Graph Homomorphisms and Long Range Action

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Abstract

We show that if a graph H is k-colorable, then $(k-1)$-branching walks on H exhibit long range action, in the sense that the position of a token at time 0 constrains the configuration of its descendents arbitrarily far into the future.

This long range action property is one of several investigated herein; all are similar in some respects to chromatic number but based on viewing H as the range, instead of the domain, of a graph homomorphism.

The properties are based on combinatorial forms of probabilistic concepts from statistical physics, although we argue that they are natural even in a purely graph-theoretic setting. They behave well in many respects, but quite a few fundamental questions remain open.

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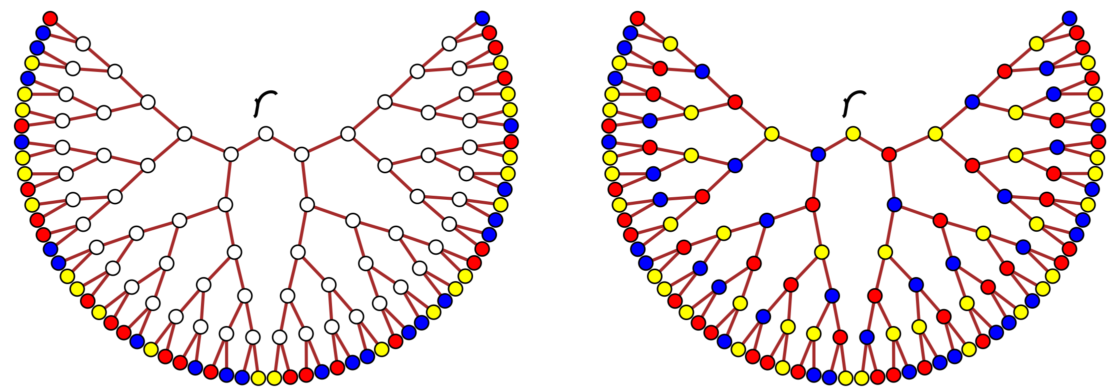

Figure 1: Reconstructing a 2-branching walk on K_3

1 Introduction

Suppose some token takes a walk on a connected graph H, stepping from node to adjacent node at each tick of a clock. If we know its position at time 0, can we deduce anything about its position at time t for large t?

Certainly we can if H is bipartite, and conversely if $\chi(H) > 2$ then for large t ($t \geq 2|H|-2$ will do) the token could be anywhere. Thus in the realm of connected graphs we could take the property to be an alternate definition of 2-colorability, and seek an extension analogous to k-colorability.

Suppose, for instance, that $H = K_3$ and imagine that the token takes a *2-branching* walk on H; at each step the token divides into two (labeled) tokens both of which step at the next tick, so that one token at time 0 yields 2^t at time t, many of which may occupy the same node. If it happens that at each time, the two "children" of each token take different steps, then it is clear that the positions of the 2^t descendents will uniquely determine the starting node. We take this property of the constraint graph K_3 as analogous to being 3-colorable.

A d-branching walk on H is nothing more than a graph homomorphism from the complete d-branching tree T^d to H. (We define T^d to be the regular Cayley tree of degree $d{+}1$, although it is often convenient to assume, as here, that the root r has degree only d). Fig. 1 illustrates the reconstruction of a particular 2-branching walk on K_3, viewed as a 3-coloring of T^2, from positions at time $t = 6$.

We denote the set of homomorphisms from a graph G to a graph H by $\mathrm{Hom}(G, H)$; later we will endow $\mathrm{Hom}(G, H)$ with its own graph structure. To avoid confusion we will call vertices of H "nodes" (usually denoted by a, b or c) and vertices of G "sites" (usually denoted by x, y or z). In this context G, which is often infinite but always countable and locally finite, will be called the "board" and the elements of $\mathrm{Hom}(G, H)$ *labelings* of G.

To simplify notation we will often confuse a graph with its set of vertices. The "constraint graph" H will always be finite and connected, and (unlike G) may have some loops; a loop at a node $a \in H$ allows a homomorphism in $\mathrm{Hom}(G, H)$ to affix the label a to adjacent sites of

G. Some of what follows will be uninteresting for looped constraint graphs, however, for the reason that a looped node causes the chromatic number to become infinite. We do require that H contain at least one edge, thus it cannot consist of a single unlooped node.

We will say that $\mathrm{Hom}(G, H)$ *exhibits long range action* if, for every $k > 0$, there are sets X and Y of sites of G with $d(X, Y) \geq k$, and maps φ and ψ in $\mathrm{Hom}(G, H)$, such that no $\theta \in \mathrm{Hom}(G, H)$ agrees with φ on X and with ψ on Y. Thus what happens in one part of G can constrain what happens far away. Our "long range action" property is precisely the *negation* of the property "strongly irreducible" of [1].

If $\mathcal{G}$ is a *class* of graphs, then $\langle \mathcal{G}, H \rangle$ is said to exhibit long range action if $\mathrm{Hom}(G, H)$ exhibits long range action for some $G \in \mathcal{G}$.

When G is a Cayley tree, we shall show that we can always take X to consist just of the root. A map φ in $\mathrm{Hom}(T^d, H)$ is said to be *cold* if there is a node a of H such that for any k, no $\psi \in \mathrm{Hom}(T^d, H)$ agrees with φ on the sites at distance k from the root r but has $\psi(r) = a$. $\mathrm{Hom}(T^d, H)$ itself is said to be cold if it contains a cold map, i.e. if some label can be forbidden at the root by values arbitrarily far away. We will see later that $\mathrm{Hom}(T^d, H)$ is cold if and only if it exhibits long range action.

It is easy to verify that if $\mathrm{Hom}(T^d, H)$ is cold then so is $\mathrm{Hom}(T^{d'}, H)$ for any $d' > d$. We say that H is $(d{+}1)$-*warm* if $\mathrm{Hom}(T^{d-1}, H)$ is *not* cold, and define the warmth $w(H)$ of H to be the greatest d such that H is d-warm; equivalently, the least d such that $\mathrm{Hom}(T^{d-1}, H)$ is cold. If H is d-warm for every d, we say $w(H) = \infty$. Every constraint graph H is 2-warm, and $w(H) = 2$ if and only if H is bipartite.

Although we hope to persuade the reader that this notion is natural and interesting within graph theory, some explanation of its origin may be in order here. A statistical system in physics is said to exhibit "long range order" if its state in one region of space gives non-disappearing information about its state in other regions far away. Thus, for example, a magnetized bar exhibits long range order because the spin of a particle at one end is correlated with the spin of a particle at the other end.

In our system the space is an infinite graph G, often the discrete Cayley tree T^d; the states are nodes of H constrained by requiring adjacent sites to be in adjacent states. Our combinatorial notion of long range action is in a sense stronger than the probabilistic notion from physics, and produces what what we call "frozen" or "semi-frozen" Gibbs measures in [3] to which, along with [2] and [4], the reader is referred for a more complete explanation.

2 Warmth

We establish first that warmth and chromatic number coincide for complete graphs.

Theorem 2.1. *For any integer $d \geq 2$, the complete graph K_d has warmth d.*

Proof. We have already observed, in effect, that $\mathrm{Hom}(T^{d-1}, K_d)$ is cold; a cold map is obtained by assigning all possible labels to the children of each site. To complete the proof we need to show that $\mathrm{Hom}(T^d, K_d)$ is *not* cold; but this is obvious because any labeling of a site and its grandchildren can be extended to its children. $\square$

It will be useful to note the very nice behavior of warmth with respect to products and retracts. The product at issue is the "categorical" graph product, in which $(a, a') \sim (b, b')$ in $H \times H'$ if and only if $a \sim b$ in H and $a' \sim b'$ in H'. A *retraction* of H is a homomorphism ρ from H to some (necessarily induced) subgraph H^- of H, called a *retract*, such that $\rho \restriction H^-$ is the identity.

Theorem 2.2. (i) *For any H and H', $w(H \times H') = \max(w(H), w(H'))$; (ii) If H^- is a retract of H then $w(H^-) \geq w(H)$.*

Proof. Both parts are just a matter of chasing down the definition. A map in $\mathrm{Hom}(T^d, H \times H')$ is nothing more or less than the pointwise product of maps in $\mathrm{Hom}(T^d, H)$ and $\mathrm{Hom}(T^d, H')$ and is cold precisely if one of the two factor maps is cold. For the second statement, let ρ retract H to H^- and suppose φ is a cold map in $\mathrm{Hom}(T^d, H^-)$ which forbids the label $a \in H^-$ at the root r. Then φ, regarded as a homomorphism to H, also forbids a at r within $\mathrm{Hom}(T^d, H)$, because if $\theta \in \mathrm{Hom}(T^d, H)$ agrees with φ on the sites at distance k from r and $\theta(x) = a$ then $\rho \circ \theta$ contradicts the coldness of φ. $\square$

We remark that the chromatic number analog of statement (ii), namely "$\chi(H^-) \geq \chi(H)$", holds with equality. However, for the analog of statement (i), "$\chi(H \times H') \leq \min(\chi(H), \chi(H'))$" is easy but equality is a notoriously open conjecture of Hedetniemi [6].

It follows from Theorem 2.2 that $w(H) \leq \chi(H)$ whenever H contains a clique of size $\chi(H)$, since a $\chi(H)$-coloring of H can then be regarded as a retraction. In fact, we will see later that the clique condition can be dropped.

If A is a subset of (the nodes of) a constraint graph H, we let $N(A) := \{b \in H \mid b \sim a \text{ for some } a \in A\}$. A collection $\{A_1, \ldots, A_s\}$ of subsets of H is said to *produce* a subset A of H if $\bigcap N(A_i) = A$. The idea is that, if what we know about the labels of s neighbors of a site x is that neighbor x_i has a label from the set A_i, then what we can deduce is exactly that x has a label from A.

Theorem 2.3. *Given a constraint graph H and a natural number $d \geq 1$, the following are equivalent:*

(i) $\mathrm{Hom}(T^d, H)$ *exhibits long range action;*

(ii) *there is a cold H-labeling of T^d (i.e., H is $(d+2)$-warm);*

(iii) *there is a d-stable family of subsets of H;*

(iv) *there is no ordering $A_1, \ldots, A_N$ of the non-trivial subsets of H such that each d-tuple $(A_{i_1}, \ldots, A_{i_d})$ of sets produces either $\emptyset$, H, or a set A_j with $j > \min\{i_k\}$.*

Proof. $(ii) \implies (i)$ is obvious, taking $X = \{r\}$ and Y the sites at distance k from the root. We are using rather degenerately the fact that H is connected since otherwise the forbidden label a might be an isolated node of H, preventing us from constructing φ.

$(iii) \implies (ii)$. Suppose there is a d-stable family $\mathcal{A}$. Then we define an H-labeling of T^d as follows. Associate the root r with a pair (A_r, a_r) where $A_r \in \mathcal{A}$ and $a_r \in A_r$. Now work out from the root. If y is associated with a pair (A_y, a_y), and $y_1, \ldots, y_d$ are the children of y, let $\{A_{y_1}, \ldots, A_{y_d}\}$ be a collection of sets from $\mathcal{A}$ producing A_y, and let a_{y_i} be a node in A_{y_i} adjacent to a_y. When all the sites of T^d have been treated in this manner, the labeling ψ where $\psi(y) = a_y$ for each y is a member of $\mathrm{Hom}(T^d, H)$. Furthermore, for any $k \in \mathbb{N}$ and any H-labeling θ such that $\theta \restriction (T^d \backslash N_k(x)) = \psi \restriction (T^d \backslash N_k(x))$, we see (by working in towards the root) that $\theta(y) \in A_y$ for every $y \in N_k(x)$. In particular, $\theta(x)$ can only take values in $A_x \neq H$, so ψ is a cold H-labeling of T^d.

$(iv) \implies (iii)$. Suppose there is no d-stable family and let $\mathcal{A}_0$ be the family of all non-trivial sets; this is not a d-stable family, so there is some set $A_1 \in \mathcal{A}_0$ that cannot be produced by d sets in $\mathcal{A}_0$. Now let $\mathcal{A}_1 = \mathcal{A}_0 \backslash \{A_1\}$, and continue, thus generating an ordering which contradicts (iv).

$(ii) \implies (iv)$. Suppose $A_1, \ldots, A_N$ is an ordering forbidden by (iv), where $N = 2^{|H|} - 2$. Let ψ be any H-labeling of T^d. For any site x at distance $\ell < N$ from r, let S_x be the set of sites y at distance N from r such that x is on the r-y path, and let C_y be the set of nodes b such that there is an extension of $\psi \restriction S_x$ in which x gets label b. Note that C_x is never empty, since it contains $\psi(x)$. Also note that C_x is exactly the set produced by $\{C_{x_1}, \ldots, C_{x_d}\}$. Therefore, by induction, either $C_x = H$ or $C_x = A_j$ for some $j \geq N - \ell + 1$. In particular, $C_r = H$, so that all labels are possible for x, and ψ is not cold.

$(i) \implies (ii)$. Suppose from now on that $\mathrm{Hom}(T^d, H)$ exhibits long range action.

Fix a $k \in \mathbb{N}$. Suppose that, for all *finite* sets X, whenever Y is a set with $d(X, Y) \geq k$, and $\varphi, \psi \in \mathrm{Hom}(T^d, H)$, then there is some $\theta \in \mathrm{Hom}(T^d, H)$ extending both $\varphi \restriction X$ and $\psi \restriction Y$. We claim that (T^d, H) fails to exhibit action at distance k, which will be a contradiction. Indeed, let $Z = \{z_1, z_2, \ldots, \}$ be an infinite set of sites, let Y be a set with $d(Z, Y) \geq k$, and let φ, ψ be homomorphisms. Set $Z_n = \{z_1, \ldots, z_n\}$, for all n. Then there are homomorphisms $\theta_1, \theta_2, \ldots$ such that each θ_n agrees with φ on Z_n and with ψ on Y. Now there is some subsequence of (θ_n) that tends to a limit, and this limit is a homomorphism that agrees with φ on all of Z and with ψ on Y, as required.

Therefore, for each $k \in \mathbb{N}$, there is a finite set X, a set Y with $d(X, Y) \geq 2k$, and homomorphisms φ, ψ such that $\varphi \restriction X$ and $\psi \restriction Y$ cannot be simultaneously extended. We can also take Y to be finite (for instance we can assume it consists of the sites at distance exactly $2k$ from X). Now let X be minimal with this property, and then take Y minimal with the property. Take any shortest path from X to Y, and let x be the site on this path at distance k from X. The site x is thus at distance at least k from $X \cup Y$, and separates $X \cup Y$. By minimality of X and Y, for each branch B from x there is some homomorphism θ_B of $\mathrm{Hom}(T^d, H)$ such that θ_B agrees with φ on $X \cap B$ and with ψ on $Y \cap B$. Since we cannot glue these homomorphisms together to make a homomorphism θ extending all of $\varphi \restriction X$ and $\psi \restriction Y$, there must be some branch B and some label $a \in H$ such that there is no

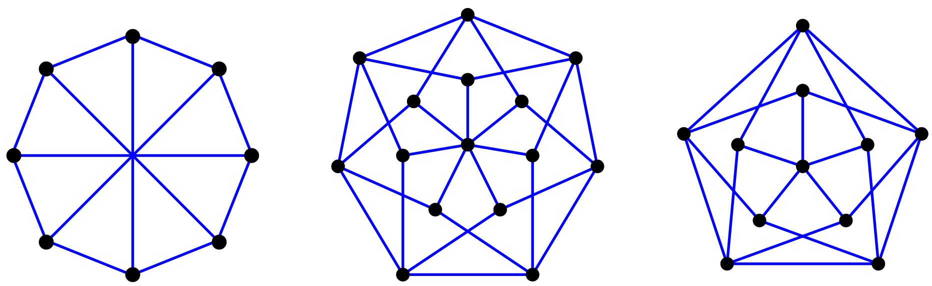

Figure 2: H_8, G_7 and G_5

homomorphism θ with $\theta(x) = a$, agreeing with θ_B on $(X \cup Y) \cap B$.

Therefore, for all k, there is a singleton set x (which we may take to be the root of T^d in each case), a set Y_k with $d(x, Y_k) \geq k$ (which we may take to be $T^d \backslash N_k(x)$), a label a_k, and a homomorphism ψ_k, such that there is no θ with $\theta \restriction Y_k = \psi_k \restriction Y_k$ and $\theta(x) = a_k$. By taking a subsequence, we may assume that the label a_k is always equal to a. Now there is some subsequence of the ψ_k that tends to a limit ψ. It is now clear that ψ is a cold H-labeling of T^d. $\square$

3 Examples

We have seen that $w(K_d) = d$; note that the singletons comprise a $(d-1)$-stable family. In fact in any d-colorable graph the color classes are candidates for being a $(d-1)$-stable family; if the graph is *uniquely* d-colorable then they indeed will be, since every node is then adjacent to nodes of all other colors. We could extend this argument to show that containing a uniquely d-colorable graph is enough to prevent $(d+1)$-warmth, but, again, we will prove a stronger result later.

Having girth at least 5, even for looped constraint graphs, already prevents 4-warmth. For, let $a_1 a_2 \cdots a_s a_1$ be a shortest cycle in H, with $s \geq 5$, and let $\mathcal{A} = \{\{a_1\}, \ldots, \{a_s\}\}$; then $\{a_i\}$ is produced by $\{\{a_{i-1}\}, \{a_{i+1}\}\}$. Indeed, this construction works whenever there is a long cycle in H not sharing two consecutive edges with a 4-cycle (or 3-cycle, when loops are present). We can use this idea to produce examples H that are not 4-warm but have arbitrarily large cliques, for instance. This in turn shows that we can have $w(H^-) > w(H)$ for H^- a retract of H (cf. Theorem 2.2).

For another example that is not 4-warm, consider the graph H_8 defined by taking an 8-cycle $a_1 a_2 \cdots a_8 a_1$, and adding the four "diagonal" edges $\{a_i a_{i+4}\}$ (see left-hand side of Fig. 2.) This graph has chromatic number 3, and is not 4-warm; the family of singletons is 2-stable, since each $\{a_i\}$ is produced by $\{\{a_{i-1}\}, \{a_{i+1}\}\}$. However, this family has no obvious connection to a 3-coloring, or to a shortest odd cycle. There is another quite different 2-stable family, namely $\{\{a_1, a_4\}, \{a_2, a_5\}, \ldots, \{a_8, a_3\}\}$; note that $\{\{a_8, a_3\}, \{a_2, a_5\}\}$ produces $\{a_1, a_4\}$.

For the next two examples we use the Grotszch construction, defined as follows: if H has vertices $a_1, \ldots, a_n$ then $\mathrm{Gr}(H)$ has vertices $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ and c; if $a_i \sim a_j$ in H then $a_i \sim a_j$ and $b_i \sim a_j$ in $\mathrm{Gr}(H)$, and c is adjacent to every b_i. One of the nice properties of the construction is that always $\chi(\mathrm{Gr}(H)) = 1 + \chi(H)$.

In the graph $G_7 := \mathrm{Gr}(C_7)$ (center, Fig. 2) the family

$$\mathcal{A} = \{\{a_1\}, \{a_2\}, \ldots, \{a_7\}, \{b_7, b_2\}, \{b_1, b_3\}, \ldots \{b_6, b_1\}, \{c\}\}$$

is 2-stable: for instance $\{a_1\}$ is produced by $\{\{a_7\}, \{b_2, b_4\}\}$, $\{b_7, b_2\}$ is produced by $\{\{a_1\}, \{c\}\}$, and $\{c\}$ is produced by $\{\{b_1, b_3\}, \{b_2, b_4\}\}$. Therefore G_7 is not 4-warm. However, there is no 2-stable family consisting of *disjoint* sets—our later proof will show that disjoint sets are always obtainable if H is 3-colorable.

The standard Grotszch graph $G_5 := \mathrm{Gr}(C_5)$ (right-hand side of Fig. 2) also fails to be 4-warm. One nice 2-stable family is

$$\mathcal{A} = \{\{a_1\}, \ldots, \{a_5\}, \{a_1, b_1\}, \ldots, \{a_5, b_5\}, \{b_1, b_3\}, \{b_2, b_4\}, \ldots, \{b_5, b_2\}, \{c\}\} \ .$$

Notice that $\mathcal{A}$ contains some pairs of sets related by inclusion, and simply taking the inclusion-minimal elements of $\mathcal{A}$ does *not* yield a 2-stable family. Note however that, for every set A in $\mathcal{A}$, there is an element $u \in A$ such that A is minimal in $\mathcal{A}$ subject to containing u—we say that A is *semi-minimal* in $\mathcal{A}$. In general, if $\mathcal{A}$ is any d-stable family in a graph H, the family of semi-minimal sets in $\mathcal{A}$ is again d-stable.

We expect that there are constraint graphs where inclusion relations in a d-stable family are unavoidable. However, for G_5, a careful pruning of the family $\mathcal{A}$ leaves the smaller 2-stable family

$$\{\{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_1, b_1\}, \{b_1, b_3\}, \{b_1, b_4\}, \{b_2, b_5\}, \{c\}\} \ ,$$

where no pair of sets is related by inclusion.

An example of a graph that *is* 4-warm is the 5-wheel W_5 (right-hand side of Fig. 3). To see this, let $a_1, \ldots, a_5$ be the nodes of the 5-cycle in W_5, each attached to the center c. Order the subsets of W_5 by size, beginning with the smaller sets, and listing those that contain c after those that do not. It is straightforward to check that each pair of sets produces a set later in this order than the earlier of the two sets.

4 Circular Chromatic Number

In the next section, we shall prove that $w(H) \leq \chi(H)$ for all unlooped constraint graphs H. To motivate our approach, we begin by considering the case of chromatic number 3, when we will be able to prove a little more. In a sense, it is not $\chi(H) \leq 3$ that forces 3-warmth but $\chi(H) < 4$.

For integers $1 \leq k \leq d$, a (k, d)*-coloring* of a loopless graph H is a function $\varphi : H \to \{0, \ldots, k-1\}$ such that, whenever $a \sim b$, $d \leq |\varphi(a) - \varphi(b)| \leq k-d$. The *circular chromatic* *number* $\chi_c(H)$ of H is the infimum of the set $\{k/d : \text{ there is a } (k,d)\text{-coloring of } H\}$. This parameter is also known as the *star-chromatic number*.

The following facts were established by Vince [14].

- For all H, $\chi(H) = \lceil \chi_c(H) \rceil$.
- The infimum in the definition of $\chi_c(H)$ is always attained; furthermore if $\chi_c(H) = k/d$ with k and d positive and relatively prime, then there is a (k,d)-coloring of H.
- If φ is a (k,d)-coloring of a graph H with $\chi_c(H) = k/d$, there is a cycle $a_0 a_1 \cdots a_{s-1} a_0$ in H such that $\varphi(a_{i+1}) \equiv \varphi(a_i) + d \pmod{k}$. We call such a cycle *tight*.

Zhu [15] has written a useful survey of work relating to the circular chromatic number.

Theorem 4.1. *If H is an unlooped constraint graph, with circular chromatic number less than 4, then there is a 2-stable family of disjoint subsets of H, and so H is not 4-warm.*

Proof. Take a (k,d)-coloring φ of H, where $\chi_c(H) = k/d < 4$ and k and d are relatively prime. Let $a_0 a_1 \cdots a_{s-1} a_0$ be a tight cycle C, with $\varphi(a_{i+1}) \equiv \varphi(a_i) + d \pmod{k}$. Without loss of generality, $\varphi(a_0) = 0$, so that $\varphi(a_i) \equiv id \pmod{k}$, and therefore s is a multiple of k.

For $i = 0, \ldots, k-1$, let C_i be the set of nodes of C that are assigned color id, so $C_i = \{a_i, a_{i+k}, a_{i+2k}, \ldots, a_{i+s-k}\}$. Now define the sets $A_i \supseteq C_i$ recursively by putting node a in A_i whenever there are nodes a^- and a^+ of H with $a^- \in A_{i-1}$ and $a^+ \in A_{i+1}$.

We claim that $\varphi(j) = id$ for every $a \in A_i$. We establish this recursively; it is true for $a \in C_i$, and if a is adjacent to nodes $a^- \in A_{i-1}$ and $a^+ \in A_{i+1}$ for which the claim is true, then $\varphi(j) \notin ((i-2)d, id)$ and $\varphi(j) \notin (id, (i+2)d)$; since $k < 4d$, id is the only color available for a.

This proves that the A_i are disjoint. By construction, $\{A_0, \ldots, A_{k-1}\}$ is a 2-stable family, with $\{A_{i-1}, A_{i+1}\}$ producing A_i for each i.

This completes the proof. □

5 Warmth and Chromatic Number

We are now in a position to connect these two parameters, one of which treats H as the range of a homomorphism, the other as the domain.

Theorem 5.1. *For every unlooped H, the warmth of H is at most its chromatic number.*

Proof. Let $\chi(H) = d+1$ with the object of constructing a cold labeling of T^d by H. We begin by finding a map Ψ from H to the unit vectors of $\mathbb{R}^d$ with the following property:

$$\text{If } u \sim v \text{ in } H \text{ then } \Psi(u) \cdot \Psi(v) < 0 \; .$$

To do this we fix a regular simplex in $\mathbb{R}^d$ which is centered at the origin and sized so that its vertices are unit vectors, then color H properly with $d{+}1$ colors and replace each color by a different vertex of the simplex.

Now that we know such a map exists, we fix such a Ψ which maximizes

$$\alpha = \min\{-\Psi(u)\cdot\Psi(v) \,\big|\, u \sim v\}$$

and then minimizes the number of edges $\{u, v\}$ of H for which $\Psi(u)\cdot\Psi(v) = -\alpha$. We call these edges "tight", likewise any node incident to a tight edge. Note that, in the case $d = 2$, a map with the property above exists if and only if $\chi_c(H) < 4$; in this case, the tight edges and nodes are those lying on some tight cycle. Thus our approach here generalizes that in Theorem 4.1.

We now digress slightly to prove a geometric lemma. We say that a set A of vectors *forces* $\vec{v}$ if $\vec{v}$ is a unit vector which satisfies $\vec{v}\cdot\vec{u} \le -\alpha$ for every $\vec{u} \in A$, but no unit vector $\vec{w}$ satisfies $\vec{w}\cdot\vec{u} < -\alpha$ for every $\vec{u} \in A$. We say that A *fixes* the unit vector $\vec{v}$ if (a) $\vec{v}\cdot\vec{u} = -\alpha$ for every $\vec{u} \in A$, and (b) if $\vec{w}\cdot\vec{u} \le -\alpha$ for every $\vec{u} \in A$ then $\vec{w} = \vec{v}$.

Lemma 5.2. *Fix α with $0 < \alpha < 1$, and let A be a finite set of unit vectors in $\mathbb{R}^d$ which forces a certain unit vector $\vec{v}$. Then there is a subset C of A with $|C| \le d$ which fixes $\vec{v}$.*

Proof. We may assume without loss of generality that $\vec{v} = (1, 0, \ldots, 0)$; let B consist of those members of A which lie on the $(d{-}2)$-sphere

$$S := \{\vec{w} \,\big|\, |\vec{w}| = 1,\ w_1 = -\alpha\}\ .$$

We claim that every closed hemisphere of S contains some $\vec{u} \in B$.

If not, we may assume the hemisphere $S^- := \{\vec{w} \in S \,\big|\, w_2 \le 0\}$ is missed, so that $u_2 > 0$ for $\vec{u} \in B$. For $\varepsilon > 0$ let

$$\vec{v}(\varepsilon) := (\sqrt{1-\varepsilon^2}, -\varepsilon, 0, 0, \ldots, 0)$$

so that

$$\vec{u}\cdot\vec{v}(\varepsilon) = -\alpha\sqrt{1-\varepsilon^2} - \varepsilon u_2 < -\alpha$$

for $\vec{u} \in B$ and sufficiently small ε. If we also take ε small enough so that $\vec{u}\cdot\vec{v}(\varepsilon) < -\alpha$ for those $u \in A \backslash B$, we have a contradiction to A forcing $\vec{v}$.

It follows by the separating hyperplane theorem that the point $\vec{z} := (-\alpha, 0, 0, \ldots, 0)$ lies in the convex hull of B. Moreover, since the dimension of the hyperplane defined by $w_1 = -\alpha$ is $d{-}1$, Carathéodory's Theorem (see e.g. [9]) tells us that there is a subset $C \subseteq B$ of size at most d such that $\vec{z}$ already lies in the convex hull of C.

If some vector $\vec{w}$ satisfies $\vec{w}\cdot\vec{u} \le -\alpha$ for each $\vec{u} \in C$ then it also satisfies $\vec{w}\cdot\vec{z} \le -\alpha$, thus $w_1 \ge 1$, and cannot be a unit vector unless $\vec{w} = \vec{v}$. Hence C fixes $\vec{v}$. $\square$

We are now ready to finish the proof of Theorem 5.1. We define a labeling of T^d by H as follows.

Choose any tight node $u \in H$ to label the root r. By Lemma 5.2 there is a set X of at most d (tight) neighbors of u whose images under Ψ fix $\Psi(u)$. Use all the elements of X to label the children of r and proceed in like fashion to label the rest of T^d.

To see that this labeling is cold, imagine that all sites at distance less than k are unlabeled, and then relabeled in some consistent fashion. We claim that at every site the old and new labels have the same image under Ψ. This can be seen by induction working in from distance $k-1$.

It follows that $\mathrm{Hom}(T_d, H)$ is cold. $\square$

The methods used in the above proof call to mind the *vector chromatic number* of a graph H, defined by Karger, Motwani and Sudan [8] as the minimum k such that there exists a labeling Ψ of $V(H)$ by unit vectors in $\mathbb{R}^{|V(H)|}$ in which adjacent nodes u and v satisfy $\langle \Psi(u), \Psi(v) \rangle \leq -\frac{1}{k-1}$. Karger, Motwani and Sudan show that the vector chromatic number of H can be approximated arbitrarily closely in randomized polynomial time. They also show that the vector chromatic number of H is at most $\vartheta(\overline{H})$, where ϑ is the Lovász theta-function (see for instance Grotschel, Lovász and Schrijver [5]). For us, the key issue is not the extremal value of the inner product, but the minimum *dimension* in which the inner product of adjacent nodes can be made negative, and we know of no connection between warmth and the Lovász theta-function.

Our particular version of "vector labeling" has occurred before in a very different context connected with the Ramsey number $R(3, 3, \ldots, 3)$; see for instance the survey article by Nešetřil and Rosenfeld [10]. In the language of that paper, we are interested in the minimum d such that the graph H is α-embeddable in $\mathbb{R}^d$ for some $\alpha > \sqrt{2}$.

We now introduce two new graph parameters, "heat" and "mobility", but warmth will remain in the picture.

6 Heat

If in condition (i) of Theorem 2.3 the d-regular tree T^{d-1} is replaced by a general graph of maximum degree d, we obtain a strengthening of the notion of warmth as follows.

Let $\mathcal{G}_d$ be the class of all (locally finite) graphs of maximum degree at most d. A constraint graph H is said to be *d-hot* if the pair $\langle \mathcal{G}_{d-1}, H \rangle$ does not exhibit long range action, i.e., if $\mathrm{Hom}(G, H)$ does not exhibit long range action for any board of maximum degree at most $d-1$. The *heat* $h(H)$ of H is the greatest d such that H is d-hot, or, equivalently, the least d such that $\mathrm{Hom}(G, H)$ exhibits long range action for some board of maximum degree d.

Theorem 6.1. *The following are equivalent for any $d \geq 2$ and any constraint graph H:*

(i) *for all $k > 0$ there is a graph $G \in \mathcal{G}_{d-1}$ such that* $\mathrm{Hom}(G, H)$ *exhibits action at distance k;*

(*ii*) *H is not d-hot (i.e., there is a single graph $G \in \mathcal{G}_{d-1}$ such that* $\mathrm{Hom}(G, H)$ *exhibits action at all distances k);*

(*iii*) *there is a graph $G \in \mathcal{G}_{d-1}$ such that, for all $k > 0$, there are* finite *witnesses $X, Y \subset G$ to action at distance k.*

Proof. Clearly we have $(iii) \implies (ii) \implies (i)$.

$(i) \implies (ii)$. Suppose that (i) holds and choose, for each $k > 0$, a $G_k \in \mathcal{G}_{d-1}$ together with subsets X_k and Y_k and maps φ_k and ψ_k in $\mathrm{Hom}(G_k, H)$ with no common extension of $\varphi_k \restriction X_k$ and $\psi \restriction Y_k$. If H is d-hot there is a distance k' which is enough to eliminate action on the disjoint union $G := \bigcup_{k=1}^{\infty} G_k$, but the sets $X_{k'}$ and $Y_{k'}$, and extensions to G of the maps $\varphi_{k'}$ and $\psi_{k'}$, testify otherwise.

$(ii) \implies (iii)$. Take a board $G \in \mathcal{G}_{d-1}$ such that $\mathrm{Hom}(G, H)$ exhibits long range action. So, for any distance k, there are sets X and Y with $d(X, Y) \geq k$, and labelings φ and ψ with no common extension of $\varphi \restriction X$ and $\psi \restriction Y$. Suppose however that no finite subset $X' \subset X$ can replace X. Then, as in the proof of Theorem 2.3, we get a contradiction by taking a nested sequence (X_i) of finite subsets of X whose union is X, and letting θ be any pointwise limit of the labelings θ_i obtained as common extensions of $\varphi \restriction X_i$ and $\psi \restriction Y$. Once X is finite we can limit Y to the (finite) set of sites at distance exactly k from X. $\square$

Thus we have a range of conditions for H equivalent to having heat less than d. However, the situation is not quite as good as for warmth; we don't know whether we can strengthen condition (iii) further to find a single finite set X in a graph $G \in \mathcal{G}_{d-1}$ which can be used for each k. It is conceivable that there is a constraint graph H of heat less than d for which the sets $X = X_k$ and $Y = Y_k$ in (iii) necessarily grow with k, whatever board $G \in \mathcal{G}_{d-1}$ is chosen, but we know of no examples of this phenomenon.

Clearly the heat $h(H)$ is always at most the warmth $w(H)$ of H. Since $\mathrm{Hom}(G, H)$ may be empty when G is not a tree, there is a *tendency* for T^{d-1} to be the easiest board in $\mathcal{G}_d$ on which to exhibit long range action, in which case heat and warmth will be equal. For instance, $\mathrm{Hom}(T^3, K_4)$ exhibits long range action, whereas $\mathrm{Hom}(\mathbb{Z}^2, K_4)$ (4-coloring the plane grid) does not. Indeed it is easily checked that $h(K_d) = w(K_d) = d$ for complete graphs K_d. Also, as for warmth, every constraint graph H is 2-warm, and H is 3-warm unless it is bipartite.

However, the 5-wheel W_5, which we served earlier as an example of a 4-warm graph, is not 4-hot. We start with a copy of T^2 having a root of degree 2, and form $G \in \mathcal{G}_3$ by replacing each site by a triangle, each vertex of which becomes incident to one of the edges incident to the original site. Let z be the lone site in G of degree 2 and suppose that $\varphi \in \mathrm{Hom}(G, W_5)$ is chosen so that in every triangle the site nearest z is labeled by the center node c of W_5 (see Fig. 3). Since in any labeling one site from each triangle must map to c, any θ consistent with φ outside some neighborhood of z must also label z by c; we have long range order for $\mathrm{Hom}(G, W_5)$.

The other examples we considered earlier, namely H_8, $\mathrm{Gr}(C_7)$ and $\mathrm{Gr}(C_5)$, all have warmth 3 and are not bipartite, and therefore have heat 3.

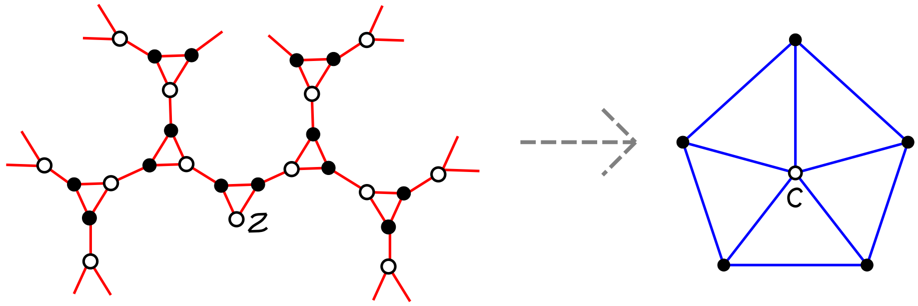

Figure 3: W_5 is not 4-hot

7 Mobility

Our third new parameter is somewhat different from heat and warmth but again motivated by considerations from statistical physics. There is also a connection to the theory of computing. It is often useful to obtain a *random sample* from $\mathrm{Hom}(G, H)$ when the board G is large but finite. This can potentially be done by "heat bath", or "single-site Glauber dynamics", in which labels are randomly changed one site at a time in accordance with the constraints imposed by H. Let us define a graph structure on $\mathrm{Hom}(G, H)$ by making two maps adjacent when they differ on one site; then we see that the heat bath can only work if $\mathrm{Hom}(G, H)$ is connected.

We will say that H is d-*mobile* if $\mathrm{Hom}(G, H)$ is connected for any finite board $G \in \mathcal{G}_{d-2}$. The *mobility* $m(H)$ of H is the greatest d for which H is d-mobile, or equivalently the least d for which $\mathrm{Hom}(G, H)$ is disconnected for some finite $G \in \mathcal{G}_{d-1}$. If H is d-mobile for every d, i.e. $\mathrm{Hom}(G, H)$ is connected for *every* finite G, then we say $m(H) = \infty$. Every constraint graph is trivially 2-mobile (graphs in $\mathcal{G}_0$ being collections of isolated points) and as with heat and warmth, $m(H) = 2$ if and only if H is bipartite.

Mobility also matches the other parameters on complete graphs.

Theorem 7.1. *For any integer $d \geq 2$, $m(K_d) = d$.*

Proof. That K_d is not $(d+1)$-mobile is clear from taking $G = K_d$ itself, since $\mathrm{Hom}(K_d, K_d)$ consists of $d!$ isolated points.

To see that K_d is d-mobile, we repeat an argument from [7] where the objective was to use a heat bath to estimate the number of d-colorings of G. For the heat bath to work in polynomial time, connectivity is not enough; "rapid mixing" of the Markov chain is also required, but has been proven only when the maximum degree of G exceeds d by a constant factor (currently 11/6 [13]).

Let φ and ψ be any two labelings by K_d (i.e. proper d-colorings) of a graph G of maximum degree at most $d-2$. We change φ sequentially to obtain ψ, as follows. Our first goal is to ensure that $\varphi(y) = 1$ whenever $\psi(y) = 1$; to do this, we look at all those z such that $\varphi(z) = 1$—these form an independent set of sites, and for all of them there is some alternative label that can be used, and we do so. Now we can use label 1 on all the sites we want; we shall not relabel these sites again. We now repeat with label 2, and so on. □

Among our earlier examples are some where mobility and warmth differ. For the 5-wheel W_5, we have $m(W_5) = 3 < 4 = w(W_5)$; to verify that W_5 is not 4-mobile, take the board to be K_3 and note (as we did when showing that $h(W_5) < 4$) that every labeling uses the center node of W_5 exactly once.

The Grotzsch graph $G_5 = \mathrm{Gr}(C_5)$ is an example of a graph whose mobility exceeds its warmth and heat. Recall that $w(G_5) = h(G_5) = 3$; we now demonstrate that G_5 is 4-mobile.

We need to show how to get from any G_5-labeling of a cycle C_n to any other. Note that there are no G_5-labelings of C_3, so we may take $n \geq 4$. Set $A = \{a_1, \ldots, a_5\}$ and $B = \{b_1, \ldots, b_5\}$, so $V(G_5) = A \cup B \cup \{c\}$.

Our first step is change any G_5-labeling to eliminate all uses of label c. If label c is used at any point in the cycle, the label to its left is some $b_i \in B$, and the label to the left of that is in $A \cup \{c\}$: in either case we have the option of changing the label b_i to at least one other b_j. There are at least three nodes of A adjacent to either b_i or b_j, and we similarly get a set of three possible nodes of A from the right side. Thus one element of A is possible from both sides, and the label c can be changed to this element by first changing its neighboring labels if necessary. Proceeding in this way, we can indeed eliminate all the uses of label c. Then of course we can replace each use of label b_i by the corresponding a_i.

We now have a homomorphism from C_n to $G_5 \restriction A$—a copy of C_5. It is easy to see that the graph $\mathrm{Hom}(C_n, C_5)$ in general falls into several connected components, with each component identified by the *winding number*, the number of times the sequence of labels winds around C_5; if n is a multiple of 5, there are also 10 isolated vertices of $\mathrm{Hom}(C_n, C_5)$, namely those homomorphisms with winding number $\pm n/5$. The winding number always has the same parity as n.

To complete the argument, it is enough to show that, working in $\mathrm{Hom}(C_n, G_5)$, we can reverse a sequence of labels such as $a_1a_2a_3a_4a_5a_1$ that winds around G_5, hence changing the winding number by 2. To do this, we step through the following labelings in turn:

$$\begin{gathered} a_1a_2a_3a_4a_5a_1,\ a_1a_2b_3a_4b_5a_1,\ a_1a_2b_3cb_5a_1,\ a_1a_2b_1cb_2a_1, \\ a_1a_5b_1cb_2a_1,\ a_1a_5b_4cb_2a_1,\ a_1a_5b_4a_3b_2a_1,\ a_1a_5a_4a_3a_2a_1. \end{gathered}$$

Our other example $\mathrm{Gr}(C_7)$ is not 4-mobile: take C_5 as a board. Figure 3 summarizes the parameter values for the various examples we have been considering.

In Section 4 we saw that if H has circular chromatic number less than 4, then H is not 4-warm; in fact it is not 4-mobile either. To see this, take a (k, d)-coloring φ of H, where $\chi_c(H) = k/d < 4$, and k and d are relatively prime, and take a tight cycle $a_0a_1 \cdots a_{s-1}a_0$ in H, so that $\varphi(a_{i+1}) = \varphi(a_i) + d \pmod{k}$. Now let the board G be an s-cycle $x_0x_1 \cdots x_{s-1}x_0$, and consider the H-labeling defined by $\varphi(x_i) = a_i$. We see that, for any ψ in the component of $\mathrm{Hom}(G, H)$ containing φ, $\psi(x_i) \in A_j$ for all i, where $j \equiv i \pmod{k}$. In particular, the H-labeling θ given by $\theta(x_i) = a_{i+1}$ is not in the same component of $\mathrm{Hom}(G, H)$ as φ.

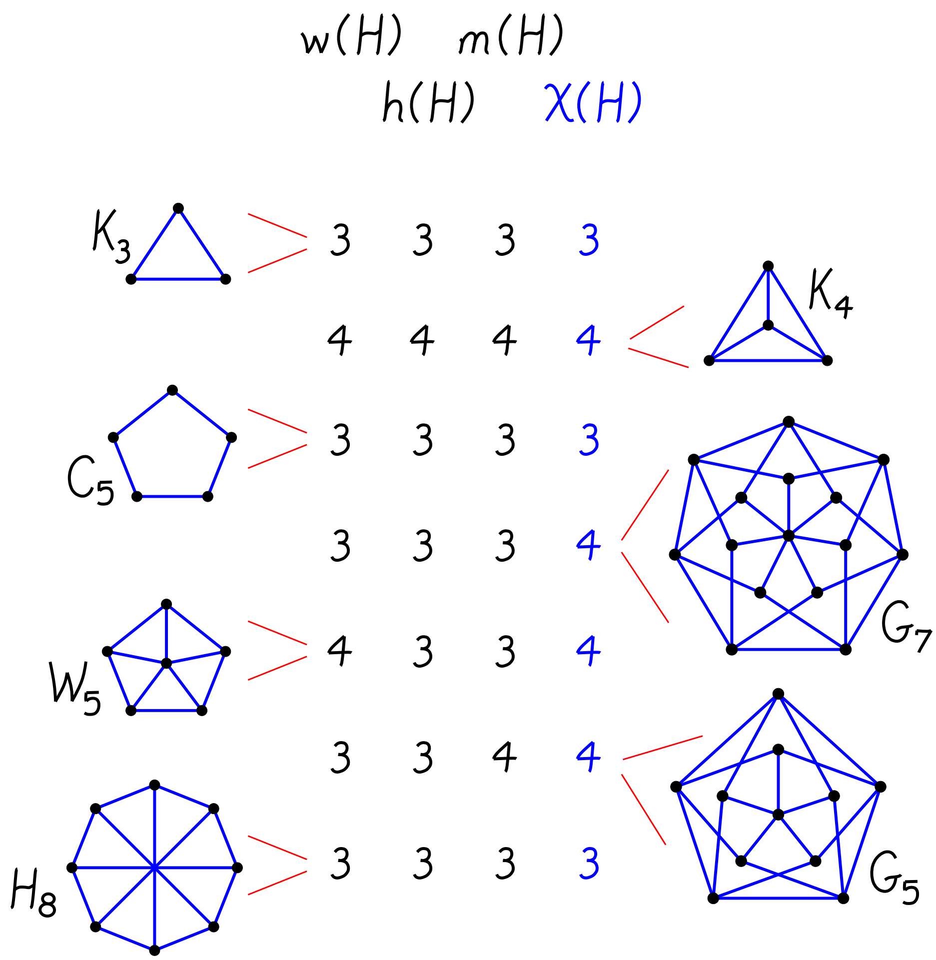

Figure 4: Warmth, heat, mobility and chromatic number

The next two results connect mobility with heat and warmth.

Theorem 7.2. *Any* $(2d-3)$*-hot constraint graph* H *is* d*-mobile.*

Proof. Let k be an integer such that, whenever $G \in \mathcal{G}_{2d-4}$, X and Y are subsets of G with $d(X,Y) \geq k$, and φ and ψ are H-labelings of G, there is an H-labeling θ simultaneously extending $\varphi \restriction X$ and $\psi \restriction Y$. The fact that H is $(2d-3)$-hot guarantees that such an integer k exists.

Let G be any finite board in $\mathcal{G}_{d-2}$, let $x_1, x_2, \ldots, x_n$ be the sites of G, and form the board G' as follows. The set of sites of G' consists of $k+1$ copies $X_0, \ldots, X_k$ of the set of sites of G, with the mth copy of site x_i labeled $x_{i,m}$ $(0 \leq m \leq k)$. For each edge $x_i x_j$ of G, with $i < j$, and each m, there are edges $x_{i,m}x_{j,m}$ and $x_{j,m}x_{i,m+1}$ of G'. The graph G' thus has maximum degree at most $2d-4$, and also $d(X_0, X_k) \geq k$.

Now let φ and ψ be two H-labelings of G. These labelings lift to G' in the obvious way: set $\varphi'(x_{i,m}) = \varphi(x_i)$ for all i, m, and similarly for ψ. Now consider an H-labeling θ of G' simultaneously extending $\varphi' \restriction X_0$ and $\psi' \restriction X_k$.

The homomorphism θ tells us how to get from φ to ψ: we start with φ as our labeling, and look at the sites of G' in the order $x_{1,1}, x_{2,1}, \ldots, x_{n,1}, x_{1,2}, \ldots, x_{n,2}, \ldots, x_{1,k}, \ldots, x_{n,k}$. At each site, interpret $\theta(x_{i,m}) = a$ as an instruction to (re)label site x_i of G with label a. The fact that θ is a homomorphism of G' with $\theta \restriction X_0 = \varphi' \restriction X_0$ tells us exactly that this procedure is legitimate; the fact that $\theta \restriction Y_k = \psi' \restriction Y_k$ tells us that the final labeling of G is ψ. $\square$

Theorem 7.3. *Any* $(2d-3)$*-mobile constraint graph* H *is* d*-warm.*

Proof. We shall suppose that H is not d-warm, take a $(d-2)$-stable family $\mathcal{A}$ of subsets of H, and construct a finite graph G in $\mathcal{G}_{2d-4}$, together with two H-labelings of G that are not connected in $\mathrm{Hom}(G, H)$.

For each pair (A, a), with $A \in \mathcal{A}$ and $a \in A$, we choose d–2 pairs $(A_1, a_1), \ldots, (A_{d-2}, a_{d-2})$ such that all the A_i are in $\mathcal{A}$ and $\{A_1, \ldots, A_{d-2}\}$ produces A, and each a_i is a neighbor of a in A_i. The fact that $\mathcal{A}$ is a $(d-2)$-stable family ensures that we can do this.

Next we form a digraph D whose vertex set consists of all 4-tuples $(A, a; B, b)$ where $A, B \in \mathcal{A}$, $a \in A$, $b \in B$. We direct arcs from $(A, a; B, b)$ to each $(A_i, a_i; B_i, b_i)$, $i = 1, \ldots, d$–2. Thus every vertex of D has outdegree $d-2$.

Let N be the total number of vertices in D, and let $\mathbf{M}$ be the $N \times N$ incidence matrix of D, so the entry $m_{\alpha\beta}$ is equal to 1 if there is an arc from α to β, and 0 otherwise.

Note that $(1, \ldots, 1)$ is an eigenvector of $\mathbf{M}$, with largest eigenvalue d–2. Therefore there is a positive rational vector $\mathbf{r}$ such that $\mathbf{M}^{\mathrm{t}}\mathbf{r} = (d-2)\mathbf{r}$. By multiplying up we can take $\mathbf{r}$ to be an integer vector. Now take r_α copies of each vertex α of D. Our intention is to form a digraph D' on this blown-up vertex set by directing one arc from each copy of α to some copy of each β with $\alpha\beta$ an arc of D. If we do this, the total number of arcs arriving at the r_β copies of β is $\sum_{\alpha \to \beta} r_\alpha = (\mathbf{M}^{\mathrm{t}}\mathbf{r})_\beta = (d-2)r_\beta$, so we can distribute the incoming arcs so that every vertex of D' has indegree $d-2$. (We can also ensure at this stage that our digraph D' has no loops.)

Now we form our graph G by forgetting the orientation of all the arcs of D'. Thus the maximum degree of G is at most $2d-4$.

There are two H-labelings φ and ψ of G given by projections: in φ, each copy of the vertex $(A, a; B, b)$ of D is given label a; in ψ, each copy of $(A, a; B, b)$ gets label b. We claim that, in the component of $\mathrm{Hom}(G, H)$ containing φ, each copy of $(A, a; B, b)$ gets a label from A. Indeed, if θ satisfies this and θ' is an adjacent labeling, differing only on a copy γ of $(A, a; B, b)$, then γ is adjacent to some copy of each $(A_i, a_i; B_i, b_i)$, which are each given a label from A_i by θ—since the A_i produce A, $\theta'(\gamma) \in A$.

Therefore, considering any vertex $(A, a; B, b)$ in which $b \notin A$, we see that ψ is not in the component of φ. (It may be that G is not connected, in which case we take a component containing such a vertex.) This completes the proof. $\square$

It is convenient to collect here the various inequalities we have been able to prove between our parameters.

$$
\begin{array}{rcll}
h(H) & \leq & w(H) & \text{(trivial)} \\
w(H) & \leq & \chi(H) & \text{for unlooped } H \text{ (Theorem 5.1)} \\
h(H) & \leq & 2m(H)-2 & \text{(Theorem 7.2)} \\
m(H) & \leq & 2w(H)-2 & \text{(Theorem 7.3)}
\end{array}
$$

Some other inequalities can be deduced from these, notably that $m(H) \leq 2\chi(H) - 2$ for unlooped H. Furthermore, if $\chi(H) \leq 3$, then $\chi_c(H) < 4$ and so, as we noted earlier, $m(H) \leq 3$. Indeed, it seems very likely to us that, as has also been suggested by Lovász, $m(H) \leq \chi(H)$ in general. This would be of particular interest as it is a statement referring only to finite boards; another way of expressing it is that, for every unlooped H with chromatic number d, there is some finite graph G of maximum degree $d-1$ such that $\mathrm{Hom}(G, H)$ is disconnected. (Alternatively, no graph of chromatic number d exhibits greater mobility than K_d.)

8 Loops and Dismantlability

We have seen that d-warmth, d-heat, and d-mobility all match for d=2 or 3; in fact a theorem from [3] shows that they match at the other end of the scale as well, that is, at $d = \infty$. We conclude with a short description and proof of this result.

The constraint graph consisting of a single looped node has infinite warmth, heat and mobility, but not every looped constraint graph is so lucky. For example, if H is a path on three nodes with a loop at each end but not in the middle, then $w(H) = 3$; to see that this H is not 4-warm, label T^2 in such a way that every node has children with two different labels.

The difference here is that this last H is not *dismantlable*. The notion of dismantlability goes back twenty years to the study of pursuit games on graphs (see e.g. [11, 12]) and

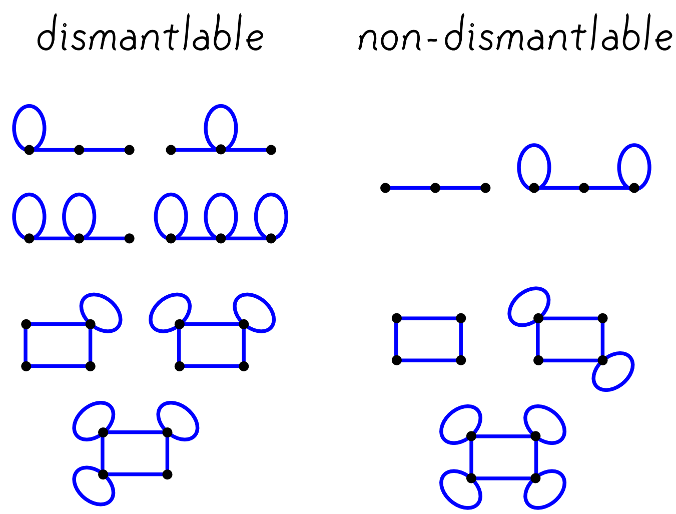

Figure 5: Some examples of dismantlable and non-dismantlable graphs

reappeared in [3], where numerous equivalent conditions are given, among them the ones which interest us here. We may define the notion recursively by saying that the graph with one node and a loop is dismantlable, and if H has two distinct nodes a and b with $N(a) \subseteq N(b)$ then $H \backslash \{a\}$ is dismantlable. Note that an unlooped graph cannot be dismantlable. Some dismantlable and non-dismantlable graphs are illustrated in Fig. 5.

In any H, if $N(a) \subseteq N(b)$, the map which sends a to b and every other node of H to itself is a retraction of H onto $H \backslash \{a\}$, which we call a *fold* and denote by ρ_{ab}.

Theorem 8.1. *If $\rho_{ab}: \ H \to H^- = H \backslash \{a\}$ is a fold then $w(H) = w(H^-)$, $h(H) = h(H^-)$ and $m(H) = m(H^-)$.*

Proof. The first two statements follow if we show that for any board G, long range action for $\mathrm{Hom}(G, H^-)$ is equivalent to long range action for $\mathrm{Hom}(G, H)$. The forward direction is easy because if X, $Y \subset G$ with $d(X,Y) \geq k$ and φ^-, $\psi^- \in \mathrm{Hom}(G, H^-)$, then we can find $\theta \in \mathrm{Hom}(G,H)$ which agrees with $\varphi^- \restriction X$ and $\rho^- \restriction Y$, and $\rho_{ab} \circ \theta$ works similarly in $\mathrm{Hom}(G, H^-)$.

The reverse implication is not much harder. Choose a distance k over which there is no action in $\mathrm{Hom}(G,H)$, and let $d(X,Y) \geq k+2$. We need to expand X and Y slightly to $X' := X \cup N(X)$ and $Y' := Y \cup N(Y)$. Then $d(X',Y') \geq k$ and for any φ and ψ in $\mathrm{Hom}(G,H)$ we can find θ^- which matches $\rho_{ab} \circ \varphi$ on X' and $\rho_{ab} \circ \psi$ on Y'. Now we claim that the map θ which agrees with φ on X, ψ on Y and θ^- everywhere else is a legal H-labeling. Indeed, the only possible problem is for an edge from, say, $x \in X$ to $z \in N(X) \backslash X$ where $\varphi(x) = a$ and $\theta^-(x) = b$. If $\varphi(z) = a$ also then a is looped, thus $a \in N(a) \subseteq N(b)$ and it is legal for θ to label x with a and z with $\rho_{ab}(\varphi(z)) = b$. Otherwise $\theta(z) = \theta^-(z) = \varphi(z) \sim \varphi(x) = a$ and we still have a homomorphism.

For mobility it suffices to show that $\mathrm{Hom}(G,H)$ is connected if and only if $\mathrm{Hom}(G,H^-)$ is; the forward implication is easy because if $\theta_1, \ldots, \theta_k$ is a path in $\mathrm{Hom}(G,H)$ then so is

$\rho_{ab} \circ \theta_1, \ldots, \rho_{ab} \circ \theta_k$ after redundant points have been discarded. For the reverse, we first convert φ and ψ to members of $\mathrm{Hom}(G, H^-)$ by changing a-labels to b-labels one by one. Then a path in $\mathrm{Hom}(G, H^-)$ completes the connection. $\square$

Now the definition of dismantlability, and induction on the number of nodes of H, gives our final result.

Corollary 8.2. *The following are equivalent for any constraint graph H:*

(i) *H is dismantlable;*

(ii) *$w(H) = \infty$;*

(iii) *$h(H) = \infty$;*

(iv) *$m(H) = \infty$.*

9 Problems

Many basic questions about warmth, heat and mobility remain open; we list here some of our favorites.

1. There are many missing bounds. Is there, for example, a function f such that $f(d)$-warm implies d-hot, or $f(d)$-warm implies d-mobile? Or such that $f(d)$-mobile implies d-hot, or $f(d)$-mobile implies d-colorable? (As far as we know, $f(d) = d+1$ could work for all of these.) Are the bounds in Theorems 7.2 and 7.3 best possible?

2. We know that girth at least 5 forces warmth at most 3, and it is similarly easy to show that it also forces mobility at most 3. There are examples of triangle-free graphs of warmth 4, but are there triangle-free graphs of arbitrarily large warmth? Heat? Mobility?

3. Suppose we say that H is *strongly* d-mobile if for any board G, finite or infinite, in $\mathcal{G}_{d-2}$, whenever φ and ψ in $\mathrm{Hom}(G, H)$ differ on only a finite number of sites, there is a finite path in $\mathrm{Hom}(G, H)$ from φ to ψ. This is what we need for the one-site condition to be equivalent to the Gibbs condition for H-labelings of boards in $\mathcal{G}_{d-2}$ (see [2]). Does mobile imply strongly mobile? If not, is there at least a function f such that $f(d)$-hot implies strongly d-mobile?

4. There are many computational issues concerning warmth, heat and mobility; we guess that the question of whether a graph is d-anything is NP-hard for any fixed $d > 3$. We do not know if the 4-warm graphs are even in NP $\cup$ co-NP; it would be nice, for example, to have a decent bound on how far action can extend on $\mathrm{Hom}(T^{d-2}, H)$ when H is an n-node, d-warm graph; or, when it is not d-warm, on the size of a minimum $(d-2)$-stable family. Worse, it is not evident that there is *any* finite algorithm for determining whether a graph is 4-hot!

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