Channel Assignment on Infinite Sets under Frequency-Distance Constraints*

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1 Introduction

Large-scale radio systems for multiple users, such as those used in mobile phone networks, are often based on a cellular structure (Hale 1980; Macario 1997; MacDonald 1979). The service area is divided into cells and each cell is serviced by one transmitter, which communicates with users within the cell using a particular radio channel or set of channels. In any particular application the available channels are uniformly spaced in the spectrum, justifying integer labellings of these channels. It is often assumed, at least as a starting position, that the transmitters all have the same power, are omnidirectional, and are laid out like the vertices of a triangular lattice: in this case, the natural cells form a hexagonal pattern.

Suppose that a radio receiver is tuned to a signal on channel c_0, broadcast by the transmitter for its cell (typically, the closest transmitter). Reception will be degraded if there is excessive interference from other transmitters in the vicinity. First there is 'co-channel' interference due to re-use of channel c_0 at nearby sites; but there are also contributions from sites using channels near c_0, since in practice neither transmitters nor receivers operate exclusively within the frequencies of their assigned channels.

One possible method to decide if an assignment of channels to transmitters is acceptable is as follows. First we agree on a suitable radio propagation model, and a collection of 'test' receiver positions. We then combine the effects of co-channel interference and other interferences at the test receiver positions, and determine whether the corresponding signal to interference ratios

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are at least some threshold value, at a suitably high proportion of the points. We shall not follow this option: instead we shall adopt the popular and much simpler engineering approach. We assume that, in order to ensure acceptable signal quality, constraints are imposed on the allowed channel separations between pairs of potentially interfering transmitters. Thus we work within the 'constraint matrix' model. As a further simplification, we assume throughout that the allowed channel separations between a pair of transmitters are uniquely determined by the distance between the transmitters.

In the most basic version of the channel assignment problem, we consider only co-channel interference, and assume that we are given a threshold 're-use distance' d (or d_0) such that interference will be acceptable as long as no channel is re-used at sites less than distance d apart. Given a set V of points in the plane and given $d > 0$, let $G(V, d)$ denote the graph with vertex set V in which distinct vertices u and v are adjacent whenever the Euclidean distance $d(u, v)$ between them is less than d. Our basic version of the channel assignment problem involves colouring such 'proximity' or 'unit disks' graphs. We need to assign colours (radio channels or frequencies) to the points in V (sites or cells or transmitters), using as few colours as possible but avoiding (excessive) interference. Recall that a colouring of the vertices of a graph G is *proper* if adjacent vertices always receive distinct colours. Thus we are interested in proper colourings of the proximity graph $G(V, d)$ using few colours. The least possible number of colours is the *chromatic number* $\chi(G(V, d))$.

The problem described in the previous paragraph will be considered in Section 3. That section is based on (McDiarmid and Reed 1999). It turns out that it is possible to make quite precise statements concerning the behaviour of $\chi(G(V, d))$ if either V is a triangular lattice or if the re-use distance d is large. (The phrase 'd is large' really refers to the asymptotic behaviour of $\chi(G(V, d))$ as $d \to \infty$. We need to assume that the set V satisfies a certain mild condition on its density.)

For more practical channel assignment problems we need to consider more than just co-channel interference, and we investigate more general trade-offs between geographical distance and channel separation. Suppose we are given a non-negative vector $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$ of $k \geq 1$ distances. An assignment (colouring) $\varphi : V \to \{1, 2, \ldots, n\}$ is called $\mathbf{d}$-feasible if it satisfies the *frequency-distance constraints*

$$d(u, v) < d_i \quad \Rightarrow \quad |\varphi(u) - \varphi(v)| \neq i$$

for each pair of distinct points u, v in V and for each $i = 0, 1, \ldots, k - 1$. When $d_0 \geq d_1 \geq \ldots \geq d_{k-1}$ we may replace '$\neq i$' above by '$> i$' and we obtain a standard model for channel assignment, see for example (Hale 1980). For many practical implementations it can indeed be assumed that $d_0 \geq d_1 \geq \cdots \geq d_{k-1}$, but there exist radio systems that can be modelled better with a description where this is not the case, see for example Section III E in (Hale 1980). The *span* $sp(V; \mathbf{d})$ is the least integer n for which there exists a $\mathbf{d}$-feasible assignment $\varphi : V \to \{1, 2, \ldots, n\}$. When $k = 1$ we write $sp(V; d_0)$; note that this is the same as $\chi(G(V, d_0))$.

The values $d_0, d_1, \ldots, d_{k-1}$ are set with the intention that any $\mathbf{d}$-feasible assignment will lead to acceptable levels of interference, and it is assumed that the vector $\mathbf{d}$ is given. As discussed above, the d_0-constraint limits co-channel interference; and similarly the d_1-constraint limits the contribution to the interference from the 'first adjacent' channels. We wish to develop an understanding of the quantity $sp(V; \mathbf{d})$, and to find methods to construct assignments that achieve the span or close to it. Thus we are interested here in the 'static channel assignment problem with frequency-distance constraints'.

In Section 4 we take a look at the behaviour of $sp(V;\mathbf{d})$ for some special cases such as the triangular lattice, and when the distances are large. To consider large distance, we suppose that $\mathbf{d} = d\,\mathbf{x}$, where $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1})$ is a fixed distance k-vector. As in the case when there were co-channel constraints only, we shall again be able to give quite precise information about the behaviour of $sp(V; d\,\mathbf{x})$ for large d. These results are based on the work in (McDiarmid 2000). We also consider briefly an extension involving demands and a co-site constraint (Gerke 2000b).

In the next section, Section 5, we return to the case of co-channel constraints only, and consider another model with demands, where each transmitter may demand a different number of channels. We restrict our discussion to the case of the triangular lattice, which is already quite interesting. This section is based on (McDiarmid and Reed 2000).

The results described in Section 3 and 4 provide reasonably accurate answers for almost any set V, provided the distances are large enough. In order to obtain more precise answers for small distances, we need to assume more structure on the set V under consideration. In Section 6 we discuss some results concerning channel assignments on sets V that are 2-dimensional lattices or have a lattice-like structure. That section is based on (Van den Heuvel 1998).

The starting point of most of the research in that section is the result from (McDiarmid and Reed 1999) (see also Section 3) that assignments that provide a good approximation to $sp(L; d_0)$ (where L is the set of points of a lattice) can be found among so-called strict tilings (see the definitions in that section). For a constraint vector $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$ let $sp_t(L;\mathbf{d})$ denote the least integer n for which there exists a $\mathbf{d}$-feasible assignment $\varphi : L \to \{1, 2, \ldots, n\}$ which is a strict tiling. Good bounds for $sp_t(L; d_0)$ follow from the results in Section 3. In Section 5 we look in some detail at the question for which $\mathbf{d}$ we can guarantee $sp_t(L;\mathbf{d}) = sp_t(L; d_0)$. In particular it follows that for these $\mathbf{d}$ we have good bounds for the span of a feasible assignment.

2 Mathematical background and definitions

Throughout this paper we will consider points as lying in the Euclidean plane $\mathbf{R}^2$. We will often think of $\mathbf{R}^2$ as a 2-dimensional vector space in the natural way. Thus we can add and scale points and sets, so that for example $x + A = \{\, x + a : a \in A \,\}$ and $\lambda\,A = \{\, \lambda\,a : a \in A \,\}$. A set of the form $x + A$ will be called a *co-set* of A.

Also we have the standard inner product $x \cdot y$ and norm $\|x\| = (x \cdot x)^{1/2}$. The (Euclidean) distance between two points is given by $d(x, y) = \|x - y\|$, and the distance between a point x and a set A is given by $d(x, A) = \inf\{\, d(x, a) : a \in A \,\}$.

Some of the definitions in this section were informally given in the previous section: we repeat them here for completeness.

Definition 2.1 *Given a set V of points in the plane and $d > 0$, $G(V, d)$ denotes the graph with vertex set V where two points are adjacent if their Euclidean distance is strictly less than d. The* chromatic number $\chi(G(V, d))$ *is the minimum number of colours needed for a proper colouring of the graph $G(V, d)$.*

Definition 2.2 *Given a set V of points and a positive integer n, an n-*labelling *of V is a function $\varphi : V \to \{1, 2, \ldots, n\}$. A* constraint vector *is a non-negative k-vector $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$ for some $k \geq 1$. An n-labelling of V is called* $\mathbf{d}$-feasible *if it satisfies the* frequency-distance constraints

$$d(u, v) \ < \ d_i \quad \Longrightarrow \quad |\varphi(u) - \varphi(v)| \ \neq \ i$$

for each pair of distinct points u, v in V and for each $i = 0, 1, \ldots, k-1$.

The span *$sp(V; \mathbf{d})$ is the minimum n such that there exists a $\mathbf{d}$-feasible n-labelling of V.*

Definition 2.3 *A* lattice *L is a subset of the plane of the form $\{\, p\, a + q\, b : p, q \in \mathrm{Z} \,\}$. Here a, b are two given linearly independent vectors, which form a* basis *of the lattice. We say that L has as* fundamental cell *the set $F = \{\, x\, a + y\, b : x, y \in \mathbf{R},\ 0 \leq x, y < 1 \,\}$.*

A generalised lattice *is a finite union of co-sets of a lattice, i.e., a set of the form $\bigcup_{u \in U} (u + L)$, where L is a lattice and U is a finite set of points.*

The triangular lattice T is the lattice with basis $(1, 0)$, $(1/2, \sqrt{3}/2)$ (see Figure 1). The square lattice S is the lattice with basis $(1, 0)$, $(0, 1)$.

Definition 2.4 *A labelling φ of a lattice L is called a* regular tiling *or just* tiling *if there exist linearly independent vectors $s, t \in L$ such that $\varphi(v + s) = \varphi(v + t) = \varphi(v)$ for all $v \in L$. We extend this definition to a labelling φ of a generalised lattice $\bigcup_{u \in U} (u + L)$ by requiring that $\varphi(v + u + s) = \varphi(v + u + t) = \varphi(v + u)$ for all $v \in L$ and $u \in U$.*

The vectors s, t that define a tiling of a (generalised) lattice form a basis for a sublattice $L^* = \{\, p\, s + q\, t : p, q \in \mathrm{Z} \,\}$ of L. And in fact we can define a tiling as a labelling ϕ such that if $v, w \in L$ with $v - w \in L^*$, then $\varphi(v) = \varphi(w)$. The sublattice L^* is the co-channel lattice *of the labelling.*

A strict tiling *of a (generalised) lattice is a tiling that has the additional property that if $v - w \notin L^*$, then $\varphi(v) \neq \varphi(w)$. Hence a strict tiling has the property that all co-sets of L^* obtain a different label.*

More special definitions will be given throughout the text when needed.

3 Co-channel constraints

A proximity graphs $G(V, d)$ can be considered as a scaled version of a 'unit disk' graphs (see Clark *et al.* 1990). When we consider channel assignment problems with only a co-channel constraint, specified by the re-use distance d or d_0, we are interested in proper colourings of such graphs. In (Hale 1980) this colouring problem is called the 'frequency-distance constrained co-channel assignment problem'.

We first consider the special case of the triangular lattice graph (the graph originating from transmitter locations with hexagonal cells), which is important in its own right and is the key to understanding the colouring problem in general. We then discuss more general sets such as 'generalised lattices' which still have some structure, and present results that are quite precise and informative, as long as the re-use distance d is not too small. When the re-use distance d

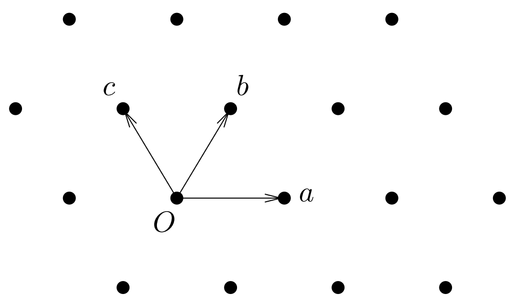

FIG. 1. The triangular lattice T

is small, small changes in d or in V can lead to large changes in the number of colours needed, and so in order to gain an overview of the problem we consider mainly the case when d is large.

After considering sets as above which have some structure, we consider arbitrary infinite sets of points in the plane, when d is large. It seems reasonable to assume that the points are well spread out. It turns out that we can give good asymptotic results for *any* set V of points in the plane as long as a mild density condition is satisfied. The hope is that such asymptotic results yield insight into finite cases with practical values for the parameters.

Let us then consider first the special case of the triangular lattice, where things work out very neatly. We assume that the lattice has the natural embedding in the plane with minimum distance 1. The origin $O = (0,0)$ and the point $a = (1,0)$ are lattice points, and so are the points $b = (1/2, \sqrt{3}/2)$ and $c = (-1/2, \sqrt{3}/2)$ – see Figure 1. Let G_T denote the corresponding 6-regular graph, with vertex set the set T of lattice points, and with two vertices adjacent whenever they are at distance 1 in the plane. The six neighbours of the origin O in G_T are then $\pm a, \pm b, \pm c$.

We are interested in colouring the proximity graph $G(T,d)$. For any $d > 0$, let d^+ be the minimum Euclidean distance between two points in T subject to that distance being at least d. Now, the distance between the origin O and the lattice point $p\,a + q\,b$, where p, q are non-negative integers, is $\left((p+q/2)^2 + (\sqrt{3}/2\,q)^2\right)^{1/2} = (p^2 + p\,q + q^2)^{1/2}$. Thus d^+ is the minimum value of $(p^2 + p\,q + q^2)^{1/2}$ such that p, q are non-negative integers and $(p^2 + p\,q + q^2)^{1/2} \geq d$. Numbers of the form $p^2 + p\,q + q^2$ are called *rhombic integers*. The first few rhombic integers are 1,3,4,7,9,12,13,16,19,21. Note that $d \leq d^+ \leq \lceil d \rceil$, and that we can compute the rhombic integer $(d^+)^2$ quickly, in $O(d)$ arithmetic operations.

Theorem 3.1 *The triangular lattice T satisfies*

$$\chi(G(T,d)) \;=\; (d^+)^2$$

for any $d > 0$, and there is an optimal colouring which is a strict tiling, with a triangular co-channel lattice.

This result is proved in (McDiarmid and Reed 1999), as indeed are all the results in this section. It appears to have been known to engineers at least since 1979 – see (MacDonald 1979; Gamst 1982). A similar result appears as Theorem 3 in (Bernstein *et al.* 1997). The above theorem is at the heart of the proof in (McDiarmid and Reed 1999) of results for more general sets which we discuss below. Since Theorem 3.1 is such a central result, we include a proof at the end of this section.

It is of interest to note as an aside that we can also work with graph distance in the triangular lattice graph G_T. Given a graph G and positive integer k, let $G^{(k)}$ denote the graph with the same vertices as G, and with distinct vertices u and v adjacent whenever their graph distance in G is at most k. (The graph distance between u and v is the least number of edges in a path joining them.) Thus $G^{(1)}$ is just G. Recall that a *clique* is a set of pairwise adjacent vertices; the *clique number* $\omega(G)$ is the maximum number of vertices in a clique. Clearly $\chi(G) \geq \omega(G)$.

Theorem 3.2 *The graph G_T of the triangular lattice satisfies*

$$\chi(G_T^{(k)}) \;=\; \omega(G_T^{(k)}) \;=\; \lceil 3/4\,(k+1)^2 \rceil$$

for any positive integer k.

This result has been proved independently in (Van den Heuvel *et al.* 1998) (where a similar result is given for the graph of the square lattice, with 3/4 replaced by 1/2), as well as in (McDiarmid and Reed 1999). It is possible to work simultaneously with Euclidean distance and graph distance: both of the last two theorems are special cases of Theorem 5 of (McDiarmid and Reed 1999).

Next we consider the more general concept of a generalised lattice. Recall that a generalised lattice is a finite union of co-sets of a lattice. Observe that these are the only sets which can possibly have a strict tiling. We have just seen that, for the triangular lattice T and any $d > 0$, the proximity graph $G(T, d)$ always has an optimal colouring which is a strict tiling. For any generalised lattice V in the plane, there is a similar approximate result, and thus of course the same holds for any lattice.

Theorem 3.3 *Let V be a generalised lattice in the plane. Then the proximity graph $G(V, d)$ has a colouring which is a strict tiling and which uses at most $\chi(G(V, d)) + O(d)$ colours.*

We shall prove some of the above results concerning lattices and strict tilings at the end of this section.

In order to consider more general sets of points, we shall need to consider densities. Let the lattice L be given as the set of all integer linear combinations of the two linearly independent vectors a and b, with fundamental cell $F = \{x\,a + y\,b : 0 \leq x, y < 1\}$. The sets $\{\, v + F : v \in L\,)$ partition the plane, and each contains exactly one point in L. It follows that any (large) ball B of radius r contains $\pi\,\sigma(L)\,r^2 + O(r)$ points of L, where $\sigma(L) = 1/area(F)$.

We use this result to guide our definition of density. We say that a set V has *density* $\sigma(V)$ if for any $\epsilon > 0$ there is an r_0 such that for every ball B of radius $r \geq r_0$,

$$\sigma(V) - \epsilon \;\leq\; \frac{|V \cap B|}{\pi\, r^2} \;\leq\; \sigma(V) + \epsilon.$$

In particular, the lattice L has density $\sigma(L) = 1/area(F)$, and a generalised lattice V which is the union of k distinct co-sets of the lattice L has density $\sigma(V) = k\,\sigma(L)$.

Now we consider sets of points that need not be as well structured as generalised lattices, but which are nevertheless still approximately uniformly spread over the plane, and which perhaps correspond well to sensible locations for transmitters (extended over the whole plane).

Let $0 < \sigma, r < \infty$. We say that a set V of points in the plane has a *cell structure* with density σ and radius r if there is a family $\{ C_v : v \in V \}$ of 'cells' indexed by V such that (a) this family partitions the plane (except perhaps for a set of measure zero); (b) each cell C_v (is measurable and) has area $1/\sigma$; and (c) $C_v \subseteq B(v, r)$ for each $v \in V$. Here $B(v, r)$ denotes the open ball with centre v and radius r. It is easily seen that such a set V has density σ. For example, the set of vertices of the square lattice with unit edge lengths has a cell structure (with square cells) with density 1 and radius $1/\sqrt{2}$. The set T of vertices of the triangular lattice has a cell structure (with hexagonal cells) with density $2/\sqrt{3}$ and radius $1/\sqrt{3}$.

We cannot now expect to give as precise answers as before. It is of interest to consider other graph parameters related to colouring. The *degree* of a vertex of G is the number of vertices adjacent to it. We denote the maximum degree of a vertex in G by $\Delta(G)$, and the minimum degree by $\delta(G)$.

Theorem 3.4 *Let the set V of points in the plane have a cell structure with density σ and radius r. Then for any $d > 0$,*

$$\pi \sigma (d - r)^2 - 1 \leq \delta(G(V, d)) \leq \Delta(G(V, d)) \leq \pi \sigma (d + r)^2 - 1; \tag{3.1}$$

$$\pi/4 \sigma d^2 \leq \omega(G(V, d)) \leq \pi/4 \sigma (d + 2r)^2; \tag{3.2}$$

and

$$\sqrt{3}/2 \sigma d^2 \leq \chi(G(V, d)) < ((\sqrt{3}/2 \sigma)^{1/2} (d + 2r) + 2/\sqrt{3} + 1)^2. \tag{3.3}$$

Thus both the minimum and the maximum degree of the graph $G(V, d)$ equal $\pi \sigma d^2 + O(d)$, $\omega(G(V, d)) = \pi/4 \sigma d^2 + O(d)$, and $\chi(G(V, d)) = \sqrt{3}/2 \sigma d^2 + O(d)$ as $d \to \infty$.

Now we move on to our greatest level of generality. A set of points in the plane is *discrete* if every bounded subset is finite, that is, every ball contains only a finite number of points of the set. Let V be any discrete set of points in the plane. We need a more general notion corresponding to density, which is well defined for any such set V. For each $r > 0$ let $f(r)$ be the supremum of the ratio $\dfrac{|V \cap B|}{\pi r^2}$ over all open balls B of radius r. The *upper density* $\sigma^+(V)$ of V is defined by setting $\sigma^+(V) = \inf_{r>0} f(r)$. If V has a density (for example if V is a lattice), then it follows that V has upper density equal to the same value. It turns out (see (McDiarmid and Reed 1999) that $f(r) \to \sigma^+(V)$ as $r \to \infty$, so that the infimum in the definition is also a limit; and it turns out also that the definition could equally well be phrased in terms of squares rather than balls, or indeed in terms of any 'reasonable' set with finite positive area.

Let us introduce one further graph invariant related to colouring. The *maximin degree* $\delta^*(G)$ is the supremum over all finite induced subgraphs of the minimum degree. This is also called the *degeneracy number* of G, and this number plus 1 is called the *colouring number* of G, since there must be a proper colouring using at most this last number of colours, see for example (Jensen and Toft 1995).

Theorem 3.5 *Let V be a discrete set of points in the plane, with upper density $\sigma = \sigma^+(V)$. For any $d > 0$, denote the clique number $\omega(G(V, d))$ by ω_d, and use χ_d, Δ_d and δ^*_d similarly for the chromatic number, maximum degree and maximin degree. Then $\omega_d/d^2 \geq \pi/4 \sigma$ and $\chi_d/d^2 \geq \sqrt{3}/2 \sigma$ for any $d > 0$; and, as $d \to \infty$, $\Delta_d/d^2 \to \pi \sigma$, $\delta^*_d/d^2 \to \pi/2 \sigma$, $\omega_d/d^2 \to \pi/4 \sigma$, and $\chi_d/d^2 \to \sqrt{3}/2 \sigma$.*

It follows for example that for any discrete set V of points in the plane with a finite positive upper density, the ratio of the chromatic number of $G(V, d)$ to its clique number tends to $2\sqrt{3}/\pi \sim 1.103$ as $d \to \infty$. It was suggested in (Gamst 1986) that such a result should hold for the triangular lattice.

We complete this section on co-channel constraints by giving proofs for the Theorems 3.1 and 3.3 above, where the set V of points has some algebraic structure. For proofs of the other theorems, the reader is referred to (McDiarmid and Reed 1999). The upper bounds on the chromatic number in these results come from choosing a suitably scaled version of the triangular lattice to 'cover' the set V, and then transferring a good colouring of the lattice over to V. All the lower bounds on the chromatic number χ come from one idea. Here it is.

Lemma 3.6 *Let V be a discrete set of points in the plane with finite upper density $\sigma = \sigma^+(G)$. Then $\chi(G(V, d)) \geq \sqrt{3}/2\,\sigma\, d^2$ for any $d > 0$.*

Proof A *packing* of disks (or balls) in the plane is a collection of disks with pairwise disjoint interiors. Let $d > 0$ be fixed, and consider disks of diameter d. We shall use Thue's classical theorem (see for example (Pach and Agarwal 1995, Rogers 1964, Schneider 1993)), that the maximum density of a packing of equal-sized disks in the plane is achieved by the hexagonal packing. Note that the hexagon with inner radius $d/2$ has area $6 \cdot \sqrt{3} \cdot (d/(2\sqrt{3}))^2 = \sqrt{3}/2\, d^2$. Given a bounded set C, let $\nu(C)$ denote the maximum number of disks with centre in C, over all packings of disks with diameter d. Thue's theorem says that $\nu(B(O, r))/(\pi r^2) \to 1/(\sqrt{3}/2\, d^2)$ as $r \to \infty$. Hence, for any $\epsilon > 0$, there is an r_0 such that for any $r \geq r_0$,

$$\nu(B(O, r)) \;\leq\; (1 + \epsilon)\,(\pi r^2)/(\sqrt{3}/2\, d^2).$$

Now recall that a *stable* (or *independent*) *set* in a graph G is a set of pairwise non-adjacent vertices; and the *stability number* $\alpha(G)$ is the maximum number of vertices in a stable set. Also, the chromatic number $\chi(G)$ is the least number of vertices in a partition of the vertex set $V(G)$ into stable sets, and so $\chi(G) \geq |V(G)|/\alpha(G)$. For any ball B_r of radius r,

$$\alpha(G(V \cap B_r, d)) \;=\; \nu(V \cap B_r) \;\leq\; \nu(B(O, r)),$$

and so

$$\alpha(G(V \cap B_r, d)) \;\leq\; (1 + \epsilon)\,(\pi r^2)/(\sqrt{3}/2\, d^2).$$

On the other hand, by the definition of upper density, there is a ball B_r of radius r with $|V \cap B_r| \geq \pi\,\sigma\, r^2$. Hence, for any $r \geq r_0$,

$$\begin{aligned} \chi(G(V, d)) \;&\geq\; \frac{|V \cap B_r|}{\alpha(G(V \cap B_r, d))} \\ &\geq\; \frac{\pi\,\sigma\, r^2}{(1 + \epsilon)\,(\pi r^2)/(\sqrt{3}/2\, d^2)} \;=\; \frac{1}{1 + \epsilon}\,\sqrt{3}/2\,\sigma\, d^2. \end{aligned}$$

Since this result holds for any $\epsilon > 0$ we obtain the desired inequality. □

Proof of Theorem 3.1 To prove the lower bound on χ, we recall that the triangular lattice with unit edge-lengths has density $\sigma = 2/\sqrt{3}$, and so by Lemma 3.6

$$\chi(G(T,d)) \;=\; \chi(G(T,d^+)) \;\geq\; (d^+)^2.$$

We shall show that there is a strict tiling with a triangular co-channel lattice which uses this number of colours.

Recall that the points $a = (1,0)$, $b = (1/2, \sqrt{3}/2)$ and $c = (-1/2, \sqrt{3}/2)$ are neighbours of the origin O in the lattice graph G_T. Recall also that the distance between O and the lattice point $x\,a + y\,b$, where x and y are non-negative integers, is $(x^2 + x\,y + y^2)^{1/2}$.

Now let x_0 and y_0 be non-negative integers such that $(d^+)^2 = x_0^2 + x_0\,y_0 + y_0^2$. Let $p = x_0\,a + y_0\,b$, and $q = x_0\,b + y_0\,c$. Then the set $L = \{\,x\,p + y\,q : x, y \text{ integers}\,\}$ is the set of lattice points of a triangular sublattice of the original triangular lattice. Also, the six points in L closest to the origin O are $\pm p$, $\pm q$, $\pm r$ where $r = x_0\,c - y_0\,a$, and each is at distance d^+. Hence the minimum distance between distinct points of L is d^+, and so the corresponding strict tiling is a proper colouring of $G(T,d)$.

How many colours does this tiling use? One way to work this out is to note that L is a copy of T scaled by d^+ (and possibly rotated), and so L has density $\sigma(L) = \sigma(T)/(d^+)^2$. Hence there must be $(d^+)^2$ co-sets of L in T. □

Theorem 3.3 will follow easily using the next lemma, from (McDiarmid and Reed 1999).

Lemma 3.7 *Let L be a lattice in the plane. Then there are constants c and D such that for any $d \geq D$, there is a sublattice $\hat{L}$ of L which has minimum distance at least d and has density at least* $(2/\sqrt{3})/(d^2 + c\,d)$.

Proof of Theorem 3.3 Let the generalised lattice V consist of k co-sets of the lattice L. If V has density $\sigma(V)$ and L has density $\sigma(L)$, then $k = \sigma(V)/\sigma(L)$. By the above lemma, there are constants c and D such that for any $d \geq D$, there is a sublattice $\hat{L}$ of L which has minimum distance at least d and has density $\hat{\sigma} = \sigma(\hat{L})$ at least $(2/\sqrt{3})/(d^2 + c\,d)$. Then V may be partitioned into $(\sigma(V)/\sigma(L)) \cdot (\sigma(L)/\hat{\sigma}) = \sigma(V)/\hat{\sigma}$ co-sets of $\hat{L}$. Thus we have a colouring of V which is a strict tiling and the number of colours used is

$$\sigma(V)/\hat{\sigma} \;=\; \sqrt{3}/2\,\sigma(V)\,d^2 + O(d).$$

□

4 General frequency-distance constraints

Let us start as in the last section by considering some particular non-asymptotic theorems, which need no lengthy introductions. The first is a simple result for the distance vector $(d, d, \ldots, d)$ and any set V. The second theorem is a general lower bound, which often provides the best known bound, even for particular cases such as the square lattice. The third and fourth theorems are special results for the square and triangular lattices respectively. After these four results, we introduce several asymptotic results, for large distances. All these results are from (McDiarmid 2000), though the first result below has been noted also for example in (Van den Heuvel 1998).

Recall that, given a set V of points and a constraint k-vector $\mathbf{d}$, the *span* $sp(V;\mathbf{d})$ is the minimum n such that there exists a $\mathbf{d}$-feasible assignment $\varphi : V \to \{1,\ldots,n\}$. Thus when $k=1$, so that we are considering co-channel constraints only, we have $sp(V;d) = \chi(G(V,d))$.

Theorem 4.1 *For the k-vector $(d,\ldots,d)$ where $d>0$,*

$$sp(V;(d,\ldots,d)) \;=\; k \cdot sp(V;d) - k + 1.$$

Let us see why this result is true. Given a d-feasible assignment $\varphi : V \to \{0,1,\ldots,s-1\}$ where $s = sp(V;d)$, the assignment $\hat{\varphi}(v) = k \cdot \varphi(v)$ for V is $(d,\ldots,d)$-feasible. Thus $sp(V;(d,\ldots,d)) \le k \cdot sp(V;d) - k + 1$. Conversely, given a $(d,\ldots,d)$-feasible assignment $\varphi : V \to \{0,1,\ldots,s-1\}$ where $s = sp(V;(d,\ldots,d))$, the assignment $\tilde{\varphi}(v) = \lfloor \varphi(v)/k \rfloor$ for V is d-feasible (since if u and v are distinct but $\tilde{\varphi}(u) = \tilde{\varphi}(v)$ then $|\varphi(u) - \varphi(v)| < k$, and so $d(u,v) \ge d$). Hence $sp(V;d) \le (sp(V;(d,\ldots,d)) - 1)/k + 1$.

Theorem 4.2 *For any set V and any non-increasing distance k-vector $\mathbf{d} = (d_0,\ldots,d_{k-1})$,*

$$sp(V;\mathbf{d}) \;\ge\; \max\{\, \sqrt{3}/2\, \sigma^+(V)\,(j+1)\, d_j^2 - j : j = 0,\ldots,k-1 \,\}.$$

This theorem follows easily from the above. For any $0 \le j \le k-1$,

$$\begin{aligned} sp(V;(d_0,d_1,\ldots,d_{k-1})) \;&\ge\; sp(V;(d_0,d_1,\ldots,d_j)) \\ &\ge\; sp(V;(d_j,d_j,\ldots,d_j)) \;=\; (j+1)\, sp(V;d_j) - j. \end{aligned}$$

But

$$sp(V;d_j) \;=\; \chi(G(V,d_j)) \;\ge\; \sqrt{3}/2\, \sigma^+(V)\, d_j^2,$$

by Lemma 3.6.

The next results need more work. In particular, the lower bound in the next theorem should look intuitively plausible or even obvious after a little thought, but it takes some time to prove in (McDiarmid 2000).

Theorem 4.3 *Let S denote the unit square lattice. Then for any positive integer d,*

$$sp(S;(\sqrt{2}\,d, d-1)) \;\le\; 2\,d^2,$$

and

$$sp(S;(\sqrt{2}\,d, d)) \;\ge\; 2\,d^2;$$

and so

$$sp(S; d\,(1, 1/\sqrt{2})) \;=\; d^2 + O(d)$$

as $d \to \infty$.

The theorem above is quite precise, but for the triangular lattice we can sometimes give the *exact* span. For any positive integer d, we have

$$sp(T;(\sqrt{3}\,d, d-1, d-1)) \;=\; sp(T;\sqrt{3}\,d) \;=\; 3\,d^2.$$

In fact we can extend this result. Recall that a *rhombic integer* is an integer of the form $x^2 + x\,y + y^2$ for (non-negative) integers x and y. These are the squares of the distances between the lattice points of the unit triangular lattice, as we noted earlier; and the first few rhombic integers are 1,3,4,7.

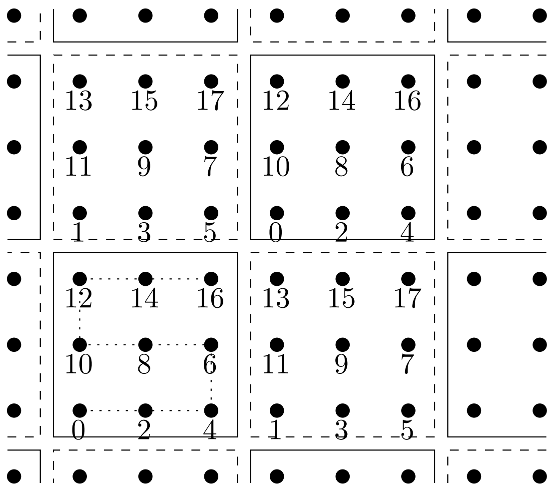

FIG. 2. A labelling of the square lattice satisfying $(\sqrt{2}\,d, d-1)$ with $d=3$

Theorem 4.4 *Let k be a rhombic integer, let T denote the unit triangular lattice, and let d be a positive integer. Then for the distance k-vector $(k^{1/2}\,d, d-1, \ldots, d-1)$ we have*

$$sp(T;(k^{1/2}\,d, d-1, \ldots, d-1)) \;=\; sp(T;k^{1/2}\,d) \;=\; k\,d^2.$$

The upper bound parts of the two theorems above (and in Theorem 4.11 below) come from a natural method for constructing assignments, which may be of practical interest. It has the general form: tile the plane, walk through a tile adding k at each step, extend the assignment over the plane and adjust using a k-colouring of the tiles. For example, the following algorithm (where $k=2$) yields the upper bound in Theorem 4.3 for the unit square lattice (see Figure 2 for the case $d=3$).

Let d be a positive integer, and consider the sublattice $d\,S$ of the unit square lattice S. Let g be the natural 2-colouring of $d\,S$; that is, $g((idx, jdy))$ is 0 if $i+j$ is even, and is 1 otherwise. The fundamental cell $\{\,(x,y) : 0 \le x < d,\ 0 \le y < d\}$ for $d\,S$ contains d^2 points of S, forming the set F say. We may trace a path through the points in F in the lattice graph on S (where points at unit distance are adjacent). Assign the values $0, 2, 4, \ldots, 2\,d^2-2$ to the points along the path, forming the assignment f say on F. The assignments f for F and g for $d\,S$ yield a natural assignment $h : S \to \{0, 1, \ldots, 2\,d^2-1\}$ for S, as follows. Each point $u \in S$ may be written uniquely as $v+w$ where $v \in F$ and $w \in d\,S$: we set $h(u) = f(v) + g(w)$. Then h is $(\sqrt{2}\,d, d-1)$-feasible.

For the asymptotic results on frequency distance constraints, we need to introduce various densities. We shall restrict our attention here to non-increasing distance vectors. Let $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1})$ be a distance k-vector, where $x_0 \ge x_1 \ge \cdots \ge x_{k-1} \ge 0$ and $x_0 > 0$. For each $i = 1, 2, \ldots$, the *i-channel density* $\alpha_i(\mathbf{x})$ is defined to be the supremum of the upper density $\sigma^+(V)$ over all sets V of points in the plane for which there is an $\mathbf{x}$-feasible assignment using channels $1, \ldots, i$. The 1-channel density $\alpha_1(1)$ is thus the maximum density of a packing of pairwise disjoint unit-diameter disks in the plane; and so $\alpha_1(1) = 2/\sqrt{3}$ and corresponds to taking V as the triangular lattice with unit edge lengths. This is the classical result of Thue on packing disks in the plane, which we met in the proof of Lemma 3.6 above. (We write $\alpha(1)$ instead of $\alpha((1))$ and so on.)

We shall be interested in particular in the 2-channel density $\alpha_2(1, x_1)$. This quantity is the solution of the following red-blue-purple disk packing problem. We wish to pack in the plane a pairwise disjoint family of red unit-diameter disks and a pairwise disjoint family of blue unit-diameter disks, where a red and a blue disk may overlap, forming a purple patch, but their centres must be at least distance x_1 apart. What is the maximum density of such a packing? (Equivalently we may think of packing unit-diameter balls in $\mathbf{R}^3$, where the balls must be in two layers, one with centres on the plane $z = 0$ and one with centres on the plane $z = (1 - x_1^2)^{1/2}$.)

The *channel density* $\alpha(\mathbf{x})$ is defined to be the infimum over all positive integers i of $\alpha_i(\mathbf{x})/i$. It is not hard to see that $\alpha(1) = \alpha_1(1)$ and so $\alpha(1) = 2/\sqrt{3}$; and that always $0 < \alpha(\mathbf{x}) < \infty$. Further, define the *inverse channel density* $\chi(\mathbf{x})$ to be $1/\alpha(\mathbf{x})$. The next theorem is the central result concerning general frequency distance constraints when the distances are large.

Theorem 4.5 *For any set V of points in the plane, and any non-increasing distance k-vector $\mathbf{x}$*

$$sp(V; d\,\mathbf{x})/d^2 \;\rightarrow\; \sigma^+(V)\,\chi(\mathbf{x}) \qquad \text{as } d \rightarrow \infty.$$

Thus in particular, for any set V of points in the plane with upper density 1, such as the set of points of the unit square lattice, the ratio $sp(V; d\,\mathbf{x})/d^2$ tends to the inverse channel density $\chi(\mathbf{x})$ as $d \rightarrow \infty$.

Following (McDiarmid 2000), let us note some basic properties of $\chi(\mathbf{x})$. Since $\alpha(1) = 2/\sqrt{3}$ we have $\chi(1) = \sqrt{3}/2$, and so the above result includes the part of Theorem 3.5 above concerning the chromatic number. By rescaling, we can see that for any $a > 0$, $\alpha_i(a\,\mathbf{x}) = \alpha_i(\mathbf{x})/a^2$ and so $\chi(a\,\mathbf{x}) = a^2\,\chi(\mathbf{x})$. This is a very useful observation: in particular, we can often set $x_0 = 1$ without loss of generality. Also note that there is an obvious monotonicity, so that if $\mathbf{x} \leq \mathbf{x}'$, then $\chi(\mathbf{x}) \leq \chi(\mathbf{x}')$. One further basic property is that the channel density $\alpha(\mathbf{x})$ and the inverse channel density $\chi(\mathbf{x})$ are continuous functions.

We looked at $\alpha_2(1, x)$ above in different ways. In fact $\chi(1, x) = 2/\alpha_2(1, x)$, which allows us to reduce a problem concerning many channels to one involving just two. Indeed, there is a general result for distance k-vectors $\mathbf{x}$ of the form $(1, x, \ldots, x)$.

Theorem 4.6 *Let $0 \leq x \leq 1$ and let $\mathbf{x}$ be the k-vector $(1, x, \ldots, x)$. Then*

$$\chi(\mathbf{x}) \;=\; k/\alpha_k(\mathbf{x}).$$

Theorems 4.1 to 4.4 above yield less precise but tidier results on letting $d \rightarrow \infty$, as follows.

Theorem 4.7 *For the k-vector of 1's we have*

$$\chi(1, \ldots, 1) \;=\; k \cdot \chi(1) \;=\; \sqrt{3}/2\,k.$$

Theorem 4.8 *For any non-increasing distance k-vector $\mathbf{x} = (x_0, \ldots, x_{k-1})$,*

$$\chi(\mathbf{x}) \;\geq\; \sqrt{3}/2 \cdot \max\{\,(j+1)\,x_j^2 : j = 0, \ldots, k-1\,\}.$$

Theorem 4.9
$$\chi(1, 1/\sqrt{2}) = 1.$$

Theorem 4.10 *Let k be a rhombic integer and let $0 \leq x \leq 1$. Then for the k-vector $(1, x, \ldots, x)$ we have*
$$\chi(1, x, \ldots, x) = \sqrt{3}/2 \cdot \max\{1, k\,x^2\}.$$

Perhaps there is most practical interest in the case $k = 2$, when we have just two distances d_0 and d_1, corresponding to co-channel and first adjacent channel constraints, with $d_0 \geq d_1$. Our current knowledge on the value of $\chi(1, x)$ is summarised in the following theorem – see also Figure 3.

Theorem 4.11 *We have the exact results from above, that $\chi(1, x) = \sqrt{3}/2$ for $0 \leq x \leq 1/\sqrt{3}$, $\chi(1, 1/\sqrt{2}) = 1$ and $\chi(1, 1) = \sqrt{3}$. We have the lower bounds, that $\chi(1, x)$ is at least*

$$\begin{array}{rl} \sqrt{3}/2, & \textit{for } 1/\sqrt{3} < x \leq 3^{1/4}/2\,(\approx 0.658); \\ 2\,x^2, & \textit{for } 3^{1/4}/2 \leq x < 1/\sqrt{2}\ (\approx 0.707); \\ 1, & \textit{for } 1/\sqrt{2} < x \leq 3^{-1/4}\ (\approx 0.760); \\ \sqrt{3}\,x^2, & \textit{for } 3^{-1/4} \leq x < 1. \end{array}$$

Finally, we have the upper bounds, that $\chi(1, x)$ is at most

$$\begin{array}{rl} 3\sqrt{3}/2\,x^2, & \textit{for } 1/\sqrt{3} < x \leq 4/\sqrt{43}\,(\approx 0.610); \\ 2\,x\,(1 - x^2)^{1/2}, & \textit{for } 4/\sqrt{43} \leq x < 1/\sqrt{2}\ (\approx 0.707); \\ 2\,(x^2 - 1/4)^{1/2}, & \textit{for } 1/\sqrt{2} < x < 1. \end{array}$$

Finally in this section, we consider an extension of the central asymptotic result above, Theorem 4.5, which involves demands. Suppose that we are given a discrete set of points V in the plane, and a non-negative integer demand w_v at each point $v \in V$, yielding the *demand vector* $\mathbf{w} = (w_v : v \in V)$. As well as a k-vector of distances $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$ as before, we now also have a positive integer c specifying the co-site constraint. We think of each point v in V as a site to which we are to assign a set of w_v radio channels, subject to frequency-distance and co-site constraints which ensure that interference is not excessive. An assignment $\varphi : V \to \mathcal{P}(\{1, 2, \ldots, t\})$ is called $(\mathbf{d}, c)$*-feasible* for $\mathbf{w}$ if the following conditions hold on demands, co-site separations and distance separations:

- for each point $v \in V$, $|\varphi(v)| = w_v$;
- for each point $v \in V$ and for each two distinct elements $a, b \in \varphi(v)$ we have $|a - b| \geq c$;
- for each $i = 0, 1, \ldots, k-1$, and for each pair of distinct points $u, v \in V$ which are at distance less than d_i, we have $|a - b| > i$ for each $a \in \varphi(u)$ and $b \in \varphi(v)$.

As before, we may assume without loss of generality that $d_0 \geq d_1 \geq \cdots \geq d_{k-1} > 0$; and since $d_{k-1} > 0$ it is natural to assume also that $c \geq k$.

The *span* $sp(\mathbf{w}; \mathbf{d}, c)$ is the least positive integer t such that there is a $(\mathbf{d}, c)$-feasible assignment $\varphi : V \to \mathcal{P}(\{1, 2, \ldots, t\})$ for $\mathbf{w}$. If the demand equals 1 at each point, we have exactly

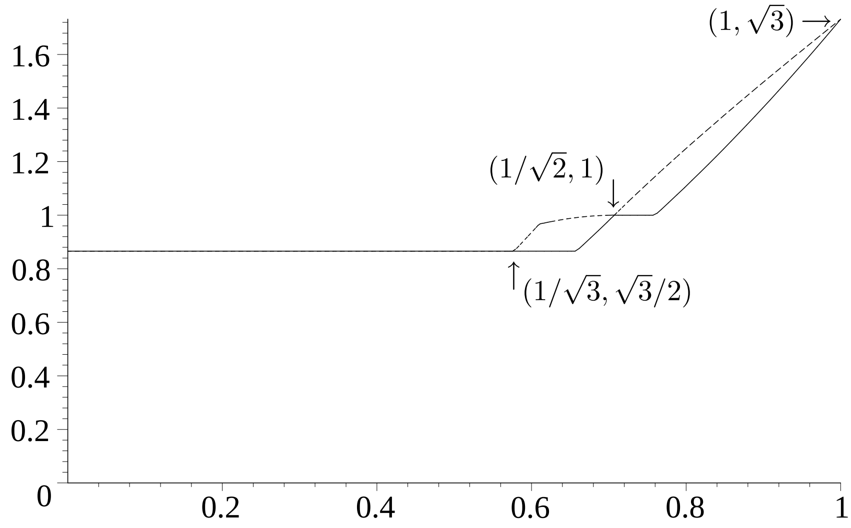

FIG. 3. Upper and lower bounds on $\chi(1, x)$

the situation discussed earlier in this section. As there we consider the span when the distances are large; that is, we suppose that $\mathbf{d} = d\,\mathbf{x}$ where $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1})$ is fixed and $d \to \infty$. We need to extend the definition for the upper density $\sigma^+(V)$ of a discrete set V of points in the plane to cover demand vectors. Let $\mathbf{w} = (w_v : v \in V)$ be a demand vector, where V is a discrete set of points in the plane. For any $x > 0$ let $f(x)$ denote the supremum of the ratio $(\sum_{v \in B} w_v)/(\pi x^2)$ over all open balls B of radius x. The *upper density* of $\mathbf{w}$ is defined to be $\sigma^+(\mathbf{w}) = \inf_{x>0} f(x)$. The definition could, as with its counterpart $\sigma^+(V)$ for sets, equally well be phrased in terms not of balls but of any reasonable sets with finite positive area. The following extension of Theorem 4.5 above (which leans heavily on that theorem) is due to S. Gerke (private communication).

Theorem 4.12 *Let $\mathbf{w} = (w_v : v \in V)$ be a demand vector, where V is a discrete set of points in the plane. Let $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1})$ where $x_0 \geq x_1 \geq \cdots \geq x_{k-1} > 0$, and let $c \geq k$. Then*

$$sp(\mathbf{w}; d\,\mathbf{x}, c)/d^2 \;\to\; \sigma^+(\mathbf{w})\,\chi(\mathbf{x}) \textit{as } d \to \infty.$$

Proof For each $\delta > 0$, let $\mathbf{x}_\delta^+$ denote the c-vector

$$(x_0 + \delta, x_1 + \delta, \ldots, x_{k-1} + \delta, \delta, \ldots, \delta),$$

and let $\mathbf{x}_\delta^-$ be the k-vector $\mathbf{x} - \delta\,\mathbf{1}$, where $\mathbf{1}$ denotes the k-vector of 1's.

Let $\epsilon > 0$. By the continuity of the function $\chi(\mathbf{x})$ (noted after Theorem 4.5 above), there is a $\delta > 0$ such that $\chi(\mathbf{x}_\delta^+) \leq \chi(\mathbf{x}) + \epsilon$, $\mathbf{x}_\delta^- \geq \mathbf{0}$, and $\chi(\mathbf{x}_\delta^-) \geq \chi(\mathbf{x}) - \epsilon$. Replace each point $v \in V$ by w_v distinct points such that the distance between each new point and v is less than $\delta/2$, and such that all the new points are distinct. We obtain a set V_δ of points in the plane with $\sigma^+(V_\delta) = \sigma^+(\mathbf{w})$. We have

$$sp(\mathbf{w}; d\,\mathbf{x}, c) \;\leq\; sp(V_\delta; d\,\mathbf{x}_\delta^+)$$

for all $d > 0$, and since $c \geq k$

$$sp(\mathbf{w}; d\,\mathbf{x}, c) \;\geq\; sp(V_\delta; d\,\mathbf{x}_\delta^-)$$

for all $d \geq 1$. But now Theorem 4.5 above yields

$$\begin{aligned}\limsup_{d\to\infty} sp(\mathbf{w}; d\,\mathbf{x}, c)/d^2 \;&\leq\; \limsup_{d\to\infty} sp(V_\delta; d\,\mathbf{x}_\delta^+)/d^2 \\ &=\; \chi(\mathbf{x}_\delta^+)\,\sigma^+(V_\delta) \;\leq\; (\chi(\mathbf{x}) + \epsilon)\cdot\sigma^+(\mathbf{w}),\end{aligned}$$

and

$$\begin{aligned}\liminf_{d\to\infty} sp(\mathbf{w}; d\,\mathbf{x}, c)/d^2 \;&\geq\; \liminf_{d\to\infty} sp(V_\delta; d\,\mathbf{x}_\delta^-)/d^2 \\ &=\; \chi(\mathbf{x}_\delta^-)\,\sigma^+(V_\delta) \;\geq\; (\chi(\mathbf{x}) - \epsilon)\cdot\sigma^+(\mathbf{w}).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\begin{aligned}\chi(\mathbf{x})\,\sigma^+(\mathbf{w}) \;&\leq\; \liminf_{d\to\infty} sp(\mathbf{w}; d\,\mathbf{x}, c)/d^2 \\ &\leq\; \limsup_{d\to\infty} sp(\mathbf{w}; d\,\mathbf{x}, c)/d^2 \;\leq\; \chi(\mathbf{x})\,\sigma^+(\mathbf{w}),\end{aligned}$$

which completes the proof. □

5 Co-channel constraints with demands

In this section we return to the simple case when there are only co-channel constraints, and when the transmitters are located at vertices of a triangular lattice (with hexagonal cells) in the plane. However, now we allow transmitters to demand more than one channel, as at the end of last section. The number w_v of channels demanded at transmitter v may vary between transmitters. We assume that vertices that are adjacent in the lattice graph must not be assigned the same channel, so as to avoid interference.

The channel assignment problem described above is a 'weighted colouring' problem on the triangular lattice, where the weights correspond to the demands. A *weight vector* for a graph G is a non-zero vector $\mathbf{w}$ of non-negative integers w_v indexed by the vertices v of G. Given a graph G and a weight vector $\mathbf{w}$, a *weighted colouring* of the pair $(G, \mathbf{w})$ is a family of stable (or independent) sets with multiplicities such that each vertex v is in w_v of these sets. The least value of the total number of sets (counting multiplicities) for which there is such a colouring is the *weighted chromatic number* of $(G, \mathbf{w})$. A weighted colouring may also be thought of as an assignment to each vertex v of a set of w_v colours such that adjacent vertices receive disjoint sets of colours. The weighted chromatic number is then the least number of colours used in such a colouring. The *weighted colouring problem* is to find a weighted colouring using 'few' colours.

There is a natural graph G_w associated with a pair $(G, \mathbf{w})$ as above, obtained by replacing each vertex v by a complete graph on w_v vertices. Weighted colourings of the pair $(G, \mathbf{w})$ correspond to usual vertex colourings of the graph G_w, and the weighted chromatic number of $(G, \mathbf{w})$ is the chromatic number $\chi(G_w)$ of G_w.

We are interested here in weighted colourings of finite induced subgraphs of the triangular lattice graph, as this corresponds precisely to the basic channel assignment problem described above. We present two theorems from (McDiarmid and Reed 2000): the first is a hardness result and the second concerns algorithmic approximation.

Theorem 5.1 *It is NP-complete to determine, on input an induced subgraph G of the triangular lattice graph together with a corresponding weight vector* $\mathbf{w}$*, if the graph G_w is 3-colourable.*

In fact the proof may be easily extended to show that, for any fixed $k \geq 3$, it is NP-complete to tell if G_w is k-colourable. Observe that the graphs G_w here are all unit disk graphs, so this result refines the recent theorem of (Gräf *et al.* 1998) that it is NP-complete to tell if a unit disk graph is k-colourable.

Although it is hard to determine $\chi(G_w)$ for a graph G_w as above, it is of course easy to find the maximum size $\omega(G_w)$ of a complete subgraph of G_w in polynomial time. The next result is also from (McDiarmid and Reed 2000). Similar results have been put forward in (Jordan and Schwabe 1996; Narayanan and Shende 2001). An extension involving also co-site constraints is given in (Schnable *et al.* 1999). For related work involving co-site constraints see (Shepherd 1999; Gerke 2000a).

Theorem 5.2 *There is a polynomial time combinatorial algorithm which, on input an induced subgraph G of the triangular lattice graph together with a corresponding weight vector* $\mathbf{w}$*, finds a weighted colouring of* $(G, \mathbf{w})$ *which uses at most* $(4\,\hat{\omega}+1)/3$ *colours, where* $\hat{\omega} = \omega(G_w)$.

The algorithm is quite simple and practical. We first compute the natural 3-colouring f of G, with values 1,2,3; and let $k = \lfloor \frac{\hat{\omega}+1}{3} \rfloor$. The algorithm then has a distributed phase, in which it constructs $3\,k$ colour sets similar to the colour sets of the colouring f; and finally a tidy-up phase, which involves colouring a weighted forest using new colours.

In the distributed phase we use the $3\,k$ colours (i, j) for $i = 1, 2, 3$ and $j = 1, \ldots, k$. For each vertex v we compute the value m_v, which is the maximum value of w_x over the neighbours x of v with $f(x) = f(v) + 1 \pmod 3$, and then compute the value $r_v = \min\{w_v - k, k - m_v\}$. We assign the 'low' colours $(f(v), 1), \ldots, (f(v), \min\{k, w_v\})$ to vertex v, and if $r_v > 0$ then we assign also the r_v 'high' colours $(f(v)+1, k-r_v+1), \ldots, (f(v)+1, k)$ borrowed from the vertices coloured $f(v)+1$. (These colours will not appear on any neighbour of v.) It turns out that the subgraph of G induced by the vertices which have not yet been fully coloured is acyclic, and so the final tidy-up phase is easy.

By using the algorithm, we can find quickly a weighted colouring for an induced subgraph of the triangular lattice such that the number of colours used is no more than about 4/3 times the corresponding clique number of G_w, and hence is no more than about 4/3 times the optimal number. Further by Theorem 5.1 above we cannot expect always to improve on this ratio.

However, perhaps we are being pessimistic. In typical radio channel assignment problems, the maximum number of channels demanded at a transmitter may be quite large. For example, the 'Philadelphia problem' described in (Gamst 1986) involves a 21 vertex subgraph of the triangular lattice with demands ranging from 8 to 77 (though it also has constraints on the channels that may be assigned to vertices at distances up to 3, and so it is not a simple weighted colouring problem). Perhaps we can improve on the ratio 4/3 if there are large demands?

We note that the 9-cycle C_9 is an induced subgraph of the triangular lattice graph. Further, for any positive integer k, if we start with a C_9 and replicate each vertex k times, we obtain a graph with clique number $2\,k$ and chromatic number $\lceil 9/4\,k \rceil$. Is this ratio 9/8 of chromatic number to clique number asymptotically worst (greatest) possible? Recent results of (Havet 2001) show that if we start with a triangle-free subgraph of the triangular lattice, then we may replace the fraction 4/3 by 7/6. For arbitrary planar triangle-free graphs, the optimal ratio is 3/2. For further discussion on this topic see (McDiarmid 2001, Gerke and McDiarmid 2001a, 2001b).

6 Assignments on lattices

In this section we return to problems concerning channel assignment in lattices. Our starting position will be Theorem 3.3 for lattices. So let L be a lattice and d a positive real number. By Theorem 3.3 we know that there exists a feasible labelling of $G(L, d)$ which is a strict tiling and which uses at most $sp(L; d) + O(d)$ labels. There is a corresponding sublattice L^* of L and a labelling of the co-sets of L^* (with all co-sets receiving a different label). Notice that this labelling of the co-sets is unimportant, apart from the fact that all labels are different, and the only conditions L^* has to satisfy is that it has minimum distance at least d and there are not too many co-sets.

Consider now the more general situation in which we have a constraint vector $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$. Since we know that we can find a strict tiling which gives a good approximation for $sp(L; d_0)$, an obvious question is if it is possible, by choosing an appropriate labelling of the co-sets of the co-channel lattice, to form a $\mathbf{d}$-feasible assignment using the same co-channel lattice L^*. A further idea behind this question is that for small values of $d_1, \ldots, d_{k-1}$ the span should be determined by the co-channel re-use distance d_0 only.

Recall that the span $sp(L; \mathbf{d})$ is the minimum n such that there exists a $\mathbf{d}$-feasible n-labelling of V. We next define $sp_t(L; \mathbf{d})$ as the minimum n such that there exists a $\mathbf{d}$-feasible n-labelling of V which is a strict tiling. This means that $sp_t(L; \mathbf{d}) \geq sp(L; \mathbf{d})$ (there are examples where $sp_t(L; \mathbf{d}) > sp(L; \mathbf{d})$, see (Van den Heuvel *et al.* 1998)). In fact, Theorem 3.3 for lattices can be formulated as $sp_t(L; d) = sp(L; d) + O(d)$. And the question above can be phrased as:

> For which constraint vectors $\mathbf{d} = (d_0, d_1, \ldots, d_{k-1})$ can we guarantee $sp_t(L; \mathbf{d}) = sp_t(L; d_0)$?

In this section we try to give a partial answer to this question. Before we can do so we take a closer look at some of the algebraic properties of lattices and strict tilings of lattices.

Let L be a lattice with basis a, b and let L^* be a sublattice of L, with basis s, t. Hence s, t is a linearly independent pair of vectors from L. As well as considering L as a geometric object, we can consider L as an infinite abelian group (with standard vector addition as the group operation) and L^* as a subgroup of L. Then the quotient group L/L^* is a finite abelian group. If $s = s_1\, a + s_2\, b$ and $t = t_1\, a + t_2\, b$, then it is an easy exercise to deduce that the order of the quotient group (also called the index of L^* in L) is the quotient of the determinants of s, t and a, b, see (Lagarias 1995). This means $|L/L^*| = |s_1\, t_2 - s_2\, t_1|$.

Using the observations above, we can view a strict tiling of L with L^* as co-channel lattice as a labelling of the quotient group L/L^* in which each element (each co-set of L^*) receives a different label. Recall that a strict tiling of L satisfying the constraint (d_0) with co-channel lattice L^* exists if and only if L^* has minimum distance at least d_0. The number of channels in such a strict tiling is $|L/L^*|$.

The following theorem gives a condition under which it is possible, starting with a strict tiling of L with co-channel lattice L^* with minimum distance at least d_0, to relabel the co-sets as to obtain a strict tiling satisfying the constraint vector $(d_0, \ldots, d_{k-1})$ (and still using $|L/L^*|$ channels).

Recall that if A and B are sets of points, λ a real number and x a point, then $\lambda A = \{\, \lambda a : a \in A \,\}$, $d(x, A) = \inf\{\, d(x, a) : a \in A \,\}$ and $d(A, B) = \inf\{\, d(a, b) : a \in A,\ b \in B \,\}$.

Theorem 6.1 *Let L be a lattice, let $\mathbf{d} = (d_0, \ldots, d_{k-1})$ be a constraint vector, and let L^* be a sublattice of L with minimum distance at least d_0. Suppose that there exists a set $B \subseteq L$ such that the following two conditions are satisfied:*

(1) The set $\{\, b + L^* : b \in B \,\}$ generates the quotient group L/L^*.

(2) $d(i\,C, L^*) \geq d_i$ for each $i = 1, \ldots, k-1$, where C denotes the convex hull of B.

Then there exists a strict tiling of L satisfying $\mathbf{d}$, using L^* as a co-channel lattice and thus with span $|L/L^*|$.

Some ideas of the proof of Theorem 6.1 will be given at the end of this section.

A drawback of Theorem 6.1 as stated above is that it assumes that a co-channel lattice is given. This is why we next give some consequences of the theorem above that make it possible to give more explicit statements on the existence of a strict tiling and on the relation between $sp_t(L; d_0)$ and $sp_t(L; \mathbf{d})$. Nevertheless, Theorem 6.1 is useful in its own right. For instance, it can form the basis for an algorithm to find channel assignments on 2-dimensional lattices.

The following is a purely geometrical consequence of the previous theorem. For a lattice L define

$$\gamma_L \;=\; \min\{\, \max\{\, \|a\|,\ \|b\|,\ \|a-b\| \,\} : a, b \text{ is a basis of } L \,\}.$$

We will give an alternative definition of γ_L later in this section. For the triangular lattice T we get $\gamma_T = 1$.

Theorem 6.2 *Let L be a lattice, let $\mathbf{d} = (d_0, \ldots, d_{k-1})$ be a constraint vector, and let L^* be a sublattice of L with minimum distance at least d_0. Suppose that there exists a point $z \in \mathbf{R}^2$ such that*

$$d(i\,z,\ L^*) \;\geq\; d_i + i\,\gamma_L \qquad \textit{for each } i = 1, \ldots, k-1.$$

Then there exists a strict tiling of L satisfying $\mathbf{d}$, using L^* as a co-channel lattice and thus with span $|L/L^*|$.

Theorem 6.2 makes it possible to derive all kinds of sufficient conditions for the existence of strict tilings, by showing that an appropriate point z exists. In order to do this, without knowing too much about the co-channel lattice L^*, we take a closer look at the geometry of lattices.

Let L be a lattice. A *minimal basis* of L is a pair $m, n \in L$ chosen such that

(1) $\|m\|$ is minimum, subject to $m \in L \setminus \{O\}$;

(2) $\|n\|$ is minimum, subject to $n \in L \setminus \{\, p\,m : p \in \mathbf{Z} \,\}$;

(3) $m \cdot n \geq 0$.

Note that a minimal basis of a lattice always exists, and is in fact a basis of the lattice. But a minimal basis is not uniquely determined: if m, n is a minimal basis, then so is the pair $-m, -n$. And lattices such as the triangular lattice with many symmetries have even more choices for a minimal basis. Also, it may be shown that if m, n is a minimal basis of a lattice L, then the constant γ_L is given by $\gamma_L = \|m - n\|$.

The problem to find a basis of a lattice satisfying certain minimality conditions such as those defined above, is a hard problem in high dimensions (see, for example, Section 5.3 of (Grötschel

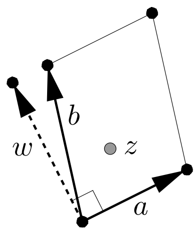

FIG. 4. Some essential points from the proof of the Theorem 6.4

et al. 1993)). But for the 2-dimensional case the so-called *Gaussian Algorithm* provides a fast way to find a minimal basis. For more information on the Gaussian Algorithm, which can be considered as a generalisation of the Euclidean Algorithm for finding greatest common divisors, see (Daudé *et al.* 1997) and references therein. In this section we need only the following property of a minimal basis which shows that if a minimal basis is known, then it is straightforward to determine the distance between a given point and the lattice.

Lemma 6.3 *Let x be a point in $\mathbf{R}^2$ and let L be a lattice with minimal basis m, n. Write $x = x_1\, m + x_2\, n$ and define $x_L = \lfloor x_1 \rfloor\, m + \lfloor x_2 \rfloor\, n$ (hence $x_L \in L$). Then*

$$d(x, L) \;=\; \min\{\, d(x, x_L),\, d(x, x_L + m),\, d(x, x_L + n),\, d(x, x_L + m + n)\,\}.$$

Now suppose we have a lattice L and a co-channel lattice L^* with minimum distance at least d. Suppose that a, b is a minimal basis of L^*. Then we know that $\|a\| \geq d$ and $\|b\| \geq d$. Let w be the projection of b on the line perpendicular to a, i.e., $w = b - (a \cdot b)\, a/\|a\|^2$, and set $z = 1/2\, a + 1/3\, w$ – see Figure 4 for a sketch of the situation. Straightforward planar geometry then shows that $d(z, 0) = d(z, a) \geq 1/\sqrt{3}\, d$, $d(z, b) \geq 1/\sqrt{3}\, d$, and $d(z, a+b) \geq \max\{d(z, 0), d(z, a), d(z, b)\}$. Then by Lemma 6.3 we can conclude $d(z, L^*) \geq 1/\sqrt{3}\, d$, which together with Theorem 6.2 proves the following result.

Theorem 6.4 *Let d be a positive real number. Then for any lattice L we have $sp_t(L; (\sqrt{3}\, d, d - \gamma_L)) = sp_t(L; \sqrt{3}\, d)$.*

By looking somewhat more closely at the arguments used in the proof of this theorem and comparing it with Theorems 3.1 and 3.3, we can show that Theorem 6.4 is in a sense best possible, in particular for the triangular lattice T.

Theorem 6.5 *Let d_0, d_1 be positive real numbers.*

For any lattice L and for all $\epsilon > 0$, if $d_0 < (\sqrt{3} - \epsilon)\, d_1$ and d_0 is large enough, then $sp_t(L; (d_0, d_1)) > sp_t(L; d_0)$.

For the triangular lattice T, if $\sqrt{3} < d_0^+ \leq \sqrt{3}\, d_1$, then $sp_t(T; (d_0, d_1)) > sp_t(T; d_0)$.

Bounds for the span of the triangular lattice similar to those in Theorems 6.4 and 6.5 appeared before in (Gamst 1982) and (Leese 1998).

Theorem 6.4 can be generalised in several directions. No further new ideas are needed to prove the following set of results.

Theorem 6.6 *Let d be a positive real number.*

Then for any lattice L, $sp_t(L; (3/\sqrt{2}\,d, d-\gamma_L, d-2\,\gamma_L)) = sp_t(L; 3/\sqrt{2}\,d)$.

And for the triangular lattice, $sp_t(T; (\sqrt{3}\,d, d-1, d-2)) = sp_t(T; \sqrt{3}\,d)$.

Theorem 6.7 *Let $\mathbf{d} = (d_0, \ldots, d_{k-1})$ be a constraint vector.*

For any lattice L, if there exists a point z such that

$$d(i\,z,\, d_0\,T) \;\geq\; \sqrt{6}/2\,(d_i + i\,\gamma_L) \qquad \text{for all } i = 1, \ldots, k-1,$$

then $sp_t(L; \mathbf{d}) = sp_t(L; d_0)$.

For the triangular lattice T, if there exists a point z such that

$$d(i\,z,\, d_0\,T) \;\geq\; d_i + i \qquad \text{for all } i = 1, \ldots, k-1,$$

then $sp_t(T; \mathbf{d}) = sp_t(T; d_0)$.

The statement about the triangular lattice in Theorem 6.6 is closely related to the formula just after Theorem 4.3 which gives a slightly better result for the case d is an integer. A similar remark holds for the relation between Theorem 4.4 and the next theorem.

Theorem 6.8 *Let $A > 2^{1/2}\,3^{-1/4}$ (≈ 1.075) and let k be a positive integer and d a positive real number such that*

$$A^2\,k\,d \;\geq\; A\,((1+\sqrt{3}/2)\,d + 2\,k - 2)\,k^{1/2} + d\,(k-2)$$

(note that this is satisfied for k and d sufficiently large). Then for the distance k-vector $(A\,k^{1/2}\,d, d, \ldots, d)$ we have

$$sp_t(T; (A\,k^{1/2}\,d, d, \ldots, d)) \;=\; sp_t(T; A\,k^{1/2}\,d).$$

We now give some ideas from the proofs of the results in this section.

Proof of Theorem 6.1 We use the terminology introduced before the statement of Theorem 6.1. Set $n = |L/L^*|$, hence n is the number of labels used in the strict tiling satisfying (d_0), and let $c_0 = O$ be the origin. Then the statement that there exists a strict tiling of L satisfying $(d_0, \ldots, d_{k-1})$ and using L^* as a co-channel lattice is equivalent to the following claim:

Claim There exists a numbering $c_0 + L^*, c_1 + L^*, \ldots, c_{n-1} + L^*$ of the elements of L/L^* such that if we define the labelling $\varphi : L \to \{1, 2, \ldots, n\}$ of L by $\varphi(v) = i + 1$ if $v \in c_i + L^*$, then this labelling satisfies $(d_0, \ldots, d_{k-1})$.

Since L^* has minimum distance at least d_0, a labelling as defined in the Claim always satisfies (d_0), no matter how the numbering is chosen. Our goal is to show that, under the assumptions of the theorem, the Claim is satisfied.

Now suppose the set $B \subseteq L$ satisfies the statements in the theorem, in particular the fact that $\{\, b + L^* : b \in B \,\}$ generates the group L/L^*. Then it follows easily, using some theory from Cayley digraphs, that we can find a sequence $b_1, b_2, \ldots, b_{n-1}$ of elements in B such that the sequence of partial sum co-sets

$$L^*,\ b_1 + L^*,\ b_1 + b_2 + L^*,\ \ldots,\ b_1 + b_2 + \cdots + b_{n-1} + L^*$$

are exactly all elements of L/L^* (see (Van den Heuvel 1998) for details). Hence the sequence $c_0 = O$, $c_1 = b_1$, $c_2 = b_1 + b_2$, …, $c_{n-1} = b_1 + b_2 + \cdots + b_{n-1}$ is a numbering of the elements of L/L^* and we will show that this numbering satisfies the Claim.

For all sequences $b_{p+1}, b_{p+2}, \ldots, b_{p+q}$, where $0 \le p < p+q \le n-1$ we have that

$$c_{p+q} - c_p \ = \ b_{p+1} + \cdots + b_{p+q} \ = \ q \cdot (\frac{1}{q}\, b_{p+1} + \cdots + \frac{1}{q}\, b_{p+q})$$

is an element of $q \cdot \mathrm{conv}(B)$. Hence by condition (2),

$$d(c_{p+q} - c_p,\, L^*) \ \ge \ d_q.$$

This last statement is equivalent to

$$d(v, w) \ \ge \ d_q \qquad \text{for all } v \in c_{p+q} + L^* \text{ and } w \in c_p + L^*.$$

Since this holds for all p, q with $0 \le p < p+q \le n-1$ the Claim, and hence the Theorem, follows. □

Proof of Theorem 6.2 Let m, n be a minimal basis of L. Let $D(z, \gamma_L) = \{\, x : d(x, z) \le \gamma_L \,\}$ be the closed ball with centre z and radius γ_L. Using the alternative definition of γ_L above Lemma 6.3, some straightforward geometry shows that there exist $b_1, b_2, b_3 \in L \cap D(z, \gamma_L)$ such that $b_2 - b_1 = m$ and $b_3 - b_1 = n$, or $b_2 - b_1 = -m$ and $b_3 - b_1 = -n$. Set $B = \{b_1, b_2, b_3\}$. It follows that both $m + L^*$ and $n + L^*$ are in the group generated by $\{b_1 + L^*, b_2 + L^*, b_3 + L^*\}$. Since $\{m, n\}$ generates the group L, $\{m + L^*, n + L^*\}$ generates the quotient group L/L^*, hence $\{b_1 + L^*, b_2 + L^*, b_3 + L^*\}$ generates L/L^* as well. This proves that B satisfies condition (1) in Theorem 6.1.

If $a \in D(z, \gamma_L)$, then $d(z, a) \le \gamma_L$ and hence

$$d(i\, a,\, L^*) \ \ge \ d(i\, z,\, L^*) - i\, \gamma_L \ \ge \ d_i \qquad \text{for all } i = 1, \ldots, k-1.$$

Since the set B lies within $D(z, \gamma_L)$, so does the convex hull of B. So certainly for every vector a in the convex hull of B the inequality above holds, which means that B satisfies condition (2) in Theorem 6.1 as well. □

Recall that a generalised lattice is a set of the form $\bigcup_{u \in U} (u + L)$, where L is a lattice and U is a finite set of points. Many of the results in this section can be generalised to generalised lattices, provided we have some additional information about the set U. The following is an extension of Theorem 6.1 which almost directly follows from the proof of that theorem.

Theorem 6.9 *Let L be a lattice, let $U = \{u_1, u_2, \ldots, u_k\}$ be a finite set of points, and let $\mathbf{d} = (d_0, \ldots, d_{k-1})$ a constraint vector. Let L^* be a sublattice of L with minimum distance at least d_0. Suppose that there exists a set $B \subseteq L$ and a set $V = \{v_2, v_3, \ldots, v_k\}$ satisfying $v_i \in u_i - u_{i-1} + L^*$ for $i = 2, \ldots, k$, such that the following two conditions are satisfied:*

(1) The set $\{ b + L^* : b \in B \}$ generates the quotient group L/L^*.

(2) $d(i\,C, L^*) \geq d_i$ for each $i = 1, \ldots, k-1$, where C denotes the convex hull of $B \cup V$.

Then there exists a strict tiling of $\bigcup_{u \in U} (u + L)$ satisfying $\mathbf{d}$, using L^* as a co-channel lattice and with span $k\,|L/L^*|$.

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