Polyhedral results for assignment problems

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CDAM Research Report LSE-CDAM-2002-01

April 19, 2002

Abstract

This paper introduces an Integer Programming model for multidimensional assignment problems and examines the underlying polytopes. It also proposes a certain hierarchy among assignment polytopes. The dimension for classes of multidimensional assignment polytopes is established, unifying and generalising previous results. The framework introduced constitutes the first step towards a polyhedral characterisation for classes of assignment problems. The generic nature of this approach is illustrated by identifying a family of facets for a certain class of multidimensional assignment problems, namely "axial" problems.

1 Introduction

Assigmnet structures are embedded in numerous combinatorial optimisation problems. An assignment occurs whenever a member of an entity must be allocated/mapped to a member of another entity. The simplest case of an assignment problem is the well known 2-index assignment, which is equivalent to the weighted bipartite matching problem. Further examples are assignment of facilities to locations or assignment of delivery points to vehicles.

Extensions of the assignment structure to more than two entities give rise to *multidimensional* (or *multi-index*) *assignment problems*, formally introduced in [21, 22]. These problems essentially ask for a minimum weight clique partition in a complete k-partite graph (see also [6]). Normally, the weighted sum of the variables constitutes the objective function, a fact that justifies the alternative term *linear sum* assignment problems ([8]). Clearly, a k-index assignment problem is defined on k sets, usually (but not always) assumed to be of the same cardinality $n.$ The goal is to identify a minimum weight collection of n disjoint tuples, each including a single element from each set. This is the class of *axial* assignment problems ([3, 28]).

A different structure appears, if the aim is, instead, to identify a collection of n^2 tuples, partitioned into n disjoint sets of n disjoint tuples. By way of illustration, consider the problem of allocating n teachers to n student groups for sessions in one of n classrooms and using one of n laboratory facilities, in such a way that all teachers teach all groups, using each time a different classroom or a different facility (for a relevant case, see [11]). These assignment problems are called *planar* and are directly linked to structures called *Mutually Orthogonal Latin Squares (MOLS)* ([17]). Generalising this concept, we could ask for an optimal collection of n^s tuples, thus defining the (k, s) assignment problem of order $n,$ hereafter denoted as $(k, s)AP_n$, which encompasses all known assignment structures.

This paper provides an Integer Programming formulation of the $(k,s)AP_n$ and examines the underlying polytope $P_I^{(k,s)}$. Previous polyhedral analysis has been conducted only for $k \leq 4$ and $s = 1, 2$ ([1, 3, 9]). Our work unifies and generalises these findings by establishing the dimension of multidimensional assignment polytopes. By introducing a generic framework, it allows for further application of the tools of polyhedral combinatorics to a whole class of assignment problems instead of a single one. In particular, it provides a non-trivial class of facets for the polytope of multidimensional (axial) assignment problem. It also defines a certain hierarchy for this class of combinatorial optimisation problems.

An excellent review for assignment problems appears in [28], where complexity and approximability issues are covered. It is known that the 2-index problem is polynomially solvable, whereas the 3-index axial and planar problems are $\mathcal{NP}$-hard (see [14] and [16], respectively). The authors of [7] prove that the multidimensional (axial) assignment problem remains $\mathcal{NP}$-hard for $k \geq 3$, if the coefficient vector of the objective function fulfills and *Anti-Monge* condition.

Apart from its theoretical significance, the $(k,s)AP_n$ possesses numerous applications. Multidimensional (axial) assignment structures have recently received substantial attention, because of their applicability in problems of data association. Such problems arise naturally in multi-target/multi-sensor tracking in satellite surveillance systems (see [23]). An application to the problem of tracking elementary particles at the Large Electron-Positron Collider of CERN is reported in [26]. The planar problems share the diverse application fields of *MOLS*, e.g. multivariate design, error correcting codes and timetabling (see also [17]).

The rest of this paper is organised as follows. Section 1.1 introduces the notation used throughout and Section 1.2 presents the mathematical formulation of the $(k,s)AP_n$. Concepts of polyhedral theory are reviewed in Section 2, where a number of boundary conditions are also exhibited. Section 3 establishes the dimension of the linear assignment polytope. Section 4 examines the convex hull of integer vectors, provides a necessary condition for the existence of a solution to the $(k,s)AP_n$ and presents a hierarchy of assignment polytopes. The dimension of the axial assignment polytopes is established in sections 5, while a family of non-trivial facets is exhibited in Section 6. Finally, Section 7 establishes the dimension of the planar assignment polytopes.

1.1 Definitions and notation

Let K denote a set of k distinct elements and assume wihout loss of generality that $K = \{1, \ldots, k\}$. Consider k disjoint n-sets $M_1, M_2, \ldots, M_k$ and define $M_K = M_1 \times M_2 \times \cdots \times M_k = \bigotimes_{t=1}^{k} M_t$. If $m^t \in M_t$, $\forall t \in K$, it follows that $(m^1, m^2, \ldots, m^k) \in M_K$. It is important to distinguish between $\bigotimes_{t=1}^{k} M_t$ and $\bigotimes_{t \in K} M_t$, since the latter could imply any set of (all) k-tuples, which are not necessarily ordered increasingly with respect to the values of index t. For example, it could be that $c = (m^k, m^{k-1}, ..., m^1) \in \bigotimes_{t \in K} M_t$, although $c \notin \bigotimes_{t=1}^{k} M_t$. Tuples can be regarded as subsets of M_K, therefore the usual set operations, e.g. union, are applicable. Two tuples are called *disjoint* if they have no common element.

Let $Q_{k,s} = \{K' \subseteq K : |K'| = s\}$. The cardinality of $Q_{k,s}$, denoted as $|Q_{k,s}|$, is equal to $\binom{k}{s}$. Let also $S \in Q_{k,s}$, $M_S = \bigotimes_{t \in S} M_t$, $M_{K \setminus S} = \bigotimes_{t \in K \setminus S} M_t$ and define the trivial mapping $\varphi_{k,s} : (M_S, M_{K \setminus S}) \to M_K$. The definition implies that an unordered s-tuple and an unordered $(k-s)$-tuple are mapped to an ordered k-tuple. Let us provide an example.

Example 1.1. *Let* $K = \{1, 2, 3, 4\}$*,* $s = 2$*,* $S = \{2, 4\}, m^t \in M_t\ \forall t \in K$*. Then*

$$\begin{aligned} \varphi_{4,2}((m^2, m^4), (m^1, m^3)) &= \varphi_{4,2}((m^2, m^4), (m^3, m^1)) = \\ \varphi_{4,2}((m^4, m^2), (m^1, m^3)) &= \varphi_{4,2}((m^4, m^2), (m^1, m^3)) = \\ &\quad (m^1, m^2, m^3, m^4) \end{aligned}$$

In order to illustrate a number of technical proofs without having to examine all symmetrical cases, we introduce a further level of abstraction by defining $i_t \in K, t = 1, .., k$, such that $\bigcup_{t=1}^{t=k} \{i_t\} \equiv K$. Let $S = \{i_1, i_2, ..., i_s\} \subseteq K$, $M_S = M_{i_1} \times M_{i_2} \times \cdots \times M_{i_s}$ and $M_{K \setminus S} = M_{i_{s+1}} \times M_{i_{s+2}} \times \cdots \times M_{i_k}$. Also, let $m^{i_t} \in M_{i_t}\ \forall t \in K$. Since i_t is not necessarily equal to t, we can write:

$$\varphi_{k,s}((m^1, ..., m^s), (m^{s+1}, ..., m^k)) = \varphi_{k,s}((m^{i_1}, ..., m^{i_s}), (m^{i_{s+1}}, ..., m^{i_k})) = (m^1, m^2, ..., m^k)$$

By definition, $\varphi_{k,s}$ is flexible enough to represent the mapping of unordered k-tuples to ordered ones. In particular, consider the cases $s = k$ and $s = 0$. In the first case, $S = K$, whereas in the second one $S = \emptyset$. Hence, if we conventionally define $M_\emptyset = \emptyset$, $\varphi_{k,k}$ (or equivalently $\varphi_{k,0}$) denotes the mapping of any unordered k-tuple, formed by $m^1, m^2, \ldots, m^k$, to the ordered tuple $(m^1, m^2, \ldots, m^k)$. Formally,

$$\varphi_{k,k}((m^{i_1}, ..., m^{i_k}), ()) = \varphi_{k,0}((), (m^{i_1}, ..., m^{i_k})) = (m^1, m^2, ..., m^k)$$

or, equivalently,

$$\varphi_{k,k}(m^{i_1}, ..., m^{i_k}) = \varphi_{k,0}(m^{i_1}, ..., m^{i_k}) = (m^1, m^2, ..., m^k) \tag{1.1}$$

It is easy to see that the mapping φ exhibits the commutative and associative properties:

$$\varphi_{k,s}((m^{i_1}, \ldots, m^{i_s}), (m^{i_{s+1}}, \ldots, m^{i_k})) = \varphi_{k,k-s}((m^{i_{s+1}}, \ldots, m^{i_k}), (m^{i_1}, \ldots, m^{i_s})), \forall s \leq k \tag{1.2}$$

$$\varphi_{k,s}((m^{i_1}, \ldots, m^{i_s}), (m^{i_{s+1}}, \ldots, m^{i_k})) = \varphi_{k,s-t}((m^{i_1}, \ldots, m^{i_{s-t}}), (m^{i_{s-t+1}}, \ldots, m^{i_k})), \forall t < s \leq k \tag{1.3}$$

For the rest of the paper, we follow the convention that m^t denotes an index with domain M_t, (thus $m^t \in M_t$), whereas $m^t_0, m^t_1, \ldots$ denote the elements of the set M_t, i.e. the values of the index m^t.

1.2 Mathematical Formulation

Definition 1.1. *Let* $K = \{1, .., k\}$ *and an integer* $s \in \{1, ..., k-1\}$*. Consider* k *disjoint* n*-sets* $M_1, M_2, \ldots, M_k$ *and select any* $(k - s + 1)$ *of these sets. Define the following sequence of powersets:*

W_0 *is the set of all* $(k - s + 1)$*-tuples* $(m^{i_1}, ..., m^{i_{k-s+1}}) \in M_{i_1} \times \cdots \times M_{i_{k-s+1}}, i_t \in K$,

W_1 *is the set of all subsets of* W_0 *consting of* n *disjoint* $(k - s + 1)$*-tuples*

for $t = 2, ..., s - 1$

W_t *is the set of all subsets of* W_{t-1} *which consist of exactly* n *disjoint members of* W_{t-1}

W_s *is the set of all subsets of* W_{s-1} *which consist of exactly* n *disjoint members of* W_{s-1}

Definition 1.2. *The* $(k, s)AP_n$ *problem asks for a minimum weight member of the set* W_s*.*

It follows that the $(k, s)AP_n$ problem asks for a minimum weight collection of n^s $(k-s+1)-$tuples. For example, the $(3,1)AP_n$ asks for a minimum weight collection of n disjoint triplets and the $(3,2)AP_n$ asks for a minimum weight collection of n^2 disjoint pairs, which can be partitioned into n sets of n disjoint pairs. Note that an equivalent representation could be expressed in terms of asking for n^{s-1} clique partitions in the complete $(k-s+1)$-partite graph.

Let us introduce an Integer Programming formulation of the $(k,s)AP_n$. A binary variable $x_{m^1,m^2,\ldots,m^k}$ and a (real) weight coefficient $w_{m^1,m^2,\ldots,m^k}$ is associated with each k-tuple $(m^1, m^2, ..., m^k)$. Any $(k-s+1)$ out of the k n-sets can be regarded as indexing the $(k-s+1)-$ tuples, while the remaining $(s-1)$ sets index the members of the powersets $W_1, \ldots, W_{s-1}$ selected. For example, $x_{m^1,m^2,\ldots,m^k} = 1$ implies selection of the tuple $c = (m^1, m^2, ..., m^{k-s+1})$. Among the n (disjoint) members of set W_{s-1} selected, tuple c belongs to the m^kth one. This member of W_{s-1} consists of n disjoint members of W_{s-2} and tuple c belongs to the m^{k-1}th one, and so on.

We examine a simple corollary of the above. Let $c, d \in M_K$ and x_c, x_d be the corresponding variables. If $|c \cap d| \geq s$, the variables x_c, x_d have at least s indices in common, hence, assume $m^{i_1}, \ldots, m^{i_s}$ to be s common indices. Then, $x_c = x_d = 1$ implies the existence of a pair of $(k-s+1)$-tuples, both belonging to the same member m^{i_t} of powerset W_{i_t}, for all $t = 1, \ldots, s-1$. But then, member m^{i_1} of powerset W_1 contains two $(k-s+1)$-tuples having index m^{i_s} in common, i.e. two non-disjoint tuples, which is a contradiction to Definition 1.1. It follows that a pair of variables, which have at least s indices in common, cannot simultaneously take value 1. This fact is reflected by the integer programming formulation of $(k,s)AP_n$, which is derived by summing over all possible subsets of $k-s$ out of k indices. In other words, there are exactly s "fixed" indices in each constraint. As an example, consider the formulation of $(k,1)AP_n$ (in [23]), which is derived by summing over all possible subsets of $k-1$ out of k indices.

$$\min \sum \{ w_{m^1,m^2,\ldots,m^k} \cdot x_{m^1,m^2,\ldots,m^k} : (m^1, m^2, \ldots, m^k) \in M_K \} \tag{1.4}$$

$$\begin{aligned} \text{s.t.} \sum \{ x_{\varphi_{k,s}(m^S, m^{K \setminus S})} : m^{K \setminus S} \in M_{K \setminus S} \} &= 1, \forall m^S \in M_S, \forall S \in Q_{k,s} && (1.5) \\ x_{m^1,m^2,\ldots,m^k} &\in \{0,1\}, \forall (m^1, m^2, \ldots, m^k) \in M_K && (1.6) \end{aligned}$$

Note that $M_{K \setminus S}$ is the set of indices appearing in the sum, whereas M_S is the set of indices common to all variables in an equality constraint. Let $A_n^{(k,s)}$ denote the $(0,1)$ matrix of the constraints (1.5). The matrix $A_n^{(k,s)}$ has n^k columns and $\binom{k}{s} \times n^s$ rows, i.e. n^s constraints for each of the $\binom{k}{s}$ distinct $S \in Q_{k,s}$. Each constraint involves n^{k-s} variables.

Under these definitions, it is obvious that $(2,1)AP_n$ refers to the 2-index assignment problem, $(3,1)AP_n$ to the 3-index *axial* assignment problem ([3, 10]), $(3,2)AP_n$ to the 3-index *planar* assignment problem ([9, 18]), and $(4,2)AP_n$ to the 4-index *planar* assignment problem ([1]). Note that parameter s is central to the type of assignment required at each problem, i.e. the *axial* problems imply $s = 1$ and the *planar* problems imply $s = 2$. An analogous formulation and classification appears in [28], where the $(k,s)AP_n$ is called the "$q-$fold kIAP".

2 Assignment polytopes and related structures

2.1 General concepts

Recall first a number of introductory definitions from polyhedral theory (see [20, 25]). A *polyhedron* is the intersection of a finite set of (affine) half spaces. A polytope is a *bounded* polyhedron. A polytope $\mathbb{P}$ is of dimension n, denoted as $dim\mathbb{P}$, if it contains $n+1$ affinely independent points. By convention, if $\mathbb{P} = \emptyset$ then $dim\mathbb{P} = -1$. Given that $\mathbb{P} \neq \emptyset$, consider an inequality $ax \leq a_0$ satisfied for all $x \in \mathbb{P}$. Then the set $F = \{x \in \mathbb{P} : ax = a_0\}$ is called a *face* of $\mathbb{P}$. A face F is called *proper* if $F \subset \mathbb{P}$. A proper, non-empty face F of $\mathbb{P}$ is a *facet* if $\dim F = \dim P - 1$. Facets are maximal faces with respect to set inclusion.

Let B be a real valued $m \times n$ matrix and $b \in \mathbb{R}^m$. If $\mathbb{P} = \{x \in \mathbb{R}^n : B^{=}x = b^{=}, B^{\leq}x \leq b^{\leq}\}$, where $B = [B^{=}, B^{\leq}]^T$, $b = [b^{=}, b^{\leq}]^T$ and $\mathbb{P} \neq \emptyset$, a core result of polyhedral theory states

$$\text{dim}\mathbb{P} = n - rankB^{=} \tag{2.1}$$

The convex hull of the integer points satisfying the constraints (1.5) is the (k, s) *assignment polytope*, denoted as $P_I^{(k,s)}$. Formally, $P_I^{(k,s)}$ =conv$\{x \in \{0,1\}^{n^k} : A_n^{(k,s)}x = e\}$, where e is a column vector of ones. The *linear relaxation* of $P_I^{(k,s)}$, also called the *linear assignment polytope*, is the polytope $P^{(k,s)} = \{x \in \mathbb{R}^{n^k} : A_n^{(k,s)}x = e, x \geq 0\}$. Obviously, $P_I^{(k,s)} \subseteq P^{(k,s)}$.

Clearly, the *assignment polytope* is a special case of the *set-partitioning* polytope defined as $P_{SP} = \{y \in \{0,1\}^q : By = e\}$, where B is a $0-1$ matrix. A close relative of P_{SP} is the *set-packing* polytope $\tilde{P}_{SP}$, defined exactly as P_{SP} but with "=" replaced by "$\leq$". In our case, $\tilde{P}_I^{(k,s)} = conv\{x \in \{0,1\}^{n^k} : A_n^{(k,s)}x \leq e\}$. A relation, inherited from the general case, is that $P_I^{(k,s)}$ is a face of $\tilde{P}_I^{(k,s)}$ implying $\dim P_I^{(k,s)} \leq \dim \tilde{P}_I^{(k,s)}$. Polytope $\tilde{P}_I^{(k,s)}$ is full dimensional, hence $\dim \tilde{P}_I^{(k,s)} = n^k$, i.e. its dimension is independent of s. For a survey on $\tilde{P}_{SP}$, P_{SP} and on related problems, see [2].

Another class of polytopes related specifically to $P_I^{(k,1)}$ is given by the *multi-index generalized assignment* problems. The simplest representative is the (two-index) generalized assignment problem *(GAP)* ([19]). This class is defined for $s = 1$, i.e. they constitute an extension of the axial assignment problems, in the sense that one of the constraint sets consists of equalities of the type "$=1$", whereas the rest consist of inequalities of the type "$\leq b_t$", where $b_t \in \mathbb{Z}_+^{|M_t|}$, $t \in K \setminus \{1\}$. Sets M_t are not necessarily of the same cardinality. Instead, $M_1 = \{1, \ldots, p_1\}$, $M_2 = \{1, \ldots, p_2\}$, ..., $M_k = \{1, \ldots, p_k\}$, where $p_1 \leq p_2 \leq \ldots \leq p_k$. The (multi-index) generalized assignment polytope is defined as

$$\begin{aligned}
\sum\{x_{\varphi_{k,1}(m^1, m^{K\setminus\{1\}})} &: m^{K\setminus\{1\}} \in M_{K\setminus\{1\}}\} = 1, \forall m^1 \in M_1 \\
\sum\{x_{\varphi_{k,1}(m^t, m^{K\setminus\{t\}})} &: m^{K\setminus\{t\}} \in M_{K\setminus\{t\}}\} \leq b_t, \forall m^t \in M_t, \forall t \in K \setminus \{1\} \\
x_{m^1, m^2, \ldots, m^k} &\in \{0,1\}, \forall (m^1, m^2, ..., m^k) \in M_K
\end{aligned}$$

Applications of this model, for the 3-index case ($k = 3$), can be found in [12, 13, 19]. Finally, we note that a similar approach can be used for modelling an extension of the transportation problem, called the *solid (multi-index) transportation problem*, introduced in [15].

2.2 Two special cases

For $s = k$, constraints (1.5) reduce to the system of trivial equalities

$$x_{m_1 m_2 \cdots m_k} = 1, \forall (m_1, m_2, \dots, m_k) \in M_K \tag{2.2}$$

whereas, for $s = 0$, constraints (1.5) result in the single equality constraint

$$\sum \{x_{m_1, m_2, \dots, m_k} : (m_1, m_2, \dots, m_k) \in M_K\} = 1 \tag{2.3}$$

Lemma 2.1. $rankA_n^{(k,k)} = n^k$ *and* $dimP^{(k,k)} = dimP_I^{(k,k)} = 0$.

Proof. It is easy to see that $A_n^{(k,k)} = I^{n^k \times n^k}$. Therefore, $rankA_n^{(k,k)} = rankI^{n^k \times n^k} = n^k$.

Note that (2.2) implies that $P^{(k,k)} \equiv P_I^{(k,k)} \neq \emptyset$ since both polytopes contain the single point $x = e$. By definition, $dimP^{(k,k)} = dimP_I^{(k,k)} = 0$. □

Lemma 2.2. $rankA_n^{(k,0)} = 1$ *and* $dimP^{(k,0)} = dimP_I^{(k,0)} = n^k - 1$.

Proof. (2.3) is a single equality constraint, thus $A_n^{(k,0)}$ is a row vector of ones. This implies $rankA_n^{(k,0)} = 1$. Clearly, $P^{(k,0)} \neq \emptyset$ since $x = (1/n^k, \dots, 1/n^k)^T \in P^{(k,0)}$. It follows from (2.1) that $dimP^{(k,0)} = n^k - 1$.

Since, $P_I^{(k,0)} \subseteq P^{(k,0)}$, it also holds that $dimP_I^{(k,0)} \leq dimP^{(k,0)}$. We prove that this bound is actually attained by exhibiting a set of n^k linearly independent points of $P_I^{(k,0)}$. For each $t \in \{1, \dots, n^k\}$ consider the vector, consisting of n^k entries, that has a one at position t and a zero elsewhere. Hence, $dimP_I^{(k,0)} = n^k - 1$. □

Corollary 2.3. $P_I^{(k,0)}$ *is a facet of* $\tilde{P}_I^{(k,0)}$.

Proof. $P_I^{(k,0)} \subset \tilde{P}_I^{(k,0)}$, since point $x = (0, \dots, 0)^T$ belongs to $\tilde{P}_I^{(k,0)}$ but not to $P_I^{(k,0)}$. By Lemma (2.2), $dimP_I^{(k,0)} = n^k - 1$ and we also know that $\dim \tilde{P}_I^{(k,0)} = n^k$. The result follows. □

3 The (k, s) linear assignment polytope

First, we propose an ordering of the rows and columns of $A_n^{(k,s)}$. Observe that each set $S \in Q_{k,s}$ is uniquely associated to a row set of $A_n^{(k,s)}$. Each such S can be regarded as a subset of s indices, written (by convention) in ascending order $(i_1, \dots, i_s)$, where $i_t < i_{t+1}$ for $t = 1, \dots, s-1$.

Definition 3.1. *Let* $S, S' \in Q_{k,s}$, *where* $S = (i_1, \dots, i_s)$, $S' = (i'_1, \dots, i'_s)$. *It holds that* $S > S'$ *if and only if there exists* $t \in \{1, .., s\}$ *such that* $i_p = i'_p$ *for* $p = 1, .., t-1$ *and* $i_t > i'_t$.

It follows that there exists a strict order of all $s-$subsets of indices, i.e. a strict order of all $S \in Q_{k,s}$. Based on this order, row sets of $A_n^{(k,s)}$ are positioned in descending order. Within a particular row set, a row corresponds to an s-tuple of the set M_S. The rows are placed in ascending order with respect to the corresponding s-tuples. Columns of $A_n^{(k,s)}$ appear in an order, where index m^1 is the slowest to vary and index m^k is the quickest. Let us give an example.

Example 3.1. *For* $k = 4, s = 2$, *there exist* $\binom{4}{2} = 6$ *row sets. They are considered in descending order with respect to the pair of "fixed" indices, i.e.* $\{(m^3, m^4), (m^2, m^4), (m^2, m^3), (m^1, m^4), (m^1, m^3), (m^1, m^2)\}$. *Rows in, say, the second row set are identified by the values of the pair of indices* (m^2, m^4). *The values of this pair of indices are* *considered in ascending order, i.e.* $(1,1),\ldots,(1,n),\ldots,(n,1),\ldots,(n,n)$*. The columns of* $A_n^{(4,2)}$ *are considered in ascending ordered w.r.t. the values of the* 4*-tuple* (m^1, m^2, m^3, m^4)*, i.e.* $(1,1,1,1)$, $\ldots$, $(1,1,1,n)$, $\ldots$, $(n,1,1,1)$, $\ldots,(n,n,n,n)$.

Let $(i_1,\ldots,i_s) \in Q_{k,s}$, where $i_1 < i_2 < \cdots < i_s$. The function f that maps, in descending order, the elements (s-tuples) of set $Q_{k,s}$ to $\{1,\ldots,|Q_{k,s}|\}$ is

$$f(i_1,\ldots,i_s) = \sum_{t=1}^{s} \left\{ \sum_{j=i_t+1}^{k-(s-t)} \binom{k-j}{s-t} \right\} + 1, \quad i_1 < \ldots < i_s,\ s,k \in \mathbb{Z} \tag{3.1}$$

To illustrate the correctness of this formula, consider the row set r_0 indexed by the s-tuple $(i_1, i_2, \ldots, i_s) \in Q_{k,s}, i_1 < \cdots < i_s, i_t \in K$ for $t = 1,\ldots,s$. Let r_1 be a row set indexed by the same indices as r_0 in positions $1,\ldots,t$ and indices $[k-(s-t)+1],\ldots,k$ in positions $t+1,\ldots,s$, i.e. r_1 is indexed by the s-tuple $(i_1,i_2,\ldots,i_t,k-(s-t)+1,\ldots,k)$. Any row set preceding r_1, which has the same indices as r_0 in positions $1,\ldots,t-1$ must have an index j in position t, such that $j \in \{i_t+1,\ldots,k-(s-t)\}$. For each such j, positions $t+1,\ldots,s$, at the s-tuple indexing the row set, can be completed by choosing $s-t$ out of $k-j$ indices. This holds because there are $s-t$ positions to be filled and $k-j$ indices with value larger than j. This results in $\binom{k-j}{s-t}$ possible row sets for each $j \in \{i_t+1,\ldots,k-(s-t)\}$. Repeating this process for all $i_t, t = 1,\ldots,s$ and taking the sum, results in the total number of row sets preceding r_0 (i.e. equation (3.1)). Therefore, the row set $(i_1,\ldots,i_s)$, $i_1 < \cdots < i_s$, includes rows $f(i_1,\ldots,i_s)\cdot n^s + 1$ to $(f(i_1,\ldots,i_s)+1)\cdot n^s$, i.e. one row for each value of the s-tuple $(m^{i_1},\ldots,m^{i_s})$. One can check that the first and the last s-tuples, indexing the row sets (in decreasing order), have $f(k-s+1,\ldots,k) = 1$ and $f(1,\ldots,s) = \binom{k}{s}$, respectively, as required.

Theorem 3.2. $rank A_n^{(k,s)} = \sum_{r=k-s}^{k} \{\binom{k}{r}(n-1)^{k-r}\}$.

Proof. For $s = k$ and $s = 0$ the theorem stands by virtue of Lemmas 2.1 and 2.2, respectively. For $1 \le s \le k-1$, the proof includes $s+1$ steps. At each step we remove a set of rows aiming, at the end of the procedure, to be left with a set of linearly independent rows. Let $r = 0,\ldots,s$ denote the step counter.

Observe first that the sum of all equalities belonging to the same row set results in an equality stating that the sum of all variables is equal to n^s. Therefore, we remove the first row from all but the row sets except for the first one. The total number of rows removed is $\binom{k}{s} - 1 = \binom{k}{0}(n-1)^0(\binom{k}{s}-1)$. This is the step $r = 0$.

At step $r = 1$, for each index $q \in K$, consider the subset of row sets $R_q = \{S \in Q_{k,s} : q \in S\}$. It is easy to see that $|R_q| = \binom{k-1}{s-1}$. Each row set belonging to R_q is further partitioned into n subsets, one for each value of the index m^q. Observe that the rows of the row sets, belonging to the same subset, form a $(k-1,s-1)AP_n$. Observe also that the n $(k-1,s-1)AP_n$ problems formed in this way, (i.e. one problem for each value of m^q), are independent of each other, i.e. each problem has a distinct set of variables. Each such problem consists of $\binom{k-1}{s-1}$ row sets, the sum of rows in each row set stating that the sum of all the n^{k-1} variables is equal to n^{s-1}. Hence, for each such problem, except for the one defined for $m^q = 1$, we remove the first row from each row set, excluding the first row set. Note that the "first" row set is the one appearing first in matrix $A_n^{(k,s)}$, according to the ordering described above. In other words, for each of the $n-1$ $(k-1,s-1)AP_n$ problems, we remove $\binom{k-1}{s-1} - 1$ rows. Note that the rows corresponding to the problem defined for $m^q = 1$, have already been made linearly independent in the previous step. Hence, for each $q \in K$, we remove $(n-1)(\binom{k-1}{s-1}-1)$ rows, yielding a total of $k(n-1)(\binom{k-1}{s-1}-1) = \binom{k}{1}(n-1)^1(\binom{k-1}{s-1}-1)$ rows removed in this step.

Similarly, at step r ($r \leq s$), sets of exactly r distinct indices are considered. Let $(i_1, \ldots, i_r)$, denote such a subset. We define the subset of row sets $R_{i_1,\ldots,i_r} = \{S \in Q_{k,s} : \bigcup_{p=1}^{r}\{i_p\} \subseteq S\}$, $|R_{i_1,\ldots,i_r}| = \binom{k-r}{s-r}$. Each row set belonging to $R_{i_1,\ldots,i_r}$ is further partitioned into n^r subsets, one for each value of the r-tuple $(m^{i_1}, \ldots, m^{i_r}) \in M_{i_1} \times \cdots \times M_{i_r}$. Again, observe that the rows of the row sets, belonging to the same subset, i.e. identified by the same value of the r-tuple $(m^{i_1}, \ldots, m^{i_r})$, form a $(k-r, s-r)AP_n$. Since there are n^r distinct subsets, we have n^r independent $(k-r, s-r)AP_n$. Each such problem consists of $\binom{k-r}{s-r}$ row sets, the sum of rows in each row set stating that the sum of all the n^{k-r} variables is equal to n^{s-r}. Hence, for each of these problems, excluding the ones defined by the r-tuples $(m^{i_1}, \ldots, m^{i_r})$ for which *at least one* of the indices is equal to 1, we remove the first row from each of its last $\binom{k-r}{s-r} - 1$ row sets. Therefore, for each $(k-r, s-r)AP_n$, defined by the r-tuple $(m^{i_1}, \ldots, m^{i_r})$, $m^{i_p} \neq 1$, $\forall p = 1, \ldots, r$, we remove $\binom{k-r}{s-r} - 1$ rows. The exact rows removed for the problem identified by $(m^{i_1}, \ldots, m^{i_r})$, are described in the following remark.

Remark 3.3. *Let t^* denote the maximal element of $R_{i_1,\ldots,i_r}$, i.e. $t^* \geq t$ for all $t \in R_{i_1,\ldots,i_r}$ (Def.3.1). For each $t \in R_{i_1,\ldots,i_r} \setminus \{t^*\}$, we remove the row which has $m^j = 1$ for every $j \in t \setminus \{i_1, \ldots, i_r\}$ and $(m^{i_1}, \ldots, m^{i_r})$ in positions $(i_1, \ldots, i_r)$, where $m^{i_p} \neq 1$, $\forall p = 1, \ldots, r$. Note that there is a $1-1$ correspondence between each $t \in R_{i_1,\ldots,i_r} \setminus \{t^*\}$ and each row removed. Obviously, $|R_{i_1,\ldots,i_r} \setminus \{t^*\}| = \binom{k-r}{s-r} - 1$.*

Because there are $(n-1)^r$ such problems, we remove $(n-1)^r \cdot (\binom{k-r}{s-r} - 1)$ rows for each distinct r-set $(i_1, \ldots, i_r)$. There are $\binom{k}{r}$ distinct such r-sets of indices, therefore, the number of rows removed at step r is at most

$$\binom{k}{r} \cdot (n-1)^r \cdot (\binom{k-r}{s-r} - 1) \tag{3.2}$$

Assume w.l.o.g. that row removal is implemented (i) in increasing order with respect to r and (ii) for a certain value of r, in increasing order with respect to row ordering. For example, the step $r = 1$ is preceding step $r = 2$ and the $(k-2, s-2)AP_n$ problems corresponding to the pair of indices (m^1, m^2) are considered before the $(k-2, s-2)AP_n$ problems corresponding to the pair (m^1, m^3). To show that the upper bound defined by (3.2) is attainable, we need to prove the following lemma.

Lemma 3.4.

(i) *No rows of the $(k-r, s-r)\mathrm{AP}_n$ problems indexed by the r-tuple r_0 were removed at a previous step involving an r-tuple $\hat{r}_0$ examined before r_0.*

(ii) *No rows of the $(k-r, s-r)\mathrm{AP}_n$ problems indexed by the r-tuple r_0 were removed at a previous step involving a v-tuple v_0 where $v < r$.*

Proof. To prove (i), let $r_0 = (i_1, \ldots, i_r)$, $\hat{r}_0 = (\hat{\imath}_1, \ldots, \hat{\imath}_r)$ be two distinct r-tuples, i.e. there exists at least one $q \in \{1, \ldots, r\}$, such that $i_q \neq \hat{\imath}_q$. Let $\hat{r}_0$ be considered first and let t be an s-tuple, indexing a row set, such that $r_0, \hat{r}_0 \subseteq t$. If $t = t^*$ (or $t = \hat{t}^*$), where t^* ($\hat{t}^*$) is the maximal element of $R_{i_1,\ldots,i_r}$ ($R_{\hat{\imath}_1,\ldots,\hat{\imath}_r}$ respectively), then no row is removed from row set t w.r.t. r_0 ($\hat{r}_0$). Thus, w.l.o.g. we assume that $t \neq t^*, \hat{t}^*$. This implies that the rows of t belong to the $(k-r, s-r)AP_n$ problems considered for both r_0 and $\hat{r}_0$. Let $m^{\hat{r}_0} = (m^{\hat{\imath}_1}, \ldots, m^{\hat{\imath}_r})$ and $m^{r_0} = (m^{i_1}, \ldots, m^{i_r})$. For each value of $m^{\hat{r}_0}$, where $m^{\hat{\imath}_p} \neq 1$, for all $p = 1, \ldots, r$, a row having $m^j = 1$, for all $j \in t \setminus \hat{r}_0$ is removed (Remark 3.3). This implies that this row has $m^j = 1$ for all $j \in r_0 \setminus (r_0 \cap \hat{r}_0)$ Recall that when considering r_0, only $(k-r, s-r)AP_n$ problems corresponding to values of m^{r_0} with no index equal to 1 are examined. Therefore, all rows removed when considering r-tuples $m^{\hat{r}_0}$ belong to $(k-r, s-r)AP_n$ problems not examined when considering r-tuples m^{r_0}.

The proof of (ii) follows the same idea. Assume $r_0 = (i_1, \dots, i_r)$ and $v_0 = (\hat{\imath}_1, \dots, \hat{\imath}_v)$, $v < r$. There exists at least one index i_t belonging to r_0 but not to v_0. Let also t be an s-tuple indexing a row set such that $r_0, v_0 \subset t$. If a row is deleted when considering the $(k-v, s-v)AP_n$ corresponding to a certain value m_{v_0}, not having any index equal to 1, then all indices of $t \setminus v_0$ will have value 1 (Remark 3.3). This includes the value of the index m^{i_t}, $i_t \in r_0 \setminus v_0$. But then this row would belong to the problem $(k-r, s-r)AP_n$ corresponding to a value of m^{r_0}, where at least one of the indices has value 1. This problem is not examined when considering r_0, since all problems defined have $m^{i_t} \neq 1$. $\square$

(Back to the proof of Theorem 3.2) Summing over all r, we get that the total number of linearly dependent rows of $A_n^{(k,s)}$ is equal to

$$\sum_{r=0}^{s} \left\{ \binom{k}{r} \cdot (n-1)^r \cdot (\binom{k-r}{s-r} - 1) \right\}$$

which yields the following upper bound on the $rankA_n^{(k,s)}$:

$$\begin{aligned} rankA_n^{(k,s)} &\leq \binom{k}{s} \cdot n^s - \sum_{r=0}^{s} \left\{ \binom{k}{r} \cdot (n-1)^r \cdot (\binom{k-r}{s-r} - 1) \right\} \\ &= \binom{k}{s} \cdot \sum_{r=0}^{s} \binom{s}{r} \cdot (n-1)^{s-r} - \sum_{r=0}^{s} \binom{k}{r}\binom{k-r}{s-r} \cdot (n-1)^r \\ &\quad + \sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r \end{aligned}$$

Replacing r by $s-r$ at the first term and using the property $\binom{k}{r}\binom{k-r}{s-r} = \binom{k}{s}\binom{s}{r}$ for the second term, we get

$$\begin{aligned} rankA_n^{(k,s)} &\leq \sum_{r=0}^{s} \binom{k}{s}\binom{s}{r} \cdot (n-1)^r - \sum_{r=0}^{s} \binom{k}{s}\binom{s}{r} \cdot (n-1)^r \\ &\quad + \sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r \end{aligned} \tag{3.3}$$

or

$$rankA_n^{(k,s)} \leq \sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r \tag{3.4}$$

To complete the proof, we need to derive an identical lower bound on the rank$A_n^{(k,s)}$ by exhibiting an equal number of affinely independent columns. Each column is presented using the notation $(m^1, \dots, m^{k-s}, m^{k-s+1}, \dots, m^k)$, i.e. it is important to differentiate between the values of the first $k-s$ indices and the values of the last s indices. Consider the following (disjoint) sets of columns.

a) Columns having the first $k-s$ indices equal to 1 and all possible values for the last s indices in increasing order. These columns are

$$(1, \dots, 1, 1, \dots, 1), ..., (1, \dots, 1, n, \dots, n)$$

There are n^s such columns.

b) Columns having any $k-s-1$ of the first $k-s$ indices equal to 1 and any possible value except for 1 for the remaining index. These columns also have $m_{k-s+1}=1$ and any possible value for the last $s-1$ indices. An example is the subset of columns

$$\begin{array}{rcl} \text{Columns} & : & (\underbrace{1,\ldots,1,2}_{k-s},1,1\ldots,1),\ldots,(1,\ldots,1,2,1,n,\ldots,n) \\ \text{No. of Indices} & : & \end{array}$$

The cardinality of a given subset of columns which has the first $k-s$ indices fixed is n^{s-1}. There are $n-1$ such subsets that have the value 1 in the same $k-s-1$ out of the first $k-s$ indices. There are $k-s$ distinct configurations, for which the $k-s-1$ out of the first $k-s$ indices have value 1 and the remaining index has a value different than one. Thus, the total number of columns in this set is $(k-s)\cdot(n-1)\cdot n^{s-1}$.

c) Columns having any $k-s-1$ of the first $k-s-1+r$ $(2\leq r\leq s-1)$ indices equal to 1 and any possible value, except for 1, for the r remaining indices $m^{i_1},\ldots,m^{i_r},i_q\in\{1,\ldots,k-s-1+r\}$, for all $q\in\{1,\ldots,r\}$. These columns also have $m^{k-s+r}=1$ and any possible value for the last $s-r$ indices. For example, let $r=2$ and let the last two, of the first $k-s-1+2$, indices have values different from 1. Then the subset of columns defined is

$$\begin{array}{rcl} \text{Columns} & : & (\underbrace{1,\ldots,1,2,2}_{k-s-1+2},1,1,\ldots,1),\ldots,(\underbrace{1,\ldots,1,n,n}_{k-s-1+2},1,n,\ldots,n) \\ \text{No. of indices} & : & \end{array}$$

There are $\binom{k-(s-r+1)}{r}$ options for selecting the indices $m^{i_1},\ldots,m^{i_r}$ and, for each such option, $n^{s-r}\cdot(n-1)^r$ columns are included.

d) Columns having any $k-s-1$ of the first $k-1$ indices equal to 1 and any possible value, except for 1, for the s remaining indices $m^{i_1},\ldots,m^{i_s},i_q\in\{1,\ldots,k-1\}$, for all $q\in\{1,\ldots,s\}$. These columns also have $m^k=1$. There are $\binom{k-1}{s}$ options for selecting the indices $m^{i_1},\ldots,m^{i_s}$ and for each option $(n-1)^s$columns are included.

The total number of columns exhibited in (a),...,(d) is

$$\begin{aligned} & \sum_{r=0}^{s}\binom{k-(s-r+1)}{r}\cdot n^{s-r}\cdot(n-1)^r \\ = \; & \sum_{r=0}^{s}\left\{\binom{k-(s-r+1)}{r}\cdot n^{s-r}\cdot\sum_{l=0}^{r}\left\{\binom{r}{l}\cdot(-1)^l\cdot n^{r-l}\right\}\right\} \end{aligned} \tag{3.5}$$

The right-hand side of (3.5) can be written in the form $\sum_{r=0}^{s}n^r\cdot g(r)$. To identify $g(r)$, observe that n^i, for all $i\in\{0,\ldots,s\}$, can be derived as $n^i\cdot n^0$, for $r=s-i$ and $l=r=s-i$. The binomial coefficients of (3.5) will then be $\binom{k-(s-(s-i)+1)}{s-i}=\binom{k-i-1}{s-i}$ and $\binom{s-i}{s-i}\cdot(-1)^{s-i}$. All the combinations that produce n^i (i.e. $n^i\cdot n^0,n^{i-1}\cdot n^1,\ldots,n^0\cdot n^i$), the corresponding values of r and l, as well as the coefficients are illustrated in Table 1.

Table 1: Identifying $g(r)$

l	r	n^i	$\binom{r}{l} \cdot (-1)^{r-i}$	$\binom{k-(s-r+1)}{r}$
$s-i$	$s-i$	$n^i \cdot n^0$	$\binom{s-i}{s-i} \cdot (-1)^{s-i}$	$\binom{k-i-1}{s-i}$
$s-i$	$s-i+1$	$n^{i-1} \cdot n^1$	$\binom{s-i+1}{s-i} \cdot (-1)^{s-i}$	$\binom{k-i}{s-i+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$s-i$	s	$n^0 \cdot n^i$	$\binom{s}{s-i} \cdot (-1)^{s-i}$	$\binom{k-1}{s}$

It follows that $g(r) = (-1)^{s-r} \cdot \sum_{l=s-r}^{s} \binom{k-(s-l+1)}{l} \binom{l}{s-r}$. Thus, (3.5) becomes

$$\begin{aligned} & \sum_{r=0}^{s} \binom{k-(s-r+1)}{r} \cdot n^{s-r} \cdot (n-1)^r \\ = & \sum_{r=0}^{s} n^r \cdot (-1)^{s-r} \cdot \sum_{l=s-r}^{s} \binom{k-(s-l+1)}{l} \binom{l}{s-r} \end{aligned} \tag{3.6}$$

The right-hand side of (3.3) is $\sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r$. Expressing term $(n-1)^r$ using Newton's polynomial, we obtain

$$\begin{aligned} \sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r &= \sum_{r=0}^{s} \binom{k}{r} \cdot \left\{ \sum_{l=0}^{r} \binom{r}{l} \cdot (-1)^l \cdot n^{r-l} \right\} \\ &= \sum_{r=0}^{s} n^r \cdot \sum_{l=r}^{k} \binom{k}{l} \cdot \binom{l}{l-r} \cdot (-1)^{l-r} \end{aligned} \tag{3.7}$$

Equality between the two last terms of (3.7) is proven through an argument analogous with the one used for proving (3.6).

By induction on s and r, it can be shown that the right-hand side of (3.7) is equal to the right-hand side of (3.6). Thus, we obtain the *independent row-column identity* for the assignment problem

$$\sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^r = \sum_{r=0}^{s} \binom{k-(s-r+1)}{r} \cdot n^{s-r} \cdot (n-1)^r$$

The submatrix of $A_n^{(k,s)}$ formed by these columns and the remaining rows is square lower triangular with each diagonal entry equal to 1, therefore non-singular. Hence, these columns are affinely independent. This provides a lower bound on $rankA_n^{(k,s)}$ equal to the upper bound illustrated in (3.4). The proof is now complete. □

Corollary 3.5. $dimP^{(k,s)} = \sum_{r=0}^{k-s-1} \binom{k}{r} \cdot (n-1)^{k-r}$.

Proof. Follows from (2.1) if term n^k is written using Newton's polynomial.

□

4 The (k, s) assignment polytope

Amongst assignment problems, the one with the longest history is the two-index assignment problem. From early on, it was known that $P_I^{(2,1)} \equiv P^{(2,1)}$. This is a direct consequence of Birkhoff's theorem on *permutation* matrices (see [8, Theorem 1.1]). Hence, $dimP_I^{(2,1)} = dimP^{(2,1)}$, which by Corollary 3.5 is equal to $(n-1)^2$. In Table 2, we state the known values of $dimP_I^{(k,s)}$ for given (k,s), other than $(2,1)$. We also state the values of n, for which the proofs are valid and the appropriate references.

Table 2: Known values of $dimP_I^{(k,s)}$.

(k,s)	n	$dimP_I^{(k,s)}$	References
$(k,0), \forall k \in Z_+$	≥ 0	$n^k - 1$	Lemma 2.2
$(3,1)$	≥ 3	$n^3 - 3n + 2$	R. Euler ([10]), E. Balas et al. ([3])
$(3,2)$	≥ 4	$(n-1)^3$	R. Euler et al.([9])
$(4,2)$	$\geq 4, \neq 6$	$n^4 - 6n^2 + 8n - 3$	G. Appa et al.([1])
$(k,k), \forall k \in Z_+$	≥ 0	0	Lemma 2.1

For general (k,s), since $P_I^{(k,s)} \subseteq P^{(k,s)}$, the result obtained in the previous section (Corollary 3.5) implies $dimP_I^{(k,s)} \leq \sum_{r=0}^{k-s-1} \binom{k}{r} \cdot (n-1)^{k-r}$. However, establishing the exact value of $dimP_I^{(k,s)}$ is not as straightforward as in the case of $dimP^{(k,s)}$. Note also that $P_I^{(k,s)} = \emptyset$ for certain values of k, s, n (for example, consider the polytope for $k = 4, s = 2, n = 2$). We will next provide a necessary condition for $P_I^{(k,s)} \neq \emptyset$ and show that the upper bound on the dimension, implied by Corollary 3.5, is attained for $s = 1$, and $s = 2$, for all $k \in \mathbb{Z}_+, k \geq s$, provided that the corresponding polytope is not empty.

Let us first take a closer look at the integer vectors of $P_I^{(k,s)}$. Definition 1.1 implies that n^s k-tuples have to be selected, therefore an integer vector must have exactly n^s variables equal to 1. Moreover, any integer vector must signify n disjoint members of powerset W_{s-1}, each one containing n^{s-1} tuples (i.e. variables). Since any index $m^t \in M_t$, $t \in K$, can be regarded as indexing powerset W_{s-1}, any of its values m_0^t must appear in exactly n^{s-1} non-zero variables. Hence, at an integer point of $P_I^{(k,s)}$, each element of any of the sets $M_1, \ldots, M_k$ appears in exactly n^{s-1} variables set to one. This remark is very useful in terms of providing a mechanism of transition from one integer point to another. Consider an arbitrary integer point $x \in P_I^{(k,s)}$, and a pair of values $m_0^t, m_1^t \in M_t, t \in K$. We set $m^t = m_1^t$ for all the $(k-s+1)$-tuples with $m^t = m_0^t$ and also $m^t = m_0^t$ for all the $(k-s+1)$-tuples with $m^t = m_1^t$. The derived structure corresponds to another integer point $\hat{x} \in P_I^{(k,s)}$, since there are again exactly n^{s-1} "new" variables set to one with $m^t = m_0^t$ and n^{s-1} "new" variables set to one with $m^t = m_1^t$. The variables, with $m^t = m_0^t$ and $m^t = m_1^t$, set to one at point x are set to zero at point $\hat{x}$ and vice versa. The values for the rest of the variables remain the same at both points.

This notion of interchanging the role of two index values is formalised with the introduction of the *interchange* operator $(\leftrightarrow)$. Hence, by setting $\hat{x} = x(m_0^t \leftrightarrow m_1^t)_t$, we imply that, at point x, we interchange the index values m_0^t and m_1^t, thus deriving point $\hat{x}$. The subscript indexing the brackets denotes the set that the interchanged elements belong to. A series of interchanges at a point $x \in P_I^{(k,s)}$ is expressed by using the operator $(\leftrightarrow)$ as many times as the number of interchanges with priority from left to right. For example, $\hat{x} = x(m_0^1 \leftrightarrow m_1^1)_1(1 \leftrightarrow m_1^2)_2$ implies that at point x we interchange m_0^1 and m_1^1, while at the derived point we interchange 1 and m_1^2.

We proceed one step further and define the *conditional interchange*, which implies that the interchange is performed only if a certain condition is met. Usually, the condition is a logical expression involving comparisons of index values. The interchange is performed only if the logical expression evaluates *true*. Since we are only going to perform conditional interchanges, for which both the logical expression and the interchange refer to the elements of a single set, we will use a common subscript for both. For example, $\hat{x} = x(m_0^t = n?m_1^t \leftrightarrow 1)_t$ implies that at point x we apply the interchange between m_1^t and 1 only if $m_0^t = n$; otherwise, $\hat{x} = x$. The interchange operator was originally introduced in [1] for integer points of $P_I^{(4,2)}$.

An important issue is, given (k, s, n), whether $P_I^{(k,s)} \neq \emptyset$. This question is difficult to answer for general (k, s, n). However, for certain values of s, we know that $P_I^{(k,s)} \neq \emptyset$. We have seen two such trivial cases in Section 2.2. Another such case is obtained for $s = 1$. It is easy to see that $P_I^{(k,1)} \neq \emptyset$, $\forall k, n \in \mathbb{Z}_+$. This is because the *diagonal* solution, noted as x^{diag} $(x_{11\cdots 1} = \cdots = x_{nn\cdots n} = 1)$, always satisfies constraints (1.5), (1.6), for $s = 1$. For general $s \geq 2$, we provide the following necessary condition for $P_I^{(k,s)}$ to be non-empty.

Proposition 4.1. *For $s \geq 2$, a necessary condition for the existence of a solution to the $(k, s)\mathrm{AP}_n$ is $k \leq n + s - 1$.*

Proof. Observe first that, according to (1.5), variables appear at the same row if and only if they have at least s indices in common (common values for indices belonging to m^S).

Given that $P_I^{(k,s)} \neq \emptyset$ and $s \geq 2$, consider the point $x' \in P_I^{(k,s)}$ having $x_{1\cdots 1m_1^s\cdots m_1^k} = x_{1\cdots 1m_2^s\cdots m_2^k} = \cdots = x_{1\cdots 1m_n^s\cdots m_n^k} = 1$. Such a point always exists, since rows $(1, \ldots, 1, m_j^s) \in M_1 \times M_2 \cdots \times M_s$, $j = 1, \ldots, n$ must have exactly one variable equal to 1. Also note that all indices $m_j^t, j = 1, \ldots, n$, must be pairwise different since otherwise a constraint would have a left-hand side equal to 2. Consider the following sequence of points:

$$
\begin{aligned}
x^s &= x'(m_1^s \neq 1?m_1^s \leftrightarrow 1)_s \cdots (m_n^s \neq n?m_n^s \leftrightarrow n)_s \\
x^{s+1} &= x^s(m_1^{s+1} \neq 1?m_1^{s+1} \leftrightarrow 1)_{s+1} \cdots (m_n^{s+1} \neq n?m_n^{s+1} \leftrightarrow n)_{s+1} \\
&\vdots \\
x^k &= x^{k-1}(m_1^k \neq 1?m_1^k \leftrightarrow 1)_k \cdots (m_n^k \neq n?m_n^k \leftrightarrow n)_k
\end{aligned}
$$

Let $x = x^k$. At point x we have $x_{1\cdots 11\cdots 1} = x_{1\cdots 12\cdots 2} = \cdots = x_{1\cdots 1n\cdots n} = 1$. The notation implies that the first $s-1$ indices are equal to 1 and the indices $s, \ldots, k$ have the same value. Point x must satisfy all constraints including row $(2, 1, \ldots, 1) \in M_1 \times M_2 \cdots \times M_s$. Consequently, there must be exactly one variable of the form $x_{21\cdots 1m^{s+1}\cdots m^k}$ with value 1. None of the indices $m^{s+1}, \ldots, m^s$ can take the value 1, because the left-hand side of a constraint would then become 2. For the same reason they must be pairwise different. Therefore, at most $n-1$ values must be allocated to $k - s$ indices. This implies $k - s \leq n - 1 \Leftrightarrow k \leq s + n - 1$. $\square$

For $s = 2$, this theorem states the known fact that there can exist no more than $n - 1$ *MOLS* of order n ([17, Theorem 2.1]).

Let us explore the relationship between the $(k, s)AP_n$ and the problems arising by decreasing by one any of the parameters k, s. Recall first that a member of powerset W_s is equivalent to n disjoint members of powerset W_{s-1}, i.e. to n sets of n^{s-1} $(k - s + 1)$ tuples each. It follows that a solution of $(k, s)AP_n$ is equivalent to n disjoint solutions of $(k-1, s-1)AP_n$. Equivalently, a vector $x \in P_I^{(k,s)}$ can be used to construct n linearly independent vectors $\bar{x}^i \in P_I^{(k-1,s-1)}$, $i = 1, \ldots, n$. The construction is quite straightforward: for a certain index $m^t, t \in K$, and $x \in P_I^{(k,s)}$, define $\bar{x}_c^i = x_{\varphi_{k,k-1}(c,m_i^t)}$ for all $c \in M_{K\setminus\{M_t\}}$, $i = 1, \ldots, n$. It is easy to verify that $\bar{x}^i \in P_I^{(k-1,s-1)}$, for $i = 1, \ldots, n$.

In an analogous way, a solution of $(k, s)AP_n$ implies a solution to $(k-1, s)AP_n$. Observe that the $(k-1, s)AP_n$ requires an integer solution with n^s variables set to one, but with one index less (i.e. n^s $(k - s)$ tuples). Let the index dropped be $m^t \in M_t, t \in K$. Assuming $x \in P_I^{(k,s)}$, a vector $x' \in P_I^{(k-1,s)}$ is derived as follows:

$$x'_c = \left\{ \begin{array}{l} 1, \text{ if } \exists\, m^t \in M_t : x_{\varphi_{k,k-1}(c,m_i^t)} = 1 \\ 0, \text{ otherwise} \end{array} \right\}, \forall c \in M_{K \setminus \{M_t\}}$$

In contrast, a solution of $(k, s)AP_n$ cannot give rise to a solution of $(k, s-1)AP_n$ and vice versa. Note also that the inverse of the above is not true, i.e. a single solution to either $(k-1, s-1)AP_n$ or $(k-1, s)AP_n$ is not always extendable to a solution of the $(k, s)AP_n$. The hierarchy implied by these observations is depicted in Figure 1.

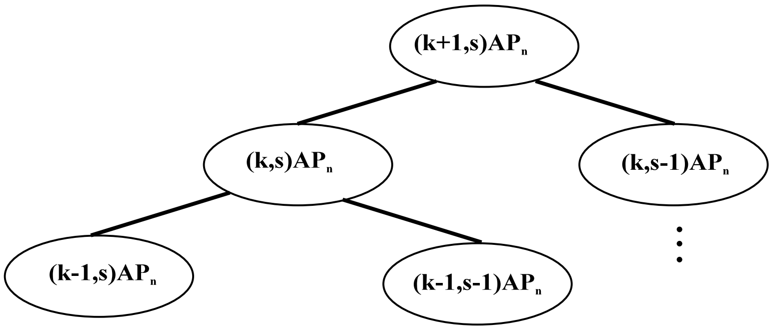

Figure 1: A hierarchy of assignment polytopes

The only exception is the case of $(k-1, 1)AP_n$, whose solutions can always be extended to solutions of $(k, 1)AP_n$. Observe that a solution to $(k-1, 1)AP_n$ is a collection of n disjoint $(k-1)$-tuples, whereas a solution to $(k-1, 1)AP_n$ requires a collection of n disjoint k-tuples. By defining an additional n-set M_k, adding a different element $m^k \in M_k$ to each of the $(k-1)$-tuples constituting the solution of the $(k-1, 1)AP_n$, results in a solution of the $(k, 1)AP_n$. This observation has obvious algorithmic implications: instead of solving the $(k, 1)AP_n$ directly (especially for large values of k), one can solve a lower dimensional axial assignment problem $(k', 1)AP_n$, for $k' << k$, and, consequently, extend the solution to a solution of the higher dimensional problem. This algorithmic aspect is exploited in [24], within a scheme based on Langrangian relaxation. It also appears in [6], the authors of which propose simple heuristics, which extend a solution of $(2, 1)AP_n$ to a solution of $(k, 1)AP_n$ by sequentially solving $(k-1)$ bipartite matchings. *Branch and bound* methods, incorporating this characteristic to extract lower bounds on the optimal solution, are proposed for the $(3, 1)AP_n$ and the $(3, 2)AP_n$ in [5] and [18], respectively. The principle is also applicable for $s \geq 2$, the difference being that solutions of $(k-1, s)AP_n$ might not be extendable to solutions of $(k, s)AP_n$. A simple case is that of pairs of *MOLS*, i.e. solutions to $(4, 2)AP_n$, which are not always extendable to triples of *MOLS*, i.e. solutions of $(5, 2)AP_n$.

A statement summarising the above, in terms of polyhedra, is the following.

Remark 4.2. *proj*$_{x \in P_I^{(k-1,1)}}(P_I^{(k,1)}) = P_I^{(k-1,1)}$ *and*

proj$_{x \in P_I^{(k-1,s)}}(P_I^{(k,s)}) \subset P_I^{(k-1,s)}$, *for* $s \geq 2$

5 The axial assignment polytopes

The two most prominent representatives of this (sub)class are the polytopes $P_I^{(2,1)}$and $P_I^{(3,1)}$. We have seen that $dimP_I^{(2,1)} = (n-1)^2$. Other basic properties of $P_I^{(2,1)}$ can be found in [8, Theorem 1.2]. The facial structure of $P_I^{(3,1)}$ has also been substantially studied. As mentioned earlier, the dimension of this polytope is established, independently, in [3, 9]. Several classes of facets, induced by cliques and odd holes of the underlying intersection graph, are identified in [3, 27]. Separation algorithms, for some of these classes, are given in [4].

To the best of our knowledge, $(2,1)AP_n$ and $(3,1)AP_n$ are the only axial assignment problems, whose underlying polyhedral structure has been studied. However, several applications of axial assignment problems, for $k > 3$, have been reported (see [8, 22] for a collection of applications). This suggests that the study of $P_I^{(k,1)}$, for general k, is of practical as well as of theoretical interest. We have already mentioned that $P_I^{(k,1)} \neq \emptyset, \forall k, n \in \mathbb{Z}_+$. Further, it can be proved, by induction on n, that the number of vertices of $P_I^{(k,1)}$ is equal to $(n!)^{k-1}$. Next, we establish the dimension of $P_I^{(k,1)}$, thus unifying and generalising the corresponding results obtained for $P_I^{(2,1)}$ and $P_I^{(3,1)}$.

Theorem 5.1. *For* $n \geq 3,\ dimP_I^{(k,1)} = \sum_{r=0}^{k-2} \binom{k}{r} \cdot (n-1)^{k-r}$.

Proof. From Corrollary 3.5, we know that $dimP_I^{(k,1)} \leq \sum_{r=0}^{k-2} \binom{k}{r} \cdot (n-1)^{k-r}$. Strict inequality holds if and only if there exists an equality $ax = a_0$ satisfied by all $x \in P_I^{(k,1)}$, which cannot be expressed as a linear combination of the equality constraints $A_n^{(k,1)}x = e$. Proving that no such equality exists, essentially proves that the system $A_n^{(k,1)}x = e$ is the minimum equality system for both $P_I^{(k,1)}$ and $P^{(k,1)}$, i.e. $dimP_I^{(k,1)} = dimP^{(k,1)}$.

Assume that every $x \in P_I^{(k,1)}$ satisfies the equality $ax = a_0$ for some $a \in \mathbb{R}^{n^k}$ and $a_0 \in \mathbb{R}$. Then there exist scalars $\lambda_{m^1}^k, \lambda_{m^2}^{k-1}, ..., \lambda_{m^k}^1, \forall m^1 \in M_1, \ldots, m^k \in M_k$, satisfying

$$a_{m^1m^2\cdots m^k} = \lambda_{m^1}^k + \lambda_{m^2}^{k-1} + \cdots + \lambda_{m^k}^1, \forall m^t \in M_t, t \in K \tag{5.1}$$

$$a_0 = \sum\{\lambda_{m^1}^k + \lambda_{m^2}^{k-1} + \cdots + \lambda_{m^k}^1 : (m^1, m^2, ..., m^k) \in M_K\} \tag{5.2}$$

We define:

$$\begin{array}{lcl} \lambda_{m^k}^1 & = & a_{1\cdots 1m^k} \\ \lambda_{m^{k-1}}^2 & = & a_{1\cdots 1m^{k-1}1} - a_{1\cdots 11} \\ \lambda_{m^{k-2}}^3 & = & a_{1\cdots 1m^{k-2}11} - a_{1\cdots 11} \\ \vdots & & \\ \lambda_{m^1}^k & = & a_{m^11\cdots 1} - a_{1\cdots 11} \end{array}$$

Note that the superscript of λs is given by the function defined in (3.1).

By substituting in (5.1), we get

$$a_{m^1m^2\cdots m^k} = a_{m^11\cdots 1} + a_{1m^21\cdots 1} + \cdots + a_{11\cdots 1m^k} - (k-1)a_{1\cdots 1} \tag{5.3}$$

By observing that $a_{1\cdots 1m^t1\cdots 1} = a_{\varphi_{k,1}((m^{i_t}),(1,\ldots,1))}$ for all $t \in K$, we re-write (5.3) as

$$a_{m^1m^2\cdots m^k} = \sum\{a_{\varphi_{k,1}((m^{i_t}),(1,\ldots,1))} : t \in K\} - (k-1)a_{1\cdots 1} \tag{5.4}$$

Observe that (5.4) is true for every k-tuple having at least $k-1$ indices equal to one.

For $n \geq 3$, consider the point $x[q] \in P_I^{(k,1)}$ which has $x_{1\cdots 1}, x_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_0^{i_q}),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))}$, where $2 \leq q \leq k$ and $1, m_0^t, m_1^t$ are three distinct elements of M_t, for all $t \in K$. We can derive $x[q]$ as follows. At x^{diag}, let x_v $(v = (m_v^1, \ldots, m_v^k))$ denote a variable set to one, other than $x_{1\cdots 1}$. Then, $x[q] = x^{diag}(m_v^{i_1} \neq m_1^{i_1} ? m_v^{i_1} \leftrightarrow m_1^{i_1})_{i_1} \cdots (m_v^{i_{q-1}} \neq m_1^{i_{q-1}} ? m_v^{i_{q-1}} \leftrightarrow m_1^{i_{q-1}})_{i_{q-1}} (m_v^{i_q} \neq m_0^{i_q} ? m_v^{i_q} \leftrightarrow m_0^{i_q})_{i_q} \cdots (m_v^{i_k} \neq m_0^{i_k} ? m_v^{i_k} \leftrightarrow m_0^{i_k})_{i_k}$

Let $x'[q] = x[q](1 \leftrightarrow m_0^{i_q})_{i_q}$. At $x'[q]$ the two, previously exhibited, variables are set to zero. In their place, we have $x'_{\varphi_{k,q}((1,\ldots,1,m_0^{i_q}),(1,\ldots,1))} = x'_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))} = 1$. Since $x[q], x'[q] \in P_I^{(k,1)}$, both points satisfy $ax = a_0$. Hence, $ax[q] = ax'[q]$. This equation, after canceling out identical terms, yields

$$\begin{aligned} & a_{1\cdots 1} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_0^{i_q}),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))} \\ = \; & a_{\varphi_{k,q}((1,\ldots,1,m_0^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))} \end{aligned} \tag{5.5}$$

Consider points $\bar{x}[q] = x'[q](m_0^{i_1} \leftrightarrow 1)_{i_1} \cdots (m_0^{i_{q-1}} \leftrightarrow 1)_{i_{q-1}}$ and $\bar{x}'[q] = \bar{x}[q](m_0^{i_q} \leftrightarrow 1)_{i_q}$. Since $\bar{x}[q], \bar{x}'[q] \in P_I^{(k,1)}$, both points satisfy $ax = a_0$. Hence, $a\bar{x}[q] = a\bar{x}'[q]$, after canceling out equivalent terms, yields

$$\begin{aligned} & a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_{q-1}},m_0^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))} \\ = \; & a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_{q-1}},1),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_0^{i_q}),(m_0^{i_{q+1}},\ldots,m_0^{i_k}))} \end{aligned} \tag{5.6}$$

Adding equations (5.5) and (5.6) results in

$$a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_q}),(1,\ldots,1))} = a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_{q-1}},1),(1,\ldots,1))} + a_{\varphi_{k,q}((1,\ldots,1,m_0^{i_q}),(1,\ldots,1))} - a_{1\cdots 1} \tag{5.7}$$

Equation (1.2) implies

$$\begin{aligned} a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_{q-1}},1),(1,\ldots,1))} &= a_{\varphi_{k,q-1}((m_0^{i_1},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))} \\ a_{\varphi_{k,q}((1,\ldots,1,m_0^{i_q}),(1,\ldots,1))} &= a_{\varphi_{k,1}((m_0^{i_q}),(1,\ldots,1))} \end{aligned}$$

Hence, (5.7) becomes

$$a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_q}),(1,\ldots,1))} = a_{\varphi_{k,q-1}((m_0^{i_1},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))} + a_{\varphi_{k,1}((m_0^{i_q}),(1,\ldots,1))} - a_{1\cdots 1} \tag{5.8}$$

Consider the recurrence relation (5.8) for $q = k$. By recursively substituting term $a_{\varphi_{k,q-1}((m_0^{i_1},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))}$ until $q = 2$, we obtain equation (5.4) for $m^t = m_0^t \in M_t \setminus \{1\}$, $t \in K$. To prove (5.4) for a k-tuple with r $(1 \leq r \leq k-2)$ indices equal to one, we perform the recursive step starting from $q = k - r$. This proves equation (5.1).

To prove equation (5.2), consider $ax^{diag} = a_0$:

$$a_0 = a_{1\cdots 1} + a_{2\cdots 2} + \cdots + a_{n\cdots n}$$

or (by substituting all terms, except $a_{1\cdots 1}$, from (5.3))

$$\begin{aligned} a_0 &= a_{1\cdots 1} + [a_{21\cdots 1} + a_{121\cdots 1} + \cdots + a_{11\cdots 2} - (k-1)a_{1\cdots 1}] \\ &\quad + \cdots + [a_{n1\cdots 1} + a_{1n1\cdots 1} + \cdots + a_{11\cdots n} - (k-1)a_{1\cdots 1}] \\ &= [(a_{21\cdots 1} - a_{1\cdots 1}) + \cdots + (a_{n1\cdots 1} - a_{1\cdots 1})] \\ &\quad + \cdots + [(a_{1\cdots 21} - a_{1\cdots 1}) + \cdots + (a_{1\cdots n1} - a_{1\cdots 1})] \\ &\quad + (a_{1\cdots 1} + a_{1\cdots 12} + \cdots + a_{1\cdots 1n}) \end{aligned}$$

Equivalently (substituting from the equations defining λs)

$$a_0 = \sum\{\lambda^k_{m^1} + \lambda^{k-1}_{m^2} + \cdots + \lambda^1_{m^k} : (m^1, m^2, ..., m^k) \in M_K\}$$

This completes the proof. $\square$

Let us examine which of the faces induced by the constraints of $P^{(k,1)}$ constitute facets of $P_I^{(k,1)}$. All the equality constraints (i.e. equalities (1.5)) are satisfied by all points of $P_I^{(k,1)}$, therefore they define improper faces of $P_I^{(k,1)}$. For $c \in M_K$, the inequalities $x_c \geq 0$ define facets of $P_I^{(k,1)}$; in contrast, the inequalities $x_c \leq 1$ are redundant, i.e. implied by the original constraint set.

Proposition 5.2. *For $n \geq 3,$ every inequality $x_c \geq 0,$ for $c \in M_K$, defines a facet of $P_I^{(k,1)}$.*

Proof. For any $c \in M_K$, consider the polytope $P_I^{(k,1)c} = \{x \in P_I^{(k,1)} : x_c = 0\}$. It is sufficient to show that $dimP_I^{(k,1)c} = dimP_I^{(k,1)} - 1$. Evidently, $dim \leq n^k - 1 - rankA_n^{(k,1)c}$ where $A_n^{(k,1)c}$ is the matrix obtained from $A_n^{(k,1)}$ by removing column c. It is not difficult to see that the rank of $A_n^{(k,1)c}$is equal to the rank of $A_n^{(k,1)}$. This is immediate, if the column a^c is not among the columns of the upper triangular matrix described in Theorem 3.2, otherwise it follows by symmetry. Therefore, $dimP_I^{(k,1)c} \leq \sum_{r=0}^{k-2} \binom{k}{r} \cdot (n-1)^{k-r} - 1$. To prove that this bound is attained, we use the same approach as in the proof of Theorem 5.1, i.e. show that any equation $ax = a_0$ (different than $x_c = 0$) satisfied for every $x \in P_I^{(k,1)c}$ is a linear combination of the system $A_n^{(k,1)c}x = e$. The proof goes through essentially unchanged. $\square$

Proposition 5.3. *For $n \geq 3,$ every inequality $x_c \leq 1,$ for $c \in M_K$, does not define a facet of $P_I^{(k,1)}$.*

Proof. For any $c \in M_K$ consider the polytope $P_I^{(k,1)c} = \{x \in P_I^{(k,1)} : x_c = 1\}$. We will show that $dimP_I^{(k,1)c} < dimP_I^{(k,1)} - 1$. We know already that $dimP_I^{(k,1)c} < dimP^{(k,1)c}$, where $P^{(k,1)c}$ is the LP-relaxation of $P_I^{(k,1)c}$. Setting x_c to one is equivalent to setting the variables, which appear at the same constraints with x_c, to zero. Note that a variable x_d appears at the same constraint with x_c, if and only if it has at least one index in common with x_c, i.e. if and only if $|c \cap d| \geq 1$. The number of variables x_d, such that $|c \cap d| = 0$ is $(n-1)^k$. It follows that the number of variables having at least one index in common with x_c, i.e. set to zero, if $x_c = 1$, is $n^k - (n-1)^k$.

Hence, $dimP^{(k,1)c} = n^k - [n^k - (n-1)^k] - rankA_n^{(k,1)c}$ where $A_n^{(k,1)c}$ is the matrix obtained from $A_n^{(k,1)}$ by removing the columns corresponding to the variables set to zero. Obviously, $rankA_n^{(k,1)c} \leq rankA_n^{(k,1)}$. It follows that

$$dimP^{(k,1)c} \leq n^k - [n^k - (n-1)^k] - rankA_n^{(k,1)} \leq dimP_I^{(k,1)} - [n^k - (n-1)^k]$$

Taking into account that $n^k - (n-1)^k > 1$ for $n \geq k \geq 3$, it holds that $dimP^{(k,1)c} < dimP_I^{(k,1)} - 1$. The result follows. $\square$

6 A family of facets for the axial assignment polytope

Let C denote the index set of columns of the $0-1$ matrix A. We refer to a column of the A matrix as a^c for $c \in C$. The *intersection* graph $G_A = (V, E)$, has a node c for every $a^c \in A$ and an edge $(c_s, c_t) \in E$ for every pair of nodes with $a^{c_s} \cdot a^{c_t} \geq 1$. Let $G_A = (C^{(k,1)}, E^{(k,1)})$ denote the intersection graph of $A_n^{(k,1)}$. Then, $C^{(k,1)} \equiv M_K$ and $(c_s, c_t) \in E^{(k,1)}$ for all $c_s, c_t \in C^{(k,1)}$ with $|c_s \cap c_t| \geq 1$. By definition, $c \in C^{(k,1)}$ refers to the k-tuple $(m_c^1, \ldots, m_c^k) \in M_K$. Hence, the variable x_c is equivalently denoted as $x_{(m_c^1,\ldots,m_c^k)}$.

Proposition 6.1. *The graph $G_A = (C^{(k,1)}, E^{(k,1)})$ is regular of degree $\sum_{t=1}^{k-1} \binom{k}{t}(n-1)^{k-t}$.*

Proof. For each $c_0 \in C^{(k,1)}$, there are exactly $(n-1)^k$ nodes with no index in common with c_0. Since c_0 is incident to all other nodes of G, the degree of node c_0 is $n^k - (n-1)^k - 1 = \sum_{t=0}^{k} \binom{k}{t}(n-1)^{k-t} - (n-1)^k - 1 = \sum_{t=1}^{k-1} \binom{k}{t}(n-1)^{k-t}$. □

Corollary 6.2. $\left|E^{(k,1)}\right| = \frac{n^k(n^k-(n-1)^k-1)}{2}$.

Proof. The number of edges is equal to the sum of the degrees of all nodes divided by 2. □

For $c_0 \in C^{(k,1)}$, define

$$Q^1(c_0) = \{c_t \in C^{(k,1)} : |c_0 \cap c_t| \geq \left\lfloor \frac{k}{2} \right\rfloor + 1\}$$

Definition 6.3. *Consider $c_0, c_t \in C^{(k,1)}$, with $|c_0 \cap c_t| \geq 1$. The complement of c_t w.r.t. c_0 is any node $\bar{c}_t(c_0) \in C^{(k,1)}$ such that $|c_t \cap \bar{c}_t(c_0)| = 0,\ |c_0 \cap c_t| + |c_0 \cap \bar{c}_t(c_0)| = k$.*

Note that $(c_t, \bar{c}_t(c_0)) \notin E^{(k,1)}$. For the rest of the section, assume $k \geq 3$. A *clique* is defined as a *maximal* complete subgraph.

Proposition 6.4. *For each $c_0 \in C^{(k,1)}$ and k odd, the node set $Q^1(c_0)$ induces a clique. There are n^k cliques of this type.*

Proof. Let $c_1, c_2 \in Q^1(c_0)$. Since both c_1, c_2 have $\lfloor \frac{k}{2} \rfloor + 1$ indices common with c_0, they must have at least one index in common with each other. It follows that $(c_1, c_2) \in E^{(k,1)}$. To show that $Q^1(c_0)$ is also maximal, consider $c_3 \in C^{(k,1)} \setminus Q^1(c_0)$. This implies $|c_0 \cap c_3| \leq \lfloor \frac{k}{2} \rfloor$. Because of k being odd, it holds that $|c_0 \cap \bar{c}_3(c_0)| \geq \lfloor \frac{k}{2} \rfloor + 1$, implying $\bar{c}_3(c_0) \in Q^1(c_0)$. Since $(c_3, \bar{c}_3(c_0)) \notin E^{(k,1)}$, the graph induced by $Q^1(c_0) \cup \{c_3\}$ is not complete, i.e. a contradiction. There is a distinct clique of this type for each node of $C^{(k,1)}$, i.e. the total number of cliques is n^k. □

Observe that $\left|Q^1(c_0)\right| = \sum_{t=\lfloor \frac{k}{2} \rfloor+1}^{k} \binom{k}{t} \cdot (n-1)^{k-t}$.

For k even, it can be verified that $Q^1(c_0)$ is not maximal, i.e it has to be augmented by introducing additional nodes. To describe the set of these nodes, a number of intermediate definitions is necessary. For $c_0 \in C^{(k,1)}$ and $S \subset K$, define:

$$C_S^{(k,1)}(c_0) = \{c_t \in C^{(k,1)} : (c_0 \cap c_t) \in M_S\}$$

It is easy to verify that $|c_t \cap c_u| = 0$ for all $c_t \in C_S^{(k,1)}(c_0), c_u \in C_{K\setminus S}^{(k,1)}(c_0)$. For k even, define $G = \left\{S \subset K : |S| = \frac{k}{2}\right\}$, $|G| = \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix}$. Observe that, for k even, $S \in G$ if and only if $K \setminus S \in G$. Therefore, the set G can be partitioned into sets G^+, G^-, such that $G = G^+ \cup G^-$ and $S \in G^+$ if and only if $K \setminus S \in G^-$. There are $2^{\frac{|G|}{2}}$ such partitions. Define, finally, the set of nodes

$$Q_{G^+}^2(c_0) = \cup\{C_S^{(k,1)}(c_0), S \in G^+\}$$

The node set $Q^2_{G^-}(c_0)$ is defined analogously. It follows that and $c_t \in Q^2_{G^+}(c_0)$ if and only if $\bar{c}_t(c_0) \in Q^2_{G^-}(c_0)$.

Proposition 6.5. *For each $c_0 \in C^{(k,1)}$ and k even, the node set $Q^{1,2}(c_0) = Q^1(c_0) \cup Q^2_{G^+}(c_0)$ induces a clique.*

Proof. Let $c_1, c_2 \in Q^{1,2}(c_0)$. If both $c_1, c_2 \in Q^1(c_0)$ or both $c_1, c_2 \in Q^2_{G^+}(c_0)$, it is easy to verify that $|c_1 \cap c_2| \geq 1$, i.e $(c_1, c_2) \in E^{(k,1)}$. If $c_1 \in Q^1(c_0)$, $c_2 \in Q^2_{G^+}(c_0)$, c_1 has $\frac{k}{2}+1$ indices in common with c_0 and c_2 has $\frac{k}{2}$ indices in common with c_0. Because of k being even, it follows that c_1, c_2 have at least one index in common.

To show that $Q^{1,2}(c_0)$ is maximal, consider $c_3 \in C^{(k,1)} \setminus Q^{1,2}(c_0)$ and note that either $|c_3 \cap c_0| \leq \frac{k}{2} - 1$ or $|c_3 \cap c_0| = \frac{k}{2}$. The first case implies that $|\bar{c}_3(c_0) \cap c_0| \geq \frac{k}{2} + 1$, i.e. $\bar{c}_3(c_0) \in Q^1(c_0)$. Hence, the graph induced by $Q^{1,2}(c_0) \cup \{c_3\}$ is not complete. In the second case, clearly, $c_3 \notin Q^2_{G^+}(c_0)$. Hence, there exists at least one element of $Q^2_{G^+}(c_0)$, namely c_4, such that $|c_3 \cap c_4| = 0$, i.e. the graph induced by $Q^{1,2}(c_0) \cup \{c_3\}$ is not complete. $\square$

For k even, it can be verified that:

$$\left|Q^{1,2}(c_0)\right| = \sum_{t=\frac{k}{2}+1}^{k} \binom{k}{t} \cdot (n-1)^{k-t} + \frac{1}{2} \cdot \binom{k}{\frac{k}{2}} \cdot (n-1)^{\frac{k}{2}}$$

Theorem 6.6. *For $k \geq 3$ and odd, $n \geq 4$ and $c \in C^{(k,1)}$, the inequality*

$$\sum \{x_q : q \in Q^1(c)\} \leq 1 \tag{6.1}$$

defines a facet of $P_I^{(k,1)}$.

Proof. Since $Q^1(c)$ is the node set of a clique of $G_A(C^{(k,1)}, E^{(k,1)})$ then the inequality (6.1) defines a facet for $\tilde{P}_I^{(k,1)}$. Since $P_I^{(k,1)} \subset \tilde{P}_I^{(k,1)}$ it follows that (6.1) is also valid for all $x \in P_I^{(k,1)}$.

Define $P_I^{(k,1)}(Q^1(c)) = \{x \in P_I^{(k,1)} : \sum\{x_q : q \in Q^1(c)\} = 1\}$.

To show that $P_I^{(k,1)}(Q^1(c)) \neq \emptyset$, consider an arbitrary point $x \in P_I^{(k,1)}$. Let x_v be one of the variables set to one at this point. Let $q = |v \cap c|$. If $q \geq \lfloor \frac{k}{2} \rfloor + 1$, $x \in P_I^{(k,1)}(Q^1(c))$. Therefore, assume that $q \leq \lfloor \frac{k}{2} \rfloor$. Let w.l.o.g. $v = \varphi_{k,q}((m_c^{i_1}, \ldots, m_c^{i_q}), (m_v^{i_{q+1}}, \ldots, m_v^{i_k}))$, where $m_v^{i_t} \neq m_c^{i_t}$ for $t = q+1, \ldots, k$. Derive point $\tilde{x} = x(m_v^{i_{q+1}} \leftrightarrow m_c^{i_{q+1}})_{i_{q+1}} \cdots (m_v^{i_k} \leftrightarrow m_c^{i_k})_{i_k}$. Since $\tilde{x}_c = 1$, $\tilde{x} \in P_I^{(k,1)}(Q^1(c))$.

To show that $P_I^{(k,1)}(Q^1(c)) \neq P_I^{(k,1)}$, consider an arbitrary point $x \in P_I^{(k,1)}(Q^1(c))$ with $x_c = 1$. At this point, for $n \geq 3$, let x_u, x_w be two variables set-to-one other than x_c. Let $\hat{x} = x(m_c^1 \leftrightarrow m_u^1)_1 \cdots (m_c^{\lfloor \frac{k}{2} \rfloor} \leftrightarrow m_u^{\lfloor \frac{k}{2} \rfloor})_{\lfloor \frac{k}{2} \rfloor}$ $(m_c^{\lfloor \frac{k}{2} \rfloor+1} \leftrightarrow m_w^{\lfloor \frac{k}{2} \rfloor+1})_{\lfloor \frac{k}{2} \rfloor+1}$. $\hat{x} \in P_I^{(k,1)} \setminus P_I^{(k,1)}(Q^1(c))$ since there are two variables, each with a distinct set of $\lfloor \frac{k}{2} \rfloor$ indices from c and one variable with the remaining index from c.

Assume w.l.o.g. that $c = c_n = (n, n, \ldots, n)$. To show that $P_I^{(k,1)}(Q^1(c_n))$ is a facet of $P_I^{(k,1)}$, we will exhibit scalars $\pi, \lambda^1_{m^k}, \lambda^2_{m^{k-1}}, \ldots, \lambda^k_{m^1}, \forall m^1 \in M_1, \ldots m^k \in M_k$ such that if $ax = a_0$ for all $x \in P_I^{(k,1)}$, then

$$a_{m^1 m^2 \cdots m^k} = \left\{ \begin{array}{ll} \lambda^1_{m^k} + \lambda^2_{m^{k-1}} + \cdots + \lambda^k_{m^1}, & (m^1, m^2, \ldots, m^k) \in C^{(k,1)} \setminus Q^1(c_n) \\ \lambda^1_{m^k} + \lambda^2_{m^{k-1}} + \cdots + \lambda^k_{m^1} + \pi, & (m^1, m^2, \ldots, m^k) \in Q^1(c_n) \end{array} \right\} \tag{6.2}$$

and

$$a_0 = \pi + \sum_{t=1}^{k} \left\{ \sum_{m^{k+1-t} \in M_{k+1-t}} \lambda^t_{m^{k+1-t}} \right\} \tag{6.3}$$

We define the scalars $\lambda^1_{m^k}, \ldots, \lambda^k_{m^1}$ as in Theorem 5.1.

Substituting the left-hand side from (6.2), for $(m^1, m^2, \ldots, m^k) \in C^{k,1} \setminus Q^1(c_n)$, we obtain

$$a_{m^1 m^2 \cdots m^k} = \sum_{t=1}^{k} a_{\varphi_{k,1}((m^t),(1,\ldots,1))} - (k-1)a_{11\cdots 1}$$

or equivalently

$$a_{m^1 m^2 \cdots m^k} = \sum_{t=1}^{k} a_{\varphi_{k,1}((m^{i_t}),(1,\ldots,1))} - (k-1)a_{11\cdots 1} \tag{6.4}$$

For $n \geq 4$ there exist $m_0^{i_t}, m_1^{i_t} \in M_{i_t} \setminus \{1, n\}, \forall t \in K$.

Consider $d \in C^{(k,1)}$. For d to belong to $C^{(k,1)} \setminus Q^1(c_n)$, we must have $0 \leq r = |d \cap c_n| \leq \left\lfloor \frac{k}{2} \right\rfloor$. W.l.o.g. assume that $m_d^{i_t} = n, t = 1, \ldots, r$ and $m_d^{i_t} = m_0^{i_t}, t = r+1, \ldots, k$. Evidently, if $r = 0$ no index of d is equal to n.

At point x^{diag} let $x_v = 1$, such that $v \neq (1, \ldots, 1)$. Let $\tilde{x}[q] = (m_v^{i_1} \neq m_1^{i_1} ? m_v^{i_1} \leftrightarrow m_1^{i_1})_{i_1} \cdots (m_v^{i_{q-1}} \neq m_1^{i_{q-1}} ? m_v^{i_{q-1}} \leftrightarrow m_1^{i_{q-1}})_{i_{q-1}} (m_v^{i_q} \neq m_d^{i_q} ? m_v^{i_q} \leftrightarrow m_d^{i_q})_{i_q} \cdots (m_v^{i_k} \neq m_d^{i_k} ? m_v^{i_k} \leftrightarrow m_d^{i_k})_{i_k}$. At point $\tilde{x}[q]$, we have

$$\tilde{x}_{1\cdots 1} = \tilde{x}_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_d^{i_q}),(m_d^{i_{q+1}},\ldots,m_d^{i_q}))} = 1$$

Let $\tilde{x}_w = 1$, $w \neq u$. Clearly, $m_w^{i_t} \neq 1, m_1^{i_t}, t = 1, \ldots, q-1$ and $m_w^{i_t} \neq 1, m_d^{i_t}, t = q, \ldots, k$. Then, $x[q] = \tilde{x}[q](m_w^{i_{r+1}} \neq n ? m_w^{i_{r+1}} \leftrightarrow n)_{i_{r+1}} \cdots (m_w^{i_k} \neq n ? m_w^{i_k} \leftrightarrow n)_{i_k}$. Note that the two previously exhibited variables are set to one, at both points $x[q]$ and $\tilde{x}[q]$. Hence, $x[q] \in P_I^{(k,1)}(Q^1(c_n))$, since there is one variable with at least $\left\lfloor \frac{k}{2} \right\rfloor + 1$ indices equal to n. The same variable is also set-to-one at point $x'[q] = x[q](m_d^{i_q} \leftrightarrow 1)_{i_q}$. Therefore, $x[q], x'[q]$ both satisfy $ax = a_0$, yielding $ax[q] = ax'[q]$ or

$$\begin{aligned} & a_{11\cdots 1} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_d^{i_q}),(m_d^{i_{q+1}},\ldots,m_d^{i_k}))} \\ = \; & a_{\varphi_{k,1}((m_d^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_d^{i_{q+1}},\ldots,m_d^{i_k}))} \end{aligned} \tag{6.5}$$

Let $\hat{x}[q] = x'[q](m_d^{i_1} \leftrightarrow 1)_{i_1} \cdots (m_d^{i_{q-1}} \leftrightarrow 1)_{i_{q-1}}$. At this point, we have

$$\hat{x}_{\varphi_{k,q}((m_d^{i_1},\ldots,m_d^{i_q}),(1,\ldots,1))} = \hat{x}_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_d^{i_{q+1}},\ldots,m_d^{i_k}))} = 1$$

Let $\hat{x}_w$ denote another variable set-to-one. Let $\bar{x}[q] = \hat{x}[q](m_w^{i_{r+1}} \neq n ? m_w^{i_{r+1}} \leftrightarrow n)_{i_{r+1}} \cdots (m_w^{i_k} \neq n ? m_w^{i_k} \leftrightarrow n)_{i_k}$. $\bar{x}[q] \in P_I^{(k,1)}(Q^1(c_n))$ since there is one variable with at least $\left\lfloor \frac{k}{2} \right\rfloor + 1$ indices equal to n. The same variable is also set to one at point $\bar{x}'[q] = \bar{x}[q](m_d^{i_q} \leftrightarrow 1)_{i_q}$. Therefore, both $\bar{x}[q], \bar{x}'[q]$ satisfy $ax = a_0$, yielding $a\bar{x}[q] = a\bar{x}'[q]$, or equivalently

$$\begin{aligned} & a_{\varphi_{k,q}((m_d^{i_1},\ldots,m_d^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},1),(m_d^{i_{q+1}},\ldots,m_d^{i_k}))} \\ = \; & a_{\varphi_{k,q-1}((m_d^{i_1},\ldots,m_d^{i_{q-1}}),(1,\ldots,1))} + a_{\varphi_{k,q}((m_1^{i_1},\ldots,m_1^{i_{q-1}},m_d^{i_q}),(m_d^{i_{q+1}},\ldots,m_d^{i_k}))} \end{aligned} \tag{6.6}$$

Adding equations (6.5) and (6.6) gives (after cancelling identical terms):

$$a_{\varphi_{k,q}((m_d^{i_1},\ldots,m_d^{i_q}),(1,\ldots,1))} = a_{\varphi_{k,q-1}((m_d^{i_1},\ldots,m_d^{i_{q-1}}),(1,\ldots,1))} + a_{\varphi_{k,1}((m_d^{i_q}),(1,\ldots,1))} - a_{11\cdots 1} \tag{6.7}$$

As in the proof of Theorem 5.1, the recurrence relation (6.7) proves (6.2) for any k-tuple belonging to $C^{(k,1)} \setminus$

$Q^1(c_n)$, since all points used for $2 \le q \le k$ belong to $P_I^{(k,1)}(Q^1(c_n))$.

Next, we consider k-tuples belonging to $Q^1(c_n)$. Let $(m^1, \dots, m^k)$ be one such tuple, with q indices equal to n. It is necessary to have $\lfloor \frac{k}{2} \rfloor + 1 \le q \le k$. Assume $m^{i_t} = n, \forall t = 1, \dots, q$, $m^{i_t} \ne n, \forall t = q+1, \dots, k$. Hence, for $\varphi_{k,q}((n, \dots, n), (m^{i_{q+1}}, \dots, m^{i_k})) \in Q^1(c_n)$, we define

$$\pi_{\varphi_{k,q}((n,\dots,n),(m^{i_{q+1}},\dots,m^{i_k}))} = a_{\varphi_{k,q}((n,\dots,n),(m^{i_{q+1}},\dots,m^{i_k}))} - \sum_{t=1}^{k-q} \lambda^t_{m^{i_{k+1-t}}} - \sum_{t=k-q+1}^{k} \lambda^t_n \tag{6.8}$$

First, we will show

$$\begin{aligned} \pi_{nn\cdots n} &= \pi_{\varphi_{k,k-1}((n,\dots,n),(m^{i_k}))} = \pi_{\varphi_{k,k-2}((n,\dots,n),(m^{i_{k-1}},m^{i_k}))} \\ &= \dots = \pi_{\varphi_{k,\lfloor \frac{k}{2} \rfloor +1}((n,\dots,n),(m^{i_{\lfloor k/2 \rfloor +2}},\dots,m^{i_k}))} = \pi \end{aligned}$$

or

$$\pi_{\varphi_{k,q}((n,\dots,n),(m^{i_{q+1}},\dots,m^{i_k}))} = \pi_{\varphi_{k,q-1}((n,\dots,n),(m^{i_q},\dots,m^{i_k}))} = \pi, \quad \left\lfloor \frac{k}{2} \right\rfloor + 2 \le q \le k \tag{6.9}$$

Let $m_2^t \in M_t \setminus \{n\}, \forall t \in K$, i.e. m_2^t may be equal to 1 for some $t \in K$. At x^{diag}, let $x_v = 1$, $v \ne (n, ..., n)$. Let $\tilde{x} = x^{diag}(m_v^1 \ne m_2^1 ? m_v^1 \leftrightarrow m_2^1)_1 \cdots (m_v^k \ne m_2^k ? m_v^k \leftrightarrow m_2^k)_k$. At $\tilde{x}$ we have $\tilde{x}_{n\cdots n} = \tilde{x}_{m_2^1 m_2^2 \cdots m_2^k} = 1$. Let $x[q] = \tilde{x}(m_2^{i_1} \leftrightarrow n)_{i_1} \cdots (m_2^{i_q} \leftrightarrow n)_{i_q}$. For $q \ge \lfloor \frac{k}{2} \rfloor + 2$ and $k \ge q$, points $x[q], x[q-1] \in P_I^{(k,1)}(Q^1(c_n))$ since at both points there exists one variable set to one with at least $\lfloor \frac{k}{2} \rfloor + 1$ indices equal to n. Both points satisfy $ax = a_0$, which implies $ax[q] = ax[q-1]$ or

$$\begin{aligned} & a_{\varphi_{k,q}((n,\dots,n),(m_2^{i_{q+1}},\dots,m_2^{i_k}))} + a_{\varphi_{k,q}((m_2^{i_1},\dots,m_2^{i_q}),(n,\dots,n))} \\ = \; & a_{\varphi_{k,q-1}((n,\dots,n),(m_2^{i_q},\dots,m_2^{i_k}))} + a_{\varphi_{k,q-1}((m_2^{i_1},\dots,m_2^{i_{q-1}}),(n,\dots,n))} \end{aligned} \tag{6.10}$$

The first terms of both sides are indexed by k-tuples belonging to $Q^1(c_n)$, whereas the other two k-tuples belong to $C^{(k,1)} \setminus Q^1(c_n)$. Therefore, substituting the first terms, of both sides, from (6.8) and the rest by (6.2), we obtain

$$\begin{aligned} & \sum_{t=1}^{k-q} \lambda^t_{m_2^{i_{k+1-t}}} + \sum_{t=k-q+1}^{k} \lambda^t_n + \pi_{\varphi_{k,q}((n,\dots,n),(m_2^{i_{q+1}},\dots,m_2^{i_k}))} + \sum_{t=1}^{k-q} \lambda^t_n + \sum_{t=k-q+1}^{k} \lambda^t_{m_2^{i_{k+1-t}}} \\ = \; & \sum_{t=k-q+2}^{k} \lambda^t_n + \sum_{t=1}^{k-q+1} \lambda^t_{m_2^{i_{k+1-t}}} + \pi_{\varphi_{k,q-1}((n,\dots,n),(m_2^{i_q},\dots,m_2^{i_k}))} + \sum_{t=k-q+2}^{k} \lambda^t_{m_2^{i_{k+1-t}}} + \sum_{t=1}^{k-q+1} \lambda^t_n \end{aligned}$$

Canceling out identical terms, yields (6.9).

Consider now two distinct k-tuples $v, w \in Q^1(c_n) \setminus \{c_n\}$, each having a different set of indices equal to n, but of the same cardinality q ($\lfloor \frac{k}{2} \rfloor + 1 \le q \le k$). We will show that $\pi_v = \pi_w$.

Let $v = \varphi_{k,q}((n, \dots, n), (m_v^{i_{q+1}}, \dots, m_v^{i_k}))$ and $w = \varphi_{k,q}((n, \dots, n), (m_w^{j_{q+1}}, \dots, m_w^{j_k}))$, where $m_v^{i_t} \in M_{i_t} \setminus \{n\}$, $m_v^{j_t} \in M_{j_t} \setminus \{n\}$, $\bigcup_{t=q+1}^k \{i_t\} \ne \bigcup_{t=q+1}^k \{j_t\}$. Let x denote the point which has $x_{n\cdots n} = x_{m_v^1 m_v^2 \cdots m_v^k} = 1$. Since $x_{n\cdots n} = 1$, $x \in P_I^{(k,1)}(Q^1(c_n))$. Let also $x[q] = x(n \leftrightarrow m_v^{i_1})_{i_1} \cdots (n \leftrightarrow m_v^{i_q})_{i_q}$. At $x[q]$ there exists one variable, set to one, with at least $\lfloor \frac{k}{2} \rfloor + 1$ indices equal to n. Therefore, $ax = ax[q]$ yields

$$\begin{aligned} & a_{nn\cdots n} + a_{m_v^1 m_v^2 \cdots m_v^k} \\ = \quad & a_{\varphi_{k,q}((n,\ldots,n),(m_v^{i_{q+1}},\ldots,m^{i_k}))} + a_{\varphi_{k,q}((m_v^{i_1},\ldots,m^{i_q}),(n,\ldots,n))} \end{aligned}$$

Substituting terms, indexed by k-tuples belonging to $Q^1(c_n)$, from (6.8) and the rest from (6.2), we obtain $\pi_{nn\cdots n} = \pi_{\varphi_{k,q}((n,\ldots,n),(m_v^{i_{q+1}},\ldots,m_v^{i_k}))} = \pi$ for $\lfloor \frac{k}{2} \rfloor + 1 \leq q \leq k$. In a similar manner, we prove $\pi_{nn\cdots n} = \pi_{\varphi_{k,q}((n,\ldots,n),(m_w^{j_{q+1}},\ldots,m_w^{j_k}))} = \pi$. The proof of (6.2) is now complete. To show (6.3), in $ax^{diag} = a_0$, we substitute terms from (6.2). The result follows. $\square$

Theorem 6.7. *For $k \geq 4$ and even, $n \geq 4$ and $c \in C^{(k,1)}$ the inequality*

$$\sum\{x_q : q \in Q^{1,2}(c)\} \leq 1 \tag{6.11}$$

defines a facet of $P_I^{(k,1)}$.

Proof. The proof differs from that of Theorem 6.6 in the points to be presented next.

First, observe that we refer to the set $P_I^{(k,1)}(Q^{1,2}(c))$ instead of $P_I^{(k,1)}(Q^1(c))$, where $P_I^{(k,1)}(Q^{1,2}(c)) = \{x \in P_I^{(k,1)} : \sum\{x_q : q \in Q^{1,2}(c)\} = 1\}$.

To prove that $P_I^{(k,1)}(Q^{1,2}(c)) \neq P_I^{(k,1)}$, consider a point $x \in P_I^{(k,1)}(Q^{1,2}(c))$ with $x_c = x_v = 1, c \neq v$. Let $x' = x(m_c^{i_1} \leftrightarrow m_v^{i_1})_{i_1} \cdots (m_c^{i_{k/2}} \leftrightarrow m_v^{i_{k/2}})_{i_{k/2}}$. At this point, we have $x'_{\varphi_{k,\frac{k}{2}}((m_v^{i_1},\ldots,m_v^{i_{k/2}}),(m_c^{i_{(k/2)+1}},\ldots,m_c^{i_k}))} = x'_{\varphi_{k,\frac{k}{2}}((m_c^{i_1},\ldots,m_c^{i_{k/2}}),(m_v^{i_{(k/2)+1}},\ldots,m_v^{i_k}))} = 1$. W.l.o.g. let x'_w be another variable set-to-one. The point $\hat{x} = x'(m_c^{i_k} \leftrightarrow m_w^{i_k})_{i_k}$ belongs to $P_I^{(k,1)} \setminus P_I^{(k,1)}(Q^{1,2}(c))$, since among the variables set-to-one three have indices from the k-tuple c, none of which belongs to $Q^{1,2}(c)$. It can be easily seen that, one of them is indexed by the complement of a k-tuple belonging to $Q^2_{G+}(c)$, one has $\frac{k}{2} - 1$ indices from c and one has exactly one index from c.

Next, we must show (6.2) and (6.3). We define the λs in exactly the same way as in the proof of Theorem 6.6. Hence, for $(m^1, m^2, \ldots, m^k) \in C^{(k,1)} \setminus Q^{1,2}(c_n)$, we must prove (6.4).

Assume that the sequence of indices $i_1, i_2, \ldots, i_k$ is such that $\varphi_{k,\frac{k}{2}((n,\ldots,n),(m^{i_{(k/2)+1}},\ldots m^{i_k}))} \notin Q^2_{G+}(c_n)$ and $\varphi_{k,\frac{k}{2}((m^{i_1},\ldots m^{i_{k/2}}),(n,\ldots,n))} \in Q^2_{G+}(c_n)$, where $m^{i_t} \in M_{i_t} \setminus \{n\}, \forall t \in K$. Observe that for any k-tuple with $\frac{k}{2}$ indices from $(n, \ldots, n)$, we can derive such a sequence of indices.

We consider a k-tuple $d \in C^{(k,1)} \setminus Q^{1,2}(c_n)$ where $m_d^{i_t} = n$, $t = 1, \ldots, r$, $m_d^{i_t} = m_0^{i_t}$, $t = r+1, \ldots, k$ and $0 \leq r = |d \cap c_n| \leq \frac{k}{2}$. The proof proceeds in the same way as in Theorem 6.6 w.r.t. the derivation of points and equations. At each of the points derived, for $r \leq \frac{k}{2} - 1$ one of the variables set-to-one has at least $\frac{k}{2} + 1$ indices equal to n whereas for $r = \frac{k}{2}$ we have $x_{\varphi_{k,\frac{k}{2}}((m^{i_1},\ldots,m^{i_{(k/2)+1}}),(n,\ldots,n))} = 1$, for some $m^{i_t} \in M_{i_t} \setminus \{n\}, t = 1, \ldots, k/2$. Hence, all points belong to $P_I^{(k,1)}(Q^{1,2}(c_n))$, for $2 \leq q \leq k$. Thus, we have proven (6.4) for an arbitrary tuple belonging to $C^{(k,1)} \setminus Q^{1,2}(c_n)$.

To prove (6.4) for a k-tuple belonging to $Q^{1,2}(c_n)$, we consider two cases: either the tuple belongs to $Q^1(c_n)$, or to $Q^2_{G+}(c_n)$. In the first case, the proof is identical to that of Theorem 6.6. In the second case, for the k-tuple

$\varphi_{k,\frac{k}{2}((m^{i_1},\ldots m^{i_{k/2}}),(n,\ldots,n))}$ belonging to $Q^2_{G^+}(c_n)$, we define

$$\pi_{\varphi_{k,\frac{k}{2}}((m^{i_1},\ldots,m^{i_{k/2}}),(n,\ldots,n))} = a_{\varphi_{k,q}((m^{i_1},\ldots,m^{i_{k/2}}),(n,\ldots,n))} - \sum_{t=1}^{k/2} \lambda^t_{m^{i_{k+1-t}}} - \sum_{t=(k/2)+1}^{k} \lambda^t_n \tag{6.12}$$

At x^{diag} let $x_v = 1$, $v \neq (n,\ldots,n)$. Let $x = x^{diag}(m^1_2 \neq m^1_v ? m^1_2 \leftrightarrow m^1_v)_1 \cdots (m^k_2 \neq m^k_v ? m^k_2 \leftrightarrow m^k_v)_k$. At x, we have $x_{n\cdots n} = x_{m^1_2 m^2_2 \cdots m^k_2} = 1$. Let $x' = x(m_2^{i_1} \leftrightarrow n)_{i_1} \cdots (m_2^{i_{k/2}} \leftrightarrow n)_{i_{k/2}}$. Obviously, $x, x' \in P_I^{(k,1)}(Q^{1,2}(c_n))$ implying $ax = ax'$ which yields

$$a_{n\cdots n} + a_{m^1\cdots m^k} = a_{\varphi_{k,\frac{k}{2}}((m_v^{i_1},\ldots,m_v^{i_{k/2}}),(n,\ldots,n))} + a_{\varphi_{k,\frac{k}{2}}((n,\ldots,n),(m_v^{i_{(k/2)+1}},\ldots,m_v^{i_k}))}$$

By substituting the second term of the right-hand side from (6.12) and the rest of the terms from (6.2), and cancelling out identical terms,.we obtain $\pi_{\varphi_{k,\frac{k}{2}}((m_v^{i_1},\ldots,m_v^{i_{k/2}}),(n,\ldots,n))} = \pi_{n\cdots n} = \pi$. $\square$

Note that, for $k = 3$, this class of facets is described also in [3].

7 The planar assignment polytopes

Each integer point of $P_I^{(3,2)}$ corresponds to a Latin square of order n (see, for example, [9]). This implies $P_I^{(3,2)} \neq \emptyset$, since there exist Latin squares of any order $n \geq 2$. The polytope $P_I^{(3,2)}$ is also referred to as *the Latin square* polytope. In [1], a more general relation between the integer points of $P_I^{(k,2)}$ and *mutually orthogonal latin squares* is presented. We briefly introduce some definitions (see [17] for details). A Latin square $\mathbb{L}$ of order n is an $n \times n$ square array consisting of n^2 entries of n different elements, each occurring exactly once in each row and column. Two Latin squares $\mathbb{L}_1 = \|a_{ij}\|$ and $\mathbb{L}_2 = \|b_{ij}\|$ are called *orthogonal,* if every ordered pair of symbols occurs exactly one among the n^2 ordered pairs (a_{ij}, b_{ij}), $i, j = 1, \ldots, n$. This definition is extended to a set of more than two Latin squares, which are said to be *mutually orthogonal,* if they are pairwise orthogonal. In [1], it is noted that an integer point of $P_I^{(k,2)}$ corresponds to a set of $k-2$ *MOLS*. Therefore, $P_I^{(k,2)}$ is called the $(k-2)$*MOLS* polytope.

The connection between *MOLS* and $P_I^{(k,2)}$ provides information on the existence of at least one integer vector in $P_I^{(k,2)}$ for certain values of the parameters k, n. For example, it is known that there cannot be more than $n-1$ *MOLS* of order n ([17, Theorem 2.1]). This implies that $k-2 \leq n-1$ or $k \leq n+1$ is a necessary condition for $P_I^{(k,2)} \neq \emptyset$. Observe that this fact is a special case of Proposition 4.1. The theory of *MOLS* provides us with further results, not implied by Proposition 4.1. For example, it is known that $P_I^{(4,2)} \neq \emptyset$, $\forall n \in \mathbb{Z}_+ \setminus \{1,2,6\}$ ([17, Theorem 2.9]).

Theorem 7.1. *For $n \geq \max\{5, k-1\}$, if $P_I^{(k,2)} \neq \emptyset$ then $dim P_I^{(k,2)} = \sum_{r=0}^{k-3} \binom{k}{r} \cdot (n-1)^{k-r}$.*

Proof. Throughout the proof we will assume that $P_I^{(k,2)} \neq \emptyset$.

We use the same technique as in the proof of Theorem 5.1, i.e. we show that, if there exists an equality $ax = a_0$, satisfied by all $x \in P_I^{(k,2)}$, then the vector $(a, a_0)^T$ can be expressed as a linear combination of the rows of $A_n^{(k,2)}$. This is equivalent to proving that $dim P_I^{(k,2)} = dim P^{(k,2)}$.

Assume that every $x \in P_I^{(k,2)}$ satisfies the equality $ax = a_0$, $a \in \mathbb{R}^{n^k}$, $a_0 \in \mathbb{R}$. Then, there exist scalars $\lambda^1_{m^{k-1}m^k}$, $\lambda^2_{m^{k-2}m^k}$, $\lambda^3_{m^{k-2}m^{k-1}}, \ldots, \lambda^{\frac{k(k-1)}{2}}_{m^1m^2}$, $\forall m^1 \in M_1, \ldots, m^k \in M_k$, such that:

$$a_{m^1\cdots m^k} = \lambda^1_{m^{k-1}m^k} + \lambda^2_{m^{k-2}m^k} + \lambda^3_{m^{k-2}m^{k-1}} + \cdots + \lambda^{\frac{k(k-1)}{2}}_{m^1m^2} \tag{7.1}$$

$$a_0 = \sum_{(m^{k-1},m^k)\in M_{k-1}\times M_k} \lambda^1_{m^{k-1}m^k} + \cdots + \sum_{(m^1,m^2)\in M_1\times M_2} \lambda^{\frac{k(k-1)}{2}}_{m^1m^2} \tag{7.2}$$

We define:

$$\begin{array}{lcl}
\lambda^1_{m^{k-1}m^k} & = & a_{1\cdots 1m^{k-1}m^k} \\
\lambda^2_{m^{k-2}m^k} & = & a_{1\cdots 1m^{k-2}1m^k} - a_{1\cdots 1m^k} \\
\lambda^3_{m^{k-2}m^{k-1}} & = & a_{1\cdots 1m^{k-2}m^{k-1}1} - a_{1\cdots 1m^{k-2}11} - a_{1\cdots 1m^{k-1}1} + a_{1\cdots 1} \\
\lambda^4_{m^{k-3}m^k} & = & a_{1\cdots 1m^{k-3}11m^k} - a_{1\cdots 1m^k} \\
\lambda^5_{m^{k-3}m^{k-2}} & = & a_{1\cdots 1m^{k-2}m^{k-3}11} - a_{1\cdots 1m^{k-3}111} - a_{1\cdots 1m^{k-2}11} + a_{1\cdots 1} \\
\vdots & & \\
\lambda^{\frac{k(k-1)}{2}}_{m^1m^2} & = & a_{m^1m^21\cdots 1} - a_{m^11\cdots 1} - a_{1m^21\cdots 1} + a_{1\cdots 1}
\end{array}$$

As in the proof of Theorem 5.1, the values of the superscript of λs are provided by the function f, defined in (3.1). We can easily verify that $f(k-1,k)=1$, $f(k-2,k)=2$, ..., $f(1,2)=\frac{k(k-1)}{2}$.

By substituting the values of the scalars in equation (7.1), we get

$$\begin{aligned}
a_{m^1m^2\cdots m^k} = \; & a_{m^1m^21\cdots 1} + a_{m^11m^31\cdots 1} + \cdots + a_{m^11\cdots 1m^k} \\
& + a_{1m^2m^31\cdots 1} + \cdots + a_{1m^21\cdots 1m^k} \\
& + \cdots + a_{1\cdots 1m^{k-1}m^k} \\
& - (k-2)\cdot [a_{m^11\cdots 1} + a_{1m^21\cdots 1} + \cdots + a_{1\cdots 1m^k}] \\
& + \frac{(k-1)(k-2)}{2}\cdot a_{1\cdots 1}
\end{aligned} \tag{7.3}$$

Writing the right-hand side of (7.3) in terms of $\varphi_{k,2}$, we get

$$\begin{aligned}
a_{m^1m^2\cdots m^k} = \; & \sum_{q=1}^{k-1}\sum_{r=q+1}^{k} a_{\varphi_{k,2}((m^q,m^r),(1,\ldots,1))} - (k-2)\sum_{p=1}^{k} a_{\varphi_{k,1}((m^p),(1,\ldots,1))} \\
& + \frac{(k-1)(k-2)}{2} a_{1\ldots 1}
\end{aligned}$$

or equivalently

$$\begin{aligned}
a_{m^1m^2\cdots m^k} = \; & \sum_{q=1}^{k-1}\sum_{r=q+1}^{k} a_{\varphi_{k,2}((m^{i_q},m^{i_r}),(1,\ldots,1))} - (k-2)\sum_{p=1}^{k} a_{\varphi_{k,1}((m^{i_p}),(1,\ldots,1))} \\
& + \frac{(k-1)(k-2)}{2} a_{1\cdots 1}
\end{aligned} \tag{7.4}$$

where $\{i_1,\ldots,i_k\}\equiv K$.

Each point of $P_I^{(k,2)}$ is illustrated a set of $(k-2)$ *MOLS*, where sets M_{i_1} and M_{i_2} are indexing the rows and the columns, respectively, of each square. The contents of the cells of the Latin square $(t-2)$ are elements of the set M_{i_t}, for all $t=3,\ldots,k$. Hence, at point x, the element lying in the cell defined by the row m^{i_1} and the column m^{i_2} of the Latin square $(t-2)$, is denoted as $m^{i_t}(m^{i_1},m^{i_2})$. The notation $m^{i_t}(x;m^{i_1},m^{i_2})$ denotes the same fact, but at a specific $x\in P_I^{(k,2)}$.

Observe that (7.4) is trivially valid for each k-tuple having at least $k-2$ indices equal to one. To prove (7.4) for every k-tuple, we need the following intermediate results.

Proposition 7.2. *Let $n \geq k-1$ and $P_I^{k,2,n} \neq \emptyset$. If $ax = a_0$ for all $x \in P_I^{(k,2)}$, it holds that*

$$
\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(m^{i_3}(m_1^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_1^{i_2})))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(m^{i_3}(m_1^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_2^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(m^{i_3}(m_2^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_1^{i_2})))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(m^{i_3}(m_2^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_2^{i_2})))} \\
& + a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(m^{i_3}(m_2^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_2^{i_2})))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(m^{i_3}(m_2^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_1^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(m^{i_3}(m_1^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_2^{i_2})))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(m^{i_3}(m_1^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_1^{i_2})))} \\
= \; & a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(m^{i_3}(m_2^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_1^{i_2})))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(m^{i_3}(m_2^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_2^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(m^{i_3}(m_1^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_1^{i_2})))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(m^{i_3}(m_1^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_2^{i_2})))} \\
& + a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(m^{i_3}(m_1^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_2^{i_2})))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(m^{i_3}(m_1^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_1^{i_1},m_1^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(m^{i_3}(m_2^{i_1},m_2^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_2^{i_2})))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(m^{i_3}(m_2^{i_1},m_1^{i_2}),\ldots,m^{i_k}(m_2^{i_1},m_1^{i_2})))}
\end{aligned}
$$

for $m_1^{i_1}, m_2^{i_1} \in M_{i_1}$ and $m_1^{i_2}, m_2^{i_2} \in M_{i_2}$.

Proof. Let $b_t, c_t, d_t, e_t \in M_{i_t}$, $\forall t \in \{3,...,k\}$. Consider the arbitrary point $x \in P_I^{(k,2)}$ as illustrated in Table 3. Let also $x' = x(m_1^{i_1} \leftrightarrow m_2^{i_1})_{i_1}$, $\bar{x} = x(m_1^{i_2} \leftrightarrow m_2^{i_2})_{i_2}$, $\bar{x}' = \bar{x}(m_1^{i_1} \leftrightarrow m_2^{i_1})_{i_1}$. Points x', $\bar{x}$, $\bar{x}'$ are illustrated at Tables 4, 6, 5 respectively.

Table 3: Point x (Proposition 7.2)

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		b_3		c_3	
$\vdots$					
$m_2^{i_1}$		d_3		e_3	
$\vdots$					

$\cdots$

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		b_k		c_k	
$\vdots$					
$m_2^{i_1}$		d_k		e_k	
$\vdots$					

Table 4: Point x' (Proposition 7.2)

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		d_3		e_3	
$\vdots$					
$m_2^{i_1}$		b_3		c_3	
$\vdots$					

$\cdots$

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		d_k		e_k	
$\vdots$					
$m_2^{i_1}$		b_k		c_k	
$\vdots$					

Since $x, x' \in P_I^{(k,2)}$, it holds that $ax = ax'$. Observe that

$$
\begin{aligned}
& a_{\varphi_{k,2}((m^{i_1},m^{i_2}),(m^{i_3}(x;m^{i_1},m^{i_2}),\ldots,m^{i_k}(x;m^{i_1},m^{i_2})))} \\
= \; & a_{\varphi_{k,2}((m^{i_1},m^{i_2}),(m^{i_3}(x';m^{i_1},m^{i_2}),\ldots,m^{i_k}(x';m^{i_1},m^{i_2})))}, \forall m^{i_1} \in M_{i_1} \setminus \{m_1^{i_1}, m_2^{i_1}\}, m^{i_2} \in M_{i_2}
\end{aligned}
$$

Table 5: Point $\bar{x}$ (Proposition 7.2)

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		c_3		b_3	
$\vdots$					
$m_2^{i_1}$		e_3		d_3	
$\vdots$					

$\cdots$

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		c_k		b_k	
$\vdots$					
$m_2^{i_1}$		e_k		d_k	
$\vdots$					

Table 6: Point $\bar{x}'$ (Proposition 7.2)

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		e_3		d_3	
$\vdots$					
$m_2^{i_1}$		c_3		b_3	
$\vdots$					

$\cdots$

	$\cdots$	$m_1^{i_2}$	$\cdots$	$m_2^{i_2}$	$\cdots$
$\vdots$					
$m_1^{i_1}$		e_k		d_k	
$\vdots$					
$m_2^{i_1}$		c_k		b_k	
$\vdots$					

Therefore, $ax = ax'$ becomes

$$
\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(b_3,..,b_k))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(c_3,..,c_k))} \\
& + \sum_{m^{i_2}\in M_{i_2}\setminus\{m_1^{i_2},m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1},m^{i_2}),(m^{i_3}(x;m_1^{i_1},m^{i_2}),..,m^{i_k}(x;m_1^{i_1},m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(d_3,\dots,d_k))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(e_3,..,e_k))} \\
& + \sum_{m^{i_2}\in M_{i_2}\setminus\{m_1^{i_2},m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1},m^{i_2}),(m^{i_3}(x;m_2^{i_1},m^{i_2}),..,m^{i_k}(x;m_2^{i_1},m^{i_2})))}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & a_{\varphi_{k,2}((m_1^{i_1},m_1^{i_2}),(d_3,\dots,d_k))} + a_{\varphi_{k,2}((m_1^{i_1},m_2^{i_2}),(e_3,\dots,e_k))} + \\
& + \sum_{m^{i_2}\in M_{i_2}\setminus\{m_1^{i_2},m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1},m^{i_2}),(m^{i_3}(x';m_1^{i_1},m^{i_2}),\dots,m^{i_k}(x';m_1^{i_1},m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1},m_1^{i_2}),(b_3,\dots,b_k))} + a_{\varphi_{k,2}((m_2^{i_1},m_2^{i_2}),(c_3,\dots,c_k))} \\
& + \sum_{m^{i_2}\in M_{i_2}\setminus\{m_1^{i_2},m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1},m^{i_2}),(m^{i_3}(x';m_2^{i_1},m^{i_2}),\dots,m^{i_k}(x';m_2^{i_1},m^{i_2})))} \qquad (7.5)
\end{aligned}
$$

We observe that $m^{i_t}(x;m_2^{i_1},m^{i_2}) = m^{i_t}(x';m_1^{i_1},m^{i_2})$ and $m^{i_t}(x;m_1^{i_1},m^{i_2}) = m^{i_t}(x';m_2^{i_1},m^{i_2})$, $\forall m^{i_t} \in$

$M_{i_2}, t \in \{3, \dots, k\}$. Thus, (7.5) becomes

$$\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(b_3,\dots,b_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(c_3,\dots,c_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m_{i_3}(x; m_1^{i_1}, m^{i_2}),\dots,m_{i_k}(x; m_1^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(d_3,\dots,d_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(e_3,\dots,e_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m_{i_3}(x; m_2^{i_1}, m^{i_2}),\dots,m_{i_k}(x; m_2^{i_1}, m^{i_2})))} \\
= \; & a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(d_3,\dots,d_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(e_3,\dots,e_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m_{i_3}(x; m_2^{i_1}, m^{i_2}),\dots,m_{i_k}(x; m_2^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(b_3,\dots,b_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(c_3,\dots,c_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m_{i_3}(x; m_1^{i_1}, m^{i_2}),\dots,m_{i_k}(x; m_1^{i_1}, m^{i_2})))} \qquad (7.6)
\end{aligned}$$

Since $\bar{x}, \bar{x}' \in P_I^{(k,2)}$ we have $a\bar{x} = a\bar{x}'$. By observing that

$$\begin{aligned}
& a_{\varphi_{k,2}((m^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}; m^{i_1}, m^{i_2})))} \\
= \; & a_{\varphi_{k,2}((m^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}'; m^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}'; m^{i_1}, m^{i_2})))}, \forall m^{i_1} \in M_{i_1} \setminus \{m_1^{i_1}, m_2^{i_1}\}, m^{i_2} \in M_{i_2}
\end{aligned}$$

$a\bar{x} = a\bar{x}'$ becomes

$$\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(c_3,\dots,c_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(b_3,\dots,b_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_1^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}; m_1^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(e_3,\dots,e_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(d_3,\dots,d_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_2^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}; m_2^{i_1}, m^{i_2})))} \\
= \; & a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(e_3,\dots,e_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(d_3,\dots,d_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}'; m_1^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}'; m_1^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(c_3,\dots,c_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(b_3,\dots,b_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}'; m_2^{i_1}, m^{i_2}),\dots,m^{i_k}(\bar{x}'; m_2^{i_1}, m^{i_2})))} \qquad (7.7)
\end{aligned}$$

We observe, again, that $m^{i_t}(\bar{x}; m_2^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}'; m_1^{i_1}, m^{i_2})$ and $m^{i_t}(\bar{x}; m_1^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}'; m_2^{i_1}, m^{i_2})$,

$\forall m^{i_2} \in M_{i_2}, t \in \{3, ..., k\}$. Hence, (7.7) becomes

$$\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(c_3,\ldots,c_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(b_3,\ldots,b_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_1^{i_1}, m^{i_2}),\ldots,m^{i_k}(\bar{x}; m_1^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(e_3,\ldots,e_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(d_3,\ldots,d_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_2^{i_1}, m^{i_2}),\ldots,m^{i_k}(\bar{x}; m_2^{i_1}, m^{i_2})))}
\end{aligned}$$

$$\begin{aligned}
= \; & a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(e_3,\ldots,e_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(d_3,\ldots,d_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_1^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_2^{i_1}, m^{i_2}),\ldots,m^{i_k}(\bar{x}; m_2^{i_1}, m^{i_2})))} \\
& + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(c_3,\ldots,c_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(b_3,\ldots,b_k))} \\
& + \sum_{m^{i_2} \in M_{i_2} \setminus \{m_1^{i_2}, m_2^{i_2}\}} a_{\varphi_{k,2}((m_2^{i_1}, m^{i_2}),(m^{i_3}(\bar{x}; m_1^{i_1}, m^{i_2}),\ldots,m^{i_k}(\bar{x}; m_1^{i_1}, m^{i_2})))} \qquad (7.8)
\end{aligned}$$

We observe that $m^{i_t}(x; m_1^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}; m_1^{i_1}, m^{i_2})$, $m^{i_t}(x; m_2^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}; m_2^{i_1}, m^{i_2})$, $m^{i_t}(x'; m_1^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}'; m_1^{i_1}, m^{i_2})$, $m^{i_t}(x'; m_2^{i_1}, m^{i_2}) = m^{i_t}(\bar{x}'; m_2^{i_1}, m^{i_2})$, $\forall m^{i_2} \in M_{i_2} \setminus \{m_1^{i_1}, m_2^{i_2}\}$. Subtracting (7.6) from (7.8) yields

$$\begin{aligned}
& a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(b_3,\ldots,b_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(c_3,\ldots,c_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(d_3,\ldots,d_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(e_3,\ldots,e_k))} \\
& + a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(e_3,\ldots,e_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(d_3,\ldots,d_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(c_3,\ldots,c_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(b_3,\ldots,b_k))} \\
& = a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(d_3,\ldots,d_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(e_3,\ldots,e_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(d_3,\ldots,d_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(e_3,\ldots,e_k))} \\
& + a_{\varphi_{k,2}((m_1^{i_1}, m_1^{i_2}),(c_3,\ldots,c_k))} + a_{\varphi_{k,2}((m_1^{i_1}, m_2^{i_2}),(b_3,\ldots,b_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_1^{i_2}),(e_3,\ldots,e_k))} + a_{\varphi_{k,2}((m_2^{i_1}, m_2^{i_2}),(d_3,\ldots,d_k))}
\end{aligned}$$

Substituting b_t, c_t, d_t, e_t by $m^{i_t}(m_1^{i_1}, m_1^{i_2})$, $m^{i_t}(m_1^{i_1}, m_2^{i_2})$, $m^{i_t}(m_2^{i_1}, m_1^{i_2})$, $m^{i_t}(m_2^{i_1}, m_2^{i_2})$, respectively for all $t \in \{3, \ldots, k\}$, we obtain the result. $\square$

The equation of Proposition 7.2, when applied to an integer point $x \in P_I^{(k,2)}$, will be denoted as $x(m_1^{i_1}, m_2^{i_1}; m_1^{i_2}, m_2^{i_2})$.

Lemma 7.3. *For $n \geq 4$ and $P_I^{(k,2)} \neq \emptyset$, let $m_0^{i_1} \in M_{i_1} \setminus \{1\}$, $m_0^{i_2} \in M_{i_2} \setminus \{1\}$. There exists the point $x \in P_I^{(k,2)}$, illustrated in Table 7 , where $m^{i_t}(x; 1, m_0^{i_2}), m^{i_t}(x; m_0^{i_1}, 1) \in M_{i_t} \setminus \{1\}$, $m^{i_t}(x; m_0^{i_1}, m_0^{i_2}) \in M_{i_t}$ for $t \in \{3, ..., k\}$*

Table 7: Point x (Lemma 7.3)

	1	$\cdots$	$m_0^{i_2}$	$\cdots$
1	$1, \ldots, 1$		$m^{i_3}(x; 1, m_0^{i_2}), \ldots, m^{i_k}(x; 1, m_0^{i_2})$	
$\vdots$				
$m_0^{i_1}$	$m^{i_3}(x; m_0^{i_1}, 1), \ldots, m^{i_k}(x; m_0^{i_1}, 1)$		$m^{i_3}(x; m_0^{i_1}, m_0^{i_2}), \ldots, m^{i_k}(x; m_0^{i_1}, m_0^{i_2})$	
$\vdots$				

Proof. Consider the (arbitrary) point $x_0 \in P_I^{(k,2)}$ illustrated in Table 8.

Table 8: Point x_0 (Lemma 7.3)

	1	$\cdots$	$m_0^{i_2}$	$\cdots$
1	$b_{i_3},\dots,b_{i_3}$		$c_{i_3},\dots,c_{i_k}$	
$\vdots$				
$m_0^{i_1}$	$d_{i_3},\dots,d_{i_k}$		$e_{i_3},\dots,e_{i_k}$	
$\vdots$				

Note that $b_{i_t},\ c_{i_t},\ d_{i_t},\ e_{i_t} \in M_{i_t}$, such that $b_{i_t}, e_{i_t} \neq c_{i_t}, d_{i_t}$ for every $t \in \{3,\dots,k\}$. It is easy to see that $x = x_0(b_{i_3} \neq 1?1 \leftrightarrow b_{i_3})_{i_3}(b_{i_4} \neq 1?1 \leftrightarrow b_{i_4})_{i_4} \cdots (b_{i_k} \neq 1?1 \leftrightarrow b_{i_k})_{i_k}$. $\square$

(Back to the proof of Theorem 7.1) For $n \geq 5$, at point x of Lemma 7.3, there exist $m_0^{i_t} \in M_{i_t} \setminus \{1, m^{i_t}(x; m_0^{i_1}, 1),$ $m^{i_t}(x; 1, m_0^{i_2}),\ m^{i_t}(x; m_0^{i_1}, m_0^{i_2})\},\ t \in \{3,...,k\}$. Let $x^3[q] = x(1 \leftrightarrow m_0^{i_3})(1 \leftrightarrow m_0^{i_4}) \cdots (1 \leftrightarrow m_0^{i_q})$, $3 \leq q \leq k$. Also let $x^2[q] = x^3[q](m_0^{i_q} = m^{i_q}(x^3_{i_q}; m_0^{i_1}, m_0^{i_2})?m_0^{i_2} \leftrightarrow m_1^{i_2})$ where $m_1^{i_2}$ is such that $m^{i_q}(x^3_{i_q}; m_0^{i_1}, m_1^{i_2}) \neq m_0^{i_q}$. Such an $m_1^{i_2}$ exists for $n \geq 5$. Hence, $m^{i_q}(x^2_{i_q}; m_0^{i_1}, m_0^{i_2}) \neq m_0^{i_q}$. Let also $x^1[q] = x^2[q](1 \leftrightarrow m_0^{i_q})_{i_q}$ and observe that

$$m^{i_t}(x^2_{i_q}; m_0^{i_1}, 1) = m^{i_t}(x^1_{i_q}; m_0^{i_1}, 1), \forall t \in \{3,...,k\} \tag{7.9}$$

$$m^{i_t}(x^2_{i_q}; 1, m_0^{i_2}) = m^{i_t}(x^1_{i_q}; 1, m_0^{i_2}), \forall t \in \{3,...,k\} \tag{7.10}$$

$$m^{i_t}(x^2_{i_q}; m_0^{i_1}, m_0^{i_2}) = m^{i_t}(x^1_{i_q}; m_0^{i_1}, m_0^{i_2}), \forall t \in \{3,...,k\} \tag{7.11}$$

Let us denote the left-hand sides of (7.9), (7.10), (7.11) as $m_1^{i_t}$, $m_2^{i_t}$, $m_3^{i_t}$ respectively. Then

$x^1[q](1, m_0^{i_1}; 1, m_0^{i_2})$:

$$a_{\varphi_{k,2}((1,1),(m_0^{i_3},\dots,m_0^{i_{q-1}},1,\dots,1))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_3^{i_3},\dots,m_3^{i_k}))}$$
$$+ a_{\varphi_{k,2}((1,1),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_0^{i_3},\dots,m_0^{i_{q-1}},1,\dots,1))}$$
$$= a_{\varphi_{k,2}((1,1),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_0^{i_3},\dots,m_0^{i_{q-1}},1,\dots,1))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_2^{i_3},\dots,m_2^{i_k}))}$$
$$+ a_{\varphi_{k,2}((1,1),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_0^{i_3},\dots,m_0^{i_{q-1}},1,\dots,1))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_1^{i_3},\dots,m_1^{i_k}))}$$

$x^2[q](1, m_0^{i_1}; 1, m_0^{i_2})$:

$$a_{\varphi_{k,2}((1,1),(m_0^{i_3},\dots,m_0^{i_q},1,\dots,1))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_3^{i_3},\dots,m_3^{i_k}))}$$
$$+ a_{\varphi_{k,2}((1,1),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_0^{i_3},\dots,m_0^{i_q},1,\dots,1))}$$
$$= a_{\varphi_{k,2}((1,1),(m_1^{i_3},\dots,m_1^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_0^{i_3},\dots,m_0^{i_q},1,\dots,1))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_2^{i_3},\dots,m_2^{i_k}))}$$
$$+ a_{\varphi_{k,2}((1,1),(m_2^{i_3},\dots,m_2^{i_k}))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_0^{i_3},\dots,m_0^{i_q},1,\dots,1))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_3^{i_3},\dots,m_3^{i_k}))} + a_{\varphi_{k,2}((m_0^{i_1},m_0^{i_2}),(m_1^{i_3},\dots,m_1^{i_k}))}$$

$x^1[q](1, m_0^{i_1}; 1, m_0^{i_2}) - x^2[q](1, m_0^{i_1}; 1, m_0^{i_2})$ results in

$$\begin{aligned} a_{\varphi_{k,2}((m_0^{i_1}, m_0^{i_2}),(m_0^{i_3},\ldots,m_0^{i_q},1,\ldots,1))} &= a_{\varphi_{k,2}((m_0^{i_1}, m_0^{i_2}),(m_0^{i_3},\ldots,m_0^{i_{q-1}},1,\ldots,1))} \\ &+ a_{\varphi_{k,2}((m_0^{i_1},1),(m_0^{i_3},\ldots,m_0^{i_q},1,\ldots,1))} + a_{\varphi_{k,2}((1,m_0^{i_2}),(m_0^{i_3},\ldots,m_0^{i_q},1,\ldots,1))} \\ &- (a_{\varphi_{k,2}((1,1),(m_0^{i_3},\ldots,m_0^{i_q},1,\ldots,1))} + a_{\varphi_{k,2}((m_0^{i_1},1),(m_0^{i_3},\ldots,m_0^{i_{q-1}},1,\ldots,1))} \\ &+ a_{\varphi_{k,2}((1,m_0^{i_2}),(m_0^{i_3},\ldots,m_0^{i_{q-1}},1,\ldots,1))}) + a_{\varphi_{k,2}((1,1),(m_0^{i_3},\ldots,m_0^{i_{q-1}},1,\ldots,1))} \end{aligned} \tag{7.12}$$

For each term of (7.12), the indices with the value 1 are grouped together, yielding:

$$\begin{aligned} a_{\varphi_{k,q}((m_0^{i_1},\ldots,m_0^{i_q}),(1,\ldots,1))} &= a_{\varphi_{k,q-1}((m_0^{i_1},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))} \\ &+ a_{\varphi_{k,q-1}((m_0^{i_1},m_0^{i_3},\ldots,m_0^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q-1}((m_0^{i_2},m_0^{i_3},\ldots,m_0^{i_q}),(1,\ldots,1))} \\ &- (a_{\varphi_{k,q-2}((m_0^{i_3},\ldots,m_0^{i_q}),(1,\ldots,1))} + a_{\varphi_{k,q-2}((m_0^{i_1},m_0^{i_3},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))} \\ &+ a_{\varphi_{k,q-2}((m_0^{i_2},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))}) + a_{\varphi_{k,q-3}((m_0^{i_3},\ldots,m_0^{i_{q-1}}),(1,\ldots,1))} \end{aligned} \tag{7.13}$$

We consider equation (7.13) for $q = k$ and recursively substitute terms until $q = 3$. As a result, we derive equation (7.4) for $m_0^{i_t} \in M_{i_t} \setminus \{1\}, t \in K$. As in Theorem 5.1, we can derive (7.4) for a k-tuple with r indices equal to one by starting the recursive substitution for $q = k - r$ $(1 \leq r \leq k - 3)$. The proof of (7.1) is now complete.

Equation (7.2) follows from the assumption that $P_I^{(k,2)} \neq \emptyset$. This assumption implies that there exists at least one $x \in P_I^{(k,2)}$ such that $A_n^{(k,2)}x = e$. Therefore, summing over all equations, weighted by the corresponding scalars λ, and taking into account equation (7.1), we obtain

$$ax = \sum_{(m^{k-1},m^k)\in M_{k-1}\times M_k} \lambda^1_{m^{k-1}m^k} + \cdots + \sum_{(m^1,m^2)\in M_1\times M_2} \lambda^{\frac{k(k-1)}{2}}_{m^1 m^2}$$

Denoting the right-hand side, of the above equation, by a_0 completes the proof. □

By similar arguments to the ones used in Propositions (5.2) and (5.3), it can be established that, for $n \geq \max\{5, k-1\}$ and $P_I^{(k,2)} \neq \emptyset$, the inequalities $x_c \geq 0$ $(x_c \leq 1)$ define (do not define) facets of $P_I^{(k,2)}$. Constraints (1.5) define improper faces of $P_I^{(k,2)}$ since they are satisfied by all points of $P_I^{(k,2)}$.

8 Concluding remarks

This work introduces an IP model for the class of linear-sum assignment problems. This model establishes a framework for unifying the polyhedral analysis of all assignment polytopes belonging to this class. In particular, the dimension of the linear relaxation of all members of this class is derived. The properties of integer points are encompassed in the definition of the interchange operator, which exploits inherent isomorphisms. A hierarchy among assignment polytopes is naturally imposed. Focusing on the classes of axial and planar assignment polytopes, through the use of the mapping φ and the derived recurrence relations, we prove that their dimension equals a sum of terms from Newton's polynomial. The potential of this unified approach is demonstrated by identifying a family of non-trivial facets for all axial assignment polytopes.

References

[1] Appa G., Magos D., Mourtos I., Janssen J.C.M., On the Orthogonal Latin Squares polytope, submitted to Discrete Mathematics.

[2] Balas E. and Padberg M., Set Partitioning: a survey, SIAM Rev., **18** (1976) 710-760.

[3] Balas E. and Saltzman M.J., Facets of the three-index assignment polytope, Discrete Applied Mathematics **23** (1989) 201-229.

[4] Balas E. and Qi L., Linear time separation algorithms for the three index assignment polytope, Discrete Applied Mathematics **23** (1989) 201 209.

[5] Balas E. and Saltzman M.J., An algorithm for the three-index assignment problem, Operations Research **39** (1991), 150-161.

[6] Bandelt H.J., Crama Y. and Spieksma C.R., Approximation algorithms for multi-dimensional assignment problems with decomposable costs, Discrete Applied Mathematics **49** (1994) 25-50.

[7] Burkard R.E., Rudolf R. and Woeginger G.J., Three-dimensional assignment problems with decomposable cost coefficients, Discrete Applied Mathematics **65** (1996) 123-139.

[8] Burkard R.E. and. Çela E, Linear Assignment problems and Extentions, in P. Pardalos and D.Z. Du, eds., *Handbook of Combinatorial Optimization* (Kluwer Academic Publishers, 1999).

[9] Euler R., Burkard R.E. and Grommes R., On latin squares and the facial structure of related polytopes, Discrete Mathematics **62** (1986) 155-181.

[10] Euler R., Odd cycle and a class of facets of the axial 3-index assignment polytope, Zastosowania Mathematyki XIX **3-4** (1987) 375-386.

[11] Euler R. and Le Verge H., Time-tables, polyhedra and the greedy algorithm, Discrete Applied Mathematics (1996) 207-221.

[12] Frieze A.M. , A Bilinear programming formulation of the 3-dimensional assignment problem, Math. Prog. **7** (1974) 376-379.

[13] Frieze A.M. and Yadegar J., An algorithm for solving 3-dimensional assignment problems with application to scheduling a teaching practice, Journal of Operational Research Society **32** (1981) 989-995.

[14] Frieze A.M., Complexity of a 3-dimensional assignment problem, Journal of the Operational Research Society **13** (1983) 161-164.

[15] Haley K.B., The multi-index problem, Operations Research **11** (1963) 368-379.

[16] Karp R.M., Reducibility among combinatorial problems, in *Complexity of Computer Computations*, R.E.Miller and J.W.Thatcher, eds., Plenum Press, New York (1972) 85-103.

[17] Laywine C. F. and Mullen G. L., *Discrete Mathematics using Latin Squares (John Wiley & Sons, 1998).*

[18] Magos D. and Miliotis P., An algorithm for the planar three-index assignment problem, European Journal of Operational Research **77** (1994) 141-153.

[19] Martello S. and Toth P., Linear assignment problems, Annals of Discrete Mathematics **31** (1987) 259-282.

[20] Newhauser G.L. and Wolsey L.A., *Integer and Combinatorial Optimization*, (John Wiley & Sons, 1988).

[21] Pierskalla W. P., The tri-substitution method for the three-dimensional assignment problem, CORS Journal **5** (1967) 71-81.

[22] Pierskalla W. P., The multidimensional assignment problem, Operations Research **16** (1968) 422-431.

[23] Poore A.B., Multidimensional assignment formulation of data association problems arising from multitarget and multisensor tracking, Computational Optimisation and Applications **3** (1994) 27-57.

[24] Poore A.B. and Robertson A.J. III, A new Langrangian relaxation based algorithm for a class of multidimensional assignment problems, Computational Optimisation and Applications **8** (1997), 129-150.

[25] Pulleyblank W.R., Polyhedral Combinatorics, in G.L. Nemhauser, A.H.G. Rinnooy Kan and M.J. Todd, eds., *Optimization* (North Holland, 1989) 371-446.

[26] Pusztaszeri J.F., Rensing P.E. and Liebling T.M., Tracking elementary particles near their primary vertex: a combinatorial approach, Journal of Global Optimization **9** (1996) 41-64.

[27] Qi L., Balas E. and Gwan G., A new class of facet defining inequalities for the three index assignment problem, in Dung-Zu Du and Jie Sun, eds., *Advances in Optimization and Approximation*, (Kluwer Academics Pub., 1993) 256-274.

[28] Spieksma F.C.R., Multi-index assignment problems: complexity, approximation, applications, in L. Pitsoulis and P. Pardalos, eds., *Nonlinear Assignment Problems, Algorithms and Applications* (Kluwer Academic Publishers, 2000) 1-12.