Rendezvous in One and More Dimensions

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Abstract

This article is mainly concerned with the rendezvous problem on the n-dimensional integer lattice. Two blind players are initially placed at nodes whose di¤erence vector has length 2 and is parallel to some coordinate axis. In each period they must move to an adjacent node. They have no common notion of locations or directions. The least expected rendezvous times R^a (using distinct strategies) and R^s (using the same mixed strategy) are shown to satisfy $\lim R^a(n)/n \leq 8/3$ and $\lim R^s(n)/n \leq 56/9$. This work extends the work of the author and S. Gal ($R^a(1) = 13/4$) and that of V. Baston ($R^s(1) \leq 4.5$), and the related 2-dimensional analysis of Anderson and Fekete. We also consider a rendezvous problem on the line where one player can see the other.

Keywords: rendezvous, search game

1 Introduction

The basic form of the rendezvous search problem [1] asks how two players, who are randomly placed in a known search space Q, can move so as to minimize the expected time taken to meet. When the players are allowed to meet beforehand and discuss which role each will take (for example one stays still while the other searches), the least expected meeting time is called the asymmetric rendezvous value $R^a(Q)$. When they cannot use distinct strategies, the common strategies they may have to use will in general be mixed, and in this case the (generally larger) least expected meeting time is called the symmetric rendezvous value $R^s(Q)$. The players are assumed to have no common labelling of points or directions in Q, to have a maximum speed of 1, and of course not to know the location of the other. Much of the literature in this area is concerned with the case where Q is the line, although Anderson and Fekete [8] and Thomas and Hulme [13] have considered rendezvous on the plane. A general survey of the rendezvous problem can be found in [3].

The main objective of this article is to extend the work done for rendezvous on the line to the n-dimensional case. For purely spatial applications, we would of course need to assume that $n \leq 3$. However the dimensions (coordinates) of search need not all be spatial. For example if two people wish to find each other on walkie-talkies, they would need to be close in two spatial dimensions, close in a radio frequency dimension, and close in the language or code convention used.

We follow the route of Anderson and Fekete (for $n = 2$) by taking the search space Q to be the integer lattice (network) with nodes $z = (z_1, \ldots, z_n) \in Z^n$ and an edge between two nodes if they have all but one coordinate identical and the remaining coordinate differs by 1. In two dimensions this is just the familiar lattice of graph paper. The most general formulation of the problem is to place the players at time $t = 0$, according to some distribution, on even nodes. (A node $z \in Z^n$ is called even if the sum of its coordinates is even; otherwise is called odd.) In each time period each player must move to an adjacent node - staying still is not allowed (but may be approximated by an oscillation between adjacent nodes). This type of placement (originating for the interval network in Howard [12]) ensures that the two players will always have the same parity, and cannot pass each other on an edge without meeting at a node.

We will be primarily concerned with a particular initial placement which is a natural generalization of the 'atomic distribution' (where the initial distance between the players is common knowledge) that has been widely studied on the line ($n = 1$) in both the asymmetric and symmetric rendezvous scenarios. To match the previous analysis for $n = 1$ with our lattice analysis

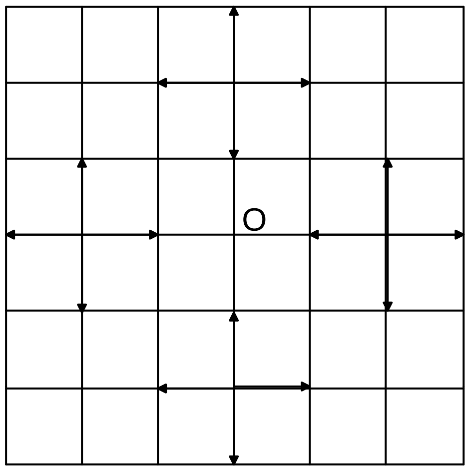

Figure 1: Starting positions of 16 agents of II.

for general n, it is best to take this known initial distance between the players as 2. If we take a coordinate system with its origin at Player I's initial location, then Player II will be initially placed at one of the $2n$ locations $\pm 2e_i$ (where e_i are the n unit vectors) and will be initially faced in one of the $2n$ directions which are parallel to the coordinate axes. Thus there are $(2n)^2 = 4n^2$ ways to start the game, and the expected values for the meeting times will be averaged over these initial events. We shall denote this n-dimensional rendezvous problem by $\Gamma(n)$. Note that in the case of the line this means there are four cases to consider, corresponding to Player II starting at 2 or -2 and calling either right or left his 'forward' direction.

The initial placements for $n = 2$ are shown in Figure 1. Player I starts at the origin, labelled 0. Player II starts at one of the four ($= 2n$) lattice points $(0,2)$, $(2,0)$, $(-2,0)$, $(0,-2)$ and initially faces in one of the four $(= 2n)$ directions. Hence there are $16 = 4n^2$ initial con…gurations, or equivalently, 16 agents of Player II. The expected meeting time of the two players is the same as the expected time for Player I to meet the 16 agents of Player II.

The asymmetric rendezvous problem on the line, where the players are allowed to use distinct strategies, was …rst considered by the author and Gal [5]. That paper gave the solution for a …xed initial distance of 2 as follows: one player moves two unit steps in his 'forward' direction and then four unit steps in the opposite 'backwards' direction (F,F,B,B,B,B); the other player moves (F,B,F,F,B,B). The players meet equiprobably at the times $t = 1, 2, 4$, or 6. Assuming that Players I's initial forward direction is taken as right, these meeting times correspond respectively to the initial events in which Player II has initial position and forward direction given by (+1,left), (+1,right), (-1,right), (-1, left). This strategy pair is optimal, in that it minimizes the expected meeting time. Consequently the asymmetric rendezvous value $R^a(n=1)$ for this problem is given by $R^a(1) = 13/4 = 3.25$. In fact it is shown in [5] that this strategy pair is uniformly optimal, in the sense that for every value of t it maximizes the probability that a meeting has occurred by time t. Our extension of this result to n-dimensions is summarized in the following result, proved in Section 2.

Theorem 1 Suppose that two players are initially placed on the n-dimensional integer lattice so that their di¤erence vector is two units long and parallel to one of the coordinate axes. Assume that the players have no common notion of location and no common labelling of the coordinates axes. The asymmetric rendezvous value $R^a(n)$ for this problem satis…es the inequality

$$R^a(n) \leq \frac{32n^3 + 12n^2 - 2n - 3}{12n^2}. \qquad (1)$$

Consequently, we have the asymptotic result

$$\lim_{n\to\infty} R^a(n)/n \leq \frac{8}{3}.$$

It is worth noting that the upper bound on the asymmetric rendezvous value given in (1) is the exact value found in [5] of $39/12 = 13/4$ for $n = 1$. The analysis of asymmetric rendezvous in this context is carried out in Section 2.

The symmetric version (where both players must use the same mixed strategy) of the rendezvous problem on the line was …rst introduced by the author in the original article [1], where it was conjectured that the so called '1F2B' strategy, of repeatedly (with independent randomization) going 1 unit in a random forward direction, followed by 2 units in the opposite backwards direction, was optimal. However the expected meeting time for 1F2B of 5 was subsequently improved by a strategy of Anderson and Essegaier [7] to about4.56, and later by a strategy of Baston [9] to about 4.418. Consequently the best estimate for $R^s(n=1) = R^s(1)$ is given by $R^s(1) \leq 4.418$. The exact value of $R^s(1)$ is not known and its determination seems to be a very di¢cult problem. While we are able to obtain some upper bounds in terms of the dimension n for $R^s(n)$, these are not very good, and our main result for symmetric n-dimension rendezvous with a …xed initial distance is restricted to the following asymptotic result, proved in Section 3.

Theorem 2 For the symmetric rendezvous problem on the n-lattice, with an initial placement two units apart in a direction parallel to a random coordinate axis, we have

$$\lim_{n\to\infty} R^s(n)/n \leq \frac{56}{9} \doteq 6.2222.$$

After carrying out the general n-dimensional analysis, we attempt to get more exact results for the planar case $n = 2$ in Section 4. In our main initial setup, where the initial difference between the players is parallel to a random coordinate axis, we obtain only a partial optimality result for our suggested strategy. We show (Theorem 3) that it maximizes the probability of a meeting by time t for $t \leq 7$.

In Section 5 we then consider another initial setup suggested earlier by Anderson and Fekete [8], where the initial difference vector is equiprobably one of the four vectors $(\pm 1, \pm 1)$. For this scenario they suggested a strategy, which we call the A-F strategy, and showed that it was optimal. We extend some of their ideas and prove (Theorem 7) that the A-F strategy is uniformly optimal and determine the set of all optimal strategies.

As the title of this article suggests, we also have some new ideas for one-dimensional rendezvous, in continuous time and space. These ideas are presented in Section 6, where we adopt an asymmetric information version of rendezvous originally suggest by Anderson and Fekete [8] for the plane. They assume that one of the players (say II) knows the initial position of the other (I), and that both know the initial distance. On the line we assume that Player I is initially at the origin, while the initial position of II is given by a known distribution F on the line. We show that this problem can be transformed into a pure search problem studied by the author and Howard [6], where two searchers located at the origins of distinct lines move alternately to find a stationary object hidden in the positive direction along one of the lines. For symmetric distributions F, we establish (Theorem 12) that a necessary and sufficient condition for staying still to be uniquely optimal for Player I is that the distribution of the initial distance between the players is strictly concave.

2 Asymmetric n-Dimensional Rendezvous

In order to analyze the asymmetric version (with distinct strategies) of the rendezvous problem $\Gamma(n)$ described in the Introduction, we will need to consider two subsidiary problems $\Gamma_1(m)$ and $\Gamma_2(m)$, for $m = 1, \ldots, 2n$. Both of these problems begin at time $t = 0$ with the placement of Players I and II respectively at a pair of nodes A and B which are two units apart

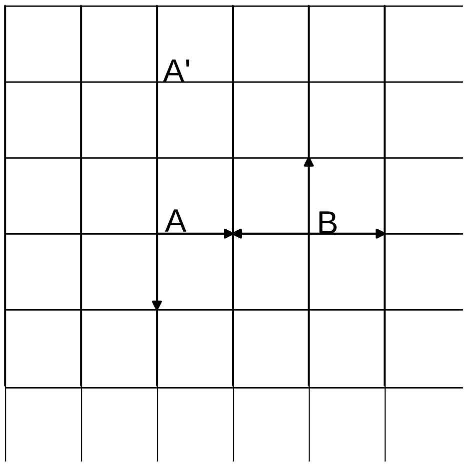

Figure 2: Start position in $\Gamma_1(3)$

along a line parallel to some coordinate axis. Then Player I is displaced to a node A' along a similar two unit line which is not the one leading to B. Player I is told the node A. In the problem $\Gamma_1(m)$, Player I is told $m-1$ directions which are certain to include the direction to B, and Player II is told m such directions. In problem $\Gamma_1(m)$, both players are told m such directions. Special cases of these problems, for $n = 2$, are drawn in Figures 2 and 3.

In order to estimate the rendezvous value $R^a(n)$ of the original problem $\Gamma(n)$, we must obtain estimates on the respective asymmetric rendezvous values $w_1(m)$ and $w_2(m)$ of $\Gamma_1(m)$ and $\Gamma_2(m)$ for various m (corresponding to the dimension n, which is implicit in our notation).

Suppose that in the problem $\Gamma_1(m)$, the …rst two moves of the players are as follows: Player I goes to the node A (which he knows) while Player II goes 2 steps randomly in one of the m indicated directions. With probability $1/m$, II will pick the direction to A, and the meeting time will be $T = 2$. Otherwise the two players will be in the initial position of the other problem $\Gamma_2(m-1)$, with the roles (of I and II) reversed. Hence we have

$$w_1(m) \leq \frac{1}{m}(2) + \frac{m-1}{m}(2 + w_2(m-1)). \tag{2}$$

Similarly, in the initial position of $\Gamma_2(m)$, the same type of strategy for the …rst two moves gives

$$w_2(m) \leq \frac{1}{m}(2) + \frac{m-1}{m}(2 + w_1(m)). \tag{3}$$

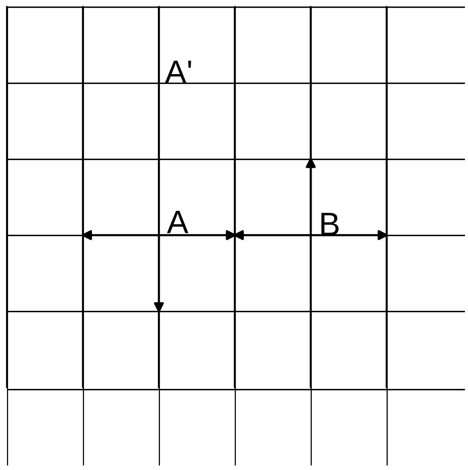

Figure 3: Start position in $\Gamma_2(3)$.

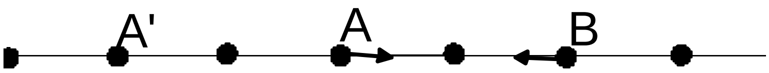

Figure 4: Start position in $\Gamma_2(1)$.

From these two inequalities we obtain upper bounds $w_i(m) \leq \bar{w}_i(m)$ by solving the associated equalities. This gives us the solutions

$$\bar{w}_1(m) = \frac{4m+1}{3}, \quad (4)$$

$$\bar{w}_2(m) = \frac{-1+3m+4m^2}{3m}. \quad (5)$$

This is consistent with the trivial base case $\Gamma_2(1)$, where I and II start 4 units apart with knowledge of the other's direction, with rendezvous value $w_2(1) = 2$ corresponding to a meeting at A. This case is illustrated in Figure 4.

We now consider the original game $\Gamma(n)$. Suppose that in their …rst two moves one player (I) goes two units in his forward direction, while the other (II) goes in some direction and the back to his start. With probability

$1/(2n)^2$ the two players will go towards each other and meet at time $T = 1.$ If Player I goes in the direction of II (probability $1/2n$) and II does not go in the direction of I (probability $(2n-1)/(2n)$) then the two will meet at time $T = 2$ at II's initial location. In the remaining case, I will …nd himself displaced two units from his start and know of one direction from his start which does not lead to II's starting location. Meanwhile, II will be at his start and not know of any of the $2n$ directions which are not correct. Consequently the situation at time $t = 2$ will be identical to that of the problem $\Gamma_1(2n)$. Therefore we have the estimate

$$R^a(n) \leq \left(\frac{1}{(2n)^2}\right)1 + \left(\frac{1}{2n}\right)\left(\frac{2n-1}{2n}\right)2 + \left(\frac{2n-1}{2n}\right)(2 + w_1(2n)).$$

Estimating w_1 by the formula (5) for $\bar{w}_1$, and simplifying, we get

$$\begin{aligned} R^a(n) &\leq \frac{32n^3 + 12n^2 - 2n - 3}{12n^2}, \text{ and hence} \\ \lim_{n\to\infty} R^a(n)/n &\leq \frac{8}{3}, \text{ which gives Theorem 1.} \end{aligned}$$

We note again that the value of the right hand side of the top inequality is $13/4$ for $n = 1$, which is the exact asymmetric rendezvous value for the line as derived in [5] for known initial distance $D = 2$. The strategy pair which gives this expected meeting time is the one which converts $\Gamma(n)$ into $\Gamma_1(2n)$ and thereafter converts each problem $\Gamma_1(m)$ into $\Gamma_2(m-1)$ and each problem $\Gamma_2(m)$ into $\Gamma_1(m)$, $m = 2n, 2n-1, \ldots, 1$ (assuming the players don't meet earlier). It may be described in the following way as an 'alternating wait for mommy' (AWFM) strategy.

De…nition (AWFM Strategy): One player (say I) arrives at $2n-1$ of the other's starting points at the times $t_i = 2 + 4i$, $i = 0, \ldots, 2n-1$, while returning to his initial position at times q_{i+1}, where $q_i = 4i$, $i = 1, \ldots, 2n-1.$ The other player (say II) returns to his initial position at times t_i (to be there when I searches) while searching out player I's initial positions at time q_i. Note that they will certainly meet before II tries to search out the last possible initial position of I. Thus the two players alternate the roles of searcher and waiter. Player I has to meet $(2n)^2 = 4n^2$ agents of Player II. With this strategy pair, he meets one agent at time $t = 1$, and $2n-1$ agents at time $t = t_0 = 2$ (all those at this initial position of II except for the one who headed towards I earlier). Thereafter, he meets exactly $2n-i$ agents at time t_i and at time q_i. The maximum meeting time for this strategy is given by $t_{2n-1} = 8n-2.$

By time $t_i = 2 + 4i$ the number of meetings is given by $N(i)$, where

$$N(i) = 2n + 2\sum_{j=1}^{i}(2n - j) = (2i + 1)(2n) - i(i + 1). \qquad (6)$$

We see that

$$N(2n - 1) = 4n^2,$$

which is the total number of agents of Player II.

2.1 Symmetric n-Dimensional Rendezvous

We now consider the symmetric version of the rendezvous problem $\Gamma(n)$, where both players are constrained to use the same strategy. Observe that pure strategies are no longer su¢cient, since if both use the same pure strategy and are initially facing in the same direction then they will always be the same distance apart and will never meet. So in this version we are forced to consider mixed strategies. Recall that even for the case $n = 1$ this is an open problem, so we will be content with getting upper bounds which will subsequently be improved. The strategies we will analyze will be of the following type. In each period of length τ, players will choose (independently of previous choices) among pure strategies which return them to their starting points. In each of these periods the probability of a meeting will be denoted by p. Given that a meeting occurs, the expected time of this meeting (counting from the beginning of the period) will be denoted by L. For such strategies the expected meeting time E satis…es the following equation

$$E = pL + (1 - p)(\tau + E), \text{with solution} \qquad (7)$$

$$E = \frac{pL + (1 - p)\tau}{p}. \qquad (8)$$

The most obvious symmetric strategy is the simple random strategy, in which each player moves one step in a random direction before returning back to his start. Here the period is $\tau = 2$, the meeting probability is $p = 1/4n^2$ and the meeting time is $L = 1$. Hence by (8) we have

$$R^s(n) \le \frac{(1/4n^2)\,1 + (1 - 1/4n^2)\,2}{1/4n^2} = 8n^2 - 1.$$

A better symmetric strategy is for the players to each follow one of the two pure strategies in the asymmetric AWFM strategy described above, choosing equiprobably between the two, and extending them to time $\tau = q_{2n} = 8n$

by returning to or waiting at the starting point in the last two time units. Neglecting the possibility that the players meet when choosing the same pure strategies (this probability goes to zero with n), we have $p = 1/2$ (the probability of choosing di¤erent strategies) and $L = \frac{32n^3+12n^2-2n-3}{12n^2}$ by (??). Since $p = 1 - p = 1/2,$ the formula (8) gives us the estimate

$$\begin{aligned} R^s(n) &\leq L + \tau = \frac{32n^3 + 12n^2 - 2n - 3}{12n^2} + 8n \\ &= \frac{128n^3 + 12n^2 - 2n - 3}{12n^2}. \end{aligned}$$

The corresponding asymptotic estimate is

$$\lim_{n\to\infty} \frac{R^s(n)}{n} \leq \frac{32}{3} = 10.666\ldots.$$

However can get a better asymptotic estimate by having the players randomize between taking roles I or II more often. Assume that k is large and n/k is a large integer. Suppose that we take $\tau = 8n + 2k$ and divide this period into k equal subperiods. At the beginning of each period the players choose the order in which they will visit the $2n$ possible starting points of the other one. In each subperiod they visit the next $2n/k$ locations, choosing equiprobably to visit them using the role of I or II, and returning to their start at the end of each subperiod. If k is large then in each period they will be choosing distinct strategies (one with role I and the other with role II) in very close to half the subperiods. So the probability of meeting in each period will be close to the probability of meeting by time $t_n \doteq \tau/2$ in the asymmetric version. Using the formula (6) for $N(i)$, this probability is given by

$$p = N(n)/4n^2 = \frac{(2n+1)(2n) - n(n+1)}{4n^2} = \frac{3n+1}{4n} \to \frac{3}{4}. \qquad (9)$$

We now seek to compute the value of $L,$ the expected meeting time given that they meet in a given period. This will be the same as computing, in the asymmetric problem, the expected meeting time given that they meet by time t_n (after checking half the possible locations). However this time will have to be doubled, because it now takes time $2t_i$ to check i possible initial locations of the other player (and meet $2ni$ possible agents of the other player), since half the time is wasted due to the players adopting the same role (both I or both II).

We observed earlier that when they adopt opposite roles (as in the asymmetric problem) I will meet $2n - i$ agents at times t_i and at times q_i. So now that they are half as e¢cient we can estimate the expected meeting time by assuming that I meets $2n-i$ agents at time $2t_i$ and at time $2q_i$. Equivalently (for expected time estimates) we may assume he meets $2(2n-i)$ agents at the averaged time $t_i + q_i = 8i + 2$. In the full period $\tau = 8n$ (for asymptotic estimates we will ignore the $2k$ term) we saw above that he will meet $3/4$ of the $4n^2$ agents, or $3n^2$ agents. Consequently, given that a meeting takes place, the expected meeting time is given by

$$\begin{aligned} L &= \frac{1}{3n^2} \sum_{i=1}^{n} 2(2n-i)(8i+2) \\ &= \frac{2}{3n^2} \left(\frac{16}{3} n^3 + 8n^2 - \frac{10}{3} n \right). \end{aligned} \tag{10}$$

If we look for the meeting time in terms of the length of the period, we get

$$\begin{aligned} \frac{L}{\tau} &= \frac{L}{8n} \\ &= \frac{2}{24n^3} \left(\frac{16}{3} n^3 + 8n^2 - \frac{10}{3} n \right) \\ &\to \frac{4}{9} \text{ as } n \to \infty. \end{aligned} \tag{11}$$

Consequently, by substituting the values (11) and (9) into (8), we get an expected meeting time of

$$\begin{aligned} E &= \frac{pL + (1-p)\tau}{p} \\ &= \frac{\frac{3}{4}\left(\frac{4\tau}{9}\right) + \frac{1}{4}\tau}{\frac{3}{4}} \\ &= \frac{7}{9}\tau = \frac{56}{9} n. \end{aligned}$$

This establishes the asymptotic estimate for symmetric n-dimensional rendezvous stated as Theorem 2 in the Introduction.

By way of comparison, we note that Baston's strategy for symmetric rendezvous on the line $(n = 2)$ when the initial distance is 2 gives an expected meeting time of about 4.4.

3 Asymmetric Rendezvous in the Plane

We have already observed that the AWFM strategy given in Section 2 is uniformly optimal in the case $n = 1$. We now establish that it also has certain optimality properties in the case $n = 2$. In particular, we will show that for all i up to 7, this strategy maximizes the probability that meeting has occurred by time i. For a general strategy, we will let x_i denote the number of agents of Player II that Player I meets (for the first time) at time $t = i$, and y_i the number he has met by time i.

Recall from our general analysis in the previous section that the AWFM strategy has $\hat{x}_1 = 1$, Note that any first move of Player I (remember that staying still is not allowed) will meet exactly one agent of Player II.$\hat{x}_2 = 2n - 1 = 3$, and then $\hat{x}_{4i} = \hat{x}_{4i+2} = 2n - i = 4 - i$, with $\hat{x}_i = 0$ for odd $i > 1$. The table below lists the sequence of $\hat{x}_i$ together with their cumulative sum $\hat{y}_i = \sum_{j=1}^{i} \hat{x}_j$. The total number of agents of II is $4n^2 = 16$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\hat{x}_i$	1	3	0	3	0	3	0	2	0	2	0	1	0	1
$\hat{y}_i$	1	4	4	7	7	10	10	12	12	14	14	15	15	16

Theorem 3 No strategy for the asymmetric rendezvous problem $\Gamma(2)$ on the 2-dimensional lattice results in more than $\hat{y}_j$ agents of Player II being met by time j, $j = 1, \ldots, 7$. In other words, the AWFM strategy outlined in the previous sections maximizes the probability of meeting by such times j.

Proof. First observe that $x_1 = 1 = y_1$ for any strategy (recall that staying still is not allowed). Next suppose that I meets $x_i \geq 2$ agents of II at some time i and node A, where A is not a starting point of either player. Then A must be equally distant (in the Manhattan, or graph metric) from x_i distinct starting points of Player II, since agents of Player II from the same starting point have distinct locations. However the only point equidistant from $x_i \geq 3$ of II's starting points is the origin. Since we have excluded this possibility it follows that $x_i = 2$ and that A is on one of the two main diagonals. This implies that i is even. If A is allowed to be a starting point (which we may assume to be of II) then we also have that i is even. Furthermore in this case we have $x_i \leq 3$ (since I would have met one of the agents starting at A by time $i - 1$) and $x_{i+1} = 0$, since the distance between starting points of II is at least 4. In any case we have shown that for any integer j, $x_{2j} + x_{2j+1} \leq 3$. The three observations $x_1 = 1$, $x_i \leq 3$, and $x_{2j} + x_{2j+1} \leq 3$ suffice to show that $y_i \leq \hat{y}_i$ up to $i = 7$. ■

4 Diagonal Start on the 2-Lattice

This technique used to establish partial optimality properties for the AWFM strategy in the previous theorem could be pushed a bit further. However

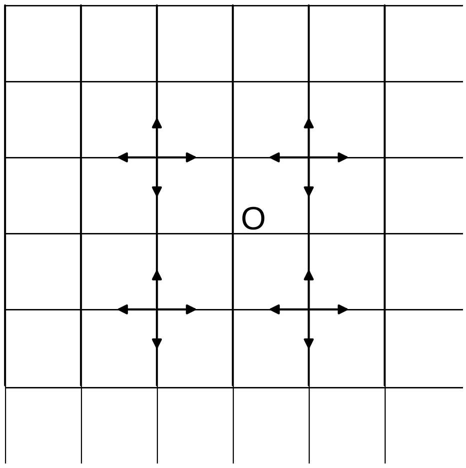

Figure 5: 16 starting types for II

to establish full uniform optimality of a strategy on the $2-$lattice, we must move to a scenario suggested by Anderson and Fekete [8]. We will call their problem the diagonal start on the 2-lattice, because Player I is initially placed at the origin and Player II is initially placed equiprobably at one of the four nodes $(\pm 1, \pm 1)$. So the two players are separated by one horizontal and one vertical edge of the lattice. These nodes are at distance 2 from the origin (starting point of Player I), however they are closer to each other, so that a traveling salesman tour for I of II's starting points is shorter than in the case of the coordinate direction start considered in most of this article. The information available to both players is that the other player is one horizontal plus one vertical step away. The 16 possible initial placements of Player II (or the 16 agents) are shown in Figure 5.

Anderson and Fekete [8] analyzed a strategy pair which we call the A-F strategy, given by $\bar{f} = [N, W, S, S, E, E, N, N]$, $\bar{g} = [N, S, N, S, N, S, N, S]$. We may consider that there are 16 equiprobable agents of Player II, and I wishes to minimize the expected time required to meet an agent. The following table indicates the meeting times corresponding to $(\bar{f}, \bar{g})$ depending on the initial direction which II calls North (labelled in terms of what I calls it) and the initial location of Player II.

Player II initial direction	$(1,1)$	Starting $(-1,1)$	Point $(-1,-1)$	$(1,-1)$
N	8	2	3	6
W	1	2	4	5
S	7	2	4	6
E	8	1	4	6

For each time $t = 1, \ldots, 8$, the number of entries of the 4×4 matrix of meeting times which are equal to t is denoted by $\bar{x}_t$ and the number which are less than or equal to t is denoted by $\bar{y}_t$. Thus in the A-F strategy Player I meets $\bar{x}_t$ of the 16 Player II agents at time t and $\bar{y}_t$ of these agents by time t. For a general strategy we will let x_t and y_t denote these numbers. For the A-F strategy we have

$$\begin{array}{ccccccccc} t & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \bar{x}_t & 2 & 3 & 1 & 3 & 1 & 3 & 1 & 2 \\ \bar{y}_t & 2 & 5 & 6 & 9 & 10 & 13 & 14 & 16 \end{array} \qquad (12)$$

The expected meeting time for this strategy is

$$\frac{1}{16}\left(1 \bullet 2 + 2 \bullet 3 + 3 \bullet 1 + 4 \bullet 3 + 5 \bullet 1 + 6 \bullet 3 + 7 \bullet 1 + 8 \bullet 2\right) = \frac{69}{16}.$$

Anderson and Fekete established that their (A-F) strategy is optimal - no strategy gives a lower expected meeting time. We will prove a stronger result, namely that it is uniformly optimal (in the sense defined in the Introduction). To do this we will need to use part of their original proof, which we give below.

Lemma 4 For any strategy pair, we have

$$\begin{aligned} x_i &\leq 3, \text{ and furthermore} \\ x_i &= 3 \text{ implies } x_{i+1} \leq 1. \end{aligned} \qquad (13)$$

Proof. Suppose that $x_i \geq 3$, which means that Player I meets at least three agents of II at time i at some location A. We first show that at time i one of the players must be back at his start. Suppose not. Then agents of Player II starting at a common node must be at distinct locations. Hence all the agents that I meets at time i must come from different starting points. Since all Player II agents are equally distant from their respective starting positions, the node A must be equally distant (in the Manhattan or graph distance) from at least three of the start points of II. The only such location is the origin, that is, Player I's initial location. So the meeting must be, as claimed, at a starting point. (Note that this implies that i is even.) By symmetry of the players, we will assume that A is one of the starting points of II. At time $i-1$ both Player I and one of the agents of II who started at A must be at the same location. Hence $x_i \leq 3$ as claimed. Since all agents of II must be at their starting points at time i, and I is at one of these, he can meet at most one agent of II at time $i+1$. ■

We will establish an extension of the class of optimal strategies. To do this, we define a mixed rendezvous strategy (f^*, g^*) as follows. The strategy f^* sends Player I cyclically around the square with corners $(\pm 1, \pm 1)$, equiprobably in one of the 8 possible ways. These ways are determined by the first two moves (e.g. N,W for the A-F pure strategy), with the second direction resulting from a left or right turn. The strategy g^* places Player II back at his starting point at all even integer times, moves in a random direction (independently of previous choices) at all odd times, except that the last odd move is the same direction as the first odd move.

To evaluate the expected meeting time for the mixed rendezvous strategy (f^*, g^*) we can assume without loss of generality that Player I follows the pure strategy $\bar{f}$ determined by the first two choices N,E. The following table gives the location of Player I at times t and the probability p_t that the first meeting time is at time t.

t	1	2	3	4	5	6	7	8
$\bar{f}(t)$	$(0,1)$	$(1,1)$	$(1,0)$	$(1,-1)$	$(0,-1)$	$(-1,-1)$	$(-1,0)$	$(-1,1)$
p_t	$\frac{1}{4}\frac{1}{4}+\frac{1}{4}\frac{1}{4}$	$\frac{1}{4}\frac{3}{4}$	$\frac{1}{4}\frac{1}{4}$	$\frac{1}{4}\frac{3}{4}$	$\frac{1}{4}\frac{1}{4}$	$\frac{1}{4}\frac{3}{4}$	$\frac{1}{4}\frac{3}{4}\frac{1}{3}$	$\frac{1}{4}\frac{3}{4}\frac{2}{3}$

This table is explained as follows. The probability p_1 arises from the possibility that II starts at $(-1,1)$ and moves E (in I's notation) or starts at $(1,1)$ and moves W. For $t = 3, 5$ the probability p_t is the probability that Player II starts at $\bar{f}(t+1)$ (just ahead of I's current position) and is lucky enough to move towards the oncoming Player I. If he was unlucky this gives the probability p_2, p_4 or p_6. The probability p_7 is the probability that II started at $(-1,1)$, did not initially go E and went S at move 7. The later probability is 1/3 for our strategy, as going E is excluded. (Note that if we used an entirely random strategy for the second player, this probability would have been 3/64 rather than $4/64$.) It follows that the expected meeting time $\Sigma_t t \bullet p_t = 69/16$. Since this is the rendezvous value established by Anderson and Fekete, we obtain the following.

Theorem 5 The mixed strategy (f^*, g^*) has one player (say I) choose one of the eight cyclic search patterns equiprobably, while the other player (say II) chooses a strategy which is back at his start at all even times, and picks a random direction at times $1, 3, 5$ (independently of previous choices), and moves in the same direction at time 7 at that chosen at time 1. This strategy is optimal, giving an expected meeting time of $69/16$, equal to the rendezvous value.

Note that if a mixed strategy is optimal for the asymmetric rendezvous problem, it follows that every pair of pure strategies (f, g) which occurs with a positive probability must be an optimal pair. Consequently we have the following.

Corollary 6 A pure strategy pair (f, g) is optimal if one of the players goes around the square $(\pm 1, \pm 1)$ in a cyclic fashion while the other moves in any direction at times 1,3,5, moves in the opposite direction from the previous at times 2,4,6, and moves in the same two directions at moves 7 and 8 as at moves 1 and 2. Such a strategy will be called a **generalized A-F strategy**.

In fact we shall establish the stronger result that these strategies constitute all the optimal strategies, and that moreover they are all uniformly optimal. (Of course if one strategy is uniformly optimal, then all optimal strategies must be uniformly optimal.) To do this we will need the condition on the x_i established in Lemma 4. It follows from that condition that for any i and j, we have

$$x_i + x_{i+1} + \cdots + x_{i+j} \leq 2 + 2 + \ldots 2 + 3.$$

In particular, we have that

$$\begin{aligned} x_i + x_{i+1} &\leq \bar{5}, \quad (14)\\ x_i + x_{i+1} + x_{i+2} &\leq \bar{7}, \\ x_i + x_{i+1} + x_{i+2} + x_{i+3} &\leq \bar{9}, \text{ and} \\ x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4} &\leq \overline{11}. \end{aligned}$$

The bars over the numbers at the right will just be used to identify where these come from in the analysis below.

Theorem 7 A pure strategy pair (f, g) is optimal if and only if it is a generalized A-F strategy. Furthermore each of these strategies is uniformly optimal.

Proof. Let (f, g) be any strategy pair for which either $y_t > \bar{y}_t$ for some t or $y_t \geq \bar{y}_t$ for all t. (Recall $\bar{y} = (2, 5, 6, 9, 10, 13, 14, 16)$). Under this assumption we will show that it must satisfy $y_t = \bar{y}_t$ for all t and must be a generalized A-F strategy, establishing both claims of the theorem. Without loss of generality we will assume that I starts by going N to $(0,1)$. Regardless of g, I will meet two agents of II at $(0,1)$, so that $x_1 = y_1 = \bar{y}_1 = 2$. This argument relies on our requirement that players cannot stay still.

Unless I turns (E or W) and II returns to his start, the largest x_2 can be is 1 (corresponding to I continuing N and meeting an agent starting at $(-1,1)$ or $(1,1)$ at location $(0,2)$). In this case $y_2 = 3$ and by (14) we have

$$y \leq (2,3,3+\bar{3},3+\bar{5},3+\bar{7},3+\bar{9},3+1\bar{1},16),$$

which violates our initial assumption. So we may assume without loss of generality that at time 2 Player I is at $(1,1)$ and Player II is back at his start. (A symmetric case arises if I chooses NW and is at $(-1,1)$).

If I does not go to $(1,-1)$ in the next two steps, with II returning to his start at time 4, the largest values for x_3 and x_4 are respectively 1 and 2 (obtained by I going to $(-1,1)$). However any strategy using these two steps is strictly dominated by the symmetric strategy going to $(1,-1)$. Any other strategy gives at most $x_3 = 1$ and $x_4 = 1$, hence at most $y_3 \leq 6$ and $y_4 \leq 7$. Consequently it has a cumulative distribution y satisfying

$$y \leq (2,3,6,7,7+\bar{3},7+\bar{5},7+\bar{7},16),$$

which again violates our assumption. So the strategy (f,g) must be as claimed (and in particular a generalized A-F strategy) up to time 4.

If I does not move W to $(0,-1)$ at time 5, then $x_5 = 0$ and $y_5 = 9$. Hence

$$y \leq (2,5,6,9,9,9+\bar{3},9+\bar{5},9+\bar{7}),$$

which again violates our assumptions. Hence I must go W to $(0,-1)$ at time 5.

If at time 6 II is not back at his start, and I at $(-1,-1)$, then $x_6 \leq 1$ and hence $y_6 \leq 11$. Consequently

$$y \leq (2,3,6,9,9,9+\bar{3},9+\bar{5},9+\bar{7}),$$

again violating our assumptions.

At time 6, I is at $(-1,-1)$ and there are three agents of II remaining at $(-1,1)$. The only way that to ensure a meeting at time 7 at $(-1,0)$ (that is, $x_7 = 1$) is for I to go to $(-1,0)$ while one of the agents of II at $(-1,1)$ also goes there. Player II must make sure that it is not the agent which I already met at time 1 which he meets there (not for the …rst time). The only way to ensure this is for II to go in the same direction at time 6 (move 7) as he went at time 0 (…rst move). So we may assume this and consequently we have

$$y \leq \bar{y},$$

so we know that the first assumption on (f, g) is impossible. This proves that any strategy with cumulative distribution function $\bar{y}$ (and hence any generalized A-F strategy) is uniformly optimal. Finally, to show that only such strategies are optimal, we observe that the only way to get $x_8 = 2$ and $y_8 = 16$ is for II to return to his start at time 8 while I continues to $(-1, 1)$. ■

5 Asymmetric Information Rendezvous

In some real situations the two players may have different information about the initial location of the other. For example if two parachutists drop at different times, the location A of the first (to drop) may be known to the second, but not the other way around. This type of problem was first considered in the plane by Anderson and Fekete [8], who also assumed that the initial distance between the players is known to both but allowed distinct speeds for the two players. We assume only that the distribution of the initial distance is known, but maintain our usual assumption of equal speeds. The analysis given here for the line is new to the literature.

In the two dimensional setting, Anderson and Fekete show that for some combinations of speeds and initial distance the optimal solution begins with both players moving to a common circle centered on A. Thereafter, both players keep a common decreasing distance to A, moving in on a spiral to A. They raise the question of when it is optimal for the player starting at A to remain there. In the 1-dimensional case it has been shown by Gal [11] that when neither player can see the other it is never (for no distribution of initial distance) optimal for a player to stay still. For the 1-dimensional asymmetric information rendezvous problem, we will determine conditions for staying still to be optimal.

We suppose that player I starts at a position A known to both, which we take as the origin 0. Player II can be assumed to know the location 0 or simply to know the direction to I. In any case it is easy to see that the following trajectory of II dominates any other motion:

$$g(t) = \begin{cases} g(0) - t, & \text{if } g(0) > 0, \\ g(0) + t, & \text{if } g(0) < 0. \end{cases}$$

Assume that the distribution of Player II's initial location $g(0)$ has a known cumulative probability distribution function F. This determines two distribution functions F_1 and F_2 of the positive initial distance to Player II along the positive and negative numbers, respectively, by

$$\begin{aligned} F_1(x) &= F(x) - F(0) = \Pr(0 \le g(0) \le x), \text{ and} \\ F_2(x) &= F(0) - F(-x) = \Pr(-x \le g(0) < 0). \end{aligned} \quad (15)$$

Since we can assume that $g(0)$ is not 0, the two cumulative probability distributions F_1 and F_2 satisfy

$$\lim_{x\to\infty} (F_1(x) + F_2(x)) = 1. \quad (16)$$

With this restriction on two distributions F_1 and F_2, we de…ne $R^a(F_1, F_2)$ to be the least expected time for the two players to meet. That is, $R^a(F) = R^a(F_1, F_2)$ is the rendezvous value for the asymmetric information rendezvous problem on the line.

Suppose that Player I follows a path $f(t)$, with $f(0) = 0$ and maximum speed 1. If $g(0) > 0$ then rendezvous will have occurred by time t if and only if $f(t) \ge g(0) - t$, or equivalently if

$$0 \le g(0) \le f(t) + t.$$

Similarly if $g(0) < 0$ then the rendezvous time T satis…es $T \le t$ if and only if

$$f(t) - t \le g(0) < 0.$$

Consequently the rendezvous probability is given by

$$F_f(t) \equiv \Pr(T \le t) = F_1(f(t) + t) + F_2(-f(t) + t), \quad (17)$$

and the expected meeting time $\hat{T}_f(F)$ and rendezvous value $R^a(F_1, F_2)$ are given by

$$\begin{aligned} \hat{T}_f(F) &= \int_0^\infty t \, dF_f(t), \\ &= \min_f \hat{T}_f(F) \end{aligned}$$

It turns out that the problem of minimizing the above expected time for f is equivalent to the Alternating Search Problem formulated by the author and Howard [6] and that the analysis of that problem can be e¤ectively applied to solve the asymmetric information rendezvous problem considered in this section. Note that this is a simpler and di¤erent reduction than that given in the Double Linear Search Problem [4].

5.1 The alternating search problem $AS(F_1, F_2)$.

We now give a brief summary of the Alternating Search Problem $AS(F_1, F_2)$ and some results on its solution given in [6]. In this search problem, there are two disjoint lines (or rays) L_1 and L_2, and a single stationary object is hidden in the positive direction along one of these lines. Its cumulative distribution along L_i is given by F_i, where the distributions F_i satisfy (16). Two searchers, S_1 and S_2, start at time 0 at the origins of their respective lines, and move with combined speed 2 in the positive directions along their lines. They may alternate moving at speed 2 or move simultaneously so the sum of their speeds is 2. Their joint motion is fully described by a single function $\alpha(t)$, where the respective positions of the two searchers at time t are given by $2\alpha(t)$ (on L_1) and $2t - 2\alpha(t)$ (on L_2). The alternation rule α is nondecreasing and satisfies the Lipshitz condition $\alpha(t) - \alpha(u) \leq t - u$, for $t > u$. The probability that the object has been found by time t if the rule α is used is given by

$$F_\alpha(t) \equiv F_1(2\alpha t) + F_2(2t - 2\alpha t). \tag{18}$$

The least expected time required to find the hidden object in this search problem is given by

$$v(F_1, F_2) = \min_\alpha \int_0^\infty t \; dF_\alpha(t),$$

and a rule α for which this minimum is achieved is called optimal.

The analysis given in [6] shows that if an optimal rule searches the two rays alternately (each at full speed 2) in consecutive time intervals, then the interval with the higher average density for the object is searched first. Furthermore, we have the following deeper results from [6], which we quote as Propositions.

Proposition 8 Suppose that the hidden object has a common distribution $F_1 = F_2$ on the two rays. If this distribution is strictly concave, then the unique optimal alternation rule is $\alpha(t) = t$, in which the two rays are searched in parallel. (Each searcher is at location t on his line at time t.) Otherwise, there are other optimal alternation rules.

Proposition 9 Suppose that F_1 and F_2 are convex on their supports. Then there is an optimal alternation rule which searches one ray (the one with the higher average density of the object) fully and then the other ray.

5.2 Application to rendezvous

We now show how the asymmetric information rendezvous problem can be reduced to the alternating search problem, and how the results obtained there can be interpreted for rendezvous. The formula (17) shows that the meeting probability in the rendezvous problem is the probability that a single stationary object placed at $g(0)$ according to F is found either by a searcher going along the positive real axis with motion $f(t)+t$ or by a searcher going along the negative real axis with motion $-f(t)+t$ (describing its distance from the origin). If we write this in terms of the alternation rule $\alpha(t)$ we find that the change of variables given by

$$\alpha(t) = \frac{f(t)+t}{2}$$

sets $F_f(t)$ in the rendezvous problem (17) equal to $F_\alpha(t)$ in the alternating search problem (18). In this equivalence the positive real axis is identified with ray 1 and the negative real axis with ray 2. Consequently we have the following.

Theorem 10 Consider the asymmetric information rendezvous problem on the line, in which I is placed at 0 and II is placed with a cumulative probability distribution F. Then the least rendezvous value $R^a(F) = R^a(F_1, F_2)$ is equal to the value $v(F_1, F_2)$ of the associated alternating search problem with the distribution F_i, given by (15).

Note that the Player I strategy $f(t) \equiv 0$ of staying still (Wait For Mommy) in the rendezvous problem corresponds to the alternation rule $\alpha(t) \equiv 1/2$ of simultaneous equal speed searching of the two rays in the alternating search problem. The rendezvous strategy of first going right to meet your partner and when this proves wrong going left ($f'(t) = 1$ first then -1) corresponds to the alternation rule of searching L_1 and then L_2.

The results from [6] cited in the previous subsection can be used to give a qualitative description of the optimal Player I motion $f(t)$ in certain cases of the rendezvous problem. Proposition 9 gives the following condition for staying still to be optimal for Player I.

Theorem 11 Suppose that Player II is symmetrically distributed in the asymmetric information rendezvous problem on the line ($F_1 = F_2$). Then a sufficient condition for 'waiting' (that is, $f(t)$ identically 0) to be optimal for Player I is that F_1 is concave. A necessary and sufficient condition for waiting to be uniquely optimal is that F_1 is strictly concave.

In some cases it is optimal for Player I to …rst move in one direction to meet on oncoming Player II until he realizes he has gone in the wrong direction, and then to move in the other direction. The following is an immediate consequence of Proposition 10.

Theorem 12 Suppose that both distributions F_1 and F_2 are convex on their supporting intervals. Then there is an optimal solution of the asymmetric information rendezvous problem on the line in which Player I goes in a single direction until the …rst moment he is sure that II was in the other direction, and then turns and goes in that direction until he meets him.

Situations in which F_1 and F_2 do not satisfy the conditions of the two previous results can be solved by the algorithms given in [6] For example, if the initial distance between the players is known to be $1,$ then Player I goes at speed 1 a distance .5 in one direction and then a distance $.5$ in the other. He meets Player II equally probably at times $.5$ and 1, so the rendezvous value is .75. If the initial location of Player II is uniformly distributed on $[-1, 1]$ (meeting the assumptions of both the previous theorems) then Player I can either wait, with average waiting time $.5;$ or he can follow the strategy of Theorem 13, in which he meets in average time .25 if he guesses the direction right and .75 if he guesses wrong, again with expected meeting time $.5.$

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