$, $d(v_2) \leq 8$, $d(v_3) \geq 12$, and $d(v_4) \geq 12$, then each of v_3 and v_4 gives u charge 1 (Fig. 3.1 (f)).

R2: Let u be a 3-vertex with neighbours v_1, v_2, v_3. By (A'), at least two of the v_i's have degree at least 12.

(a) If $d(v_i) \geq 12$ for all $i = 1, 2, 3$, then each v_i gives u charge 1 (Fig. 3.2 (a)).

(b) If $d(v_1) \leq 11$, then due to (A') both v_2 and v_3 are B-vertices. Then each of v_2 and v_3 gives u charge 3/2 (Fig. 3.2 (b)).

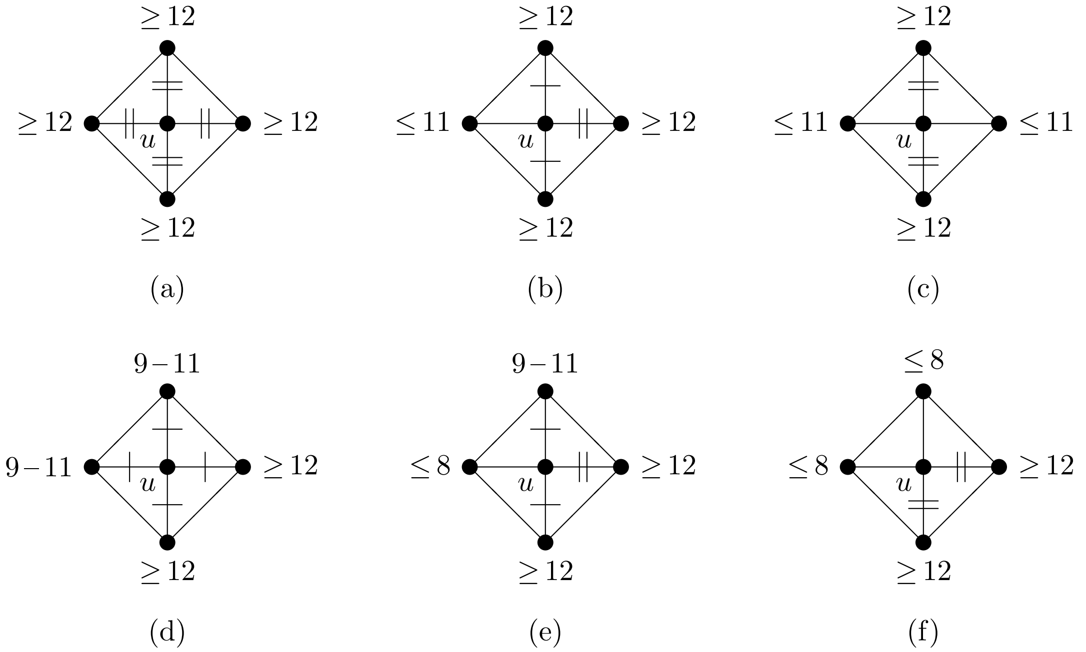


Fig. 3.1 : The rules of R1

R3 : Let u be a 5-vertex, and let $v_1, \ldots, v_5$ be its neighbours in a cyclic order. By (A'), at least two of the v_i's have degree at least 9.

(a) If each of v_{i-1}, v_i and v_{i+1} is a B-vertex, then v_i gives u charge 1. Suppose this is not the case, but v_i has degree at least 9 and at least one of v_{i-1} and v_{i+1} also has degree at least 9, then v_i gives u charge 1/2 (Fig. 3.3 (a)).

(b) If u has no two consecutive neighbours of degree at least 9, then each of its two neighbours of degree at least 9 gives u charge 1/2 (Fig. 3.3 (b)).

Remark. Rule R3 implies that each neighbour v_1 of degree at least 9 of a 5-vertex u always gives a positive charge to u, unless $d(v_2) \leq 8$, $d(v_3) \geq 9$, $d(v_4) \geq 9$, and $d(v_5) \leq 8$, in which case

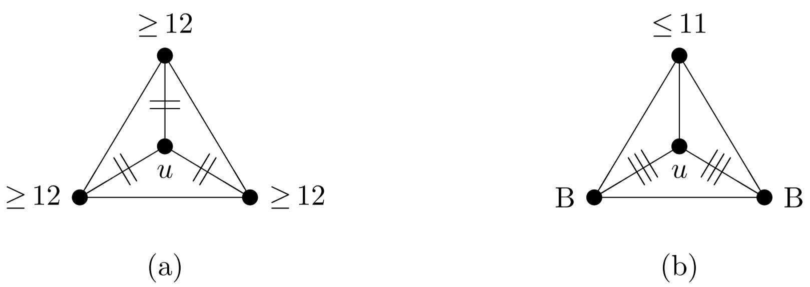


Fig. 3.2 : The rules of R2

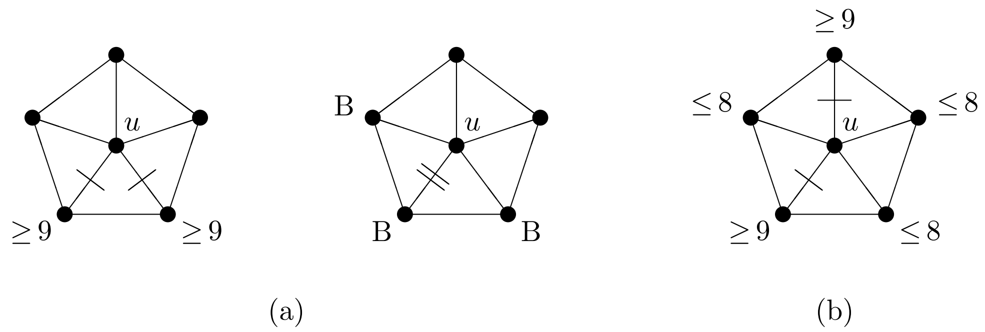


Fig. 3.3 : The rules of R3

each of v_3 and v_4 gives $1/2$ to u by a), while v_1 is exempted from transferring charge.

The new charge of $v \in V$ after applying R1 − R3 is denoted by $\mu^*(v)$.

Lemma 3.1 (on L-vertices)
Each L-vertex $v \in V$ satisfies $\mu^*(v) \geq 0$.

Proof It follows directly from R1 − R3 that $\mu^*(v) \geq 0$ if $d(v) \leq 8$.

Our next goal is to prove $\mu^*(v) \geq 0$ if $9 \leq d(v) \leq 25$. We estimate the total donation of v according to R1 − R3 by means of a simple averaging argument. The *generous* donation of v to its minor neighbour u_i is defined as follows :

Let $\lambda(v) = \mu(v)/d(v)$. Then v (generously) gives u_i the following charge :

- $2\,\lambda(v)$ if none of u_{i-1}, u_{i+1} is minor;
- $3\,\lambda(v)/2$ if precisely one of u_{i-1}, u_{i+1} is minor, and
- $\lambda(v)$ if both u_{i-1} and u_{i+1} are minor.

Clearly, v gives to all its minor neighbours at most $\mu(v)$ in total. To see this, imagine that v first sends $\lambda(v)$ to each neighbour, i.e., precisely $\mu(v)$ in total, and then the donation to a non-minor neighbour u_k is shared by $\lambda(v)/2$ between u_{k-1} and u_{k+1}. As a result, each minor neighbour gets from v in this imaginary experiment exactly what is prescribed by the generous scheme.

It remains to show that in practice, i.e., according to R1 − R3, each minor neighbour gets from v not more than under the generous scheme.

Observe that $\lambda(v) \geq 1/3$ if $d(v) \geq 9$, $\lambda(v) \geq 1/2$ if $d(v) \geq 12$, and $\lambda(v) \geq 2/3$ if $d(v) \geq 18$. This clearly implies that the generous donation of our v is not less than by R1 − R3 everywhere except for possibly in R1 (c), R1 (f) and R3 (b).

Let us prove the same for the remaining cases. First consider R1 (c). If neither v_1 nor v_3 is minor, the statement is obvious. Suppose that precisely one of v_1 and v_3 is minor. By (A') (for u), we have $d(v_2) > 18$ and $d(v_4) > 18$, so that v_2 and v_4 give at least as much as required by R1 − R3. Finally, if both v_1 and v_3 are minor, then both v_2 and v_4 are B-vertices by (A').

Now let us consider R1 (f). By (A'), we have $d(v_3) > 18$ and $d(v_4) > 18$, and the same argument works. Finally, in the case R3 (b), using (A') again, we deduce that $d(v_1) > 12$ and $d(v_3) > 12$, whence the statement follows. This completes the proof of Lemma 3.1. □

Now suppose that G has a B-vertex b such that $\mu^*(b) < 0$. Let b have degree d, and denote its neighbours in a cyclic order by $v_1, v_2, \ldots, v_d$.

Lemma 3.2 (structural)

(a) *If b is incident with a sesquialteral edge bv_2, then precisely one of v_1 and v_3 is a B-vertex (so that the corresponding edge bv_1 or bv_3 is zero).*

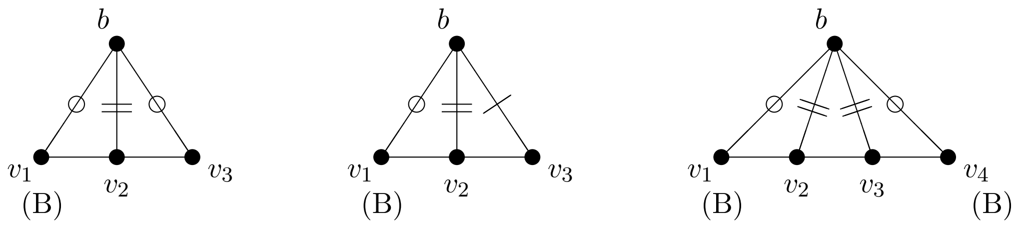
(b) *If b is incident with a half edge bv_2, then at most one of v_1 and v_3 is a B-vertex.*

(c) *Suppose b has a B-neighbour v_1, while the edge bv_2 is unit. Then one of the following statements is true:*

(i) *bv_3 is zero (Fig. 3.4 (a));*

(ii) *bv_3 is a half edge (Fig. 3.4 (b));*

(iii) *bv_3 is unit and v_4 is a B-vertex (whence bv_4 is zero) (Fig. 3.4 (c)).*



(a) (b) (c)

Fig. 3.4

Proof For (a), see R2 (b); for (b), apply R1 (b,d,e) and R3.

Let us prove (c). Assume that neither (i) nor (ii) hold. Then we show that (iii) should hold. Indeed, bv_2 is unit, while bv_3 is either unit or sesquialteral. It follows that both v_2 and v_3 are minor.

We now prove $d(v_2) = 4$. Observe that v_2 has two B-neighbours, b and v_1, and a minor neighbour v_3. Hence if $d(v_2) = 3$, then bv_2 is sesquialteral by R2 (b), contrary to the hypotheses of the lemma. Suppose $d(v_2) = 5$. Then due to R3, bv_2 is unit only if each of v_1 and v_3 is a B-vertex. Since v_3 is already known to be minor, it follows that $d(v_2) = 4$.

Let t be the neighbour of v_2 other than b, v_1 or v_3 (Fig. 3.5 (a)). Note that $d(t) \leq 8$, for otherwise bv_2 would be a half edge by R1 (b,e). Hence, $d(v_2) + d(t) \leq 12$, and due to (A') for v_3 we have $d(v_3) \neq 3$. It follows from R3 that $d(v_3) \neq 5$, because v_2 is an L-vertex. Thus, $d(v_3) = 4$. Furthermore, the vertex v_4 adjacent to v_3 is a B-vertex, for otherwise v_3 would have a precomplete star on v_2, v_4 and t of weight at most 37, contrary to (A') (Fig. 3.5 (b)). Finally, observe that bv_3 is unit by R1 (f). This implies part (iii) of (c), and completes the proof of Lemma 3.2. □

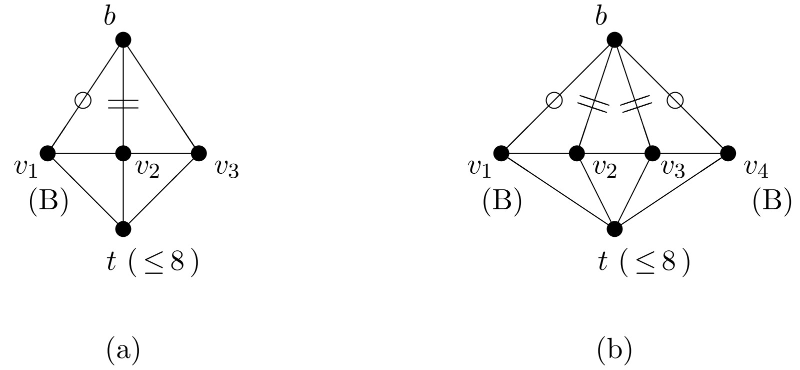


Fig. 3.5

To estimate the total donation of b along incident edges, we introduce the following averaging rule AR.

AR (averaging rule): Let bv_i transfer charge $\lambda_i \neq 0$, and let v_{i+1} be a B-vertex. Then b shifts charge $1/2$ from bv_i to bv_{i+1} (Fig. 3.6 (a)). As a result, bv_{i+1} becomes at least a half edge (it becomes unit if it gets $1/2$ also from bv_{i+2}). If v_{i-1} is an L-vertex, then bv_i now takes away charge $\lambda_i - 1/2$ (Fig. 3.6 (b)). However, if v_{i-1} is a B-vertex, then bv_i also shifts $1/2$ to bv_{i-1}, so that bv_i finally transfers $\lambda_i - 1$ (Fig. 3.6 (c)).

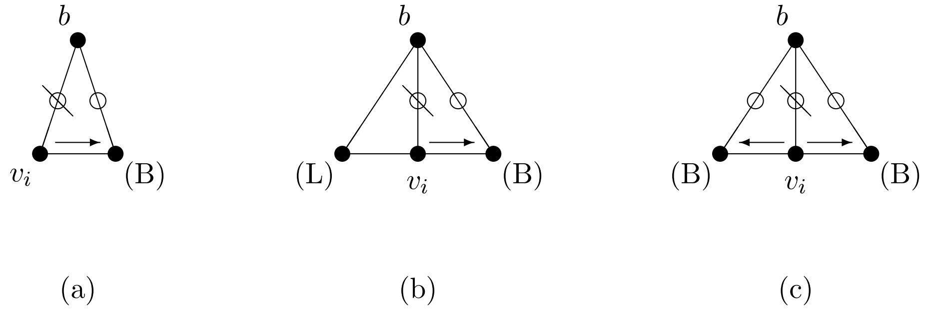


Fig. 3.6

To see that AR is well-defined if bv_i is a half edge, use Lemma 3.2 (b). It says that a half edge bv_i can shift $1/2$ to at most one of the two immediate neighbour edges. From Lemma 3.2 (a) it follows that each sesquialteral edge incident with b is made by AR into unit. Hence, b has no more sesquialteral edges after applying AR. Also observe that the shift from unit edge bv_2 to zero edge bv_1, results in one of the cases (i) – (iii) described in Lemma 3.2 (c). Finally, if b is incident with a zero edge leading to an L-vertex, then this edge remains zero after applying AR. As for zero edges leading to B-vertices, they clearly become either half or unit edges, depending on the number of $1/2$'s obtained. Let us formulate another useful consequence of AR.

Claim 3.3 (on unit edges)
If the edge bv_i becomes unit after applying AR, then v_i is either a B-vertex or has degree 3 or 4.

Proof Observe that if $d(v_i) = 5$ and bv_i was unit before averaging, then bv_i becomes zero by R3 (a). If $d(v_i) = 5$ and bv_i was half or zero, then it cannot become unit. It follows that if v_i is minor, then its degree is 3 or 4.

Now suppose $d(v_i) > 5$. Then bv_i was zero initially, and gets $1/2$ from each of bv_{i-1} and bv_{i+1}. This is only possible if v_i is a B-vertex. This completes the proof. □

From now on, by zero, half and unit edges in the vicinity of b we mean those AFTER averaging. Denote their numbers by e_0, $e_{1/2}$, and e_1, respectively. From $\mu^*(b) < 0$ it follows that

$$2\,e_0 + e_{1/2} \;\leq\; 11. \tag{3.3}$$

By a *prebunch* of length k we mean any maximal (non-extendable) sequence of k consecutive unit edges in the vicinity of b. (It cannot be extended either because its boundary edges are not unit, or if $k = d$.) A *separator* of length ℓ is a sequence of ℓ consecutive non-unit edges in the vicinity of b bounded from both sides by unit edges (here, $\ell = d$ or $\ell = d - 1$ is impossible due to (3.3)).

Thus, the set of edges in the vicinity of b is split into disjoint and alternating prebunches and separators. Clearly, their numbers are the same, unless all edges incident with b are unit. Sometimes we shall refer to a non-unit edge as *separating*. It follows from (3.3) that b sees at most 11 separating edges.

Claim 3.4 (on the boundary of a separator)
If bv_1 shifts $1/2$ to bv_2 by AR, then either bv_2 becomes unit after applying AR, or each of bv_2 and bv_3 becomes separating. In particular, if bv_1 is separating, then bv_2 cannot be a boundary edge in a separator that contains bv_1.

Proof If bv_3 was not zero before averaging, then it shifts $1/2$ to bv_2, so that bv_2 becomes unit. If bv_3 was zero, it cannot become unit because it receives nothing from bv_2. In this case, both bv_2 and bv_3 become separating, as claimed. □

Our next lemma explains the role of prebunches and shows how helpful AR is.

Lemma 3.5 (on prebunches)
Each prebunch of length $k \geq 3$ in the vicinity of b is a part of a bunch of length at least $k+2$ with one pole at b and the other at a B-vertex t. Moreover, all the edges of the prebunch are consecutive vertical edges of the bunch, and none of them is a boundary edge in the bunch.

Proof Let our prebunch consist of the edges $bv_1, bv_2, \ldots, bv_k$. Claim 3.3 implies that v_i is either a B-vertex or has degree 3 or 4 whenever $i = 1, \ldots, k$. We next prove that if $d(v_i) = 3$ (where $i = 1, \ldots, k$), then v_i has precisely two B-neighbours in G. It suffices to prove that bv_i transfers charge according to R2 (b). Assume otherwise that R2 (a) takes place. Then before averaging, bv_i was unit, and both bv_{i-1} and bv_{i+1} were zero. Note that none of the two latter can become unit, because there is no shift from bv_i. This contradiction proves that v_i has two B-neighbours (one of which is b).

Next, we define an "other pole" map $\pi : \{v_1, \dots, v_k\} \longrightarrow V$ as follows. If v_i is a B-vertex, we put $\pi(v_i) = v_i$. If $d(v_i) = 3$, then let $\pi(v_i)$ be the only B-neighbour of v_i other than b. If $d(v_i) = 4$, then $\pi(v_i)$ is defined to be the vertex opposite to b in the vicinity of v_i.

Let us prove that the image of all π consists of one point; i.e., that G has a B-vertex t such that $\pi(v_i) = t$ for all $i = 1, \dots, k$. To this end, it suffices to prove $\pi(v_i) = \pi(v_{i+1})$ whenever $i = 1, \dots, k-1$.

If $d(v_i) = 3$, then one of v_{i-1} and v_{i+1} was proved to be a B-vertex. If this B-vertex is v_{i+1}, then the definition of π implies $\pi(v_i) = \pi(v_{i+1}) = v_{i+1}$. Suppose this B-vertex is v_{i-1}. Then as proved above, v_{i+1} has degree 3 or 4. If $d(v_{i+1}) = 3$, then G has two adjacent 3-vertices v_i and v_{i+1}, which is impossible in a triangulation without loops and multiple edges. Thus $d(v_{i+1}) = 4$, and by the definition of π we have $\pi(v_i) = \pi(v_{i+1}) = v_{i-1}$ (Fig. 3.7).

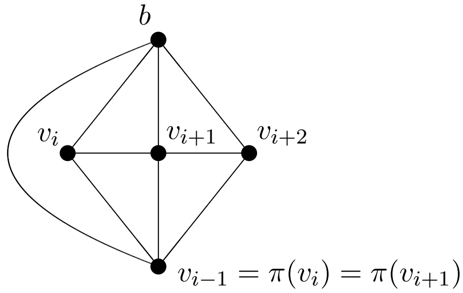


Fig. 3.7

Suppose $d(v_i) = 4$. If $d(v_{i+1}) = 3$, then the only B-neighbour of v_{i+1} other than b is v_{i+2}. Now the definition of π implies that $\pi(v_i) = \pi(v_{i+1}) = v_{i+2}$. If $d(v_{i+1}) = 4$, then since the faces of G are triangles, it follows that $\pi(v_i) = \pi(v_{i+1})$. Let us prove that v_{i+1} cannot be a B-vertex. Indeed, otherwise before averaging bv_{i+1} had to be zero and bv_i unit. However, then bv_i shifts 1/2 to bv_{i+1} by AR. As a result, bv_i could not become unit ultimately, contrary to the assumption of the lemma.

Let v_i be a B-vertex. If $d(v_{i+1}) = 3$, then the definition of π implies that $\pi(v_i) = \pi(v_{i+1}) = v_i$. As proved above, $d(v_{i+1}) = 4$ is impossible. Furthermore, v_{i+1} cannot be a B-vertex, since in this case each of bv_i and bv_{i+1} was initially zero, and none of them can become unit in the end. So, we have proved that G has a vertex t such that $\pi(v_i) = t$ for all $i = 1, \dots, k$.

Now it follows from the definition of π that $bv_1, \dots, bv_k$ are consecutive vertical edges of a bunch of length at least k with poles b and t. Let us prove that t is a B-vertex. If for some $i \in \{1, \dots, k\}$ the degree of v_i is not 4, then $\pi(v_i) = t$ must be a B-vertex. If $d(v_i) = 4$ for all $i = 1, \dots, k$, then v_2 is adjacent to two 4-vertices v_1 and v_3 (and also to b and t). By (A'), it follows that t is a B-vertex.

Since the length of the bunch found at the B-vertex b is at least k, it follows from property (B') that $k < d - 1$. The last inequality means that there are two different edges bv_d and bv_{k+1} not belonging to the prebunch. Let us prove that both of them belong to the bunch considered. It suffices to prove this for bv_d. If v_1 is a B-vertex, then $t = v_1$ by the definition of π. In this case, bv_1 is the parental edge of our bunch, and the path bv_dv_1 extends the bunch. If $d(v_1) = 4$, then since G is triangular, it follows that the path bv_dt also extends the bunch. Suppose $d(v_1) = 3$. If $t = \pi(v_1) = v_d$, then the edge bv_d is parental in the bunch. Finally, if $t = \pi(v_1) = v_2$, then the bunch is extended by the path $bv_dv_2 = bv_dt$. This completes the proof of Lemma 3.5. □

Lemma 3.6 (on half separators)
After averaging, the vicinity of b does not contain separators consisting of one half edge.

Proof Suppose there is a separator of one half edge bv_3. By definition, both bv_2 and bv_4 are unit edges.

CASE 1. bv_3 was a half edge before averaging.

It follows from R1 − R3 that $d(v_3) \in \{4, 5\}$. If $d(v_3) = 4$, then by R1 at least one of bv_2 and bv_4 must be zero initially. Moreover, this edge cannot become unit because it does not receive charge from bv_3.

Suppose $d(v_3) = 5$. Since bv_2 and bv_4 became unit, and did not receive 1/2 from bv_3, it follows that each of them was unit or sesquialteral before averaging.

First assume that both of them were unit. Then Claim 3.3, R1, and R2 imply that $d(v_2) = d(v_4) = 4$ (see Fig. 3.8). Let us prove that precisely one of x, y in Fig. 3.8 has degree at least 9.

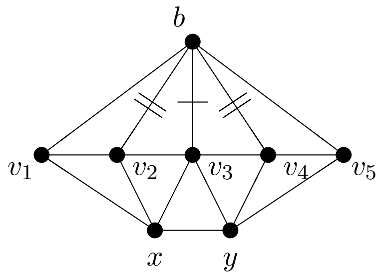


Fig. 3.8

Indeed, if $d(x) \le 8$ and $d(y) \le 8$, then in the vicinity of the 5-vertex v_3 there is a precomplete star defined by v_2, v_4, x and y, whose weight is at most 24. However, if $d(x) \ge 9$ and $d(y) \ge 9$, then bv_3 is zero by R3 (a), contrary to the assumption. Hence we may assume by symmetry, that $d(x) \ge 9$ and $d(y) \le 8$. Now by (A'), it follows that v_5 is a B-vertex, and bv_5 was initially zero, for otherwise we have a precomplete star at v_4 of weight at most 38. So, bv_4 shifts 1/2 to bv_5, and bv_4 becomes unit, contrary to the assumption.

Now suppose bv_2 was sesquialteral and bv_4 unit. Then $d(v_2) = 3$ and $d(v_4) = 4$. In the notation of Fig. 3.8, it means that x coincides with v_1 and is a B-vertex (see R2 (b)). As above, we see that bv_3 is zero if $d(y) \ge 9$, or bv_4 becomes a half edge if $d(y) \le 8$.

Finally, suppose that each of bv_2 and bv_4 was sesquialteral. Now both v_2 and v_4 are 3-vertices, and it follows from R2 (b) that each of x, y is a B-vertex. Again, we conclude that bv_3 is zero, a contradiction.

Case 2. bv_3 becomes half after averaging.

First suppose that bv_3 was initially zero and obtained $1/2$ from bv_2. Then v_3 is a B-vertex, and bv_2 was sesquialteral. If bv_4 was not zero, then it also shifted $1/2$ to bv_3, so that bv_3 becomes unit. But if bv_4 was zero, then it cannot become unit because it does not receive $1/2$ from bv_3.

Now suppose bv_3 was unit before averaging and becomes half due to shifting $1/2$ to (the initially zero edge) bv_2. Then Lemma 3.2 (c) implies that bv_4 becomes zero or half, contrary to the assumption. This completes the proof of Lemma 3.6. $\square$

Corollary 3.7
The vicinity of b consists of at most 5 separators and at most 5 prebunches.

Proof We can say that each zero edge saves a unit of charge for b (as compared to a unit edge), and a half edge saves $1/2$. Since $\mu^*(b) < 0$, the total saving at b is at most $5\frac{1}{2}$. Since by Lemma 3.6 each separator saves at least 1, the statement follows. $\square$

Lemma 3.8 (on triple separators)
After averaging, there is no separator S at b consisting of three half edges.

Proof Suppose the contrary, and consider two cases.

Case 1. All three edges of S were half before averaging.

We first observe that each edge of S leads to a 5-vertex. Indeed, by R1 each half edge bu going to a 4-vertex lies in a 3-face $buvb$ with an initially zero edge bv. Moreover, if $bu \in S$, then bv cannot become unit because it does not receive $1/2$ from bu. But if bv becomes half after averaging, then it also belongs to S, contrary to the assumption.

Thus, we may assume that S consists of the edges bv_2, bv_3 and bv_4 as shown in Fig. 3.9 (a).

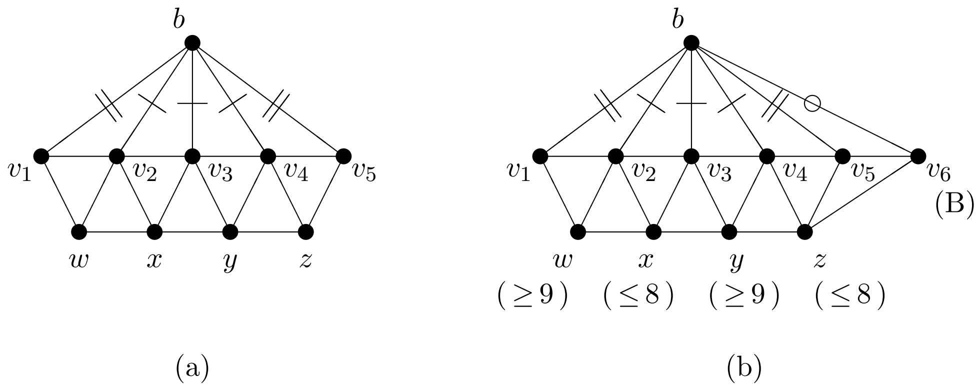


Fig. 3.9

The same argument as in the proof of Lemma 3.6 implies that precisely one of the vertices x, y in Fig. 3.9 (a) has degree at least 9 (for otherwise bv_3 is not half). The same is true for y and z, and also for x and w. By symmetry, we may assume that $d(w) \geq 9$, $d(x) \leq 8$, $d(y) \geq 9$ and $d(z) \leq 8$ (Fig. 3.9 (b)).

Observe that the unit edge bv_5 could not be zero before averaging because it did not receive 1/2 from bv_4. Now Claim 3.3 implies that $d(v_5) \le 4$. Since $d(v_4) + d(z) \le 13$, it follows from (A') that v_5 has degree 4 and v_6 adjacent to v_5 is a B-vertex. By R1 (f), the edge bv_5 was initially unit (Fig. 3.9 (b)). It means that bv_5 gave 1/2 to the initially zero edge bv_6, so that bv_5 became a half edge, contrary to the assumption.

CASE 2. An edge of S became half after averaging.

If bv_i was zero and became half due to the shift of 1/2 from the half edge bv_{i+1}, then bv_{i+1} becomes zero after averaging. Hence, bv_{i+1} cannot belong to S.

Let bv_2 belong to S and become half due to the shift from the sesquialteral edge bv_1. Then v_2 must be a B-vertex. Since bv_1 became unit, it follows that S consists of bv_2, bv_3 and bv_4. If bv_3 was not zero, then it also shifted 1/2 to bv_2, so that bv_2 becomes unit. Suppose that bv_3 was zero and becomes half due to the shift from bv_4. Since both bv_3 and bv_4 became half, the edge bv_4 was initially unit and v_3 was a B-vertex (Fig. 3.10). Note that the shift from the unit edge bv_4

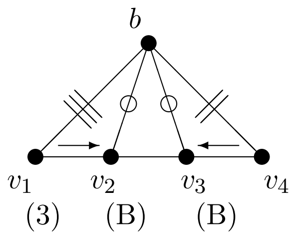


Fig. 3.10

to the zero edge bv_3 can occur only in one of the cases (i) – (iii) described in Lemma 3.2 (c). It is not hard to see that in none of these cases bv_5 can become unit nor bv_4 can become zero. In all these cases, S cannot consist of the three half edges bv_2, bv_3 and bv_4.

It remains to consider the case that a certain edge of S became half by shifting 1/2 from the unit edge bv_3 to the zero edge bv_2. If bv_2 became half, then S must consist of bv_1, bv_2 and bv_3, or of bv_2, bv_3 and bv_4. If bv_2 becomes unit after averaging, then S consists of bv_3, bv_4 and bv_5. Remark that S cannot consist of bv_2, bv_3 and bv_4 due to Claim 3.4. From Lemma 3.2 (c) it follows that S also cannot consist of bv_1, bv_2 and bv_3.

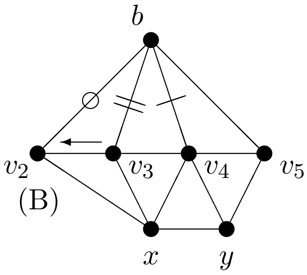
Suppose that S consists of bv_3, bv_4 and bv_5, and that bv_2 becomes unit. According to Lemma 3.2 (c), the edge bv_4 was initially zero, half or unit. Moreover, in the last case bv_5 was zero and v_5 was a B-vertex.

If bv_4 was zero, then v_4 must be an L-vertex, for otherwise bv_3 shifts 1/2 to bv_4, and bv_3 becomes zero. This means that bv_4 remains zero, contrary to the assumptions.

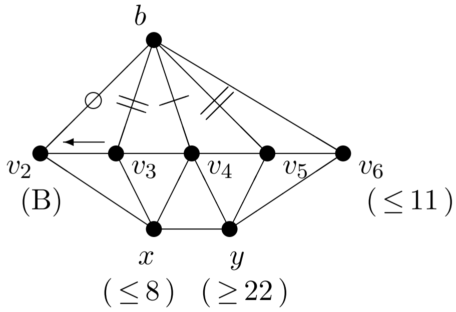
Suppose that bv_4 was unit. By Lemma 3.2 (c), bv_4 shifts 1/2 to the initially zero edge bv_5. Then by Claim 3.4, bv_5 cannot be a boundary edge in S, contrary to the assumptions.

Now suppose bv_4 was initially half. Since bv_3 was unit and v_4 is minor, we conclude by R1 – R3 that $d(v_3) = 4$. By assumption, bv_4 is a half edge both before and after averaging; hence $d(v_4) \in \{4, 5\}$. Assume $d(v_4) = 4$. Then R1 implies that $d(v_5) \ge 12$ and that bv_5 was zero before averaging. Now if v_5 is an L-vertex, then bv_5 remains zero after averaging. But if v_5 is a B-vertex, then bv_4 shifts 1/2 to bv_5, so that bv_4 becomes zero.

Now assume $d(v_4) = 5$ (see Fig. 3.11 (a)). Since bv_3 was unit, it follows from R1 that the



(a)



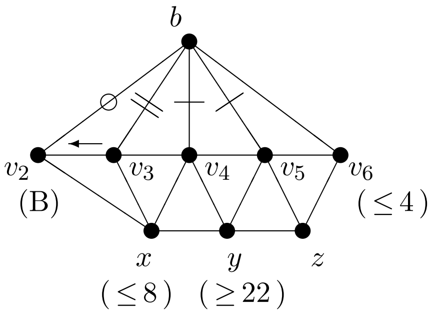
(b)

Fig. 3.11

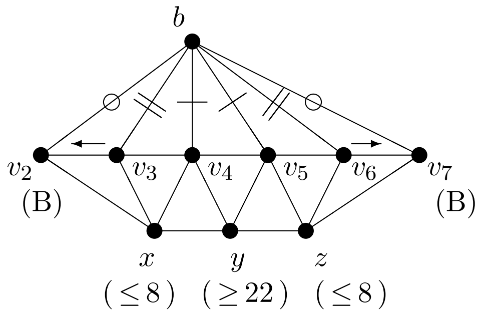
degree of the vertex x in Fig. 3.11 (a) is at most 8. As bv_4 does not shift 1/2 to bv_5, v_5 is an L-vertex. This implies that bv_5 was half or unit before averaging, so that v_5 is minor. As seen above, $d(v_3) = 4$ and $d(x) \leq 8$, while (A') applied to v_4 yields $d(y) \geq 22$.

As shown above, bv_5 may be either half or unit. If it was unit, then it follows from R1–R3 that $d(v_5) = 4$ (because the common neighbour v_4 of v_5 and b has degree 5). Moreover, the fact that bv_5 is unit, by R1 implies that the degree of the vertex v_6 adjacent to v_5 is at most 11 (Fig. 3.11 (b)). In this case, bv_5 does not shift 1/2 to bv_6, so that bv_5 remains unit, a contradiction.

Now suppose that bv_5 was initially half. If $d(v_5) = 4$, then bv_6 was zero by R1. Moreover, it cannot become unit because of the lack of shift from bv_5. We must have $d(v_5) = 5$. Since bv_6 becomes unit and there was no shift to it from bv_5, we see from Claim 3.3 that $d(v_6) \leq 4$ (Fig. 3.12 (a)). Since bv_5 was initially half, it follows from R3 that the degree of z in Fig. 3.12



(a)



(b)

Fig. 3.12

is at most 8. Hence, by applying (A') to v_6 we conclude that $d(v_6) \neq 3$, as it was proved above that $d(v_5) + d(z) \leq 13$. Thus $d(v_6) = 4$ and, moreover, the vertex v_7 adjacent to v_6 is different from z and is a B-vertex. Finally, bv_6 was initially unit by R1 (f) (Fig. 3.12 (b)). It means that there was a shift from the unit edge bv_6 to the zero edge bv_7, so that bv_6 becomes half. This contradiction completes the proof of Lemma 3.8. $\square$

Lemma 3.9 (main)
The vicinity of b consists of precisely 5 separators and precisely 5 bunches, bounded by these separators. Moreover, each separator consists of two edges, and at most one separating edge is zero.

Proof By Corollary 3.7, the number of separators is at most 5. Due to (3.3), there are at most 11 separating edges, whence the number of unit edges is at least $d - 11$. Suppose that the number of separators, and hence that of prebunches, is at most 4. Then there is a prebunch of length at least $\lceil (d-11)/4 \rceil$. By Lemma 3.5, this prebunch forms a part of a bunch of length at least $\lceil (d-11)/4 \rceil + 2$ with a pole at b. Since b is a B-vertex, it follows that $\lceil (d-11)/4 \rceil + 2 > d/5$, which contradicts property (B'). Hence, b is incident with precisely 5 separators and 5 prebunches.

We now prove that each prebunch at b is a part of a bunch. By Lemma 3.5, it suffices to prove that each prebunch has length at least 3. If a shorter prebunch exists, then the other 4 prebunches contain at least $d - 13$ unit edges in total, and there is a prebunch of length at least $\lceil (d-13)/4 \rceil$. By Lemma 3.5, this prebunch is contained in a bunch of length at least $\lceil (d-13)/4 \rceil + 2$ with a pole at b. Since b is a B-vertex, it follows that $\lceil (d-13)/4 \rceil + 2 > d/5$, contrary to (B'). Hence, b is incident with precisely 5 separators and 5 bunches.

Next, we prove that each separator has length two. Recall the notion of saving used in proving Corollary 3.7. Lemma 3.8 implies that each separator of length at least three saves for b at least two units of charge. If such a separator exists, the by Lemma 3.6 the total saving of all five separators is at least 6, which contradicts the assumption $\mu^*(b) < 0$. Hence, the length of each separator is at most two. If there were a separator of length one (i.e., consisting of just one edge), then by Lemma 3.5 the total length of all five bunches at b would be greater than d, and there is a bunch of length greater than $d/5$, contrary to (B').

It remains to observe that, by (3.3), among the 10 separating edges at b there may exist at most one zero edge. This completes the proof of Lemma 3.9. $\square$

By Lemma 3.9, the vicinity of b consists of at most 5 bunches, whose total length is at least d. If not all the bunches have the same length, then there is a bunch of length greater than $d/5$, contrary to (B'). So, we can assume that each bunch has length precisely $d/5$, and no bunch has a parental edge (for otherwise the second part of statement (B) in Theorem 2.1 holds). Moreover, each parental edge in the vicinity of b is a boundary edge for precisely one bunch.

It follows from the proof of Lemma 3.5 that each unit edge at b, leading to a B-vertex, is a parental edge for a certain bunch. It also follows that if a bunch contains a bunch vertex of degree 3, then it contains a parental edge too. Hence, we may assume that each unit edge at b leads to a 4-vertex. Then R1 and AR imply that each unit edge at b was also unit before averaging. Let us prove a stronger fact; namely, no shift by AR happens at b. Indeed, it was just proved for unit edges, and for separating edges it follows from Lemma 3.9 and the definition of AR.

Let us prove that each half edge at b leads to a 5-vertex. Suppose there is a separator consisting of edges bv_1 and bv_2, where bv_2 is half and $d(v_2) = 4$. Since G is a triangulation, it follows that v_2 is an interior vertex in a bunch that contains v_3 as a bunch vertex (see the proof of Lemma 3.5). Then bv_2 is not a boundary edge of this bunch, and the length of this bunch is greater than $d/5$. This contradiction implies that each half edge at b leads to a 5-vertex and this vertex is an end vertex for one bunch. We have thus proved that if each separator edge at b is half, then the last part of statement (B) of Theorem 2.1 holds.

Finally, due to Lemma 3.9, we have to consider a separator at b that consists of a half edge bv_1 and a zero edge bv_2. If $d(v_2) \leq 11$, then the last part of statement (B) of Theorem 2.1 holds. Therefore, suppose $d(v_2) \geq 12$. As seen above, we now have $d(v_1) = 5$ and $d(v_3) = d(v_4) = 4$. Moreover, v_3 and v_4 are interior vertices and v_2 is an end vertex in a bunch with one pole at b and the other at a B-vertex t (Fig. 3.13).

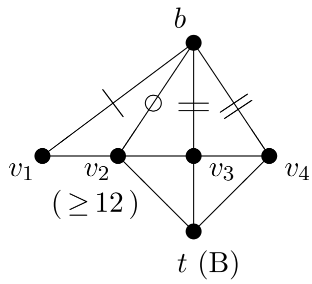


Fig. 3.13

It remains to observe that due to R1 (b), the edge bv_3 must be half, contrary to the assumptions. This completes the proof of Theorem 2.1. ■

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