Clique Facets of the Orthogonal Latin Squares Polytope*

By

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Abstract

Since 1782, when Euler addressed the question of existence of a pair of Orthogonal Latin Squares *(OLS)* by stating his famous conjecture ([8, 9, 13]), these structures have remained an active area of research due to their theoretical properties as well as their applications in a variety of fields. In the current work we consider the polyhedral aspects of *OLS*. In particular we establish the dimension of the *OLS* polytope, describe all cliques of the underlying intersection graph and categorize them into three classes. For two of these classes we show that the related inequalities have Chvátal rank two and both are facet defining. For each such class, we give a separation algorithm of the lowest possible complexity, i.e. linear in the number of variables.

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1 Introduction

A Latin square L of order n is a $n \times n$ square matrix having n^2 entries of n different elements each occurring exactly once in every row and column. W.l.o.g. we can assume that the n different elements are the integers $0, 1, ..., n-1$. Two latin squares $L_1 = \| a_{ij} \|$, $L_2 = \| b_{ij} \|$ on n symbols are called *orthogonal* if every ordered pair of symbols occurs exactly once among n^2 pairs $(a_{ij,} b_{ij})\, i, j = 0, 1, ..., n-1$. An example of a pair of orthogonal latin squares *(OLS)* of order 4 is illustrated in Table 1.

Table 1: An *OLS* configuration of order 4

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

An alternative definition of *OLS* can be given with respect to disjoint sets and transversals. A *transversal* of a latin square of order n is a set of n cells, no two of which are in the same row or the same column or contain the same symbol. Consider three disjoint sets $I, J, K,\ | I | = | J | = | K | = n$. Let I be the set of rows, J the set of columns and K the set of symbols in the cells of a latin square. Then the latin square has an orthogonal mate if and only if it has n disjoint transversals ([8, Theorem 5.1.1]). Each transversal has n triplets, each representing a different row, column, and symbol. Therefore the transversal partitions the union of the n-sets of rows, columns, and symbols. For example, the four transversals, numbered from 0 to 3, of the first latin square illustrated in Table 1 are:

$((0,0,0),\ (1,2,3),\ (2,3,1),\ (3,1,2))_0$

$((0,1,1),\ (1,3,2),\ (2,2,0),\ (3,0,3))_1$

$((0,2,2),\ (1,0,1),\ (2,1,3),\ (3,3,0))_2$

$((0,3,3),\ (1,1,0),\ (2,0,2),\ (3,2,1))_3$

Let L, $(| L | = n)$ denote the index-set of the transversals. The orthogonal mate can be constructed if we set at each cell, defined by a row index and a column index, the value of the l index which indicates the transversal that this cell belongs to. This definition reveals another property of *OLS*: *Latin squares* $L_1,\ L_2$ *of order* n *are orthogonal if and only if for each symbol the* n *cells that contain it in* L_1 *correspond to* n *cells of a transversal in* L_2 *and vice-versa*. This implies that each pair of symbols $(a_{ij}, b_{ij})\ i, j = 0, ..., n-1$ appears exactly once *(the orthogonal property)*.

Now consider all possible triplets with one element from each of the three disjoint sets $I,\ J,\ K$ and a weight associated with each triplet. The *planar four-index assignment* problem $(4PAP_n)$ refers to identifying a minimum weight collection of n^2 triplets, which form n disjoint subsets (transversals) of n disjoint triplets. Each such subset forms a partition of the union of the three *n*-sets. This is the *weighted OLS* problem. It is analogous to the *planar three-index assignment* problem

$(3PAP_n)$ which asks for a minimum weight collection of n^2 pairs which form n disjoint subsets of n disjoint pairs. The $3PAP_n$ is equivalent to the problem of finding a minimum weight latin square (see [14]), while $4PAP_n$ is equivalent to identifying a minimum weight pair of *OLS*.

OLS have attracted substantial attention from very early on. As noted in [8, p. 156], Euler in 1779 had proposed the 36-officers problem which asked for a pair of *OLS* of order 6 (see also [13]). Having failed to find such a configuration and probably misguided by the nonexistence of a pair of OLS for $n = 2$, he conjectured that there exists no pair of *OLS* of order $n = 2(mod\, 4)$ ([8, 13]), i.e. odd multiple of two. At the beginning of the 20th century his conjecture was proven for $n = 6$ ([18]). However, it took sixty more years to prove that his conjecture for $n > 6$ was wrong ([5]). Even today *OLS* remain a very active area of research due to their theoretical properties and their applications in diverse fields. In terms of theoretical interest, these structures are strongly related to the theory of affine and projective planes and to orthogonal hypercubes and (t, m, s)-nets. Their practical applications include the problem of multi-variate experimental design, the problem of designing optimal error correcting codes and that of encryption. An extensive study of the theory of *OLS* and related structures as well as a variety of applications can be found in [8, 9, 13].

In spite of the early availability of a 0-1 integer programming formulation (due to D. Gale as noted in [7]), *OLS* have not been analyzed through mathematical programming. The current work is a step in that direction. In this paper we focus on the *OLS* polytope, study its intersection graph and obtain facets induced by clique constraints. In Section 2 we present the mathematical formulation of the problem and discuss related problems. The associated intersection graph and its cliques are described in Section 3. In Section 4 we prove the dimension of the underlying polytope and show that two of the three classes of clique inequalities induce facets of this polytope. We also provide proofs that these inequalities are of Chvátal rank 2. A separation algorithm for each of these two facet-defining clique classes is given in Section 5.

Throughout the rest of the paper we will assume $n > 1$ since for $n = 1$ *OLS* reduces to a trivial single-variable problem.

2 The OLS polytope and related structures

Appa ([1]) gives several different mathematical programming formulations for the *OLS* problem and indicates that the one attributed to D. Gale (in [7]) and reproduced below is the most suitable for theoretical and computational work:

$$\sum\{x_{ijkl} : i \in I, j \in J\} = 1, \forall k \in K, l \in L \tag{2.1}$$

$$\sum\{x_{ijkl} : j \in J, k \in K\} = 1, \forall i \in I, l \in L \tag{2.2}$$

$$\sum\{x_{ijkl} : i \in I, k \in K\} = 1, \forall j \in J, l \in L \tag{2.3}$$

$$\sum\{x_{ijkl} : k \in K, l \in L\} = 1, \forall i \in I, j \in J \tag{2.4}$$

$$\sum\{x_{ijkl} : i \in I, l \in L\} = 1, \forall j \in J, k \in K \tag{2.5}$$

$$\sum\{x_{ijkl} : j \in J, l \in L\} = 1, \forall i \in I, k \in K \tag{2.6}$$

$$x_{ijkl} \in \{0,1\} \forall i \in I, j \in J, k \in K, l \in L \tag{2.7}$$

where I, J, K, L are disjoint sets with $\mid I \mid = \mid J \mid = \mid K \mid = \mid L \mid = n$.

Given real weights c_{ijkl} for every $(i,j,k,l,) \in I \times J \times K \times L$ the problem of minimizing (maximizing) the function $\sum\{c_{ijkl}x_{ijkl} : i \in I, j \in J, k \in K, l \in L\}$ over the polytope described by constraints (2.1),..., (2.7) is the $4PAP_n$. This formulation requires n^4 binary variables and $6n^2$ equality constraints.

Let A denote the coefficient matrix of constraints (2.1),..., (2.6). Then we define the polytope P_L as $P_L = \{x \in \mathbb{R}^{n^4} Ax = e, x \geq 0\}$ where $e = \{1,1,...,1\}^T$. The convex hull of integer points of P_L is defined as $P_I = \text{conv}\{x \in \{0,1\}^{n^4} Ax = e\}$. This is the *OLS* polytope since every integer point $x \in P_I$ is an *OLS*. P_L is also called the *linear relaxation* of P_I. Clearly $P_I \subset P_L$. We will sometimes refer to P_I as P_I^n so as to include the concept of order in the notation. Thus $P_I^6 = \emptyset$ is another way of stating Euler's conjecture for $n = 6$.

Substituting $(=)$ by $(\leq)$ in constraints (2.1),..., (2.6) yields the polytope $\tilde{P}_I = \text{conv}\{x \in \{0,1\}^{n^4} : Ax \leq e\}$. The polytopes $P_I, \tilde{P}_I$ are related since $P_I \subset \tilde{P}_I$. Let D denote a matrix of zeros and ones. Then P_I is a special case of the set partitioning polytope $P_{SPP} = \{x \in \{0,1\}^q : Dx = e\}$ whereas $\tilde{P}_I$ is a special case of the set packing polytope $P_{SP} = \{x \in \{0,1\}^q : Dx \leq e\}$ (see [2, 15] for details).

There are two problems, each involving three disjoint $n-$ sets, that are highly related to the *OLS* problem: $(3PAP_n)$ and the *axial three-index assignment problem* $(3AAP_n)$. We have referred to the former in the previous section. The latter is defined with respect to three disjoint $n-$ sets, namely I, J, K, and a weight coefficient c_{ijk} for each triplet $(i,j,k) \in I \times J \times K$. $3AAP_n$ is the problem of finding n disjoint triplets of minimum weight, i.e. finding a single transversal of minimum weight. The constraints of $3AAP_n$ are

$$\sum\{x_{ijk} : i \in I, j \in J\} = 1, \forall k \in K$$
$$\sum\{x_{ijk} : j \in J, k \in K\} = 1, \forall i \in I$$
$$\sum\{x_{ijk} : i \in I, k \in K\} = 1, \forall j \in J$$
$$x_{ijk} \in \{0, 1\}, \forall i \in I, j \in J, k \in K$$

Remark 2.1. Constraints (2.1), (2.2), (2.3), for a given value of the l index, are equivalent to the constraints of *a* $3AAP_n$.

The above remark will be very useful for establishing the dimension of P_L and P_I. Research work on $3AAP_n$ and $3PAP_n$ polytopes can be found in [3, 4, 11, 17] and [10].

3 The intersection graph and its cliques

Let R and C denote the index sets of rows and columns respectively of the 0-1 A matrix. We refer to a column of the A matrix as a^c for $c \in C$. The *intersection* graph $G_A(V, E)$ has a node c for every column a^c of A and an edge (c_s, c_t) if and only if $a^{c_s} \cdot a^{c_t} \geq 1$, i.e., both columns c_s and c_t of A have a $+1$ entry in at least one common row.

Let $G_A(C, E_C)$ denote the intersection graph of *OLS*, where $C = I \times J \times K \times L$. It is convenient to label the n^4 columns of the *OLS* A matrix, not from 1 to n^4, but with four indices i, j, k and l ranging from 1 to n. This leads to an equivalent definition of $G_A(C, E_C)$ where c_s represents the index set of column s.

Definition 3.1. The intersection graph of *OLS* $G_A(C, E_C)$ has a node c, for every $c \in C$, and an edge (c_s, c_t) for every pair of nodes $c_s, c_t \in C$ such that $\mid c_s \cap c_t \mid= 2$ or 3.

Note that an edge $(c_s, c_t) \in E_C$ corresponds to columns a^{c_s}, a^{c_t} with $a^{c_s} \cdot a^{c_t} = 1$ or 3. The row set of the *OLS* A matrix is defined as $R = (K \times L) \cup (I \times L) \cup (J \times L) \cup (I \times J) \cup (J \times K) \cup (I \times K)$. Since $\mid I \mid=\mid J \mid=\mid K \mid=\mid L \mid= n$, $\mid C \mid= n^4$ and $\mid R \mid= 6n^2$.

Proposition 3.2. *The graph* $G_A(C, E_C)$ *is regular of degree* $2(3n - 1)(n - 1)$.

Proof. Consider any $c \in C$. There are $(n-1)^4$ elements of C, which have no index in common with c. For each of the four indices of c there are $(n-1)^3$ elements of C, which share the same value for this index but have different values for the other three. Therefore, there are $4(n-1)^3$ elements of C which have exactly one index in common with c. By definition 3.1, c is connected only to nodes that have two or three indices in common with it, so it is connected to all but $(n-1)^4 + 4(n-1)^3$ nodes. Therefore the degree of each $c \in C$ is $n^4 - 1 - ((n-1)^4 + 4(n-1)^3) = 6n^2 - 8n + 2 = 2(3n-1)(n-1)$.

□

Corollary 3.3. $\mid E_C \mid = n^4(3n-1)(n-1)$.

Proof. Since the number of edges of a graph equals the sum of the degrees of its nodes divided by 2, we have $\mid E_C \mid = 0.5 \times n^4 \times 2(3n-1)(n-1) = n^4(3n-1)(n-1)$.

□

A *maximal complete* subgraph of a graph $G(V,E)$ is called a *clique* ([3, 10, 15]). Let $Q \subseteq V$ denote the node set of a clique. The *cardinality* of a clique is the cardinality of its node set Q, denoted $\mid Q \mid$. Cliques of the intersection graph $G_A(V,E)$ define inequalities of the form $\sum\{x_q : q \in Q\} \leq 1$, which are highly relevant to the description of the set packing polytope. Next we will examine the cliques of $G_A(C, E_C)$.

Let a_r^c denote the entry of the A matrix at row r and column c. Then we define the set $R(r) = \{c \in C : a_r^c = 1\}$. So $R(r)$ denotes the set of columns with a non-zero entry in row r.

Proposition 3.4. *For each $r \in R$, the node set $R(r)$ induces a clique in $G_A(C, E_C)$ of cardinality n^2. There are $6n^2$ cliques of this type.*

Proof. The subgraph induced by the node set $R(r)$ is complete since all its elements have two indices in common. To prove that it is also maximal w.l.o.g. assume that $r = (i_1, j_1) \in I \times J$ and consider $c_0 = (i_0, j_0, k_0, l_0) \in C \setminus R(r)$ where $i_0 \neq i_1$ and $j_0 \neq j_1$. Since $R(r)$ contains all n^2 elements of C whose first two indices are i_1 and j_1, it contains an element $c_1 = (i_1, j_1, k_1, l_1)$ with $\mid c_0 \cap c_1 \mid = 0$. Next consider $c_0 = (i_1, j_0, k_0, l_0) \in C \setminus R(r)$. But then there exists $c_1 \in C$ (for example $c_1 = (i_1, j_1, k_1, l_1)$) so that $\mid c_0 \cap c_1 \mid = 1$. The same happens if $c_0 = (i_0, j_1, k_0, l_0)$. Therefore there is no c_0 such that the subgraph induced by $R(r) \cup \{c_0\}$ is complete. Consequently, the subgraph having $R(r)$ as its node set is maximal. There are as many cliques of this type as the number of rows of the A matrix, i.e. $6n^2$.

□

Proposition 3.5. *For each $c \in C$ the set $Q(c) = \{c\} \cup \{s \in C : \mid c \cap s \mid = 3\}$ induces a clique of cardinality $4n-3$ in $G_A(C, E_C)$. There are n^4 cliques of this type.*

Proof. W.l.o.g. consider $c = c_0 = (i_0, j_0, k_0, l_0) \in C$ and $c_1, c_2 \in Q(c_0)$ with $c_1 \neq c_2 \neq c_0 \neq c_1$. Since c_1, c_2 have three indices in common with c_0, at least two of their indices coincide. Therefore $(c_1, c_2) \in E_C$ for any $c_1, c_2 \in Q(c_0)$, thus $Q(c_0)$ is complete. To show that the subgraph is also maximal, consider $c_3 = (i_3, j_3, k_3, l_3) \in C \setminus Q(c_0)$ and $(c_0, c_3) \in E_C$. Then c_3 has exactly two indices in common with c_0, by definition. If $\mid c_0 \cap c_3 \mid = 2$, w.l.o.g. consider $i_0 = i_3$, $j_0 = j_3$ and $k_0 \neq k_3$, $l_0 \neq l_3$. By definition, $Q(c_0)$ contains two elements, namely $c_s = (i_s, j_0, k_0, l_0)$ and $c_t = (i_0, j_t, k_0, l_0)$ such that $i_0 \neq i_s$ and $j_0 \neq j_t$. But then $\mid c_3 \cap c_s \mid = \mid c_3 \cap c_t \mid = 1$, thus the graph with node set $Q(c_0) \cup \{c_3\}$ is not complete. So $Q(c_0)$ is maximal.

The set $Q(c_0)$ includes node $c_0 = (i_0, j_0, k_0, l_0)$ and all nodes with exactly one index different from c_0. So $\mid Q(c_0) \mid = 4(n-1) + 1 = 4n-3$.

There are n^4 elements belonging to the set C, each of which can play the role of c_0. Therefore, there are n^4 cliques of this type.

$\square$

Proposition 3.6. *Let $c, s \in C$ such that $\mid c \cap s \mid= 1$. Then the set $Q(c,s) = \{c\} \cup \{t \in C :\mid c \cap t \mid= 2, \mid s \cap t \mid= 3\}$ induces a 4-clique in $G_A(C, E_C)$.*

Proof. W.l.o.g. let $c = c_0 = (i_0, j_0, k_0, l_0)$ and $s = (i_0, j_1, k_1, l_1)$. We can uniquely define three elements $t_1 = (i_0, j_0, k_1, l_1)$, $t_2 = (i_0, j_1, k_0, l_1)$, $t_3 = (i_0, j_1, k_1, l_0)$, satisfying $\mid c \cap t_i \mid= 2$ and $\mid s \cap t_i \mid= 3$ for $i = 1, 2, 3$. It is obvious that the node set $\{c, t_1, t_2, t_3\}$ induces a complete subgraph of $G_A(C, E_C)$. To show that it is also maximal, consider $c_2 = \{i_2, j_2, k_2, l_2\} \in C \setminus Q(c,s)$. If $i_2 \neq i_0$ then for an edge (c, c_2) to exist in $G_A(C, E_C)$ we must have $\mid c \cap c_2 \mid\geq 2$, which implies that $\mid c_2 \cap t_i \mid\leq 1$ for $i = 1, 2, 3$. Therefore $Q(c,s)$ cannot be extended to include c_2, since the resulting graph is not complete. If $i_2 = i_0$ either c_2 has another element common with c and the remaining two with s, in which case it coincides with one of the t_i's, or it has three elements in common with c and one with s. In the latter case, w.l.o.g. let $j_2 = j_0$ and $k_2 = k_1$. Then we have $\mid c_2 \cap t_2 \mid= 1$. Hence, in this case as well $Q(c,s)$ cannot be extended. Therefore the subgraph induced by $Q(c,s)$ is complete and maximal.

$\square$

Concerning the cardinality of the set of cliques of this type, every ordered pair (c, s) such that $\mid c \cap s \mid= 1$ can be used to create a clique of this type. Considering that $\mid C \mid= n^4$ and that for each $c \in C$ there are $4(n-1)^3$ possible s such that $\mid c \cap s \mid= 1$, the number of such ordered pairs is $4n^4(n-1)^3$. Note, however, that the 4-clique $Q(c,s) = (c, t_1, t_2, t_3)$ is also generated as $Q(c_i, s_i)$ for $i = 1, 2, 3$ where

$c_1 = t_1 = (i_0, j_0, k_1, l_1)$ and $s_1 = (i_0, j_1, k_0, l_0)$,

$c_2 = t_2 = (i_0, j_1, k_0, l_1)$ and $s_2 = (i_0, j_0, k_1, l_0)$,

$c_3 = t_3 = (i_0, j_1, k_1, l_0)$ and $s_3 = (i_0, j_0, k_0, l_1)$.

Proposition 3.7. $Q(c,s) = Q(c_i, s_i),\ i = 1, 2, 3$

It is also obvious that the 4-clique $Q(c,s) = (c, t_1, t_2, t_3)$ cannot arise from any other choice of c and s.

Corollary 3.8. *The number of distinct 4-cliques is $n^4(n-1)^3$.*

Proof. Each 4-clique arises from four different ordered pairs of C and there exist $4n^4(n-1)^3$ such pairs.

$\square$

Cliques described in Propositions 3.4, 3.5 and 3.6 will be called cliques of type I, II and III respectively. Next we will show that these are the only types of cliques in $G_A(C, E_C)$.

Theorem 3.9. *The cliques of type I, II and III are the only cliques in $G_A(C, E_C)$.*

Proof. Let Q be the node set of a clique in $G_A(C, E_C)$. Let $c = (i_0, j_0, k_0, l_0) \in Q$. Every other $q \in Q$ must have at least two indices in common with c. So there has to be a $q_s \in Q$ such that $| c \cap q_s |= 2$. W.l.o.g. let $q_s = (i_0, j_0, k_1, l_1)$. If every other element of Q has the same values i_0, j_0 for the first two indices then Q is a node set of a clique of type I. If not, then there exists a $q_t = (i_t, j_t, k_t, l_t) \in Q$ which must satisfy the following relationships:

(i) Either $i_t = i_0$ or $j_t = j_0$. If both $i_t \neq i_0$ and $j_t \neq j_0$ then $k_t = k_0$ and $l_t = l_0$ in order for q_t to be connected to c. But then $| q_s \cap q_t |= 0$, which means that Q does not induce a clique.

(ii) Either $k_t = k_0$ and $l_t = l_1$ or $k_t = k_1$ and $l_t = l_0$ then, together with (i), $| c \cap q_t |= 1$ while if $k_t = k_0$ and $l_t = l_0$ then $| q_s \cap q_t |= 1$. In both cases Q does not induce a clique.

W.l.o.g. assume that $q_t = (i_0, j_1, k_0, l_1)$. If $(i_0, j_1, k_1, l_0) \in Q$ then $Q \equiv Q(c, (i_0, j_1, k_1, l_1))$, in which case Q is a node set of a clique of type III. If $(i_0, j_1, k_1, l_0) \notin Q$ then there is a $q_r \in Q$ such that $| q_r \cap (i_0, j_1, k_1, l_0) |\leq 1$, $| q_r \cap c |\geq 2$, $| q_r \cap q_s |\geq 2$ and $| q_r \cap q_t |\geq 2$. Then one can check by enumeration of case that every such q_r must have at least three indices in common with (i_0, j_0, k_0, l_1), in which case $Q \equiv Q(i_0, j_0, k_0, l_1)$ i.e. Q induces a clique of type II.

□

Corollary 3.10. *The total number of cliques in $G_A(C, E_C)$ is $n^4((n-1)^3 + 1) + 6n^2$.*

Proof. As shown above, there are $6n^2$, n^4, and $n^4(n-1)^3$ cliques of type I, II and III respectively.

□

4 Facets induced by clique inequalities

We briefly summarize some basic concepts and definitions of polyhedral theory (for a short but succinct presentation of this theory see [12, 16]). A *polyhedron* is the intersection of a finite set of half spaces. A *polytope* is a bounded polyhedron. A polytope P is of *dimension* n, denoted as dim$(P) = n$, if it contains $n + 1$ affinely independent points. By convention, if $P = \emptyset$ then dim$(P) = -1$. If $P = \{x \in \mathbb{R}^n : B^= x = b^=, B^\leq x \leq b^\leq\}$ then dim$(P) = n -$ rank$(B^=)$. Here $B^=$ and $B^\leq$ denote the matrix of co-efficients of equality and less than or equal to type inequality constraints respectively, while $b^=$ and $b^\leq$ denote the corresponding right-hand side vectors for the linear system defining P. An inequality $ax \leq a_0$ is called *valid* for P if it is satisfied by all $x \in P$. It is called *supporting* if it is valid and there exist some $\hat{x} \in P$ satisfying $a\hat{x} = a_0$. The set of points which satisfy $ax \leq a_0$ as equality ($F = \{x \in P : ax = a_0\}$) is called a *face* of P. A face F of a polytope P is said to be *improper* if $ax = a_0$ for all $x \in P$. A proper, non-empty face F of P is called a *facet* if dim$(F) =$ dim$(P) - 1$. Facets are important since they provide a minimal inequality representation of a polyhedron. Our main interest here is in the facets of the convex hull P_I of integer points in P_L defined in Section 2. Conditions *(c)* and *(d)* of the following theorem usually provide the two basic tools for proving that a given inequality $ax \leq a_0$ induces a facet.

Theorem 4.1. *(see [16, Theorem 3.16], [12, Theorem1])*

Let $P \subseteq \mathbb{R}^n$ be a polyhedron and assume that B is a real valued $m \times n$ matrix and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n : B^{=}x = b^{=}, B^{\leq}x \leq b^{\leq}\}$ where $B = (B^{=}, B^{\leq})^T$ and $b = (b^{=}, b^{\leq})^T$. Let F be a non-empty face of P, then the following statements are equivalent:

(a) F *is a facet of* P.

(b) F *is a maximal proper face of* P.

(c) $dim(F) = dim(P) - 1$.

(d) There exists an inequality $ax \leq a_0$ valid with respect to P with the following three properties:

(i) $F = \{x \in P : ax = a_0\}$.

(ii) There exists $\bar{x} \in P$ with $a\bar{x} < a_0$, i.e. the inequality is proper

(iii) If any other inequality $dx \leq d_0$, valid with respect to P satisfies $F = \{x \in P : dx = d_0\}$, then (d, d_0) can be expressed as a linear (affine) combination of $(B^{=}, b^{=})$ and (a, a_0).

In this paper, we will use *(d)* to prove that cliques of type II and III induce facets of P_I. The dimension of P_I will also be established through the same approach. The same technique has been used for proving facet-defining inequalities and the dimension of the $3AAP_n$, and $3PAP_n$ polytopes in [3] and [10] respectively.

First, we discuss some properties of $\tilde{P}_I$. In [15] it is shown that for the general packing polytope (P_{SP}) the cliques of the underlying intersection graph induce facet-defining inequalities. However, no similar result has been proven for P_{SPP}. Since the *OLS* packing polytope $\tilde{P}_I$ is a special case of P_{SP}, the inequalities $\sum\{x_q : q \in Q\} \leq 1$, where Q is the node set of a clique of $G_A(C, E_C)$, define facets of the polytope $\tilde{P}_I$. Other properties of $\tilde{P}_I$ arising from its relation to P_{SP} are:

(i) $\tilde{P}_I$ is full-dimensional, i.e. $dim(\tilde{P}_I) = n^4$. The $n^4 + 1$ independent points of $\tilde{P}_I$ are the zero vector and all the n^4 unit vectors.

(ii) $\tilde{P}_I$ is down monotone, i.e. $x \in \tilde{P}_I \Rightarrow y \in \tilde{P}_I$ for all y such that $0 \leq y \leq x$.

(iii) The non-negativity constraints $x_{ijkl} \geq 0$ define facets of $\tilde{P}_I$.

Although we know quite a few things about the facial structure of $\tilde{P}_I$, the same cannot be said with respect to P_I. Since P_I is a face of $\tilde{P}_I$ we know that $dim(P_I) \leq dim(\tilde{P}_I)$. However, the structure of P_I presents irregularities that do not appear in $\tilde{P}_I$. For example, we know that $P_I = \emptyset$ for $n = 2$ and $n = 6$. $P_I^2 = \emptyset$ can be easily verified since there are only two latin squares for $n = 2$. As stated previously, $P_I^6 = \emptyset$ was proven in [18]. Fortunately, $P_I^n \neq \emptyset$ for $n \neq 2, 6$ as shown in [6] (see also [13, Theorem 2.9]). Before establishing the dimension P_I we prove the dimension of P_L.

Theorem 4.2. *The rank of the system $Ax = e$ is $6n^2 - 8n + 3$*

Proof. Order the n^4 columns of the A matrix, denoted by x_{ijkl}, so that indices k, j, i and l vary in that order. For $n = 2$, the order of the column indices is:

$(1,1,1,1),(1,1,2,1),(1,2,1,1),(1,2,2,1),(2,1,1,1),(2,1,2,1),(2,2,1,1),(2,2,2,1),$

$(1,1,1,2),(1,1,2,2),(1,2,1,2),(1,2,2,2),(2,1,1,2),(2,1,2,2),(2,2,1,2),(2,2,2,2)$

As to the $6n^2$ rows, we divide them into six sets of n^2 rows each, as defined by equalities (2.1)..(2.6).

Figure 1 illustrates the matrix for $n = 2$ (each constraint set is separated from the next by a horizontal line).

Figure 1: *OLS* A matrix for $n = 2$

$$
\left[\begin{array}{cccccccc|cccccccc}
1 & & 1 & & 1 & & 1 & & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & & & 1 & & 1 & & 1 & & 1 \\
\hline
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & & & 1 & 1 & & & & & & & & & & \\
 & & 1 & 1 & & & 1 & 1 & & & & & & & & \\
 & & & & & & & & 1 & 1 & & & 1 & 1 & & \\
 & & & & & & & & & & 1 & 1 & & & 1 & 1 \\
\hline
1 & 1 & & & & & & & 1 & 1 & & & & & & \\
 & & 1 & 1 & & & & & & & 1 & 1 & & & & \\
 & & & & 1 & 1 & & & & & & & 1 & 1 & & \\
 & & & & & & 1 & 1 & & & & & & & 1 & 1 \\
\hline
1 & & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1 & & & & 1 \\
\hline
1 & & 1 & & & & & & 1 & & 1 & & & & & \\
 & 1 & & 1 & & & & & & 1 & & 1 & & & & \\
 & & & & 1 & & 1 & & & & & & 1 & & 1 & \\
 & & & & & 1 & & 1 & & & & & & 1 & & 1
\end{array}\right]
$$

To find the rank of the A matrix, we follow five steps, the last four of which identify a $3AAP_n$ substructure, exactly as at Remark 2.1.

Step I: It is obvious that the sum of all the rows of each set is the same, i.e. $\sum\{x_{ijkl} : i \in I, j \in J, k \in K, l \in L\} = n^2$. Therefore, any one constraint can be removed from any of the six sets as being linearly dependent. We choose to keep the row set (2.1) intact and remove the first row of all the remaining sets. Table 2 shows the outcome.

Table 2: *Linearly dependent rows removed at Step I*

Row set	*Rows removed*	*Rows removed for n=2*
(2.1)	$-$	$-$
(2.2)	n^2+1	5
(2.3)	$2n^2+1$	9
(2.4)	$3n^2+1$	13
(2.5)	$4n^2+1$	17
(2.6)	$5n^2+1$	21

Step II: Consider row sets (2.1), (2.2) and (2.3). Observe that, as noted at Remark 2.1, they form n independent $3AAP_n$ problems, one for each value of the index l. For $l = l_0$, the corresponding $3AAP_n$ involves the n^3 variables x_{ijkl_0}, for $i, j, k = 1, ..., n$, and the $3n$ rows $(l_0 - 1) \cdot n + t$, $n^2 + (l_0 - 1) \cdot n + t$, $2n^2 + (l_0 - 1) \cdot n + t$, for $t = 1, ..., n$. Balas and Saltzman show in [3] that the rank of a $3n \times n^3$ $3AAP_n$ matrix is $3n - 2$. So, we can remove up to 2 rows from each of the n $3AAP_n$ problems. Note that, having removed rows $n^2 + 1$, $2n^2 + 1$ at Step I, for $l_0 = 1$, the corresponding $3AAP_n$ includes no linearly dependent rows. For the remaining $n - 1$ independent $3AAP_n$ problems, we can remove the two linearly dependent rows. We choose to remove rows numbered $n^2+(t-1)\cdot n+1$, $2n^2+(t-1)\cdot n+1$, for $t = 2, ..., n$, a total of $2(n - 1)$ rows. Table 3 gives a complete list of rows removed so far.

Table 3: *Linearly dependent rows removed at Steps I & II*

Row set	*Rows removed*	*Rows removed for n=2*
(2.1)	$-$	$-$
(2.2)	$\{n^2 + (t-1) \cdot n + 1, t = 1..n\}$	$5, 7$
(2.3)	$\{2n^2 + (t-1) \cdot n + 1, t = 1..n\}$	$9, 11$
(2.4)	$3n^2+1$	13
(2.5)	$4n^2+1$	17
(2.6)	$5n^2+1$	21

Step III: Consider row sets (2.1), (2.5) and (2.6). Observe that they form n independent $3AAP_n$ problems, one for each value of the index k. For $k = k_0$, the corresponding $3AAP_n$ involves variables x_{ijk_0l}, for $i, j, l = 1...n$, and rows $k_0 + (t - 1) \cdot n$, $4n^2 + k_0 + (t - 1) \cdot n$, $5n^2 + k_0 + (t - 1) \cdot n$, for $t = 1, ..., n$. Again, for $k_0 = 1$, the corresponding $3AAP_n$ includes no linearly dependent rows, since rows $4n^2 + 1$, $5n^2 + 1$ have already been removed. All the rows of the remaining $n - 1$ independent $3AAP_n$ problems are present. We choose to remove two linearly dependent rows from each problem, namely rows $4n^2 + t$, $5n^2 + t$, for $t = 2, ..., n$,i.e. $2(n - 1)$ rows in total. Table 4 gives a complete list of rows removed so far.

Step IV: Consider row sets (2.3), (2.4) and (2.5). Observe that they form n independent $3AAP_n$ problems, one for each value of the index j. For $j = j_0$, the corresponding $3AAP_n$ involves variables x_{ij_0kl}, for $i, k, l = 1, ..., n$ and rows $2n^2 + j_0 + (t-1) \cdot n$, $3n^2 + j_0 + (t-1) \cdot n$, $4n^2 + (j_0 - 1) \cdot n + t$, for $t = 1, ..., n$. For $j_0 = 1$, the corresponding $3AAP_n$ includes no linearly dependent rows, since rows $2n^2 + 1$, $4n^2 + t$, $5n^2 + t$, for $t = 1, ..., n$, have already been removed. The

Table 4: *Linearly dependent rows removed at Steps I - III*

Row set	*Rows removed*	*Rows removed for n=2*
(2.1)	$-$	$-$
(2.2)	$\{n^2+(t-1)\cdot n+1,\, t=1..n\}$	$5,7$
(2.3)	$\{2n^2+(t-1)\cdot n+1,\, t=1..n\}$	$9,11$
(2.4)	$3n^2+1$	13
(2.5)	$4n^2+1, \{4n^2+t, t=1..n\}$	$17,18$
(2.6)	$5n^2+1, \{5n^2+t, t=1..n\}$	$21,22$

remaining $n-1$ independent $3AAP_n$ problems have been left intact. We choose to remove two linearly dependent rows from each problem, namely rows $3n^2+t$, $4n^2+(t-1)\cdot n$, for $t=2...n$, i.e. $2(n-1)$ rows. Table 5 gives a complete list of rows removed so far.

Table 5: *Linearly dependent rows removed at Steps I - IV*

Row set	*Rows removed*	*Rows removed for n=2*
(2.1)	$-$	$-$
(2.2)	$\{n^2+(t-1)\cdot n+1,\, t=1..n\}$	$5,7$
(2.3)	$\{2n^2+(t-1)\cdot n+1,\, t=1..n\}$	$9,11$
(2.4)	$3n^2+1, \{3n^2+t, t=2..n\}$	$13,14$
(2.5)	$\{4n^2+(t-1)\cdot n+1, t=1..n\}, \{4n^2+t, t=2..n\}$	$17,18,19$
(2.6)	$5n^2+1, \{5n^2+t, t=2..n\}$	$21,22$

Step V: Consider row sets (2.2), (2.4) and (2.6). Observe that they form n independent $3AAP_n$ problems one for each value of the index i. For $i=i_0$, the corresponding $3AAP_n$ involves variables x_{i_0jkl}, for $j,k,l=1,...,n$, and rows $n^2+i_0+(t-1)\cdot n$, $3n^2+(i_0-1)\cdot n+t$, $5n^2+(i_0-1)\cdot n+t$, for $t=1,...,n$. Again for $i_0=1$, the corresponding $3AAP_n$ includes no linearly dependent rows, since rows $n^2+(t-1)\cdot n+1$, $3n^2+t$, for $t=1,...,n$ and $5n^2+1$ have already been removed. All the rows of the remaining $n-1$ independent $3AAP_n$ problems are present. We choose to remove two linearly dependent rows from each problem, namely rows $3n^2+(t-1)\cdot n$, $5n^2+(t-1)\cdot n$, for $t=2,...,n$ i.e. $2(n-1)$ rows. Table 6 gives a complete list of rows removed so far.

Table 6: *Linearly dependent rows removed at Steps I - V*

Row set	*Rows removed*	*Rows removed for n=2*
(2.1)	$-$	$-$
(2.2)	$\{n^2+(t-1)\cdot n+1,\, t=1..n\}$	$5,7$
(2.3)	$\{2n^2+(t-1)\cdot n+1,\, t=1..n\}$	$9,11$
(2.4)	$\{3n^2+(t-1)\cdot n+1, t=1..n\}, \{3n^2+t, t=2..n\}$	$13,14,15$
(2.5)	$\{4n^2+(t-1)\cdot n+1, t=1..n\}, \{4n^2+t, t=2..n\}$	$17,18,19$
(2.6)	$\{5n^2+(t-1)\cdot n+1, t=1..n\}, \{5n^2+t, t=2..n\}$	$21,22,23$

In total, $4\cdot 2\cdot(n-1)+5=8n-3$ rows have been removed. Therefore, $6n^2-8n+3$ is an upper bound on the rank of A. We will complete the proof by exhibiting $6n^2-8n+3$ affinely independent columns.

Consider the columns:

$$(1,1,1,1), ..., (1,1,n,1), ..., (1,1,1,n), ..., (1,1,n,n) \qquad (n^2 \text{columns})$$
$$(2,1,1,1), ..., (n,1,1,1), ..., (2,1,1,n), ..., (n,1,1,n) \qquad (n(n-1) \text{ columns})$$
$$(1,2,1,1), ..., (1,n,1,1), ..., (1,2,1,n), ..., (1,n,1,n) \qquad (n(n-1) \text{ columns})$$
$$(2,2,1,1), ..., (n,n,1,1), ..., (2,2,1,n-1), ..., (n,n,1,n-1) \qquad ((n-1)^2 \text{ columns})$$
$$(1,2,2,1), ..., (1,n,n,1), ..., (1,2,2,n-1), ..., (1,n,n,n-1) \qquad ((n-1)^2 \text{ columns})$$
$$(2,1,2,1), ..., (n,1,n,1), ..., (2,1,2,n-1), ..., (n,1,n,n-1) \qquad ((n-1)^2 \text{ columns})$$

The matrix formed by these columns and the $6n^2 - 8n + 3$ remaining rows of A is upper triangular, with each diagonal element equal to one.

□

Corollary 4.3. $dim(P_L) = n^4 - 6n^2 + 8n - 3$.

We now describe the tools needed to obtain the dimension and the clique facets of P_I.

Unless otherwise stated, we will illustrate a pair of *OLS* as points of P_I expressed in terms of four sets of indices, viz. I for the row set, J for the column set, and K and L for the set of elements of the first and the second latin square respectively. The elements of all four sets are the integers from 1 to n. We will further use $k(i,j)$ (respectively $l(i,j)$) to denote the value of the cell in row i, column j of the first (second) latin square. Thus $k(i,j) \in K$ and $l(i,j) \in L$. The following remark reveals a very useful property of the points of P_I corresponding to a pair of *OLS*.

Remark 4.4. Given an *OLS* structure and $m_1, m_2 \in M$, where M can be any one of the disjoint $n-$ sets $I,\ J,\ K,\ L$ then (inter)changing all m_1 values to m_2 and all m_2 values to m_1 yields another *OLS* structure. These two structures are called *equivalent* ([8, p. 168]).

If the interchange is carried out for elements of set I (i.e. $M = I$) we will call it a first index interchange, for elements of set J a second index interchange, etc. To facilitate a study of interchanges, we define the *interchange* operator $(\leftrightarrow)$. Thus, by setting $x^* = x(i_1 \leftrightarrow i_2)_1$ we imply that at point $x \in P_I$ we apply a first index interchange between rows $i_1, i_2 \in I$ deriving point $x^* \in P_I$. Note that brackets must also have an index for denoting the set of the indices that are interchanged. Notation without this subscript in cases like $(1 \leftrightarrow n)$ becomes ambiguous. It is easy to see that $x(m_1 \leftrightarrow m_2)_t = x(m_2 \leftrightarrow m_1)_t$ and $x = x(m_1 \leftrightarrow m_2)_t(m_1 \leftrightarrow m_2)_t$. A series of interchanges at a point $x \in P$ is expressed by using the operator $(\leftrightarrow)$ as many times as the number of interchanges with priority from left to right. For example, $x^* = x(i_1 \leftrightarrow i_2)_1(1 \leftrightarrow n)_3$ is taken to mean that at point x we apply a first index interchange between $i_1, i_2 \in I$ and then at the derived point we apply a third index interchange between $1, n \in K$, thus yielding point x^*.

We additionally define a *conditional* interchange as the interchange to be performed only when a certain condition is met. The condition refers to a logical expression consisting of values of an index set, at a given point x. If this expression evaluates *true* then the interchange will be applied to point x. Since we are going to use only conditional interchanges for which both the logical expression and the interchange refer to elements of the same set we will use a common subscript for both. Thus to denote this type of interchange at point x we will use the expression $x(condition?interchange)_t$, where the subscript refers to the index set. For example, $x_2 = x_1(i_1 = n?1 \leftrightarrow i_1)_1$ implies that if $i_1 = n$ at point x_1 then we derive point x_2 by applying at point x_1 the first index interchange between $1, i_1$. If $i_1 \neq n$ then $x_2 = x_1$. As in the previous case we can have more than one conditional interchange in the same expression. Again priority is considered from left to right. As we will see shortly, the interchanges will be used extensively for proving the dimension of P_I and facet defining inequalities.

An additional complication for proving the dimension of P_I comes from the fact that it is not easy to exhibit a pair of *OLS* for every value of n, i.e. for every n it is difficult to demonstrate a 0-1 vector feasible w.r.t. constraints (2.1),...,(2.6) that will have specific variables set to one. In contrast, both for $3AAP_n$ and $3PAP_n$ such "trivial" points exist. For the $3AAP_n$ the trivial point has $x_{ijk} = 1$ for $i = j = k$ and $x_{ijk} = 0$ for $i \neq j \neq k \neq i$. For the $3PAP_n$ the trivial solution is defined w.r.t. $\bar{k} = i + j - 1 \mod n$, i.e. the trivial point has $x_{ij\bar{k}} = 1$ for $k_0 > 0$, $x_{ijn} = 1$ for $k_0 = 0$ and all other variables set to zero. To overcome this difficulty, in the following lemma we establish, for $n \geq 4$ and $n \neq 6$, the existence of an *OLS* structure with four specific variables set to one.

Lemma 4.5. *For $n \geq 4$ and $n \neq 6$ let $i_0 \in I \setminus \{1\}$, $j_0 \in J \setminus \{1\}$, $k_0, k_1, k_2 \in K \setminus \{1\}$ with $k_2 \neq k_0, k_1$, $l_0, l_1 \in L \setminus \{1\}$. Then there exists a point $x_0 \in P_I$ with four particular variables taking value one as illustrated in Table 7.*

Table 7: Point x_0

	1	$\cdots$	j_0	$\cdots$
1	1		k_1	
$\vdots$				
i_0	k_0		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	1		l_1	
$\vdots$				
i_0	l_0		1	
$\vdots$				

Proof. Consider an arbitrary point $x \in P_I$ as illustrated in Table 8.

Table 8: An arbitrary point $x \in P_I$ (Lemma 4.5)

	1	$\cdots$	j_0	$\cdots$
1	k_b		k_c	
$\vdots$				
i_0	k_d		k_e	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_b		l_c	
$\vdots$				
i_0	l_d		l_e	
$\vdots$				

where $k_b, k_c, k_d, k_e \in K$, $l_b, l_c, l_d, l_e \in L$. For x to be a valid *OLS* structure we must have $k_b \neq k_c, k_d$; $k_e \neq k_c, k_d$; $l_b \neq l_c, l_d$; $l_e \neq l_c, l_d$. We consider two cases:

case 1: $k_b \neq k_e$

Let $x^* = x(k_b \neq 1?1 \leftrightarrow k_b)_3(l_b \neq 1?1 \leftrightarrow l_b)_4$. If $l_e = 1$ then we are done, i.e. x^* is x_0 if we denote k_d as k_0, k_c as k_1, l_c as l_1 and l_d as l_0. If $l_e \neq 1$ then let i_1 ($i_1 \neq 1$) be the row for which $l(i_1, j_0) = 1$ at point x^*. By labeling $k(i_1, 1)$ as k_0, $k(i_1, j_0)$ as k_2, k_c as k_1, $l(i_1, 1)$ as l_0 and l_c as l_1 then $x_0 = x^*(i_0 \leftrightarrow i_1)_1$.

case 2: $k_b = k_e$

In this case $l_b \neq l_e$ (orthogonal property). Again we set $x^* = x(k_b \neq 1?1 \leftrightarrow k_b)_3(l_b \neq 1?1 \leftrightarrow l_b)_4$. At point x^* if we denote k_d as k_0, k_c as k_1, l_d as l_0, l_c as l_1 and l_e as l_2 and exchange the roles of sets K and L we have point x_0.

□

There might be more than one points of P_I with these four variables set to one. However, one point suffices to carry out the proofs that follow.

Since $P_I \subset P_L$, dim$(P_I) \leq$ dim(P_L). Moreover, dim$(P_I) <$ dim(P_L) if and only if there exists an equation $ax = a_0$ satisfied by all $x \in P_I$ such that it is not implied by (i.e. cannot be expressed as a linear combination of) the equations $Ax = e$. We now show that no such equation exists.

Theorem 4.6. *Let $n \geq 4$ and $n \neq 6$, and suppose every $x \in P_I^n$ satisfies $ax = a_0$ for some $a \in \mathbb{R}^{n^4}$, $a_0 \in \mathbb{R}$. Then there exist scalars λ^1_{kl}, λ^2_{il}, λ^3_{jl}, λ^4_{ij}, λ^5_{jk}, λ^6_{ik}, $i \in I$, $j \in J$, $k \in K$, $l \in L$, satisfying*

$$a_{ijkl} = \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik} \tag{4.1}$$

$$\begin{aligned} a_0 &= \sum\{\lambda^1_{kl} : k \in K, l \in L\} + \sum\{\lambda^2_{il} : i \in I, l \in L\} \\ &+ \sum\{\lambda^3_{jl} : j \in J, l \in L\} + \sum\{\lambda^4_{ij} : i \in I, j \in J\} \\ &+ \sum\{\lambda^5_{jk} : j \in J, k \in K\} + \sum\{\lambda^6_{ik} : i \in I, k \in K\} \end{aligned} \tag{4.2}$$

Proof. Define

$$\lambda^1_{kl} = a_{11kl}$$

$$\lambda^2_{il} = a_{i11l} - a_{111l}$$

$$\lambda^3_{jl} = a_{1j1l} - a_{111l}$$

$$\lambda^4_{ij} = a_{ij11} - a_{i111} - a_{1j11} + a_{1111}$$

$$\lambda^5_{jk} = a_{1jk1} - a_{1j11} - a_{11k1} + a_{1111}$$

$$\lambda^6_{ik} = a_{i1k1} - a_{i111} - a_{11k1} + a_{1111}$$

Note that these $6n^2$ scalars are defined in such a way so that exactly $8n-3$ of them, corresponding to the dependent rows of the A matrix, equal zero.

By substituting the λs in equation (4.1) we get:

$$\begin{aligned} a_{ijkl} &= a_{11kl} + a_{i11l} + a_{1j1l} + a_{ij11} + a_{1jk1} + a_{i1k1} \\ &\quad -2a_{i111} - 2a_{1j11} - 2a_{11k1} - 2a_{111l} + 3a_{1111} \end{aligned} \tag{4.3}$$

Substitution alone is enough to show that (4.3) is true for a_{1111} and for all cases where at least two of the indices are equal to one. For all cases where only one of the indices equals one, equation (4.3) becomes

$$a_{ijk1} = a_{ij11} + a_{i1k1} + a_{1jk1} - a_{i111} - a_{1j11} - a_{11k1} + a_{1111} \tag{4.4}$$

$$a_{ij1l} = a_{ij11} + a_{i11l} + a_{1j1l} - a_{i111} - a_{1j11} - a_{111l} + a_{1111} \tag{4.5}$$

$$a_{i1kl} = a_{i1k1} + a_{i11l} + a_{11kl} - a_{i111} - a_{11k1} - a_{111l} + a_{1111} \tag{4.6}$$

$$a_{1jkl} = a_{1jk1} + a_{1j1l} + a_{11kl} - a_{1j11} - a_{11k1} - a_{111l} + a_{1111} \tag{4.7}$$

Before proving that (4.3) to (4.7) hold, we prove the following proposition.

Proposition 4.7. *For $n \geq 3$ and $n \neq 6$, it can be shown that*

$$\begin{aligned}
& a_{i_1 j_1 k(i_1,j_1) l(i_1,j_1)} + a_{i_1 j_2 k(i_1,j_2) l(i_1,j_2)} + a_{i_2 j_1 k(i_2,j_1) l(i_2,j_1)} + a_{i_2 j_2 k(i_2,j_2) l(i_2,j_2)} \\
& + a_{i_1 j_1 k(i_2,j_2) l(i_2,j_2)} + a_{i_1 j_2 k(i_2,j_1) l(i_2,j_1)} + a_{i_2 j_1 k(i_1,j_2) l(i_1,j_2)} + a_{i_2 j_2 k(i_1,j_1) l(i_1,j_1)} \\
= & \, a_{i_1 j_1 k(i_2,j_1) l(i_2,j_1)} + a_{i_1 j_2 k(i_2,j_2) l(i_2,j_2)} + a_{i_2 j_1 k(i_1,j_1) l(i_1,j_1)} + a_{i_2 j_2 k(i_1,j_2) l(i_1,j_2)} \\
& + a_{i_1 j_1 k(i_1,j_2) l(i_1,j_2)} + a_{i_1 j_2 k(i_1,j_1) l(i_1,j_1)} + a_{i_2 j_1 k(i_2,j_2) l(i_2,j_2)} + a_{i_2 j_2 k(i_2,j_1) l(i_2,j_1)}
\end{aligned} \tag{4.8}$$

for $i_1, i_2 \in I$, $i_1 \neq i_2$ and $j_1, j_2 \in J$, $j_1 \neq j_2$.

Proof. Let $b, c, d, e \in K$ and $p, q, r, s \in L$ and an arbitrary point $x \in P_I$ as illustrated in Table 9.

Table 9: Point x (Proposition 4.7)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		b		c	
$\vdots$					
i_2		d		e	
$\vdots$					

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		p		q	
$\vdots$					
i_2		r		s	
$\vdots$					

Let $x' = x(i_1 \leftrightarrow i_2)_1$ (Table 10).

Table 10: Point x' (Proposition 4.7)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		d		e	
$\vdots$					
i_2		b		c	
$\vdots$					

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		r		s	
$\vdots$					
i_2		p		q	
$\vdots$					

Let $\bar{x} = x(j_1 \leftrightarrow j_2)_2$ (Table 11).

Let $\bar{x}' = \bar{x}(i_1 \leftrightarrow i_2)_1$ (Table 12).

Let $k^x(i_s, j_t)$ ($l^x(i_s, j_t)$) denote the value of the $k(l)$ index for $i = i_s$, $j = j_t$ at point x. $k^{x'}(i_s, j_t)$, $k^{\bar{x}}(i_s, j_t)$, $k^{\bar{x}'}(i_s, j_t)$ and $l^{x'}(i_s, j_t)$, $l^{\bar{x}}(i_s, j_t)$, $l^{\bar{x}'}(i_s, j_t)$ are defined accordingly for points x', $\bar{x}$ and $\bar{x}'$.

Since $x, x' \in P_I$ we have $ax = ax'$. By observing that all a_{ijkl} terms for every $i \in I \setminus \{i_1, i_2\}$ are canceled out,

Table 11: Point $\bar{x}$(Proposition 4.7)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		c		b	
$\vdots$					
i_2		e		d	
$\vdots$					

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		q		p	
$\vdots$					
i_2		s		r	
$\vdots$					

Table 12: Point $\bar{x}'$ (Proposition 4.7)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		e		d	
$\vdots$					
i_2		c		b	
$\vdots$					

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$
$\vdots$					
i_1		s		r	
$\vdots$					
i_2		q		p	
$\vdots$					

$ax = ax'$ becomes

$$\begin{aligned}
& a_{i_1 j_1 bp} + a_{i_1 j_2 cq} + \sum\{a_{i_1 j k^x(i_1,j) l^x(i_1,j)} : j \in J \setminus \{j_1, j_2\}\} \\
+ \quad & a_{i_2 j_1 dr} + a_{i_2 j_2 es} + \sum\{a_{i_2 j k^x(i_2,j) l^x(i_2,j)} : j \in J \setminus \{j_1, j_2\}\} \\
= \quad & a_{i_1 j_1 dr} + a_{i_1 j_2 es} + \sum\{a_{i_1 j k^{x'}(i_1,j) l^{x'}(i_1,j)} : j \in J \setminus \{j_1, j_2\}\} \\
+ \quad & a_{i_2 j_1 bp} + a_{i_2 j_2 cq} + \sum\{a_{i_2 j k^{x'}(i_2,j) l^{x'}(i_2,j)} : j \in J \setminus \{j_1, j_2\}\} \qquad (4.9)
\end{aligned}$$

We observe that $k^x(i_2, j) = k^{x'}(i_1, j)$, $k^x(i_1, j) = k^{x'}(i_2, j)$ and $l^x(i_2, j) = l^{x'}(i_1, j)$, $l^x(i_1, j) = l^{x'}(i_2, j)$ for $j \in J \setminus \{j_1, j_2\}$. Writing (4.9) in terms of point x, we derive

$$\begin{aligned}
& a_{i_1 j_1 bp} + a_{i_1 j_2 cq} + \sum\{a_{i_1 j k^x(i_1,j) l^x(i_1,j)} : j \in J \setminus \{j_1, j_2\}\} \\
+ \quad & a_{i_2 j_1 dr} + a_{i_2 j_2 es} + \sum\{a_{i_2 j k^x(i_2,j) l^x(i_2,j)} : j \in J \setminus \{j_1, j_2\}\} \\
= \quad & a_{i_1 j_1 dr} + a_{i_1 j_2 es} + \sum\{a_{i_1 j k^x(i_2,j) l^x(i_2,j)} : j \in J \setminus \{j_1, j_2\}\} \\
+ \quad & a_{i_2 j_1 bp} + a_{i_2 j_2 cq} + \sum\{a_{i_2 j k^x(i_1,j) l^x(i_1,j)} : j \in J \setminus \{j_1, j_2\}\} \qquad (4.10)
\end{aligned}$$

Similarly, since $\bar{x}, \bar{x}' \in P_I$ we have $a\bar{x} = a\bar{x}'$. Terms a_{ijkl}, for every $i \in I \setminus \{i_1, i_2\}$, are canceled out, so $a\bar{x} = a\bar{x}'$ becomes

$$\begin{aligned}
& a_{i_1j_1cq} + a_{i_1j_2bp} + \sum\{a_{i_1jk^{\bar{x}}(i_1,j)l^{\bar{x}}(i_1,j)} : j \in J \setminus \{j_1,j_2\}\} \\
+ \quad & a_{i_2j_1es} + a_{i_2j_2dr} + \sum\{a_{i_2jk^{\bar{x}}(i_2,j)l^{\bar{x}}(i_2,j)} : j \in J \setminus \{j_1,j_2\}\} \\
= \quad & a_{i_1j_1es} + a_{i_1j_2dr} + \sum\{a_{i_1jk^{\bar{x}'}(i_1,j)l^{\bar{x}'}(i_1,j)} : j \in J \setminus \{j_1,j_2\}\} \\
+ \quad & a_{i_2j_1cq} + a_{i_2j_2bp} + \sum\{a_{i_2jk^{\bar{x}'}(i_2,j)l^{\bar{x}'}(i_2,j)} : j \in J \setminus \{j_1,j_2\}\} \qquad (4.11)
\end{aligned}$$

We observe that $k^{\bar{x}}(i_2,j) = k^{\bar{x}'}(i_1,j)$, $k^{\bar{x}}(i_1,j) = k^{\bar{x}'}(i_2,j)$ and $l^{\bar{x}}(i_2,j) = l^{\bar{x}'}(i_1,j)$, $l^{\bar{x}}(i_1,j) = l^{\bar{x}'}(i_2,j)$ for $j \in J \setminus \{j_1,j_2\}$. Writing (4.11) in terms of point $\bar{x}$, we derive

$$\begin{aligned}
& a_{i_1j_1cq} + a_{i_1j_2bp} + \sum\{a_{i_1jk^{\bar{x}}(i_1,j)l^{\bar{x}}(i_1,j)} : j \in J \setminus \{j_1,j_2\}\} \\
+ \quad & a_{i_2j_1es} + a_{i_2j_2dr} + \sum\{a_{i_2jk^{\bar{x}}(i_2,j)l^{\bar{x}}(i_2,j)} : j \in J \setminus \{j_1,j_2\}\} \\
= \quad & a_{i_1j_1es} + a_{i_1j_2dr} + \sum\{a_{i_1jk^{\bar{x}}(i_2,j)l^{\bar{x}}(i_2,j)} : j \in J \setminus \{j_1,j_2\}\} \\
+ \quad & a_{i_2j_1cq} + a_{i_2j_2bp} + \sum\{a_{i_2jk^{\bar{x}}(i_1,j)l^{\bar{x}}(i_1,j)} : j \in J \setminus \{j_1,j_2\}\} \qquad (4.12)
\end{aligned}$$

Subtracting (4.10) from (4.12) and observing that for $i \in \{i_1,i_2\}$ and $j \in J \setminus \{j_1,j_2\}$ we have $k^x(i,j) = k^{\bar{x}}(i,j)$ and $l^x(i,j) = l^{\bar{x}}(i,j)$, we obtain

$$\begin{aligned}
& a_{i_1j_1bp} + a_{i_1j_2cq} + a_{i_2j_1dr} + a_{i_2j_2es} - (a_{i_1j_1cq} + a_{i_1j_2bp} + a_{i_2j_1es} + a_{i_2j_2dr}) \\
= \; & a_{i_1j_1dr} + a_{i_1j_2es} + a_{i_2j_1bp} + a_{i_2j_2cq} - (a_{i_1j_1es} + a_{i_1j_2dr} + a_{i_2j_1cq} + a_{i_2j_2bp})
\end{aligned}$$

If we eliminate the negative sign by moving terms in brackets to the other side of the equation and write the elements of sets K and L using the notation $k(i,j)$ and $l(i,j)$ respectively, we obtain equation (4.8).

□

In Proposition 4.7, the role of the sets I, J for the row and column set, respectively, is purely conventional. Any pair of sets from I, J, K, L can be used for the role of row/column set. Hence, for the rest of the paper, the notation $x((m_1,m_2)_{t_1};(n_1,n_2)_{t_2})$ implies equation (4.8), derived by applying Proposition 4.7 at point x, for rows m_1,m_2 and columns n_1,n_2. In this expression, the first pair denotes the rows whereas the second denotes the columns. The subscripts t_1,t_2 declare the sets that index the rows and the columns, respectively. Following the same convention as for the interchanges, 1 is used to denote set I, 2 is used to denote set J, etc. For example, $x((1,i_1)_1;(1,n)_2)$ denotes equation (4.8) written for rows $1, i_1$ and columns $1, n$ at point x, where elements of the first pair belong to set I and of the second pair to set J.

(Back to the proof of Theorem 4.6). We will show (4.3),...,(4.7) for $i = i_0$, $j = j_0$, $k = k_0$, $l = l_0$.

At point x_0 of Lemma 4.5 we distinguish two cases, viz. $k_0 = k_1$, $k_0 \neq k_1$.

case 1: $k_0 = k_1$

Let $x_1^* = x_0$ (see Table 13). In this case $l_0 \neq l_1$ (orthogonal property). For $n \geq 4$ there exists (another) $k_1 \in$

Table 13: Point x_1^*(Theorem 4.6, case 1)

	1	$\cdots$	j_0	$\cdots$
1	1		k_0	
$\vdots$				
i_0	k_0		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	1		l_1	
$\vdots$				
i_0	l_0		1	
$\vdots$				

$K \setminus \{1, k_0, k_2\}$. Let $x_1 = x_1^*(k_0 \leftrightarrow k_1)_3$ (see Table 14).

Table 14: Point x_1(Theorem 4.6, case 1)

	1	$\cdots$	j_0	$\cdots$
1	1		k_1	
$\vdots$				
i_0	k_1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	1		l_1	
$\vdots$				
i_0	l_0		1	
$\vdots$				

$x_1((1, i_0)_1; (1, j_0)_2) \Rightarrow$

$$a_{1111} + a_{1j_0k_1l_1} + a_{i_01k_1l_0} + a_{i_0j_0k_21} + a_{11k_21} + a_{1j_0k_1l_0} + a_{i_01k_1l_1} + a_{i_0j_011}$$
$$= a_{11k_1l_0} + a_{1j_0k_21} + a_{i_0111} + a_{i_0j_0k_1l_1} + a_{11k_1l_1} + a_{1j_011} + a_{i_01k_21} + a_{i_0j_0k_1l_0}$$

Let $x_2 = x_1(1 \leftrightarrow k_0)_3$ (see Table 15).

Table 15: Point x_2 (Theorem 4.6, case 1)

	1	$\cdots$	j_0	$\cdots$
1	k_0		k_1	
$\vdots$				
i_0	k_1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	1		l_1	
$\vdots$				
i_0	l_0		1	
$\vdots$				

$x_2((1,i_0)_1;(1,j_0)_2)$⇒

$$a_{11k_01}+a_{1j_0k_1l_1}+a_{i_01k_1l_0}+a_{i_0j_0k_21}+a_{11k_21}+a_{1j_0k_1l_0}+a_{i_01k_1l_1}+a_{i_0j_0k_01}$$
$$=a_{11k_1l_0}+a_{1j_0k_21}+a_{i_01k_01}+a_{i_0j_0k_1l_1}+a_{11k_1l_1}+a_{1j_0k_01}+a_{i_01k_21}+a_{i_0j_0k_1l_0}$$

$x_2((1,i_0)_1;(1,j_0)_2)$-$x_1((1,i_0)_1;(1,j_0)_2)$ yields (4.4).

This completes the proof of case 1 for (4.4). Before proceeding to case 2, we derive a further relationship for case 1 which will be used later on, for proving (4.3).

Let $x_3 = x_1(1 \leftrightarrow l_0)_4$ and $x_4 = x_3(1 \leftrightarrow k_0)_3$. Points x_3 and x_4 are illustrated in Tables 16, 17 respectively.

Table 16: Point x_3 (Theorem 4.6, case 1)

	1	$\cdots$	j_0	$\cdots$
1	k_0		k_1	
$\vdots$				
i_0	k_1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		l_1	
$\vdots$				
i_0	1		l_0	
$\vdots$				

Table 17: Point x_4 (Theorem 4.6, case 1)

	1	$\cdots$	j_0	$\cdots$
1	1		k_1	
$\vdots$				
i_0	k_1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		l_1	
$\vdots$				
i_0	1		l_0	
$\vdots$				

$x_3((1,i_0)_1;(1,j_0)_2)$⇒

$$a_{11k_0l_0}+a_{1j_0k_1l_1}+a_{i_01k_11}+a_{i_0j_0k_2l_0}+a_{11k_2l_0}+a_{1j_0k_11}+a_{i_01k_1l_1}+a_{i_0j_0k_0l_0}$$
$$=a_{11k_11}+a_{1j_0k_2l_0}+a_{i_01k_0l_0}+a_{i_0j_0k_1l_1}+a_{11k_1l_1}+a_{1j_0k_0l_0}+a_{i_01k_2l_0}+a_{i_0j_0k_11}$$

$x_4((1,i_0)_1;(1,j_0)_2)$⇒

$$a_{111l_0}+a_{1j_0k_1l_1}+a_{i_01k_11}+a_{i_0j_0k_2l_0}+a_{11k_2l_0}+a_{1j_0k_11}+a_{i_01k_1l_1}+a_{i_0j_01l_0}$$
$$=a_{11k_11}+a_{1j_0k_2l_0}+a_{i_011l_0}+a_{i_0j_0k_1l_1}+a_{11k_1l_1}+a_{1j_01l_0}+a_{i_01k_2l_0}+a_{i_0j_0k_11}$$

$x_3((1,i_0)_1;(1,j_0)_2)$-$x_4((1,i_0)_1;(1,j_0)_2)\Rightarrow$

$$a_{i_0j_0k_0l_0} = (a_{i_0j_01l_0} + a_{i_01k_0l_0} + a_{1j_0k_0l_0}) - a_{i_011l_0} - a_{1j_01l_0} - a_{11k_0l_0} + a_{111l_0} \tag{4.13}$$

(4.13) will be used later for proving (4.3).

case 2: $k_0 \neq k_1$

Let $x_1 = x_0$. Thus point x_1 is point x_0 of Lemma 4.5, exactly as illustrated in Table 7.

$x_1((1,i_0)_1;(1,j_0)_2)\Rightarrow$

$$\begin{aligned} & a_{1111} + a_{1j_0k_1l_1} + a_{i_01k_0l_0} + a_{i_0j_0k_21} + a_{11k_21} + a_{1j_0k_0l_0} + a_{i_01k_1l_1} + a_{i_0j_011} \\ = {} & a_{11k_0l_0} + a_{1j_0k_21} + a_{i_0111} + a_{i_0j_0k_1l_1} + a_{11k_1l_1} + a_{1j_011} + a_{i_01k_21} + a_{i_0j_0k_0l_0} \end{aligned}$$

Let $x_2 = x_1(1 \leftrightarrow k_0)_3$ (see Table 18).

Table 18: Point x_2 (Theorem 4.6, case 2)

	1	$\cdots$	j_0	$\cdots$
1	k_0		k_1	
$\vdots$				
i_0	1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	1		l_1	
$\vdots$				
i_0	l_0		1	
$\vdots$				

$x_2((1,i_0)_1;(1,j_0)_2)\Rightarrow$

$$\begin{aligned} & a_{11k_01} + a_{1j_0k_1l_1} + a_{i_01k_0l_0} + a_{i_0j_0k_21} + a_{11k_21} + a_{1j_0k_0l_0} + a_{i_01k_1l_1} + a_{i_0j_0k_01} \\ = {} & a_{111l_0} + a_{1j_0k_21} + a_{i_01k_01} + a_{i_0j_0k_1l_1} + a_{11k_1l_1} + a_{1j_0k_01} + a_{i_01k_21} + a_{i_0j_01l_0} \end{aligned}$$

$x_1((1,i_0)_1;(1,j_0)_2)$-$x_2((1,i_0)_1;(1,j_0)_2)\Rightarrow$

$$\begin{aligned} & a_{1111} + a_{1j_0k_01} + a_{i_01k_01} + a_{i_0j_011} - (a_{i_0j_0k_01} + a_{i_0111} + a_{1j_011} + a_{11k_01}) \\ = \quad & a_{i_0j_0k_0l_0} + a_{i_011l_0} + a_{1j_01l_0} + a_{11k_0l_0} - (a_{i_0j_01l_0} + a_{i_01k_0l_0} + a_{1j_0k_0l_0} + a_{111l_0}) \end{aligned} \tag{4.14}$$

We refer to equation (4.14) as (4.14)$_{l_0}$ to distinguish it from (4.14) with l_1 in the place of l_0 denoted as (4.14)$_{l_1}$. We observe that we can derive (4.14)$_{l_1}$ by applying Proposition 4.7 at points $\dot{x}_1 = x_1(l_1 \leftrightarrow l_0)_4$ and $\dot{x}_2 = x_2(l_1 \leftrightarrow l_0)_4$.

This is also true for the case where $l(1, j_0) = l_0$ (i.e. $l_1 = l_0$) at points x_0, x_1 and x_2 since for $n \geq 3$ there exists (another) $l_1 \in L \setminus \{1, l_0\}$ such that we can derive $\dot{x}_1, \dot{x}_2$. In both cases, (4.14)$_{l_0}$+(4.14)$_{l_1}$⇒

$$\begin{aligned} & 2(a_{1111} + a_{1j_0k_01} + a_{i_01k_01} + a_{i_0j_011} - (a_{i_0j_0k_01} + a_{i_0111} + a_{1j_011} + a_{11k_01})) \\ = \; & \sum\{a_{i_0j_0k_0l} + a_{i_011l} + a_{1j_01l} + a_{11k_0l} - (a_{i_0j_01l} + a_{i_01k_0l} + a_{1j_0k_0l} + a_{111l}) : l \in \{l_0, l_1\}\} \end{aligned} \tag{4.15}$$

At point x_2 we distinguish two cases viz. $l(1, j_0) = l_0$, $l(1, j_0) = l_1 \neq l_0$

case 2.1: $l(1, j_0) = l_0$.

For $n \geq 3$ we have already establish the existence of $l_1 \in L \setminus \{1, l_0\}$. Let $x_3 = x_2(1 \leftrightarrow l_0)_4(1 \leftrightarrow l_1)_4$ (Table 19) and $x_4 = x_3(1 \leftrightarrow k_0)_3$ (Table 20).

Table 19: Point x_3 (Theorem 4.6, case 2.1)

	1	$\cdots$	j_0	$\cdots$
1	k_0		k_1	
$\vdots$				
i_0	1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		l_1	
$\vdots$				
i_0	l_1		l_0	
$\vdots$				

Table 20: Point x_4 (Theorem 4.6, case 2.1)

	1	$\cdots$	j_0	$\cdots$
1	1		k_1	
$\vdots$				
i_0	k_0		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		l_1	
$\vdots$				
i_0	l_1		l_0	
$\vdots$				

$x_3((1, i_0)_1; (1, j_0)_2)$⇒

$$\begin{aligned} & a_{11k_0l_0} + a_{1j_0k_1l_1} + a_{i_011l_1} + a_{i_0j_0k_2l_0} + a_{11k_2l_0} + a_{1j_01l_1} + a_{i_01k_1l_1} + a_{i_0j_0k_0l_0} \\ = \; & a_{111l_1} + a_{1j_0k_2l_0} + a_{i_01k_0l_0} + a_{i_0j_0k_1l_1} + a_{11k_1l_1} + a_{1j_0k_0l_0} + a_{i_01k_2l_0} + a_{i_0j_01l_1} \end{aligned}$$

$x_4((1, i_0)_1; (1, j_0)_2)$⇒

$$\begin{aligned} & a_{111l_0} + a_{1j_0k_1l_1} + a_{i_01k_0l_1} + a_{i_0j_0k_2l_0} + a_{11k_2l_0} + a_{1j_0k_0l_1} + a_{i_01k_1l_1} + a_{i_0j_01l_0} \\ = \; & a_{11k_0l_1} + a_{1j_0k_2l_0} + a_{i_011l_0} + a_{i_0j_0k_1l_1} + a_{11k_1l_1} + a_{1j_01l_0} + a_{i_01k_2l_0} + a_{i_0j_0k_0l_1} \end{aligned}$$

$x_4((1, i_0)_1; (1, j_0)_2)$-$x_3((1, i_0)_1; (1, j_0)_2)$$\Rightarrow$

$$\sum\{a_{i_0 j_0 k_0 l} + a_{i_0 1 1 l} + a_{1 j_0 1 l} + a_{1 1 k_0 l} - (a_{i_0 j_0 1 l} + a_{i_0 1 k_0 l} + a_{1 j_0 k_0 l} + a_{1 1 1 l}) : l \in \{l_0, l_1\}\} = 0 \qquad (4.16)$$

case 2.2: $l(1, j_0) = l_1 \neq l_0$.

Let $x_3 = x_2(1 \leftrightarrow l_0)_4(1 \leftrightarrow l_1)_4$ (Table 21) and $x_4 = x_3(1 \leftrightarrow k_0)_3$ (Table 22).

Table 21: Point x_3 (Theorem 4.6, case 2.2)

	1	$\cdots$	j_0	$\cdots$
1	k_0		k_1	
$\vdots$				
i_0	1		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		1	
$\vdots$				
i_0	l_1		l_0	
$\vdots$				

Table 22: Point x_4 (Theorem 4.6, case 2.2)

	1	$\cdots$	j_0	$\cdots$
1	1		k_1	
$\vdots$				
i_0	k_0		k_2	
$\vdots$				

	1	$\cdots$	j_0	$\cdots$
1	l_0		1	
$\vdots$				
i_0	l_1		l_0	
$\vdots$				

$x_3((1, i_0)_1; (1, j_0)_2)$$\Rightarrow$

$$\begin{aligned} & a_{1 1 k_0 l_0} + a_{1 j_0 k_1 1} + a_{i_0 1 1 l_1} + a_{i_0 j_0 k_2 l_0} + a_{1 1 k_2 l_0} + a_{1 j_0 1 l_1} + a_{i_0 1 k_1 1} + a_{i_0 j_0 k_0 l_0} \\ = {} & a_{1 1 1 l_1} + a_{1 j_0 k_2 l_0} + a_{i_0 1 k_0 l_0} + a_{i_0 j_0 k_1 1} + a_{1 1 k_1 1} + a_{1 j_0 k_0 l_0} + a_{i_0 1 k_2 l_0} + a_{i_0 j_0 1 l_1} \end{aligned}$$

$x_4((1, i_0)_1; (1, j_0)_2)$$\Rightarrow$

$$\begin{aligned} & a_{1 1 1 l_0} + a_{1 j_0 k_1 1} + a_{i_0 1 k_0 l_1} + a_{i_0 j_0 k_2 l_0} + a_{1 1 k_2 l_0} + a_{1 j_0 k_0 l_1} + a_{i_0 1 k_1 1} + a_{i_0 j_0 1 l_0} \\ = {} & a_{1 1 k_0 l_1} + a_{1 j_0 k_2 l_0} + a_{i_0 1 1 l_0} + a_{i_0 j_0 k_1 1} + a_{1 1 k_1 1} + a_{1 j_0 1 l_0} + a_{i_0 1 k_2 l_0} + a_{i_0 j_0 k_0 l_1} \end{aligned}$$

$x_4((1, i_0)_1; (1, j_0)_2)$-$x_3((1, i_0)_1; (1, j_0)_2)$ yields (4.16).

Hence (4.16) is valid for case 2 (both for case 2.1 and 2.2). Substituting the right-hand side of equation (4.15) from (4.16) yields (4.4).

Our proof with respect to (4.4) is now complete. Equations (4.5),(4.6) and (4.7) follow by symmetry. To show (4.3) we consider (4.13). This equation is valid for case 2 as well; the right-hand side of equation (4.14) is equal to 0 due to equation (4.4) thus yielding (4.13). Therefore, if we substitute in equation (4.13) terms in brackets from (4.5), (4.6) and (4.7) we get (4.3).

Finally, (4.2) is true since for $n \geq 3$ and different than 6, $P_I^n \neq \emptyset$ (see [13, Theorem 2.9]). This implies that there exists at least one 0-1 vector x for which (2.1),...,(2.6) are satisfied, thus by multiplying these equations with the corresponding scalars and summing over all rows we get:

$$\begin{aligned} ax &= \sum\{\lambda^1_{kl} : k \in K, l \in L\} + \sum\{\lambda^2_{il} : i \in I, l \in L\} \\ &+ \sum\{\lambda^3_{jl} : j \in J, l \in L\} + \sum\{\lambda^4_{ij} : i \in I, j \in J\} \\ &+ \sum\{\lambda^5_{jk} : j \in J, k \in K\} + \sum\{\lambda^6_{ik} : i \in I, k \in K\} \end{aligned}$$

This completes the proof for general n. The theorem holds for $n \geq 4$ and $n \neq 6$ since the proof requires four distinct values for one of the indices (index k). In particular, we have used values $(1, i_0)$ for index i, $(1, j_0)$ for index j, $(1, k_0, k_1, k_2)$ for index k , $(1, l_0, l_1)$ for index l . Hence, the result holds for $n \geq 4$ and $n \neq 6$.

□

Corollary 4.8. *For $n \geq 4$ and $n \neq 6$, $dim(P_I) = n^4 - 6n^2 + 8n - 3$.*

Next we will examine which of the constraints defining P_L are facet defining for P_I.

Proposition 4.9. *For $n \geq 4$ and $n \neq 6$ every inequality of the type $x_c \geq 0$ for $c \in C$ defines a facet of P_I.*

Proof. For any $c \in C$ consider the polytope $P_I^c = \{x \in P_I : x_c = 0\}$. We need to show that $\dim(P_I^c) = \dim(P_I) - 1$. Evidently, $\dim(P_I^c) \leq n^4 - 1 - \text{rank}(A^c)$ where A^c is the matrix obtained from A if we remove column a^c. It is not hard to see that the rank of A^c is equal to the rank of A. This is immediate, if the column a^c is not among the columns of the upper triangular matrix described in Theorem 4.2, otherwise it follows by symmetry. Therefore, $\dim(P_I^c) \leq n^4 - 6n^2 + 8n - 4$. To prove that this bound is attained, we use the same approach as in the proof of Theorem 4.6, i.e. show that any equation $ax = a_0$ (different than $x_c = 0$) satisfied for every $x \in P_I^c$ is a linear combination of $A^c x = e$. The proof goes through essentially unchanged.

□

Proposition 4.10. *For $n \geq 3$ and $n \neq 6$ every inequality $x_c \leq 1$ for $c \in C$ does not define a facet of P_I*

Proof. For any $c \in C$ consider the polytope $P_I^c = \{x \in P_I : x_c = 1\}$. We will show that $\dim(P_I^c) < \dim(P_I) - 1$. We know that $\dim(P_I^c) \leq \dim(P_L^c)$ where P_L^c is the linear relaxation of P_I^c. Setting x_c to one is equivalent to setting the variables belonging to the common constraints with x_c to zero. The number of these variables is $2(3n-1)(n-1)$ (Proposition 3.2). Thus, $\dim(P_L^c) = n^4 - 2(3n-1)(n-1) - \text{rank}(A_n^c)$ where A_n^c is the matrix obtained from A by removing the columns corresponding to the variables set to zero. Obviously $\text{rank}(A_{n-1}) \leq \text{rank}(A_n^c)$ where A_{n-1} is the constraint matrix of the *OLS* of order $n-1$. By Theorem 4.2 $\text{rank}(A_{n-1}) = 6(n-1)^2 - 8(n-1) + 3$ so

$$\begin{aligned}\dim(P_I^c) \leq \dim(P_L^c) &\leq n^4 - 2(3n-1)(n-1) - (6(n-1)^2 - 8(n-1) + 3) \\ &= n^4 - 12n^2 + 28n - 19\end{aligned}$$

which is less than $\dim(P_I) - 1 = n^4 - 6n^2 + 8n - 4$ for $n \geq 3$.

□

It is easy to see that cliques of type I do not induce facets of P_I. For each of these cliques the coefficient vector of the corresponding inequality is identical to a row of the A matrix. Thus, each of this inequalities is satisfied as equality by all $x \in P_I$ and therefore defines an improper face of P_I.

Next we consider the inequalities induced by cliques of type II.

Theorem 4.11. *Let $Q(c)$ denote the node set of a clique of type II. Then for $n \geq 5$ and $n \neq 6$ the inequality*

$$\sum\{x_q : q \in Q(c)\} \leq 1 \tag{4.17}$$

defines a facet of P_I for every $c \in C$.

Proof. As usual, we will assume that $n \neq 2, 6$. Let $P_I^{Q(c)} = \{x \in P_I : \sum\{x_q : q \in Q(c)\} = 1\}$. We will show that $P_I^{Q(c)}$ is a facet of P_I.

First we note that (4.17) is a valid inequality for all $x \in P_I$ because $Q(c)$ is the node set of a clique in the intersection graph $G_A(C, E_C)$.

W.l.o.g. let $c = (n, n, n, n)$. Then

$$\begin{aligned}Q(n,n,n,n) = \; &\{(n,n,n,n), (1,n,n,n), ..., (n-1,n,n,n), (n,1,n,n), ..., (n,n-1,n,n), \\ &(n,n,1,n), ..., (n,n,n-1,n), (n,n,n,1), ..., (n,n,n,n-1)\}\end{aligned}$$

It is easy to show that the face $P_I^{Q(n,n,n,n)}$ is not empty. Clearly for any point $x \in P_I$ there exist $i_0 \in I$ and $j_0 \in J$ such that $x_{i_0 j_0 nn} = 1$. Let $x^* = x(i_0 \neq n?i_0 \leftrightarrow n)_1(j_0 \neq n?j_0 \leftrightarrow n)_2$. Point x^* belongs to $P_I^{Q(n,n,n,n)}$ since it has $x_{nnnn} = 1$.

Now consider an arbitrary point $x \in P_I^{Q(n,n,n,n)}$. We will show that there exists at least one point in $P_I \setminus P_I^{Q(n,n,n,n)}$.

W.l.o.g. suppose that at point x we have $x_{njnn} = 1$, $j \neq n$. Then we apply a first index interchange between n and i for any $i \in I \setminus \{n\}$. The resulting point belongs to P_I but not to $P_I^{Q(n,n,n,n)}$. Similarly at point x $x_{innn} = 1$ and $i \neq n$ or $x_{nnkn} = 1$ and $k \neq n$ or $x_{nnnl} = 1$ and $l \neq n$ then a corresponding one index interchange will give a point in $P_I \setminus P_I^{Q(n,n,n,n)}$. Finally, if at point x we have $x_{nnnn} = 1$ then we apply a first index interchange between n and $i \in I \setminus \{n\}$ and then a second index interchange between n and $j \in J \setminus \{n\}$. The resulting point belongs to P_I but not to $P_I^{Q(n,n,n,n)}$. Thus $P_I^{Q(n,n,n,n)}$ is a proper face of P_I.

To show that $P_I^{Q(n,n,n,n)}$ is a facet of P_I we will use the same approach as in Theorem 4.6. Thus we will exhibit $6n^2$ scalars λ^1_{kl}, λ^2_{il}, λ^3_{jl}, λ^4_{ij}, λ^5_{jk}, λ^6_{ik} for $i \in I$, $j \in J$, $k \in K$, $l \in L$ and an additional scalar π for the clique inequality, such that if $ax = a_0$ for all $x \in P_I^{Q(n,n,n,n)}$, then

$$a_{ijkl} = \begin{cases} \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik}, & (i,j,k,l) \in C \setminus Q(n,n,n,n) \\ \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik} + \pi, & (i,j,k,l) \in Q(n,n,n,n) \end{cases} \tag{4.18}$$

and

$$\begin{aligned} a_0 &= \sum\{\lambda^1_{kl} : k \in K, l \in L\} + \sum\{\lambda^2_{il} : i \in I, l \in L\} \\ &+ \sum\{\lambda^3_{jl} : j \in J, l \in L\} + \sum\{\lambda^4_{ij} : i \in I, j \in J\} \\ &+ \sum\{\lambda^5_{jk} : j \in J, k \in K\} + \sum\{\lambda^6_{ik} : i \in I, k \in K\} + \pi \end{aligned} \tag{4.19}$$

Again we define

$$\lambda^1_{kl} = a_{11kl}$$

$$\lambda^2_{il} = a_{i11l} - a_{111l}$$

$$\lambda^3_{jl} = a_{1j1l} - a_{111l}$$

$$\lambda^4_{ij} = a_{ij11} - a_{i111} - a_{1j11} + a_{1111}$$

$$\lambda^5_{jk} = a_{1jk1} - a_{1j11} - a_{11k1} + a_{1111}$$

$$\lambda^6_{ik} = a_{i1k1} - a_{i111} - a_{11k1} + a_{1111}$$

Thus for $(i,j,k,l) \in C \setminus Q(n,n,n,n)$ we have to show

$$\begin{aligned} a_{ijkl} &= a_{11kl} + a_{i11l} + a_{1j1l} + a_{ij11} + a_{1jk1} + a_{i1k1} \\ &\quad -2a_{i111} - 2a_{1j11} - 2a_{11k1} - 2a_{111l} + 3a_{1111} \end{aligned} \tag{4.20}$$

This is clearly true for a_{1111} and for all cases where at least two of the indices equal one. When one of the indices equals one, we denote equation (4.20) as (4.20)$_{m=1}$ where m is any of i, j, k, l. For example equation (4.20)$_{l=1}$ is:

$$a_{ijk1} = a_{ij11} + a_{i1k1} + a_{1jk1} - a_{i111} - a_{1j11} - a_{11k1} + a_{1111}$$

For each of the cases to be examined next, when at most one of the indices is equal to one, the proof of (4.20) will be carried out in a manner analogous to that of Theorem 4.6 except that this time we will exclusively use points belonging to $P_I^{Q(n,n,n,n)}$. For any point $x \in P_I$ we denote as X the collection of points $(x, x', \bar{x}, \bar{x}')$ (notation and derivation of points from x as introduced in Proposition 4.7). $X \in P_I^{Q(n,n,n,n)}$ implies that $x, x', \bar{x}, \bar{x}' \in P_I^{Q(n,n,n,n)}$. Thus for all of the following cases, equation (4.20)$_{l=1}$ is derived explicitly by the application of Proposition 4.7 to points $x_t \in X_t$ where $X_t \in P_I^{Q(n,n,n,n)}$ for $t = 1, ..., 4$. Equations (4.20)$_{i=1}$, (4.20)$_{j=1}$, (4.20)$_{k=1}$ follow by symmetry. On the other hand, when none of the indices is equal to one, (4.20) is shown by substituting these equations to an equation, corresponding to (4.13) of Theorem 4.6, which is derived in the course of proving (4.20)$_{l=1}$.

The indices i, j, k, l of a_{ijkl} give rise to the cases where none, one or two of them are equal to n. In each of these cases we make use of the point x_q which is established in Lemma 4.12.

Lemma 4.12. *For $n \geq 5$ and $n \neq 6$ let $i_q \in I \setminus \{1, n\}$, $j_q \in J \setminus \{1, n\}$, $k_q, k_1, k_2 \in K \setminus \{1, n\}$ with $k_2 \neq k_1, k_q$, $l_q, l_1 \in L \setminus \{1, n\}$ with $l_q \neq l_1$. Then there exists the point $x_q \in P_I$ as illustrated in Table 23.*

Table 23: Point x_q (Lemma 4.12)

	1	$\cdots$	j_q	$\cdots$
1	1		k_1	
$\vdots$				
i_q	k_q		k_2	
$\vdots$				

	1	$\cdots$	j_q	$\cdots$
1	1		l_1	
$\vdots$				
i_q	l_q		n	
$\vdots$				

Proof. Consider the *OLS* that has in natural order the elements of the a) first row of the two latin squares and b) the elements of the first column of one of the two squares . A pair of *OLS* with this property is said to be a *standardized* set. If $P_I \neq \emptyset$ then there always exists an *OLS* structure of this type ([8, p. 159]). W.l.o.g. assume the elements of the first column of the second latin square (latin square consisting of the values of set L) to be in natural order. At this point we have $k(n,1) = k_q \neq n$. Let $j_3 \in J \setminus \{1\}$ be such that $k(n, j_3) = 1$. For $n \geq 5$ there exist $l_q \in L \setminus \{1, n\}$ and $l_1 \in L \setminus \{1, l_q, n\}$ such that $l(1, j_1) = l_q$, $l(1, j_2) = l_1$ and $l(n, j_2) = l_q$ where $j_1, j_2 \in L \setminus \{1, n\}$ with $j_1 \neq j_2$, $j_2 \neq j_3$. We denote $k(1, j_2)$ as k_1 and $k(n, j_2)$ as k_2. Note that since k_1, k_2 lie in the same column we have $k_1 \neq k_2$. Combining this with $k(1,1) = 1$, $k(1,n) = n$, we obtain that $k_1 \in K \setminus \{1, k_2, n\}$. In a similar manner, because $k(n,1) = k_q$, $k(n, j_3) = 1$ and $j_3 \neq j_2$, $k_1 \neq k_2$ we derive that $k_2 \in K \setminus \{1, k_1, k_q\}$. We apply a fourth index interchange between l_q and n. The resulting point namely x_q^* is illustrated in

Table 24. If $k(n,j_2)=n$ then for $n\geq 5$ there exists $k_2\in K\setminus\{1,k_1,k_q,n\}$. Let $x_q^{**}=x_q^*(k(n,j_2)=n?k_2\leftrightarrow n)_3$. If at

Table 24: Point x_q^*(Lemma 4.12)

	1	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
1	1				k_1		n
$\vdots$							
i_q							
$\vdots$							
n	k_q						

	1	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
1	1		n		l_1		l_q
$\vdots$							
i_q							
$\vdots$							
n	l_q				n		

point x_q^* $k(n,j_2)\neq n$ then we denote $k(n,j_2)$ as k_2. In any case at point x_q^{**} we have $k(n,j_2)=k_2\in K\setminus\{1,k_1,k_q,n\}$. Let $i_q\in I\setminus\{1,n\}$ and $j_q\in J\setminus\{1,n\}$. Then $x_q=x_q^{**}(i_q\leftrightarrow n)_1(j_q\neq j_2?j_q\leftrightarrow j_2)_2$.

□

(Back to the proof of Theorem 4.11). Now we are ready to examine each case separately.

case 1: None of the indices is equal to n.

W.l.o.g. let $i=i_q\neq n$, $j=j_q\neq n$, $k=k_q\neq n$, $l=l_q\neq n$. Thus we must prove

$$\begin{aligned}a_{i_qj_qk_ql_q}&=a_{11k_ql_q}+a_{i_q11l_q}+a_{1j_q1l_q}+a_{i_qj_q11}+a_{1j_qk_q1}+a_{i_q1k_q1}\\&\quad-2a_{i_q111}-2a_{1j_q11}-2a_{11k_q1}-2a_{111l_q}+3a_{1111}\end{aligned}\tag{4.21}$$

At point x_q of Lemma 4.12 we distinguish two cases, viz. $k_q=k_1$, $k_q\neq k_1$.

case 1.1: $k_q=k_1$.

Let $x_q^*=x_q(k_q\leftrightarrow n)_3$ (Table 25). At point x_q^*, since $k(1,j_q)=n$ and $k(i_q,1)=n$ there exist $i_1\in I\setminus\{1,i_q\}$ and $j_1\in J\setminus\{1,j_q\}$ such that $x_{i_1j_1nn}=1$. Let $x_1=x_q^*(i_1\neq n?i_1\leftrightarrow n)_1(j_1\neq n?j_1\leftrightarrow n)_2$. Then $X_1\in P_I^{Q(n,n,n,n)}$ $(x_{nnnn}=1)$. Let $x_2=x_1(1\leftrightarrow k_q)_3$ (Table 26). Again $X_2\in P_I^{Q(n,n,n,n)}$ $(x_{nnnn}=1)$.

Table 25: Point x_q^*(Theorem 4.11, case 1.1)

	1	$\cdots$	j_q	$\cdots$
1	1		n	
$\vdots$				
i_q	n		k_2	
$\vdots$				

	1	$\cdots$	j_q	$\cdots$
1	1		l_1	
$\vdots$				
i_q	l_q		n	
$\vdots$				

Table 26: Point x_2. (Theorem 4.11, case 1.1)

	1	$\cdots$	j_q	$\cdots$	n
1	k_q		n		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

$x_1((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{1111}+a_{1j_qnl_1}+a_{i_q1nl_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qnl_q}+a_{i_q1nl_1}+a_{i_qj_q11} \\ = \quad & a_{11nl_q}+a_{1j_qk_2n}+a_{i_q111}+a_{i_qj_qnl_1}+a_{11nl_1}+a_{1j_q11}+a_{i_q1k_2n}+a_{i_qj_qnl_q} \end{aligned}$$

$x_2((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{11k_q1}+a_{1j_qnl_1}+a_{i_q1nl_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qnl_q}+a_{i_q1nl_1}+a_{i_qj_qk_q1} \\ = \quad & a_{11nl_q}+a_{1j_qk_2n}+a_{i_q1k_q1}+a_{i_qj_qnl_1}+a_{11nl_1}+a_{1j_qk_q1}+a_{i_q1k_2n}+a_{i_qj_qnl_q} \end{aligned}$$

$x_1((1,i_q)_1;(1,j_q)_2)$-$x_2((1,i_q)_1;(1,j_q)_2)$ yields (4.21)$_{l_q=1}$. To show (4.21) consider the points $x_3 = x_2(1 \leftrightarrow l_q)_4$ and $x_4 = x_1(1 \leftrightarrow l_q)_4$. $X_3, X_4 \in P_I^{Q(n,n,n,n)}$ $(x_{nnnn} = 1)$. Points x_3, x_4 are illustrated in Table 27 and 28 respectively.

Table 27: Point x_3. (Theorem 4.11, case 1.1)

	1	$\cdots$	j_q	$\cdots$	n
1	k_q		n		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_q		l_1		
$\vdots$					
i_q	1		n		
$\vdots$					
n					n

$x_3((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{11k_ql_q}+a_{1j_qnl_1}+a_{i_q1n1}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qn1}+a_{i_q1nl_1}+a_{i_qj_qk_ql_q} \\ = \quad & a_{11n1}+a_{1j_qk_2n}+a_{i_q1k_ql_q}+a_{i_qj_qnl_1}+a_{11nl_1}+a_{1j_qk_ql_q}+a_{i_q1k_2n}+a_{i_qj_qn1} \end{aligned}$$

Table 28: Point x_4. (Theorem 4.11, case 1.1)

	1	$\cdots$	j_q	$\cdots$	n
1	1		n		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_q		l_1		
$\vdots$					
i_q	1		n		
$\vdots$					
n					n

$x_4((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{111l_q}+a_{1j_qnl_1}+a_{i_q1n1}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qn1}+a_{i_q1nl_1}+a_{i_qj_q1l_q} \\ = \quad & a_{11n1}+a_{1j_qk_2n}+a_{i_q11l_q}+a_{i_qj_qnl_1}+a_{11nl_1}+a_{1j_q1l_q}+a_{i_q1k_2n}+a_{i_qj_qn1} \end{aligned}$$

$x_3((1,i_q)_1;(1,j_q)_2)$-$x_4((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$a_{i_qj_qk_ql_q}=a_{111l_q}-a_{11k_ql_q}-a_{1j_q1l_q}-a_{i_q11l_q}+(a_{i_qj_q1l_q}+a_{i_q1k_ql_q}+a_{1j_qk_ql_q}) \tag{4.22}$$

(4.22) will be used to derive (4.21).

case 1.2: $k_q\neq k_1$.

Let $x_q^*=x_q(k_q\leftrightarrow n)_3(l_1\leftrightarrow n)_4$. At point x_q^*because $k(i_q,1)=n$ and $l(1,j_q)=n$ there exist $i_1\in I\setminus\{1,i_q\}$ and $j_1\in J\setminus\{1,j_q\}$ such that $x_{i_1j_1nn}=1$. Let $x_1=x_q^*(i_1\neq n?i_1\leftrightarrow n)_1(j_1\neq n?j_1\leftrightarrow n)_2$. Let $x_2=x_1(1\leftrightarrow k_q)_3$. $X_1,X_2\in P_I^{Q(n,n,n,n)}$ $(x_{nnnn}=1)$. The points x_1, x_2 are illustrated in Table 29 and 30 respectively.

Table 29: Point x_1. (Theorem 4.11, case 1.2)

	1	$\cdots$	j_q	$\cdots$	n
1	1		k_1		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		n		
$\vdots$					
i_q	l_q		l_1		
$\vdots$					
n					n

$x_1((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{1111}+a_{1j_qk_1n}+a_{i_q1nl_q}+a_{i_qj_qk_2l_1}+a_{11k_2l_1}+a_{1j_qnl_q}+a_{i_q1k_1n}+a_{i_qj_q11} \\ = \quad & a_{11k_1n}+a_{1j_q11}+a_{i_q1k_2l_1}+a_{i_qj_qnl_q}+a_{11nl_q}+a_{1j_qk_2l_1}+a_{i_q111}+a_{i_qj_qk_1n} \end{aligned}$$

Table 30: Point x_2. (Theorem 4.11, case 1.2)

	1	$\cdots$	j_q	$\cdots$	n
1	k_q		k_1		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		n		
$\vdots$					
i_q	l_q		l_1		
$\vdots$					
n					n

$x_2((1, i_q)_1; (1, j_q)_2) \Rightarrow$

$$\begin{aligned} & a_{11k_q1} + a_{1j_qk_1n} + a_{i_q1nl_q} + a_{i_qj_qk_2l_1} + a_{11k_2l_1} + a_{1j_qnl_q} + a_{i_q1k_1n} + a_{i_qj_qk_q1} \\ = \ & a_{11k_1n} + a_{1j_qk_q1} + a_{i_q1k_2l_1} + a_{i_qj_qnl_q} + a_{11nl_q} + a_{1j_qk_2l_1} + a_{i_q1k_q1} + a_{i_qj_qk_1n} \end{aligned}$$

$x_1((1, i_q)_1; (1, j_q)_2)$-$x_2((1, i_q)_1; (1, j_q)_2)$ yields (4.21)$_{l_q=1}$.

Let $x_3 = x_2(1 \leftrightarrow l_q)_4$ and $x_4 = x_1(1 \leftrightarrow l_q)_4$ (see Table 31 and 32 respectively). $X_3, X_4 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$).

Table 31: Point x_3. (Theorem 4.11, case 1.2)

	1	$\cdots$	j_q	$\cdots$	n
1	k_q		k_1		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_q		n		
$\vdots$					
i_q	1		l_1		
$\vdots$					
n					n

Table 32: Point x_4. (Theorem 4.11, case 1.2)

	1	$\cdots$	j_q	$\cdots$	n
1	1		k_1		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_q		n		
$\vdots$					
i_q	1		l_1		
$\vdots$					
n					n

$x_3((1, i_q)_1; (1, j_q)_2) \Rightarrow$

$$\begin{aligned} & a_{11k_ql_q} + a_{1j_qk_1n} + a_{i_q1n1} + a_{i_qj_qk_2l_1} + a_{11k_2l_1} + a_{1j_qn1} + a_{i_q1k_1n} + a_{i_qj_qk_ql_q} \\ = \ & a_{11k_1n} + a_{1j_qk_ql_q} + a_{i_q1k_2l_1} + a_{i_qj_qn1} + a_{11n1} + a_{1j_qk_2l_1} + a_{i_q1k_ql_q} + a_{i_qj_qk_1n} \end{aligned}$$

$x_4((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$
\begin{aligned}
& a_{111l_q}+a_{1j_qk_1n}+a_{i_q1n1}+a_{i_qj_qk_2l_1}+a_{11k_2l_1}+a_{1j_qn1}+a_{i_q1k_1n}+a_{i_qj_q1l_q} \\
= \; & a_{11k_1n}+a_{1j_q1l_q}+a_{i_q1k_2l_1}+a_{i_qj_qn1}+a_{11n1}+a_{1j_qk_2l_1}+a_{i_q11l_q}+a_{i_qj_qk_1n}
\end{aligned}
$$

$x_3((1,i_q)_1;(1,j_q)_2)$-$x_4((1,i_q)_1;(1,j_q)_2)$ yields equation (4.22).

For all cases examined we have derived (4.21)$_{l_q=1}$ explicitly, as well as equation (4.22). It is easy to see that equations (4.21)$_{i_q=1}$, (4.21)$_{j_q=1}$, (4.21)$_{k_q=1}$ follow by symmetry for all these cases. Substituting in equation (4.22) terms in brackets from (4.21)$_{i_q=1}$, (4.21)$_{j_q=1}$, (4.21)$_{k_q=1}$ yields equation (4.21).

case 2: One of the indices is equal to n.

W.l.o.g. assume $i=i_q\neq n$, $j=j_q\neq n$, $k=n$, $l=l_q\neq n$. Equation (4.20) becomes

$$
\begin{aligned}
a_{i_qj_qnl_q} = {} & a_{11nl_q}+a_{i_q11l_q}+a_{1j_q1l_q}+a_{i_qj_q11}+a_{1j_qn1}+a_{i_q1n1} \\
& -2a_{i_q111}-2a_{1j_q11}-2a_{11n1}-2a_{111l_q}+3a_{1111}
\end{aligned}
\tag{4.23}
$$

As in the previous case, first we show explicitly (4.23)$_{l_q=1}$. At point x_q of Lemma 4.12 we apply a third index interchange between k_q and n. At the derived point, namely x_q^2 (Table 33), we distinguish two cases, viz. $k(1,j_q)=n$, $k(1,j_q)=k_1\neq n$. The first case can occur if $k_q=k_1$ at point x_q of Lemma 4.12.

Table 33: Point x_q^2 (Theorem 4.11, case 2)

	1	$\cdots$	j_q	$\cdots$
1	1		$k(1,j_q)$	
$\vdots$				
i_q	n		k_2	
$\vdots$				

	1	$\cdots$	j_q	$\cdots$
1	1		l_1	
$\vdots$				
i_q	l_q		n	
$\vdots$				

case 2.1: $k(1,j_q)=n$.

Point x_q^2 is the point x_q^* of case 1.1. Therefore we can derive point x_1 as in case 1.1. This implies that we have the same X_1 collection and the same $x_1(1,i_q;1,j_q)$ equation both for this case and case 1.1.

Let $x_2^*=x_1(1\leftrightarrow n)_3$. At this point because $k(1,1)=n$ and $l(i_q,j_q)=n$ there exist $i_1\in I\setminus\{1,i_q\}$ and $j_1\in\setminus\{1,j_q\}$ such that $x_{i_1j_1nn}=1$. Let $x_2=x_2^*(i_1\neq n?i_1\leftrightarrow n)_1(j_1\neq n?j_1\leftrightarrow n)_2$ (Table 34). Then $X_2\in P_I^{Q(n,n,n,n)}$ ($x_{nnnn}=1$).

Table 34: Point x_2 (Theorem 4.11, case 2.1)

	1	$\cdots$	j_q	$\cdots$	n
1	n		1		
$\vdots$					
i_q	1		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

$x_2((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{11n1}+a_{1j_q1l_1}+a_{i_q11l_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_q1l_q}+a_{i_q11l_1}+a_{i_qj_qn1} \\ = \quad & a_{111lq}+a_{1j_qk_2n}+a_{i_q1n1}+a_{i_qj_q1l_1}+a_{111l_1}+a_{1j_qn1}+a_{i_q1k_2n}+a_{i_qj_q1l_q} \end{aligned}$$

$x_1((1,i_q)_1;(1,j_q)_2)$-$x_2((1,i_q)_1;(1,j_q)_2)\Rightarrow$

$$\begin{aligned} & a_{1111}+a_{i_qj_q11}+a_{1j_qn1}+a_{i_q1n1}-(a_{i_qj_qn1}+a_{i_q111}+a_{1j_q11}+a_{11n1}) \\ = \quad & \sum\{a_{i_qj_qnl}+a_{11nl}+a_{1j_q1l}+a_{i_q11l}-(a_{i_qj_q1l}+a_{111l}+a_{1j_qnl}+a_{i_q1nl}):l\in\{l_q,l_1\}\} \end{aligned} \tag{4.24}$$

For $n\geq 5$ there exists $l_2\in L\setminus\{1,l_1,l_q,n\}$ such that we can derive two additional equations of the type (4.24); one by substituting (l_q,l_1) by (l_1,l_2) and the other by substituting (l_q,l_1) by (l_2,l_q) at points x_1,x_2. By summing these two equations and (4.24) we get

$$\begin{aligned} & 3[a_{1111}+a_{i_qj_q11}+a_{1j_qn1}+a_{i_q1n1}-(a_{i_qj_qn1}+a_{i_q111}+a_{1j_q11}+a_{11n1})] \\ = \quad & 2\sum\{a_{i_qj_qnl}+a_{11nl}+a_{1j_q1l}+a_{i_q11l}-(a_{i_qj_q1l}+a_{111l}+a_{1j_qnl}+a_{i_q1nl}):l\in\{l_q,l_1,l_2\}\} \end{aligned} \tag{4.25}$$

Let $x_3=x_2(1\leftrightarrow l_2)_4$ (Table 35) and $x_4=x_1(1\leftrightarrow l_2)_4$ (Table 36). $X_3,X_4\in P_I^{Q(n,n,n,n)}$ ($x_{nnnn}=1$).

Table 35: Point x_3 (Theorem 4.11, case 2.1)

	1	$\cdots$	j_q	$\cdots$	n
1	n		1		
$\vdots$					
i_q	1		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_2		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

$x_3((1,i_q)_1;(1,j_q)_2)$-$x_4((1,i_q)_1;(1,j_q)_2)$ yields the right-hand side of (4.25) to zero proving thus (4.23)$_{l_q=1}$.

Table 36: Point x_4 (Theorem 4.11, case 2.1)

	1	$\cdots$	j_q	$\cdots$	n
1	1		n		
$\vdots$					
i_q	n		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	l_2		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

Now consider the equation (4.24) derived for (l_1, l_2) instead of (l_q, l_1). Multiply by 2 and subtract from (4.25). The left-hand side of the derived equation is equal to zero (due to (4.23)$_{l_q=1}$) yielding

$$a_{i_q j_q n l_q} = (a_{i_q j_q 1 l_q} + a_{1 j_q n l_q} + a_{i_q 1 n l_q}) + a_{111 l_q} - a_{11 n l_q} - a_{1 j_q 1 l_q} - a_{i_q 11 l_q} \tag{4.26}$$

We will show that equation (4.26) is valid for all sub-cases of case 2.2, as well.

case 2.2: $k(1, j_q) = k_1 \neq n$.

We investigate the location of pair $(n, n) \in K \times L$ at point x_q^2. That is, we examine variable $x_{i_2 j_2 nn} = 1$ for all feasible values of i_2, j_2. Clearly $i_2 \in I \setminus \{i_q\}$ since $k(i_q,1) = n$ and $l(i_q, 1) = l_q \neq n$ (by definition). For the same reason $j_2 \neq 1$. Additionally $j_2 \neq j_q$ since $l(i_q, j_q) = n$ and $k(i_q, j_q) = k_2 \neq n$. Therefore $j_2 \in J \setminus \{1, j_q\}$. We distinguish two cases w.r.t. index i_2 of $x_{i_2 j_2 nn} = 1$ viz. $i_2 \neq 1$, $i_2 = 1$.

case 2.2.1: $i_2 \neq 1$.

Let $x_1 = x_q^2(i_2 \neq n?i_2 \leftrightarrow n)_1(j_2 \neq n?j_2 \leftrightarrow n)_2$. $X_1 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$). Let $x_2^* = x_1(1 \leftrightarrow n)_3$. At this point since $k(1, 1) = n$ and $l(i_q, j_q) = n$ there exist $i_1 \in I \setminus \{1, i_q\}$ and $j_1 \in J \setminus \{1, j_q\}$ such that $x_{i_1 j_1 nn} = 1$. Let $x_2 = x_2^*(i_1 \neq n?i_1 \leftrightarrow n)_1(j_1 \neq n?j_1 \leftrightarrow n)_2$ (Table 37). $X_2 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$).

Table 37: Point x_2. (Theorem 4.11, case 2.2.1)

	1	$\cdots$	j_q	$\cdots$	n
1	n		k_1		
$\vdots$					
i_q	1		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

$x_1((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned}&a_{1111}+a_{1j_qk_1l_1}+a_{i_q1nl_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qnl_q}+a_{i_q1k_1l_1}+a_{i_qj_q11}\\ =\;&a_{11k_1l_1}+a_{1j_q11}+a_{i_q1k_2n}+a_{i_qj_qnl_q}+a_{11nl_q}+a_{1j_qk_2n}+a_{i_q111}+a_{i_qj_qk_1l_1}\end{aligned}$$

$x_2((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned}&a_{11n1}+a_{1j_qk_1l_1}+a_{i_q11l_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_q1l_q}+a_{i_q1k_1l_1}+a_{i_qj_qn1}\\ =\;&a_{11k_1l_1}+a_{1j_qn1}+a_{i_q1k_2n}+a_{i_qj_q1l_q}+a_{111l_q}+a_{1j_qk_2n}+a_{i_q1n1}+a_{i_qj_qk_1l_1}\end{aligned}$$

$x_1((1,i_q)_1;(1,j_q)_2)$-$x_2((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned}&a_{1111}+a_{i_qj_q11}+a_{i_q1n1}+a_{1j_qn1}-(a_{i_qj_qn1}+a_{i_q111}+a_{1j_q11}+a_{11n1})\\ =\;&a_{i_qj_qnl_q}+a_{11nl_q}+a_{i_q11l_q}+a_{1j_q1l_q}-a_{111l_q}-(a_{i_qj_q1l_q}+a_{i_q1nl_q}+a_{1j_qnl_q})\end{aligned}\tag{4.27}$$

We derive points $\dot{x}_1 = x_1(l_q \leftrightarrow l_1)_4$ and $\dot{x}_2 = x_2(l_q \leftrightarrow l_1)_4$. $\dot{X}_1, \dot{X}_2 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$). $\dot{x}_1((1,i_q)_1;(1,j_q)_2)$-$\dot{x}_2((1,i_q)_1;(1,j_q)_2)$ yields equation (4.27) with l_1 in the place of l_q. By adding this equation to (4.27) we get

$$\begin{aligned}&2[a_{1111}+a_{i_qj_q11}+a_{i_q1n1}+a_{1j_qn1}-(a_{i_qj_qn1}+a_{i_q111}+a_{1j_q11}+a_{11n1})]\\ =\;&\sum\{a_{i_qj_qnl}+a_{11nl}+a_{i_q11l}+a_{1j_q1l}-a_{111l}-(a_{i_qj_q1l}+a_{i_q1nl}+a_{1j_qnl}):l\in\{l_q,l_1\}\}\end{aligned}\tag{4.28}$$

Let $x_3 = x_2(1 \leftrightarrow l_1)_4$ and $x_4 = x_1(1 \leftrightarrow l_1)_4$. $X_3, X_4 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$). $x_3((1,i_q)_1;(1,j_q)_2)$-$x_4((1,i_q)_1;(1,j_q)_2)$ yields the right-hand side of (4.28) to zero thus implying $(4.23)_{l_q=1}$. Substituting term $a_{i_qj_qn1}$ from $(4.23)_{l_q=1}$ to (4.27) yields equation (4.26).

case 2.2.2: $i_2 = 1$.

Let $x_1 = x_q^2(j_2 \neq n?j_2 \leftrightarrow n)_2$. $X_1 \in P_I^{Q(n,n,n,n)}$ ($x_{1nnn} = 1$). Point x_2 is derived from point x_1 exactly as in case 2.2.1. $X_2 \in P_I^{Q(n,n,n,n)}$ ($x_{nnnn} = 1$). The rest of the points are derived from x_1 and x_2 exactly as in the previous case. The only difference is that collections derived from x_1 belong to $P_I^{Q(n,n,n,n)}$ because they have $x_{1nnn} = 1$ for all of their points. The equations of the previous case apply throughout this case as well.

Thus, for case 2 we have shown $(4.23)_{l_q=1}$ and (4.26). Cases $(4.23)_{i_q=1}$ and $(4.23)_{j_q=1}$ follow by symmetry. Substituting in (4.26) terms in brackets from $(4.23)_{i_q=1}$, $(4.23)_{j_q=1}$ and $(4.21)_{k_q=1}$ yields equation (4.23).

By reversing the roles of the sets, w.r.t. the row set, the column set and the set of the elements of the first and the second latin square of an *OLS* structure, we can show equation (4.20) with any other index being equal to n.

case 3: Two of the indices are equal to n.

W.l.o.g. assume $i = i_q \neq n$, $j = n$, $k = n$, $l = l_q \neq n$. Equation (4.20) becomes

$$a_{i_q n n l_q} = a_{11nl_q} + a_{i_q 11 l_q} + a_{1n1l_q} + a_{i_q n11} + a_{1nn1} + a_{i_q 1n1}$$
$$-2a_{i_q 111} - 2a_{1n11} - 2a_{11n1} - 2a_{111l_q} + 3a_{1111} \tag{4.29}$$

As in the previous case, first we show explicitly $(4.29)_{l_q=1}$. At point x_q^2 of the previous case we apply a second index interchange between j_q and n. At the derived point, namely x_q^3 (Table 38), we distinguish two cases, viz. $k(1,n) = n$, $k(1,n) = k_1 \neq n$.

Table 38: Point x_q^3 (Theorem 4.11, case 3)

	1	$\cdots$	n
1	1		$k(1,n)$
$\vdots$			
i_q	n		k_2
$\vdots$			

	1	$\cdots$	n
1	1		l_1
$\vdots$			
i_q	l_q		n
$\vdots$			

case 3.1: $k(1,n) = n$.

At point x_q^3 there exist $i_1 \in I\backslash\{1, i_q\}$ and $j_1 \in J\backslash\{1, n\}$ such that $x_{i_1 j_1 nn} = 1$. Let $x_1 = x_2^3(i_1 \neq n?i_1 \leftrightarrow n)_1$. Clearly $X_1 \in P_I^{Q(n,n,n,n)}$ ($x_{nj_1nn} = 1$). Let $x_2^* = x_1(1 \leftrightarrow n)_3$. At point x_2^* there exist $i_2 \in I \setminus \{1, i_q\}$ and $j_2 \in J \setminus \{1, n\}$ such that $x_{i_2 j_2 nn} = 1$. Therefore let $x_2 = x_2(i_2 \neq n?i_2 \leftrightarrow n)_1$. Clearly $X_2 \in P_I^{Q(n,n,n,n)}$ ($x_{nj_2nn} = 1$). Point x_2 is illustrated at Table 39. Note that $j_2 \neq j_1$.

Table 39: Point x_2 (Theorem 4.11, case 3.1)

	1	$\cdots$	j_2	$\cdots$	n
1	n				1
$\vdots$					
i_q	1				k_2
$\vdots$					
n			n		

	1	$\cdots$	j_2	$\cdots$	n
1	1				l_1
$\vdots$					
i_q	l_q				n
$\vdots$					
n			n		

The rest of the points are derived from x_1 and x_2 exactly as in case 2.1 (same interchanges). The points and the related X collections derived from x_1 belong to $P_I^{Q(n,n,n,n)}$ since they have $x_{nj_1nn} = 1$ whereas those derived from x_2 belong to $P_I^{Q(n,n,n,n)}$ since they have $x_{nj_2nn} = 1$. All equations of case 2.1, with n in the place of j_q, are derived in a similar manner for this case as well. Therefore, we prove $(4.29)_{l_q=1}$ which is $(4.23)_{l_q=1}$ with n replaced by j_q and derive the equation corresponding to (4.26):

$$a_{i_qnnl_q} = (a_{i_qn1l_q} + a_{1nnl_q} + a_{i_q1nl_q}) + a_{111l_q} - a_{11nl_q} - a_{1n1l_q} - a_{i_q11l_q} \tag{4.30}$$

We will show that (4.30) is valid for the following sub-cases.

case 3.2: $k(1,n) = k_1 \neq n$.

Let $x_1^* = x_q^3(k_1 \leftrightarrow n)_3(l_1 \leftrightarrow l_q)_4$ (Table 40). At this point since $k(1,n) = n$, $l(1,n) = l_q \neq n$ and

Table 40: Point x_1^* (Theorem 4.11, case 3.2)

	1	$\cdots$	n
1	1		n
$\vdots$			
i_q	k_1		k_2
$\vdots$			

	1	$\cdots$	n
1	1		l_q
$\vdots$			
i_q	l_1		n
$\vdots$			

$k(i_q,n) = k_2 \neq n$, $l(i_q,n) = n$ there exist $i_1 \in I \setminus \{1, i_q\}$ and $j_1 \in J \setminus \{n\}$ such that $x_{i_1j_1nn} = 1$. Let $x_1 = x_1^*(1 \leftrightarrow i_1)_1$. $X_1 \in P_I^{Q(n,n,n,n)}$ because if $j_1 \neq 1$ then $x_{nj_1nn} = 1$ for all points of X_1, else if $j_1 = 1$ then $x_{n1nn} = 1$ at points x_1, x_1' and $x_{nnnn} = 1$ at points $\bar{x}_1$, $\bar{x}_1'$. Let $x_2^* = x_1(1 \leftrightarrow n)_3$. At x_2^* since $k(1,1) = n$, $l(1,1) = 1$ and $k(i_q,n) = k_2 \neq n$, $l(i_q,n) = n$ there exist $i_2 \in I \setminus \{1, i_q\}$ and $j_2 \in J \setminus \{1, n\}$ such that $x_{i_2j_2nn} = 1$. Note that $j_1 \neq j_2$. Let $x_2 = x_2^*(i_2 \leftrightarrow n)_1$ (Table 41). $X_2 \in P_I^{Q(n,n,n,n)}$ ($x_{nj_2nn} = 1$).

Table 41: Point x_2 (Theorem 4.11, case 3.2)

	1	$\cdots$	j_2	$\cdots$	n
1	n				1
$\vdots$					
i_q	k_1				k_2
$\vdots$					
n			n		

	1	$\cdots$	j_2	$\cdots$	n
1	1				l_q
$\vdots$					
i_q	l_1				n
$\vdots$					
n			n		

$x_1((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{1111} + a_{1nnl_q} + a_{i_q1k_1l_1} + a_{i_qnk_2n} + a_{11k_2n} + a_{1nk_1l_1} + a_{i_q1nl_q} + a_{i_qn11} \\ = \; & a_{11nl_q} + a_{1n11} + a_{i_q1k_2n} + a_{i_qnk_1l_1} + a_{11k_1l_1} + a_{1nk_2n} + a_{i_q111} + a_{i_qnnl_q} \end{aligned}$$

$x_2((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{11n1}+a_{1n1l_q}+a_{i_q1k_1l_1}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_1l_1}+a_{i_q11l_q}+a_{i_qnn1} \\ = \quad & a_{111l_q}+a_{1nn1}+a_{i_q1k_2n}+a_{i_qnk_1l_1}+a_{11k_1l_1}+a_{1nk_2n}+a_{i_q1n1}+a_{i_qn1l_q} \end{aligned}$$

$x_1((1,i_q)_1;(1,n)_2)$-$x_2((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{1111}+a_{i_qn11}+a_{i_q1n1}+a_{1nn1}-(a_{i_qnn1}+a_{i_q111}+a_{1n11}+a_{11n1}) \\ = \quad & a_{11nl_q}+a_{1n1l_q}+a_{i_q11l_q}+a_{i_qnnl_q}-(a_{i_qn1l_q}+a_{1nnl_q}+a_{i_q1nl_q}+a_{111l_q}) \end{aligned} \tag{4.31}$$

By interchanging l_q, l_1 in points x_1 and x_2, we derive equation (4.31) with l_1 in the place of l_q. By adding this equation to (4.31) we get

$$\begin{aligned} & 2[a_{1111}+a_{i_qn11}+a_{i_q1n1}+a_{1nn1}-(a_{i_qnn1}+a_{i_q111}+a_{1n11}+a_{11n1})] \\ = \quad & \sum\{a_{11nl}+a_{1n1l}+a_{i_q11l}+a_{i_qnnl}-(a_{i_qn1l}+a_{1nnl}+a_{i_q1nl}+a_{111l}) : l\in\{l_q,l_1\}\} \end{aligned} \tag{4.32}$$

Let $x_3=x_2(1\leftrightarrow l_1)_4$ and $x_4=x_1(1\leftrightarrow l_1)_4$. Points x_3 and x_4 are illustrated in Table 42 and 43

Table 42: Point x_3 (Theorem 4.11, case 3.2)

	1	$\cdots$	j_2	$\cdots$	n
1	n				1
$\vdots$					
i_q	k_1				k_2
$\vdots$					
n			n		

	1	$\cdots$	j_2	$\cdots$	n
1	l_1				l_q
$\vdots$					
i_q	1				n
$\vdots$					
n			n		

Table 43: Point $x4$ (Theorem 4.11, case 3.2)

	1	$\cdots$	j_1	$\cdots$	n
1	1				n
$\vdots$					
i_q	k_1				k_2
$\vdots$					
n			n		

	1	$\cdots$	j_1	$\cdots$	n
1	l_1				l_q
$\vdots$					
i_q	1				n
$\vdots$					
n			n		

respectively (note that we can have $j_1=1$). $X_3\in P_I^{Q(n,n,n,n)}$ and $X_4\in P_I^{Q(n,n,n,n)}$due to the same reasoning that $X_2\in P_I^{Q(n,n,n,n)}$ and $X_1\in P_I^{Q(n,n,n,n)}$ respectively.

$x_3((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{11nl_1}+a_{1n1l_q}+a_{i_q1k_11}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_11}+a_{i_q11l_q}+a_{i_qnnl_1} \\ = \quad & a_{111l_q}+a_{1nnl_1}+a_{i_q1k_2n}+a_{i_qnk_11}+a_{11k_11}+a_{1nk_2n}+a_{i_q1nl_1}+a_{i_qn1l_q} \end{aligned}$$

$x_4((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{111l_1}+a_{1nnl_q}+a_{i_q1k1}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_11}+a_{i_q1nl_q}+a_{i_qn1l_1} \\ = \quad & a_{11nl_q}+a_{1n1l_1}+a_{i_q1k_2n}+a_{i_qnk_11}+a_{11k_11}+a_{1nk_2n}+a_{i_q11l_1}+a_{i_qnnl_q} \end{aligned}$$

$x_3((1,i_q)_1;(1,n)_2)$-$x_4((1,i_q)_1;(1,n)_2)$ yields the right-hand side of equation (4.32) to zero, proving thus (4.29)$_{l_q=1}$.

Substituting term a_{i_qnn1} in equation (4.31) from (4.29)$_{l_q=1}$, we derive equation (4.30).

Therefore, for case 3, we have derived (4.29)$_{l_q=1}$ and (4.30). Equation (4.29)$_{i_q=1}$ follows by symmetry. By replacing to equation (4.30) term $a_{i_qn1l_q}$ from (4.20) with $i=i_q$, $j=n$, $k=1$, $l=l_q$ (shown implicitly in case 2), term a_{1nnl_q} from (4.29)$_{i_q=1}$ and term $a_{i_q1nl_q}$ from (4.23)$_{j_q=1}$ we derive equation (4.29).

Again, by reversing the roles of the sets we derive equation (4.20) with any other pair of indices being equal to n.

Our proof with respect to $(i,j,k,l)\in C\setminus Q(n,n,n,n)$ is now complete.

For $(i,j,k,l)\in Q(n,n,n,n)$ we define

$$\pi_{ijkl}=a_{ijkl}-(\lambda^1_{kl}+\lambda^2_{il}+\lambda^3_{jl}+\lambda^4_{ij}+\lambda^5_{jk}+\lambda^6_{ik})$$

To prove (4.18) for $(i,j,k,l)\in Q(n,n,n,n)$ we must prove that all π_{ijkl} are equal. Note that for $(i,j,k,l)\in Q(n,n,n,n)$ we have $4n-3$ terms:

$$\pi_{nnnn}=\pi_{tnnn}=\pi_{ntnn}=\pi_{nntn}=\pi_{nnnt},\ \forall t\in\{1,...,n-1\}$$

First we show $\pi_{nnk_pn}=\pi_{nnk_rn}$ for $k_p,k_r\in K\setminus\{n\}$ and $k_p\neq k_r$.

For $k_0,k_1,k_p,n\in K$ and $l_0,l_1,n\in L$ consider the point x_{k_p} as illustrated in Table 44. At this point we have $k_p\neq k_0,k_1,n$ and $l_0\neq n$, $l_1\neq n$. To show that for $n\geq 4$ and $n\neq 6$ such a point exists, consider the point x illustrated in Table 45. Let $j_0\in J\setminus\{1,n\}$ be such that $l(1,j_0)=n$. Denote $l(n,j_0)$ as l_0, $k(1,j_0)$ as k_p and $k(n,j_0)$ as k_0. Then $x_{k_p}=x(1\leftrightarrow j_0)_2$.

Table 44: Point x_{k_p} (Theorem 4.11)

	1	$\cdots$	n
1	k_p		k_1
$\vdots$			
n	k_0		n

	1	$\cdots$	n
1	n		l_1
$\vdots$			
n	l_0		n

Table 45: Point x (Theorem 4.11)

	1	$\cdots$	n
1	k_t		k_1
$\vdots$			
n	k_s		n

	1	$\cdots$	n
1	l_t		l_1
$\vdots$			
n	l_s		n

For $n \geq 5$ there is $k_r \in K \setminus \{k_0, k_1, k_p, n\}$. Let $x_{k_r} = x_{k_p}(k_p \leftrightarrow k_r)_3$. Note that $X_{k_p}, X_{k_r} \in P_I^{Q(n,n,n,n)}$. $x_{k_p}((1,n)_1;(1,n)_2)$ -$x_{k_r}((1,n)_1;(1,n)_2) \Rightarrow$

$$a_{nnk_pn} + a_{11k_pn} + a_{n1k_rn} + a_{1nk_rn} - (a_{n1k_pn} + a_{1nk_pn} + a_{11k_rn} + a_{nnk_rn}) = 0$$

By substituting each term from (4.18) we get $\pi_{nnk_pn} = \pi_{nnk_rn}$.

Exchanging the roles of the sets we obtain the corresponding results for sets I, J, L, i.e.

$$\begin{aligned} \pi_{i_pnnn} &= \pi_{i_rnnn}, \quad \forall i_p, i_r \in I \setminus \{n\}, i_p \neq i_r, \\ \pi_{nj_pnn} &= \pi_{nj_rnnn}, \quad \forall j_p, j_r \in J \setminus \{n\}, j_p \neq j_r, \\ \pi_{nnnl_p} &= \pi_{nnnl_r}, \quad \forall l_p, l_r \in L \setminus \{n\}, l_p \neq l_r \end{aligned} \tag{4.33}$$

Now we will show that $\pi_{nnkn} = \pi_{nnnl}$ for $k \in K \setminus \{n\}, l \in L \setminus \{n\}$.

We consider points x_k, x_l illustrated in Table 46 and 47 respectively.

Table 46: Point x_k (Theorem 4.11)

	1	$\cdots$	n
1	k		k_1
$\vdots$			
n	k_0		n

	1	$\cdots$	n
1	n		l_1
$\vdots$			
n	l_0		n

Again note that $X_k, X_l \in P_I^{Q(n,n,n,n)}$. $x_k((1,n)_1;(1,n)_2)$-$x_l((1,n)_1;(1,n)_2) \Rightarrow$

$$a_{11kn} + a_{nnkn} + a_{n1nl} + a_{1nnl} - (a_{n1kn} + a_{1nkn} + a_{11nl} + a_{nnnl}) = 0$$

Table 47: Point x_l (Theorem 4.11)

	1	$\cdots$	n
1	n		k_1
$\vdots$			
n	k_0		n

	1	$\cdots$	n
1	l		l_1
$\vdots$			
n	l_0		n

Again substituting terms from (4.18) we obtain $\pi_{nnkn} = \pi_{nnnl}$.

Thus by exchanging the roles of the sets and taking into account equations (4.33) we obtain

$$\pi_{innn} = \pi_{njnn} = \pi_{nnkn} = \pi_{nnnl} = \pi, \forall i \in I \setminus \{n\}, j \in J \setminus \{n\}, k \in K \setminus \{n\}, l \in L \setminus \{n\}$$

Consider again $x_k((1,n)_1;(1,n)_2) \Rightarrow$

$$\begin{aligned} & a_{11kn} + a_{1nk_1l_1} + a_{n1k_0l_0} + a_{nnnn} + a_{11nn} + a_{1nk_0l_0} + a_{n1k_1l_1} + a_{nnkn} \\ = \; & a_{11k_0l_0} + a_{1nnn} + a_{n1kn} + a_{nnk_1l_1} + a_{11k_1l_1} + a_{1nkn} + a_{n1nn} + a_{nnk_0l_0} \end{aligned}$$

By substituting terms from (4.18) we get

$$\pi_{nnnn} + \pi_{nnkn} = \pi_{1nnn} + \pi_{n1nn}$$

thus yielding $\pi_{nnnn} = \pi$.

Finally, using the same argument as in Theorem 4.6, (4.19) is true since $P_I^{Q(c)} \neq \emptyset$ for all $c \in C$ for $n \neq 2, 6$.

$\square$

Proposition 4.13. *The inequalities (4.17) are of rank 2.*

Proof. We will first derive a lower bound for the rank of (4.17) by showing that the inequality cannot be of rank 1. We will proceed by illustrating how the inequality can be derived within two steps of the Chvátal-Gomory procedure. This will provide an upper bound of 2 on its rank.

W.l.o.g. assume $c = (i_0, j_0, k_0, l_0)$. Then the node set of the clique is

$$
\begin{aligned}
Q(c) \quad = \quad & \{(i_0, j_0, k_0, l_0), \\
& (i_0, j_0, k_0, 1), ..., (i_0, j_0, k_0, l_0 - 1), ..., (i_0, j_0, k_0, l_0 + 1), ..., (i_0, j_0, k_0, n), \\
& (i_0, j_0, 1, l_0), ..., (i_0, j_0, k_0 - 1, l_0), ..., (i_0, j_0, k_0 + 1, l_0), ..., (i_0, j_0, n, l_0), \\
& (i_0, 1, k_0, l_0), ..., (i_0, j_0 - 1, k_0, l_0), ..., (i_0, j_0 + 1, k_0, l_0), ..., (i_0, n, k_0, l_0), \\
& (1, j_0, k_0, l_0), ..., (i_0 - 1, j_0, k_0, l_0), ..., (i_0 + 1, j_0, k_0, l_0), ..., (n, j_0, n, l_0)\}
\end{aligned}
$$

The induced inequality is:

$$
\begin{aligned}
x_{i_0 j_0 k_0 l_0} \quad + \quad & \sum\{x_{i j_0 k_0 l_0} : i \in I \setminus \{i_0\}\} + \sum\{x_{i_0 j k_0 l_0} : j \in J \setminus \{j_0\}\} \\
+ \quad & \sum\{x_{i_0 j_0 k l_0} : k \in K \setminus \{k_0\}\} + \sum\{x_{i_0 j_0 k_0 l} : l \in L \setminus \{l_0\}\} \leq 1 \qquad (4.34)
\end{aligned}
$$

If the inequality (4.34) is of Chvátal rank 1, then there exists $0 < \epsilon < 1$, such that

$$
\begin{aligned}
x_{i_0 j_0 k_0 l_0} \quad + \quad & \sum\{x_{i j_0 k_0 l_0} : i \in I \setminus \{i_0\}\} + \sum\{x_{i_0 j k_0 l_0} : j \in J \setminus \{j_0\}\} \\
+ \quad & \sum\{x_{i_0 j_0 k l_0} : k \in K \setminus \{k_0\}\} + \sum\{x_{i_0 j_0 k_0 l} : l \in L \setminus \{l_0\}\} \leq 2 - \epsilon \qquad (4.35)
\end{aligned}
$$

Any solution $x \in P_L$ having $x_{i_0 j_0 k_0 l_0} = 0$ and $x_{i j_0 k_0 l_0} = x_{i_0 j k_0 l_0} = x_{i_0 j_0 k l_0} = x_{i_0 j_0 k_0 l} = \frac{1}{2(n-1)}$, $\forall i \in I \setminus \{i_0\}, j \in J \setminus \{j_0\}, k \in K \setminus \{k_0\}, l \in L \setminus \{l_0\}$, violates (4.35) since its left-hand side has $4(n-1)$ variables equal to $\frac{1}{2(n-1)}$, and therefore adding to 2. Such a solution exists, and its coordinates are:

- $x_{i j k_0 l_0} = x_{i j_0 k l_0} = x_{i j_0 k_0 l} = x_{i_0 j k l_0} = x_{i_0 j k_0 l} = x_{i_0 j_0 k l} = 0,$
- $x_{i_0 j k l} = x_{i j_0 k l} = x_{i j k_0 l} = x_{i j k l_0} = \frac{2n-3}{2(n-1)^3},$
- $x_{ijkl} = \frac{(n-2)^2}{(n-1)^4}$

for all $i \in I \setminus \{i_0\}, j \in J \setminus \{j_0\}, k \in K \setminus \{k_0\}, l \in L \setminus \{l_0\}$.

To see that $x \in P_L$, assume row labeled $(m_1, m_2) \in M_1 \times M_2$, where M_1, M_2 can be any of I, J, K, L with $M_1 \neq M_2$. Then recalling that $c = \{i_0, j_0, k_0, l_0\}$:

case 1: If $m_1, m_2 \in c$, the row has $2(n-1)$ variables equal to $\frac{1}{2(n-1)}$, and $(n-1)^2 + 1$ variables equal to 0.

case 2: If $m_1 \in c$, $m_2 \notin c$ or $m_1 \notin c$, $m_2 \in c$, the row has one variable equal to $\frac{1}{2(n-1)}$, $(n-1)^2$ variables equal to $\frac{2n-3}{2(n-1)^3}$ and $2(n-1)$ variables equal to 0.

case 3: If $m_1, m_2 \notin c$, the row has $2(n-1)$ variables equal to $\frac{2n-3}{2(n-1)^3}$, $(n-1)^2$ variables equal to $\frac{(n-2)^2}{(n-1)^4}$ and one variable equal to 0.

Therefore, inequality (4.34) is of Chvátal rank at least 2.

We now show that the Chvátal rank is at most 2. To this end, note that adding rows (i_0, l_0), (j_0, l_0), (k_0, l_0), dividing the resulting inequality by 2 and rounding down both sides gives

$$x_{i_0j_0k_0l_0} + \sum\{x_{ij_0k_0l_0} : i \in I \setminus \{i_0\}\} + \sum\{x_{i_0jk_0l_0} : j \in J \setminus \{j_0\}\} + \sum\{x_{i_0j_0kl_0} : k \in K \setminus \{k_0\}\} \leq 1 \tag{4.36}$$

Adding rows (i_0, k_0), (j_0, k_0), (k_0, l_0), dividing the resulting inequality by 2 and rounding down both sides gives

$$x_{i_0j_0k_0l_0} + \sum\{x_{ij_0k_0l_0} : i \in I \setminus \{i_0\}\} + \sum\{x_{i_0jk_0l_0} : j \in J \setminus \{j_0\}\} + \sum\{x_{i_0j_0k_0l} : l \in L \setminus \{l_0\}\} \leq 1 \tag{4.37}$$

Adding rows (i_0, j_0), (j_0, k_0), (j_0, l_0), dividing the resulting inequality by 2 and rounding down both sides gives

$$x_{i_0j_0k_0l_0} + \sum\{x_{ij_0k_0l_0} : i \in I \setminus \{i_0\}\} + \sum\{x_{i_0j_0kl_0} : k \in K \setminus \{k_0\}\} + \sum\{x_{i_0j_0k_0l} : l \in L \setminus \{l_0\}\} \leq 1 \tag{4.38}$$

Adding rows (i_0, j_0), (i_0, k_0), (i_0, l_0), dividing the resulting inequality by 2 and rounding down both sides gives

$$x_{i_0j_0k_0l_0} + \sum\{x_{i_0jk_0l_0} : j \in J \setminus \{j_0\}\} + \sum\{x_{i_0j_0kl_0} : k \in K \setminus \{k_0\}\} + \sum\{x_{i_0j_0k_0l} : l \in L \setminus \{l_0\}\} \leq 1 \tag{4.39}$$

Finally, adding (4.36)-(4.39) yields

$$\begin{aligned} 4x_{i_0j_0k_0l_0} \quad &+ \quad 3\sum\{x_{ij_0k_0l_0} : i \in I \setminus \{i_0\}\} + 3\sum\{x_{i_0jk_0l_0} : j \in J \setminus \{j_0\}\} \\ &+ \quad 3\sum\{x_{i_0j_0kl_0} : k \in K \setminus \{k_0\}\} + 3\sum\{x_{i_0j_0k_0l} : l \in L \setminus \{l_0\}\} \leq 4 \end{aligned} \tag{4.40}$$

Dividing (4.40) by 3 and rounding down both sides gives inequality (4.34). Therefore, inequality (4.34) is of rank at most 2 and the proof is complete.

□

Theorem 4.14. *For $n \geq 5$ and $n \neq 6$ the inequality*

$$\sum\{x_q : Q(c,s)\} \leq 1 \tag{4.41}$$

defines a facet of P_I for every $c, s \in C$ such that $\mid c \cap s \mid= 1$.

Proof. Let $P_I^{Q(c,s)} = \{x \in P_I : \sum\{x_q : q \in Q(c,s)\} = 1\}$. For $n \geq 5$ and $n \neq 6$ we will show that $P_I^{Q(c,s)}$ is a facet of P_I.

First, we note that (4.41) is a valid inequality for all $x \in P_I$ because $Q(c,s)$ is a node set of a clique in the intersection graph $G_A(C, E_C)$.

W.l.o.g. let $c = (n,n,n,n)$. Then s can be any of the following four quadruples: (n,j,k,l), (i,n,k,l), (i,j,n,l), (i,j,k,n) for $i \in I\backslash\{n\}$, $j \in J\backslash\{n\}$, $k \in K\backslash\{n\}$ and $l \in L\backslash\{n\}$. Since all cases are symmetrical we will analytically carry out the proof only for $s = (n,j,k,l)$. Thus by setting $j = j_0$, $k = k_0$, $l = l_0$ with $j_0 \in J \setminus \{1,n\}$, $k_0 \in K \setminus \{1,n\}$ and $l_0 \in L \setminus \{1,n\}$ we derive $Q(c,s) = Q((n,n,n,n),(n,j_0,k_0,l_0)) = \{(n,n,n,n),(n,n,k_0,l_0),(n,j_0,k_0,n),(n,j_0,n,l_0)\}$.

It is easy to see that $P_I^{Q(c,s)} \neq \emptyset$. Clearly $x_{ijnn} = 1$ for every $x \in P_I$. If pair (i,j) for this variable has $i \neq n$ and/or $j \neq n$ we apply a first and/or second index interchange between n, i and/or n, j. The point derived satisfies $x_{nnnn} = 1$, therefore it belongs to $P_I^{Q(c,s)}$.

Next we will show that if $x \in P_I^{Q(c,s)}$ there exists at least one point which belongs to $P_I \setminus P_I^{Q(c,s)}$. For this purpose, we will assume point $x \in P_I^{Q(c,s)}$ having $x_{i_1 n k_0 l_0} = x_{i_2 j_0 k_0 n} = x_{i_3 j_0 n l_0} = 1$ for $i_1 \neq i_2 \neq i_3 \neq i_1$, all other cases being easier to handle. Note that the constraints of the problem also impose the condition $i_1, i_2, i_3 \in I \setminus \{n\}$. Point x is illustrated in Table 48. For $n \geq 5$ there exists $i_0 \in I \setminus \{i_1, i_2, i_3, n\}$ such that $x_{i_0 j_0 k_1 l_1} = 1$ for $k_1 \neq k_0, n$, $l_1 \neq l_0, n$

Table 48: A point $x \in P_I^{Q(c,s)}$ for $c = (n,n,n,n)$, $s = (n, j_0, k_0, l_0)$

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_1				k_0
$\vdots$				
i_2		k_0		
$\vdots$				
i_3		n		
$\vdots$				
n				n

	$\cdots$	j_0	$\cdots$	n
$\vdots$				
i_1				l_0
$\vdots$				
i_2		n		
$\vdots$				
i_3		l_0		
$\vdots$				
n				n

and $x_{i_0 n k_2 l_2} = 1$ for $k_2 \neq k_0, n$, $l_2 \neq l_0, n$. We apply a first index interchange between n, i_0. At the derived point $x_{i_2 j_0 k_0 n} = x_{i_3 j_0 n l_0} = x_{n n k_2 l_2} = 1$ implying $x_{n j_0 k_0 n} = x_{n j_0 n l_0} = x_{n n k_0 l_0} = x_{nnnn} = 0$. Thus the point belongs to $P_I \setminus P_I^{Q(c,s)}$. We observe that if pairs $(k_0, n), (n, l_0) \in K \times L$ lie at the same row $(i_2 = i_3)$ we need only $n \geq 4$. The same is true for pairs $(k_0, l_0), (n, n) \in K \times L$. The argument goes essentially unchanged if $x_{n j_0 n l_0} = 1$ or $x_{n j_0 k_0 n} = 1$ or $x_{n n k_0 l_0} = 1$ instead of $x_{nnnn} = 1$ at point x.

To show that $P_I^{Q(c,s)}$ is a facet of P_I we will exhibit scalars λ^1_{kl}, λ^2_{il}, λ^3_{jl}, λ^4_{ij}, λ^5_{jk}, λ^6_{ik}, $\pi \in \mathbb{R}$ for $i \in I$, $j \in J$, $k \in K$,

$l \in L$, such that if $ax = a_0$ for all $x \in P_I^{Q(c,s)}$ then

$$a_{ijkl} = \begin{cases} \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik}, & (i,j,k,l) \in C \setminus Q(c,s) \\ \lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik} + \pi, & (i,j,k,l) \in Q(c,s) \end{cases} \tag{4.42}$$

and

$$\begin{aligned} a_0 &= \sum\{\lambda^1_{kl} : k \in K, l \in L\} + \sum\{\lambda^2_{il} : i \in I, l \in L\} \\ &+ \sum\{\lambda^3_{jl} : j \in J, l \in L\} + \sum\{\lambda^4_{ij} : i \in I, j \in J\} \\ &+ \sum\{\lambda^5_{jk} : j \in J, k \in K\} + \sum\{\lambda^6_{ik} : i \in I, k \in K\} + \pi \end{aligned} \tag{4.43}$$

As in Theorem 4.6 we define

$$\begin{aligned} \lambda^1_{kl} &= a_{11kl} \\ \lambda^2_{il} &= a_{i11l} - a_{111l} \\ \lambda^3_{jl} &= a_{1j1l} - a_{111l} \\ \lambda^4_{ij} &= a_{ij11} - a_{i111} - a_{1j11} + a_{1111} \\ \lambda^5_{jk} &= a_{1jk1} - a_{1j11} - a_{11k1} + a_{1111} \\ \lambda^6_{ik} &= a_{i1k1} - a_{i111} - a_{11k1} + a_{1111} \end{aligned}$$

By replacing λs from these equations to (4.42) for $(i,j,k,l) \in Q \setminus Q(c,s)$ we end up with equation (4.20) which must be proven for any $(i,j,k,l) \in Q \setminus Q(c,s)$. It is easy to see that this equation is valid for a_{1111} and for all cases in which at least two of the indices equal 1. For cases where at most one of the indices is equal to one, we will show (4.20) using exclusively points from $P_I^{Q(c,s)}$ for $c = (n,n,n,n)$ and $s = (n, j_0, k_0, l_0)$.

W.r.t. the indices $(i,j,k,l) \in Q \setminus Q(c,s)$ we distinguish four cases viz. none, one, two, three of the indices are equal to n. For the first three cases we follow the same approach as in Theorem 4.11 i. e., we show explicitly (4.20)$_{l=1}$. Equations (4.20)$_{i=1}$, (4.20)$_{j=1}$, (4.20)$_{k=1}$ follow by symmetry whereas (4.20), when none of the indices is equal to one, is shown by substituting terms from these equations to an equation derived explicitly in the course of proving (4.20)$_{l=1}$. For the last case equation (4.42) will be shown explicitly.

case 1 None of the indices is equal to n.

W.l.o.g. let $i = i_q \neq n$, $j = j_q \neq n$, $k = k_q \neq n$, $l = l_q \neq n$. The proof is exactly the same as in case 1 of Theorem 4.11. We observe that all the points, on which we have applied Proposition 4.7, in case 1 of Theorem 4.11

have $x_{nnnn} = 1$. So these points belong not only to $P_I^{Q(n,n,n,n)}$ but to $P_I^{Q(c,s)}$ as well. Hence, all the equations derived for the case 1 of Theorem 4.11 are valid for this case as well. W.l.o.g. we consider $k_0 \neq k_1, k_2$ and $l_0 \neq l_1$ at all points used. Thus, for $n \geq 5$ we could have $k_q = k_0$ and/or $j_q = j_0$ and/or $l_q = l_0$. This means that (4.20), when none of the indices equals n, is valid when some (or none or all) of the corresponding indices are set to values j_0 , k_0, l_0 respectively.

case 2: One of the indices is equal to n.

W.l.o.g. assume $i = i_q \neq n$, $j = j_q \neq n$, $k = n$, $l = l_q \neq n$. Equation (4.20) becomes equation (4.23). For all cases except case 2.2.2, points used in the proof of case 2 have $x_{nnnn} = 1$ which implies that they belong to $P_I^{Q(c,s)}$ as well. Thus, the proof followed in all sub-cases of case 2 of Theorem 4.11, but case 2.2.2, is valid for this case as well. Next we will show (4.23) for case 2.2.2. using points of $P_I^{Q(c,s)}$.

For case 2.2.2 at point x_q^2 we have $x_{1j_2nn} = 1$ with $j_2 \neq j_q$ and $k_0 \neq k_1, k_2$, $l_1 \neq l_0$. Let $x_1^* = x_q^2(k_0 \leftrightarrow n)_3(l_q \neq l_0?l_q \leftrightarrow l_0)_4(l_0 \leftrightarrow l_1)_4$ (Table 49). At point x_1^* since $k(i_q, 1) = k_0$, $l(i_q, 1) = l_1 \neq l_0$ and $k(1, j_q) = k_1 \neq k_0$,

Table 49: Point x_1^* (Theorem 4.11, case 2)

	1	$\cdots$	j_q	$\cdots$
1	1		k_1	
$\vdots$				
i_q	k_0		k_2	
$\vdots$				

	1	$\cdots$	j_q	$\cdots$
1	1		l_0	
$\vdots$				
i_q	l_1		n	
$\vdots$				

$l(1, j_q) = l_0$ there exist $i_1 \in I \setminus \{1, i_q\}$, $j_1 \in J \setminus \{1, j_q\}$ such that $x_{i_1 j_1 k_0 l_0} = 1$. Let $x_1 = x_1^*(i_1 \neq n?i_1 \leftrightarrow n)_1(j_1 \neq n?j_1 \leftrightarrow n)_2$. $X_1 \in P_I^{Q(c,s)}$ ($x_{nnk_0l_0} = 1$). Let $x_2 = x_1(1 \leftrightarrow n)_3$ (Table 50). $X_2 \in P_I^{Q(c,s)}$ ($x_{nnk_0l_0} = 1$).

Table 50: Point x_2 (Theorem 4.11, case 2)

	1	$\cdots$	j_q	$\cdots$	n
1	n		k_1		
$\vdots$					
i_q	k_0		k_2		
$\vdots$					
n					k_0

	1	$\cdots$	j_q	$\cdots$	n
1	1		l_0		
$\vdots$					
i_q	l_1		n		
$\vdots$					
n					l_0

$x_1((1, i_q)_1; (1, j_q)_2) \Rightarrow$

$$\begin{aligned} & a_{1111} + a_{1j_qk_1l_0} + a_{i_q1k_0l_1} + a_{i_qj_qk_2n} + a_{11k_2n} + a_{1j_qk_0l_1} + a_{i_q1k_1l_0} + a_{i_qj_q11} \\ = \quad & a_{11k_1l_0} + a_{1j_q11} + a_{i_q1k_2n} + a_{i_qj_qk_0l_1} + a_{11k_0l_1} + a_{1j_qk_2n} + a_{i_q111} + a_{i_qj_qk_1l_0} \end{aligned}$$

$x_2((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned} & a_{11n1}+a_{1j_qk_1l_0}+a_{i_q1k_0l_1}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_qk_0l_1}+a_{i_q1k_1l_0}+a_{i_qj_qn1} \\ = \quad & a_{11k_1l_0}+a_{1j_qn1}+a_{i_q1k_2n}+a_{i_qj_qk_0l_1}+a_{11k_0l_1}+a_{1j_qk_2n}+a_{i_q1n1}+a_{i_qj_qk_1l_0} \end{aligned}$$

$x_1((1,i_q)_1;(1,j_q)_2)$-$x_2((1,i_q)_1;(1,j_q)_2)$ yields (4.23)$_{l_q=1}$. Equations (4.23)$_{i_q=1}$, (4.23)$_{j_q=1}$ follow by symmetry.

Let $x_3^* = x_q^2(1 \leftrightarrow n)_4$. At point x_3^* since $k(1,1)=n$, $l(1,1)=1$ and $k(i_q,j_q)=1$, $l(i_q,j_q)=n$ there exist $i_1 \in I\setminus\{1,i_q\}$, $j_1 \in J\setminus\{1,j_q\}$ such that $x_{i_1j_1nn}=1$. Let $x_3 = x_3^*(i_1 \neq n?i_1 \leftrightarrow n)_1(j_1 \neq n?j_1 \leftrightarrow n)_2$ (Table 51). Let $x_4 = x_3(1 \leftrightarrow l_q)_4$. $X_3, X_4 \in P_I^{Q(c,s)}$ ($x_{nnnn}=1$).

Table 51: Point x_3 (Theorem 4.11, case 2)

	1	$\cdots$	j_q	$\cdots$	n
1	n		k_1		
$\vdots$					
i_q	1		k_2		
$\vdots$					
n					n

	1	$\cdots$	j_q	$\cdots$	n
1	1		l_1		
$\vdots$					
i_q	l_q		n		
$\vdots$					
n					n

$x_3((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned} & a_{11n1}+a_{1j_qk_1l_1}+a_{i_q11l_q}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_q1l_q}+a_{i_q1k_1l_1}+a_{i_qj_qn1} \\ = \quad & a_{11k_1l_1}+a_{1j_qn1}+a_{i_q1k_2n}+a_{i_qj_q1l_q}+a_{111l_q}+a_{1j_qk_2n}+a_{i_q1n1}+a_{i_qj_qk_1l_1} \end{aligned}$$

$x_4((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned} & a_{11nl_q}+a_{1j_qk_1l_1}+a_{i_q111}+a_{i_qj_qk_2n}+a_{11k_2n}+a_{1j_q11}+a_{i_q1k_1l_1}+a_{i_qj_qnl_q} \\ = \quad & a_{11k_1l_1}+a_{1j_qnl_q}+a_{i_q1k_2n}+a_{i_qj_q11}+a_{1111}+a_{1j_qk_2n}+a_{i_q1nl_q}+a_{i_qj_qk_1l_1} \end{aligned}$$

$x_3((1,i_q)_1;(1,j_q)_2)$-$x_4((1,i_q)_1;(1,j_q)_2)$⇒

$$\begin{aligned} a_{i_qj_qnl_q} \quad = \quad & (a_{i_qj_qn1}+a_{i_q1nl_q}+a_{1j_qnl_q}-a_{i_qj_q1l_q})+a_{1j_q1l_q}+a_{i_q11l_q}-a_{111l_q}-a_{11nl_q} \\ & +a_{11n1}+a_{i_qj_q11}+a_{1111}-a_{i_q1n1}-a_{1j_qn1}-a_{i_q111}-a_{1j_q11} \end{aligned} \tag{4.44}$$

By replacing to equation (4.44) terms $a_{i_qj_qn1}, a_{i_q1nl_q}, a_{1j_qnl_q}, a_{i_qj_q1l_q}$ from (4.23)$_{l_q=1}$. (4.23)$_{j_q=1}$, (4.23)$_{i_q=1}$ and

$(4.21)_{k_q=1}$ respectively we derive equation (4.23).

Our proof w.r.t. case 2 is now complete. We observe again that j_q and l_q could be equal to values j_0 and/or l_0, respectively, without affecting the proof. Thus, (4.23) is valid for $j = j_0$ and/or $l = l_0$.

By reversing the roles of the sets, we derive equation (4.20) with any other index being equal to n.

case 3: Two of the indices are equal to n.

We observe that not all cases of $(i, j, k, l) \in Q \setminus Q(c, s)$ for which two of the indices are equal to n, are symmetric. Consider for example (i_q, n, n, l_q) and (n, j_q, n, l_q). In the first case the only restriction w.r.t. i_q and l_q is that they must be different than n. In the second case j_q and l_q have the additional restriction that they cannot be equal to j_0 and l_0 respectively, simultaneously, because then $(n, j_q, n, l_q) = (n, j_0, n, l_0) \in Q(c, s)$. This is due to the fact that (4.41) for $c = (n, n, n, n)$ and $s = (n, j_0, k_0, l_0)$ is not symmetric w.r.t. index i and the rest of the indices. Consequently, we consider two cases. The first (second) case refers to a pair of indices, each being equal to n, not including (respectively including) index i.

case 3.1: W.l.o.g. assume $i = i_q \neq n$, $j = n$, $k = n$, $l = l_q \neq n$. Equation (4.20) becomes equation (4.29). At point x_q^3 we distinguish two cases, viz $k(1, n) = n$, $k(1, n) = k_1 \neq n$.

case 3.1.1: $k(1, n) = n$.

Let $x_1^* = x_q^3(k_0 \leftrightarrow n)_3$. At point x_1^* since $k(1, n) = k_0$, $l(1, n) = l_1 \neq n$ and $k(n, 1) = k_0$, $l(n, 1) = l_q \neq n$ there exist $i_1 \in I \setminus \{1, i_q\}$, $j_1 \in J \setminus \{1, n\}$ such that $x_{i_1 j_1 k_0 n} = 1$. Let $x_1 = x_1^*(i_1 \neq n ? i_1 \leftrightarrow n)_1(j_1 \neq j_0 ? j_1 \leftrightarrow j_0)_2$. Let $x_2 = x_1(1 \leftrightarrow n)_3$. $X_1, X_2 \in P_I^{Q(c,s)}$ ($x_{n j_0 k_0 n} = 1$). Points x_1 and x_2 are illustrated in Table 52 and 53 respectively.

Table 52: Point x_1 (Theorem 4.11, case 3.1.1)

	1	$\cdots$	j_0	$\cdots$	n
1	1				k_0
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	1				l_1
$\vdots$					
i_q	l_q				n
$\vdots$					
n			n		

$x_1((1, i_q)_1; (1, n)_2) \Rightarrow$

$$
\begin{aligned}
& a_{1111} + a_{1nk_0l_1} + a_{i_q1k_0l_q} + a_{i_qnk_2n} + a_{11k_2n} + a_{1nk_0l_q} + a_{i_q1k_0l_1} + a_{i_qn11} \\
= \; & a_{11k_0l_1} + a_{1n11} + a_{i_q1k_2n} + a_{i_qnk_0l_q} + a_{11k_0l_q} + a_{1nk_2n} + a_{i_q111} + a_{i_qnk_0l_1}
\end{aligned}
$$

Table 53: Point x_2 (Theorem 4.11, case 3.1.1)

	1	$\cdots$	j_0	$\cdots$	n
1	n				k_0
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	1				l_1
$\vdots$					
i_q	l_q				n
$\vdots$					
n			n		

$x_2((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{11n1}+a_{1nk_0l_1}+a_{i_q1k_0l_q}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_0l_q}+a_{i_q1k_0l_1}+a_{i_qnn1} \\ = \; & a_{11k_0l_1}+a_{1nn1}+a_{i_q1k_2n}+a_{i_qnk_0l_q}+a_{11k_0l_q}+a_{1nk_2n}+a_{i_q1n1}+a_{i_qnk_0l_1} \end{aligned}$$

$x_1((1,i_q)_1;(1,n)_2)$-$x_2((1,i_q)_1;(1,n)_2)$ yields (4.29)$_{l_q=1}$.

Let $x_3=x_2(1\leftrightarrow l_q)_4$ (Table 54) and $x_4=x_1(1\leftrightarrow l_q)_4$ (Table 55). $X_3,X_4\in P_I^{Q(c,s)}$ ($x_{nj_0k_0n}=1$).

Table 54: Point x_3 (Theorem 4.11, case 3.1.1)

	1	$\cdots$	j_0	$\cdots$	n
1	n				k_0
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				l_1
$\vdots$					
i_q	1				n
$\vdots$					
n			n		

Table 55: Point x_4 (Theorem 4.11, case 3.1.1)

	1	$\cdots$	j_0	$\cdots$	n
1	1				k_0
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				l_1
$\vdots$					
i_q	1				n
$\vdots$					
n			n		

$x_3((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{11nl_q}+a_{1nk_0l_1}+a_{i_q1k_01}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_01}+a_{i_q1k_0l_1}+a_{i_qnnl_q} \\ = \; & a_{11k_0l_1}+a_{1nnl_q}+a_{i_q1k_2n}+a_{i_qnk_01}+a_{11k_01}+a_{1nk_2n}+a_{i_q1nl_q}+a_{i_qnk_0l_1} \end{aligned}$$

$x_4((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{111l_q}+a_{1nk_0l_1}+a_{i_q1k_01}+a_{i_qnk_2n}+a_{11k_2n}+a_{1nk_01}+a_{i_q1k_0l_1}+a_{i_qn1l_q} \\ = \; & a_{11k_0l_1}+a_{1n1l_q}+a_{i_q1k_2n}+a_{i_qnk_01}+a_{11k_01}+a_{1nk_2n}+a_{i_q11l_q}+a_{i_qnk_0l_1} \end{aligned}$$

$x_3((1,i_q)_1;(1,n)_2)$-$x_4((1,i_q)_1;(1,n)_2)$ yields equation (4.30). We will show that this equation is valid for the next sub-case.

case 3.1.2: $k(1,n)=k_1$.

Let $x_1^*=x_q^3(k_0\leftrightarrow n)_3(l_1\leftrightarrow n)_4$. At point x_1^* since $k(i_q,1)=k_0$, $l(i_q,1)=l_q\neq n$ and $k(1,n)=k_1\neq k_0$, $l(1,n)=n$ there exist $i_1\in I\setminus\{1,i_q\}$, $j_1\in J\setminus\{1,n\}$ such that $x_{i_1j_1k_0n}=1$. Let $x_1=x_1^*(i_1\neq n?i_1\leftrightarrow n)_1(j_1\neq j_0?j_1\leftrightarrow j_0)_2$. Let $x_2=x_1(1\leftrightarrow n)_3$. $X_1,X_2\in P_I^{Q(c,s)}$ ($x_{nj_0k_0n}=1$). Points x_1, x_2 are illustrated in Table 56 and 57 respectively.

Table 56: Point x_1 (Theorem 4.11, case 3.1.2)

	1	$\cdots$	j_0	$\cdots$	n
1	1				k_1
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	1				n
$\vdots$					
i_q	l_q				l_1
$\vdots$					
n			n		

Table 57: Point x_2 (Theorem 4.11, case 3.1.2)

	1	$\cdots$	j_0	$\cdots$	n
1	n				k_1
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	1				n
$\vdots$					
i_q	l_q				l_1
$\vdots$					
n			n		

$x_1((1,i_q)_1;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{1111}+a_{1nk_1n}+a_{i_q1k_0l_q}+a_{i_qnk_2l_1}+a_{11k_2l_1}+a_{1nk_0l_q}+a_{i_q1k_1n}+a_{i_qn11} \\ = \; & a_{11k_1n}+a_{1n11}+a_{i_q1k_2l_1}+a_{i_qnk_0l_q}+a_{11k_0l_q}+a_{1nk_2l_1}+a_{i_q111}+a_{i_qnk_1n} \end{aligned}$$

$x_2((1,i_q)_1;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{11n1}+a_{1nk_1n}+a_{i_q1k_0l_q}+a_{i_qnk_2l_1}+a_{11k_2l_1}+a_{1nk_0l_q}+a_{i_q1k_1n}+a_{i_qnn1} \\
= \quad & a_{11k_1n}+a_{1nn1}+a_{i_q1k_2l_1}+a_{i_qnk_0l_q}+a_{11k_0l_q}+a_{1nk_2l_1}+a_{i_q1n1}+a_{i_qnk_1n}
\end{aligned}
$$

$x_1((1,i_q)_1;(1,n)_2)$-$x_2((1,i_q)_1;(1,n)_2)$ yields (4.29)$_{l_q=1}$.

Let $x_3 = x_2(1 \leftrightarrow l_q)_4$ (Table 58) and $x_4 = x_1(1 \leftrightarrow l_q)_4$ (Table 59). $X_3, X_4 \in P_I^{Q(c,s)}$ $(x_{nj_0k_0n} = 1)$.

Table 58: Point x_3 (Theorem 4.11, case 3.1.2)

	1	$\cdots$	j_0	$\cdots$	n
1	n				k_1
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				n
$\vdots$					
i_q	1				l_1
$\vdots$					
n			n		

Table 59: Point x_4 (Theorem 4.11, case 3.1.2)

	1	$\cdots$	j_0	$\cdots$	n
1	1				k_1
$\vdots$					
i_q	k_0				k_2
$\vdots$					
n			k_0		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				n
$\vdots$					
i_q	1				l_1
$\vdots$					
n			n		

$x_3((1,i_q)_1;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{11nl_q}+a_{1nk_1n}+a_{i_q1k_01}+a_{i_qnk_2l_1}+a_{11k_2l_1}+a_{1nk_01}+a_{i_q1k_1n}+a_{i_qnnl_q} \\
= \quad & a_{11k_1n}+a_{1nnl_q}+a_{i_q1k_2l_1}+a_{i_qnk_01}+a_{11k_01}+a_{1nk_2l_1}+a_{i_q1nl_q}+a_{i_qnk_1n}
\end{aligned}
$$

$x_4((1,i_q)_1;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{111l_q}+a_{1nk_1n}+a_{i_q1k_01}+a_{i_qnk_2l_1}+a_{11k_2l_1}+a_{1nk_01}+a_{i_q1k_1n}+a_{i_qn1l_q} \\
= \quad & a_{11k_1n}+a_{1n1l_q}+a_{i_q1k_2l_1}+a_{i_qnk_01}+a_{11k_01}+a_{1nk_2l_1}+a_{i_q11l_q}+a_{i_qnk_1n}
\end{aligned}
$$

$x_3((1,i_q)_1;(1,n)_2)$-$x_4((1,i_q)_1;(1,n)_2)$ yields equation (4.30).

For case 3.1. we have shown (4.29)$_{l_q=1}$ and (4.30) using points of $P_I^{Q(c,s)}$. (4.29)$_{i_q=1}$ follows by symmetry.

By substituting terms in brackets in (4.30) as in the case 3 of Theorem 4.11 we obtain (4.29). Note that l_q is allowed to take the value l_0.

case 3.2: W.l.o.g. assume $i = n$, $j = n$, $k = k_q \neq n$, $l = l_q \neq n$. Equation (4.20) becomes

$$
\begin{aligned}
a_{nnk_ql_q} = a_{11k_ql_q} + a_{n11l_q} + a_{1n1l_q} + a_{nn11} + a_{1nk_q1} + a_{n1k_q1} \\
-2a_{n111} - 2a_{1n11} - 2a_{11k_q1} - 2a_{111l_q} + 3a_{1111}
\end{aligned} \tag{4.45}
$$

The proof of equation (4.45) will be carried out by adopting an alternative representation of points of P_I. Consider sets I and K in reverse roles. That is, set K is the set of rows and set I is the set of elements of the first square. Points x' and $\bar{x}'$ of a collection X are derived by exchanging the elements of two rows (of the *OLS* structures representing points x and $\bar{x}$ respectively) indexed by elements of set K. We define $i(k, j)$ to be the element of the cell at row k and column j of the first latin square. Analogously we define $l(k, j)$ for the second latin square. Thus, equation (4.8) of Proposition 4.7 is modified to

$$
\begin{aligned}
& a_{i(k_1,j_1)j_1k_1l(k_1,j_1)} + a_{i(k_1,j_2)j_2k_1l(k_1,j_2)} + a_{i(k_2,j_1)j_1k_2l(k_2,j_1)} + a_{i(k_2,j_2)j_2k_2l(k_2,j_2)} \\
+ \quad & a_{i(k_2,j_2)j_1k_1l(k_2,j_2)} + a_{i(k_2,j_1)j_2k_1l(k_2,j_1)} + a_{i(k_1,j_2)j_1k_2l(k_1,j_2)} + a_{i(k_1,j_1)j_2k_2l(k_1,j_1)} \\
= \quad & a_{i(k_2,j_1)j_1k_1l(k_2,j_1)} + a_{i(k_2,j_2)j_2k_1l(k_2,j_2)} + a_{i(k_1,j_1)j_1k_2l(k_1,j_1)} + a_{i(k_1,j_2)j_2k_2l(k_1,j_2)} \\
+ \quad & a_{i(k_1,j_2)j_1k_1l(k_1,j_2)} + a_{i(k_1,j_1)j_2k_1l(k_1,j_1)} + a_{i(k_2,j_2)j_1k_2l(k_2,j_2)} + a_{i(k_2,j_1)j_2k_2l(k_2,j_1)}
\end{aligned}
$$

The above equation for a pair of rows (k_1, k_2) and a pair of columns (j_1, j_2) at point x, with $k_1, k_2 \in K$ and $j_1, j_2 \in J$, is denoted as $x((k_1, k_2)_3; (j_1, j_2)_2)$.

First we need to establish the existence of the point $x(k_q)$.

Lemma 4.15. *For $n \geq 5$ and $n \neq 6$ let $i_q, i_1 \in I \setminus \{1, n\}$ with $i_q \neq i_1$, $i_2 \in I \setminus \{i_q, i_1, n\}$, $k_q \in K \setminus \{1, n\}$, $l_0 \in L \setminus \{1, n\}$, $l_1 \in L \setminus \{1, l_0, n\}$. Then there exists the point $x(k_q) \in P_I$ as illustrated in Table 60.*

Table 60: Point $x(k_q)$ (Lemma 4.15)

	1	$\cdots$	n
1	1		i_1
$\vdots$			
k_q	i_q		i_2
$\vdots$			

	1	$\cdots$	n
1	1		l_0
$\vdots$			
k_q	l_0		l_2
$\vdots$			

Proof. Consider *OLS* structures with K as the set of rows, J the set of columns, I the set of the elements of the first square and L the set of the elements of the second square. Consider again the *OLS* pair with the elements of the first row of both squares in natural order and the elements of the first column of the second square in natural order. At this point we apply a fourth index interchange between l_0 and n thus deriving point x^*. Point x^* is illustrated in Table 61. We observe that $i_1, i_2 \neq n$. For $n \geq 5$ there exists $i_q \in I \setminus \{1, i_1, i_2, n\}$. Let

Table 61: Point x^* (Lemma 4.15)

	1	$\cdots$	n
1	1		i_1
$\vdots$			
n	n		i_2

	1	$\cdots$	n
1	1		l_0
$\vdots$			
n	l_0		l_1

$x(k_q) = x^*(i_q \leftrightarrow n)_1(k_q \leftrightarrow n)_3$ with $k_q \in K \setminus \{1, n\}$.

$\square$

(Back to the proof of Theorem 4.14, case 3.2) We observe that k_q can be equal to k_0. At point $x(k_q)$ since $x_{i_1 n 1 l_0} = x_{i_q 1 k_q l_0} = 1$ there exist $k_1 \in K \setminus \{1, k_q\}$, $j_1 \in J \setminus \{1, n\}$ such that $x_{n j_1 k_1 l_0} = 1$. Let $x_1 = x(k_q)(j_1 \neq j_0 ? j_1 \leftrightarrow j_0)_2(k_1 \neq n ? k_1 \leftrightarrow n)_3$. $X_1 \in P_I^{Q(c,s)}$ ($x_{n j_0 n l_0} = 1$). Point x_1 is illustrated in Table 62. Let $x_2^* = x_1(1 \leftrightarrow n)_3$. At point x_2^* we have $x_{i_1 n 1 l_0} = x_{i_q 1 k_q l_0} = 1$ again. So there exist $k_2 \in K \setminus \{1, k_q\}$,

Table 62: Point x_1 (Theorem 4.14, case 3.2)

	1	$\cdots$	j_0	$\cdots$	n
1	1				i_1
$\vdots$					
k_q	i_q				i_2
$\vdots$					
n			n		

	1	$\cdots$	j_0	$\cdots$	n
1	1				l_0
$\vdots$					
k_q	l_0				l_1
$\vdots$					
n			l_0		

$j_2 \in J \setminus \{1, n\}$ such that $x_{n j_2 k_2 l_0} = 1$. Note that $j_2 \neq j_1$. Let $x_2 = x_2^*(j_2 \neq j_0 ? j_2 \leftrightarrow j_0)_2(k_2 \neq n ? k_2 \leftrightarrow n)_3$. $X_2 \in P_I^{Q(c,s)}$ ($x_{n j_0 n l_0} = 1$). Point x_2 is illustrated in Table 63.

Table 63: Point x_2 (Theorem 4.14, case 3.2)

	1	$\cdots$	j_0	$\cdots$	n
1	n				i_1
$\vdots$					
k_q	i_q				i_2
$\vdots$					
n			n		

	1	$\cdots$	j_0	$\cdots$	n
1	1				l_0
$\vdots$					
k_q	l_0				l_1
$\vdots$					
n			l_0		

$x_1((1,k_q)_3;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{1111}+a_{i_1n1l_0}+a_{i_q1k_ql_0}+a_{i_2nk_ql_1}+a_{i_211l_1}+a_{i_qn1l_0}+a_{i_11k_ql_0}+a_{1nk_q1}\\
= \quad & a_{i_111l_0}+a_{1n11}+a_{i_21k_ql_1}+a_{i_qnk_ql_0}+a_{i_q11l_0}+a_{i_2n1l_1}+a_{11k_q1}+a_{i_1nk_ql_0}
\end{aligned}
$$

$x_2((1,k_q)_3;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{n111}+a_{i_1n1l_0}+a_{i_q1k_ql_0}+a_{i_2nk_ql_1}+a_{i_211l_1}+a_{i_qn1l_0}+a_{i_11k_ql_0}+a_{nnk_q1}\\
= \quad & a_{i_111l_0}+a_{nn11}+a_{i_21k_ql_1}+a_{i_qnk_ql_0}+a_{i_q11l_0}+a_{i_2n1l_1}+a_{n1k_q1}+a_{i_1nk_ql_0}
\end{aligned}
$$

$x_1((1,k_q)_3;(1,n)_2)$-$x_2((1,k_q)_3;(1,n)_2)$ yields (4.45)$_{l_q=1}$. (4.45)$_{k_q=1}$ follows by symmetry.

To show (4.45) we consider $l_q \in L\setminus\{1,l_0,l_1,n\}$. Obviously for $n\geq 5$ such an l_q exists. Let $x_3 = x_2(1\leftrightarrow l_q)_4$ (Table 64) and $x_4 = x_1(1\leftrightarrow l_q)_4$ (Table 65). $X_3, X_4 \in P_I^{Q(c,s)}$ $(x_{nj_0nl_0}=1)$.

Table 64: Point x_3 (Theorem 4.14, case 3.2)

	1	$\cdots$	j_0	$\cdots$	n
1	n				i_1
$\vdots$					
k_q	i_q				i_2
$\vdots$					
n			n		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				l_0
$\vdots$					
k_q	l_0				l_1
$\vdots$					
n			l_0		

Table 65: Point x_4 (Theorem 4.14, case 3.2)

	1	$\cdots$	j_0	$\cdots$	n
1	1				i_1
$\vdots$					
k_q	i_q				i_2
$\vdots$					
n			n		

	1	$\cdots$	j_0	$\cdots$	n
1	l_q				l_0
$\vdots$					
k_q	l_0				l_1
$\vdots$					
n			l_0		

$x_3((1,k_q)_3;(1,n)_2)\Rightarrow$

$$
\begin{aligned}
& a_{n11l_q}+a_{i_1n1l_0}+a_{i_q1k_ql_0}+a_{i_2nk_ql_1}+a_{i_211l_1}+a_{i_qn1l_0}+a_{i_11k_ql_0}+a_{nnk_ql_q}\\
= \quad & a_{i_111l_0}+a_{nn1l_q}+a_{i_21k_ql_1}+a_{i_qnk_ql_0}+a_{i_q11l_0}+a_{i_2n1l_1}+a_{n1k_ql_q}+a_{i_1nk_ql_0}
\end{aligned}
$$

$x_4((1,k_q)_3;(1,n)_2)\Rightarrow$

$$\begin{aligned} & a_{111l_q}+a_{i_1n1l_0}+a_{i_q1k_ql_0}+a_{i_2nk_ql_1}+a_{i_211l_1}+a_{i_qn1l_0}+a_{i_11k_ql_0}+a_{1nk_ql_q} \\ = \quad & a_{i_111l_0}+a_{1n1l_q}+a_{i_21k_ql_1}+a_{i_qnk_ql_0}+a_{i_q11l_0}+a_{i_2n1l_1}+a_{11k_ql_q}+a_{i_1nk_ql_0} \end{aligned}$$

$x_3((1,k_q)_3;(1,n)_2)$-$x_4((1,k_q)_3;(1,n)_2)\Rightarrow$

$$a_{nnk_ql_q}=(a_{nn1l_q}+a_{n1k_ql_q}+a_{1nk_ql_q})+a_{111l_q}-a_{n11l_q}-a_{1n1l_q}-a_{11k_ql_q} \tag{4.46}$$

By substituting terms in brackets from (4.45)$_{k_q=1}$, (4.20) with $(i,j,k,l)=(n,1,k_ql_q)$ and (4.20) with $(i,j,k,l)=(1,n,k_ql_q)$ (the two last equations were shown implicitly in case 2) we derive equation (4.45).

The proof of case 3 is now complete. Note that in case 3.2 where k_q can be equal to k_0, l_q is restricted from taking the value l_0 due to $l(1,n)=l_0$, $l(k_q,1)=l_0$ for all x_1,x_2,x_3,x_4.

Conclusively, by reversing the roles of the sets, we can show equation (4.20) with any other pair of indices being equal to (n,n), including cases where $i=n$.

case 4: Three of the indices are equal to n.

W.l.o.g. assume $i=n$, $j=n$, $k=k_q\neq n$, $l=n$. Unlike the previous cases, we will explicitly show equation (4.42). Let $x_1=x_q^3(1\leftrightarrow j_0)_2(i_q\leftrightarrow n)_1(k_q\leftrightarrow k_2)_3(l_q\neq l_0?l_q\leftrightarrow l_0)_4$. We distinguish two cases, viz. $k(1,n)=n$, $k(1,n)=k_1\neq n$.

case 4.1: $k(1,n)=n$.

W.l.o.g. let $l(1,j_1)=n$, and $l(n,j_2)=1$ with $j_1,j_2\in J\setminus\{j_0,n\}$. We note that we can have $j_1=j_2$. Let $x_2=x_1(1\leftrightarrow n)_4$. Points x_1, x_2 are illustrated in Table 66 and 67 respectively.

Table 66: Point x_1 (Theorem 4.11, case 4.1)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1						1		n
$\vdots$								
n						n		k_q

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1		n				1		l_1
$\vdots$								
n				1		l_0		n

Since $x_1,x_2\in P_I^{Q(c,s)}$ ($x_{nj_0nl_0}=1$) we have $ax_1=ax_2$. We will write this equation by adopting the following notation. At point x_1 let $j(i,l)$ denotes the column that defines the cell, at row i, which contains element l at the second latin square . The corresponding element at the first latin square is defined as $k(i,l)$. In this format

Table 67: Point x_2 (Theorem 4.11, case 4.1)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1						1		n
$\vdots$								
n						n		k_q

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1		1				n		l_1
$\vdots$								
n				n		l_0		1

$j(1,n) = j_1$ and $j(n,1) = j_2$. Thus $ax_1 = ax_2$, after canceling out identical terms, becomes

$$\begin{aligned}
& \sum\{a_{ij(i,1)k(i,1)1} : i \in I \setminus \{1,n\}\} \quad + a_{1j_011} + a_{nj(n,1)k(n,1)1} \\
+ \quad & \sum\{a_{ij(i,n)k(i,n)n} : i \in I \setminus \{1,n\}\} \quad + a_{1j(1,n)k(1,n)n} + a_{nnk_qn} \\
= \quad & \\
& \sum\{a_{ij(i,1)k(i,1)n} : i \in I \setminus \{1,n\}\} \quad + a_{1j_01n} + a_{nj(n,1)k(n,1)n} \\
+ \quad & \sum\{a_{ij(i,n)k(i,n)1} : i \in I \setminus \{1,n\}\} \quad + a_{1j(1,n)k(1,n)1} + a_{nnk_q1}
\end{aligned} \tag{4.47}$$

Equation (4.47) includes only one term with three indices equal to n (a_{nnk_qn}) since both at points x_1 and x_2 we cannot have pair $(n,n) \in K \times L$ at the row n or column n ($x_{nj_0nl_0} = x_{1nnl_1} = 1$ at both points). We also observe that since $j(n,1) = j_2 \neq n$ term $a_{nj(n,1)k(n,1)n} \neq a_{nj_0k_0n}$. Therefore, the set of indices of every term of (4.47), except a_{nnk_qn}, has been considered in one of the previous cases. That is, we have shown (4.42) for every term of (4.47), except a_{nnk_qn}. Substituting only terms in summands and $a_{nj(n,1)k(n,1)1}$, $a_{1j(1,n)k(1,n)n}$, $a_{nj(n,1)k(n,1)n}$, $a_{1j(1,n)k(1,n)1}$ from (4.42) in (4.47) and canceling out identical terms yields:

$$\begin{aligned}
& \sum\{\lambda^1_{k(i,1)1} + \lambda^1_{k(i,n)n} + \lambda^3_{j(i,1)1} + \lambda^3_{j(i,n)n} : i \in I\} \\
- \quad & (\lambda^1_{k(1,1)1} + \lambda^1_{k(n,n)n} + \lambda^3_{j(1,1)1} + \lambda^3_{j(n,n)n}) + \lambda^2_{n1} + \lambda^2_{1n} \quad + a_{1j_011} + a_{nnk_qn} \\
= \quad & \sum\{\lambda^1_{k(i,1)n} + \lambda^1_{k(i,n)1} + \lambda^3_{j(i,1)n} + \lambda^3_{j(i,n)1} : i \in I\} \\
- \quad & (\lambda^1_{k(n,n)1} + \lambda^1_{k(1,1)n} + \lambda^3_{j(n,n)1} + \lambda^3_{j(1,1)n}) + \lambda^2_{nn} + \lambda^2_{11} \quad + a_{1j_01n} + a_{nnk_q1}
\end{aligned} \tag{4.48}$$

(4.48) is derived by adding and subtracting to (4.47) terms in brackets, so that the index of summands runs for all values of the set I. It is easy to see that summands cancel out since

$$\begin{aligned}
\sum\{\lambda^t_{k(i,1)1} : i \in I\} &= \sum\{\lambda^t_{k(i,n)1} : i \in I\},\ t \in \{1,3\} &(4.49)\\
\sum\{\lambda^t_{k(i,1)n} : i \in I\} &= \sum\{\lambda^t_{k(i,n)n} : i \in I\}\ t \in \{1,3\} &(4.50)\\
\sum\{\lambda^t_{j(i,1)1} : i \in I\} &= \sum\{\lambda^t_{j(i,n)1} : i \in I\},\ t \in \{1,3\} &(4.51)\\
\sum\{\lambda^t_{j(i,1)n} : i \in I\} &= \sum\{\lambda^t_{j(i,n)n} : i \in I\}\ t \in \{1,3\} &(4.52)
\end{aligned}$$

This is because the n-tuple $(k(i,1))_{i\in I}$ ($(j(i,1))_{i\in I}$) is an array containing all elements of $\{1,...,n\}$ in some order. The same is true for $(k(i,n))_{i\in I}$ ($(j(i,n))_{i\in I}$). If we consider the two n-tuples as unordered we have $(k(i,1))_{i\in I} \equiv (k(i,n))_{i\in I} \equiv (1,...,n)$ ($(j(i,1))_{i\in I} \equiv (j(i,n))_{i\in I} \equiv (1,...,n)$). Hence equations (4.49),..., (4.52) are valid. Canceling out identical terms and dropping the $j(i,l)$ and $k(i,l)$ notation by substituting from the "actual" elements of x_1, x_2, (4.48) becomes

$$\begin{aligned}
a_{nnk_qn} &= a_{1j_01n} + a_{nnk_q1} - a_{1j_011}\\
&+ \lambda^1_{11} + \lambda^1_{k_qn} + \lambda^3_{j_01} + \lambda^3_{nn} + \lambda^2_{nn} + \lambda^2_{11}\\
&- (\lambda^1_{k_q1} + \lambda^1_{1n} + \lambda^3_{n1} + \lambda^3_{j_0n}) - \lambda^2_{n1} - \lambda^2_{1n} \qquad (4.53)
\end{aligned}$$

By substituting terms $a_{1j_01n}, a_{nnk_q1}, a_{1j_011}$ from (4.42) and canceling out identical terms (4.53) yields $a_{nnk_qn} = \lambda^1_{k_qn} + \lambda^2_{nn} + \lambda^3_{nn} + \lambda^4_{nn} + \lambda^5_{nk_q} + \lambda^6_{nk_q}$ which is (4.42) for $(i,j,k,l) = (n,n,k_q,n)$.

case 4.2: $k(1,n) = k_1 \neq n$.

W.l.o.g. let $l(1,j_1) = n$, and $l(n,j_2) = l_1$ with $j_1, j_2 \in J \setminus \{j_0, n\}$. We note that we can have $j_1 = j_2$. Let $x_2 = x_1(l_1 \leftrightarrow n)_4$. Points x_1, x_2 are illustrated in Table 68 and 69 respectively.

Table 68: Point x_1 (Theorem 4.11, case 4.2)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1						1		k_1
$\vdots$								
n						n		k_q

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1		n				1		l_1
$\vdots$								
n				l_1		l_0		n

As in the previous case since $x_1, x_2 \in P_I^{Q(c,s)}$ ($x_{nj_0nl_0} = 1$) we have $ax_1 = ax_2$. By adopting the same

Table 69: Point x_2 (Theorem 4.11, case 4.2)

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1						1		k_1
$\vdots$								
n						n		k_q

	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	j_0	$\cdots$	n
1		l_1				1		n
$\vdots$								
n				n		l_0		l_1

notation as in case 4.1 and after canceling out identical terms, $ax_1 = ax_2$ becomes

$$\begin{aligned}
& \textstyle\sum\{a_{ij(i,l_1)k(i,l_1)l_1} : i \in I \setminus \{1,n\}\} && +a_{1nk_1l_1} + a_{nj(n,l_1)k(n,l_1)l_1} \\
+ \quad & \textstyle\sum\{a_{ij(i,n)k(i,n)n} : i \in I \setminus \{1,n\}\} && +a_{1j(1,n)k(1,n)n} + a_{nnk_qn} \\
= \quad & \\
& \textstyle\sum\{a_{ij(i,l_1)k(i,l_1)n} : i \in I \setminus \{1,n\}\} && +a_{1nk_1n} + a_{nj(n,l_1)k(n,l_1)n} \\
+ \quad & \textstyle\sum\{a_{ij(i,n)k(i,n)l_1} : i \in I \setminus \{1,n\}\} && +a_{1j(1,n)k(1,n)l_1} + a_{nnk_ql_1}
\end{aligned} \tag{4.54}$$

We observe again that all terms of (4.54), except a_{nnk_qn}, can be substituted from (4.42). As in the previous case this yields the desired result.

Note that in both sub-cases of case 4 k_q can be equal to k_0. Reversing the roles of the sets and following the same procedure we prove (4.42) for the remaining three cases: $(i,j,k,l) = (n,n,n,l_q)$, $(i,j,k,l) = (n,j_q,n,n)$ and $(i,j,k,l) = (i_q,n,n,n)$ where $i_q \in I \setminus \{n\}$, $j_q \in J \setminus \{n\}$, $k_q \in K \setminus \{n\}$, $l_q \in L \setminus \{n\}$.

Our proof with respect to $(i,j,k,l) \in C \setminus Q(c,s)$ is now complete. We proceed by considering $(i,j,k,l) \in Q(c,s)$. We define

$$\pi_{ijkl} = a_{ijkl} - (\lambda^1_{kl} + \lambda^2_{il} + \lambda^3_{jl} + \lambda^4_{ij} + \lambda^5_{jk} + \lambda^6_{ik})$$

To prove (4.42) we have to show that all π_{ijkl} are equal. Note that for $(i,j,k,l) \in Q((n,n,n,n),(n,j_0,k_0,l_0))$ we only have four terms, i.e. π_{nnnn}, $\pi_{nnk_0l_0}$, $\pi_{nj_0nl_0}$, $\pi_{nj_0k_0n}$. From point x_0 of Lemma 4.5 we derive point x as follows: $x = x_0(i_0 \leftrightarrow n)_1(1 \leftrightarrow i_0)_1(j_0 \leftrightarrow n)_2(1 \leftrightarrow j_0)_2(1 \leftrightarrow n)_4(1 \leftrightarrow k_0)_3(1 \leftrightarrow n)_3$. We consider two cases

case 5: $k_0 \neq k_1$ at point x_0.

Then at point x let $j_1, j_2 \in J \setminus \{j_0, n\}$ with $j_1 \neq j_2$ such that $k(i_0,j_1) = k_2$ and $k(i_0,j_2) = n$. Let also $l(i_0,j_1) = l_t$ and $l(i_0,j_2) = l_s$. Point x is illustrated in Table 70. Let $x' = x(i_0 \leftrightarrow n)_1$ (see Table 71). Clearly $x \in P_I^{Q(c,s)}$ since

Table 70: Point x (Theorem 4.14, case 5)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		k_0		k_2		n		k_1
$\vdots$								
n		n						k_2

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		n		l_t		l_s		l_1
$\vdots$								
n		l_0						n

Table 71: Point x' (Theorem 4.14, case 5)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		n						k_2
$\vdots$								
n		k_0		k_2		n		k_1

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		l_0						n
$\vdots$								
n		n		l_t		l_s		l_1

$x_{nj_0nl_0} = 1$ and $x' \in P_I^{Q(c,s)}$ since $x_{nj_0k_0n} = 1$. Thus $ax = ax' \Rightarrow$

$$\begin{aligned} &\sum\{a_{i_0jk^x(i_0,j)l^x(i_0,j)} + a_{njk^x(n,j)l^x(n,j)} : j \in J\} \\ = &\sum\{a_{i_0jk^x(n,j)l^x(n,j)} + a_{njk^x(i_0,j)l^x(i_0,j)} : j \in J\} \end{aligned} \tag{4.55}$$

Let $\bar{x} = x(k_2 \leftrightarrow n)_3$ (see Table 72). Let $\bar{x}' = \bar{x}(i_0 \leftrightarrow n)_1$ (see Table 73). Note that point $\bar{x} \in P_I^{Q(c,s)}$ since

Table 72: Point $\bar{x}$ (Theorem 4.14, case 5)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		k_0		n		k_2		k_1
$\vdots$								
n		k_2						n

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		n		l_t		l_s		l_1
$\vdots$								
n		l_0						n

$x_{nnnn} = 1$ and $\bar{x}' \in P_I^{Q(c,s)}$ since $x_{nj_0k_0n} = 1$. Therefore $a\bar{x} = a\bar{x}' \Rightarrow$

$$\begin{aligned} &\sum\{a_{i_0jk^{\bar{x}}(i_0,j)l^{\bar{x}}(i_0,j)} + a_{njk^{\bar{x}}(n,j)l^{\bar{x}}(n,j)} : j \in J\} \\ = &\sum\{a_{i_0jk^{\bar{x}}(n,j)l^{\bar{x}}(n,j)} + a_{njk^{\bar{x}}(i_0,j)l^{\bar{x}}(i_0,j)} : j \in J\} \end{aligned} \tag{4.56}$$

We observe that $k^x(i,j) = k^{\bar{x}}(i,j)$, $l^x(i,j) = l^{\bar{x}}(i,j)$ for $i = i_0,\ j \in J \setminus \{j_1, j_2\}$ and $i = n,\ j \in J \setminus \{j_0, n\}$. Thus

Table 73: Point $\bar{x}'$ (Theorem 4.14, case 5)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		k_2						n
$\vdots$								
n		k_0		n		k_2		k_1

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	j_2	$\cdots$	n
$\vdots$								
i_0		l_0						n
$\vdots$								
n		n		l_t		l_s		l_1

(4.55)-(4.56)$\Rightarrow$

$$\begin{aligned} & a_{i_0j_1k_2l_t} + a_{i_0j_2nl_s} + a_{nj_0nl_0} + a_{nnk_2n} + a_{i_0j_0k_2l_0} + a_{i_0nnn} + a_{nj_1nl_t} + a_{nj_2k_2l_s} \\ = & a_{i_0j_1nl_t} + a_{i_0j_2k_2l_s} + a_{nj_0k_2l_0} + a_{nnnn} + a_{i_0j_0nl_0} + a_{i_0nk_2n} + a_{i_0j_1k_2l_t} + a_{nj_2nl_s} \end{aligned}$$

If we substitute in the equation above all terms from (4.42) we derive $\pi_{nnnn} = \pi_{nj_0nl_0} = \pi$.

Then we write (4.55) as

$$\begin{aligned} & \sum\{a_{i_0jk^x(i_0,j)l^x(i_0,j)} : j \in J\} + \sum\{a_{njk^x(n,j)l^x(n,j)} : j \in J \setminus \{j_0\}\} + a_{nj_0nl_0} \\ = & \sum\{a_{i_0jk^x(n,j)l^x(n,j)} : j \in J\} + \sum\{a_{njk^x(i_0,j)l^x(i_0,j)} : j \in J \setminus \{j_0\}\} + a_{nj_0k_0n} \end{aligned}$$

By substituting from (4.42) and canceling out terms we obtain

$$\begin{aligned} & \sum\{\lambda^2_{i_0l(i_0,j)} + \lambda^2_{nl(n,j)} + \lambda^6_{i_0k(i_0,j)} + \lambda^6_{nk(n,j)} : j \in J\} + \pi_{nj_0nl_0} \\ = & \sum\{\lambda^2_{i_0l(n,j)} + \lambda^2_{nl(i_0,j)} + \lambda^6_{i_0k(n,j)} + \lambda^6_{nk(i_0,j)} : j \in J\} + \pi_{nj_0k_0n} \end{aligned}$$

Since i_0, n indicate rows of a latin square, and considering each such row as an unordered n-tuple with respect to the values of the cells of that row, then these unordered n-tuples are equivalent, i.e. $(k(i_0,1),...,k(i_0,n)) \equiv (k(n,1),...,k(n,n))$ and $(l(i_0,1),...,l(i_0,n)) \equiv (l(n,1),...,l(n,n))$. Therefore, the above equation becomes $\pi_{nj_0nl_0} = \pi_{nj_0k_0n} = \pi$.

Let $x_1 = \bar{x}(l_0 \leftrightarrow n)_4(k_0 \leftrightarrow n)_3$ and $x_2 = \bar{x}'(l_0 \leftrightarrow n)_4(k_0 \leftrightarrow n)_3$. Clearly $x_1, x_2 \in P_I^{Q(c,s)}$ since for x_1 we have $x_{nnk_0l_0} = 1$ and for x_2 we have $x_{nj_0nl_0} = 1$. Therefore, if we substitute in $ax_1 = ax_2$ terms from (4.42) and take into account the equivalent terms derived from the latin square property of the rows we get $\pi_{nnk_0l_0} = \pi_{nj_0nl_0} = \pi$.

case 6: $k_0 = k_1$ at point x_0.

Let $j_1 \in J \setminus \{j_0, n\}$ be such that $k(i_0, j_1) = k_2$. We denote $l(i_0, j_1)$ as l_t and point x is illustrated in Table 74. Let $x' = x(i_0 \leftrightarrow n)_1$ (see Table 75). Then $x, x' \in P_I^{Q(c,s)}$ since for point x we have $x_{nj_0nl_0} = 1$ and for point x' we

Table 74: Point x (Theorem 4.14, case 6)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		k_0		k_2		n
$\vdots$						
n		n				k_2

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		n		l_t		l_1
$\vdots$						
n		l_0				n

Table 75: Point x' (Theorem 4.14, case 6)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		n				k_2
$\vdots$						
n		k_0		k_2		n

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		l_0				n
$\vdots$						
n		n		l_t		l_1

have $x_{nj_0k_0n} = 1$.

Let $\bar{x} = x(k_2 \leftrightarrow n)_3$ and $\bar{x}' = \bar{x}(i_0 \leftrightarrow n)_1$. Points $\bar{x}, \bar{x}'$ are illustrated in Table 76 and Table 77 respectively.

Table 76: Point $\bar{x}$ (Theorem 4.14, case 6)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		k_0		n		k_2
$\vdots$						
n		k_2				n

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		n		l_t		l_1
$\vdots$						
n		l_0				n

Clearly $\bar{x}, \bar{x}' \in P_I^{Q(c,s)}$ since for point $\bar{x}$ we have $x_{nnnn} = 1$ and for point $\bar{x}'$ we have $x_{nj_0k_0n} = 1$. Therefore $ax - a\bar{x} = ax' - a\bar{x}' \Rightarrow$

$$a_{i_0j_1k_2l_t} + a_{i_0nnl_1} + a_{nj_0nl_0} + a_{nnk_2n} + a_{i_0j_0k_2l_0} + a_{i_0nnn} + a_{nj_1nl_t} + a_{nnk_2l_1}$$
$$= a_{i_0j_1nl_t} + a_{i_0nk_2l_1} + a_{nj_0k_2l_0} + a_{nnnn} + a_{i_0j_0nl_0} + a_{i_0nk_2n} + a_{nj_1k_2l_t} + a_{nnnl_1}$$

Again by substituting all terms from (4.42) we get $\pi_{nnnn} = \pi_{nj_0nl_0} = \pi$. In a similar manner as in case 5 we derive $\pi_{nnk_0l_0} = \pi_{nj_0k_0n} = \pi_{nj_0nl_0} = \pi$.

Finally, (4.43) is true since we have shown that for $n \geq 5$ and $n \neq 6$ $P_I^{Q(c,s)} \neq \emptyset$.

$\square$

Proposition 4.16. *The inequalities (4.41) are of rank 2.*

Table 77: Point $\bar{x}'$ (Theorem 4.14, case 6)

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		k_2				n
$\vdots$						
n		k_0		n		k_2

	$\cdots$	j_0	$\cdots$	j_1	$\cdots$	n
$\vdots$						
i_0		l_0				n
$\vdots$						
n		n		l_t		l_1

Proof. The proof will essentially proceed in the same way as the proof of Proposition 4.13. Assume w.l.o.g. $c = (i_0, j_0, k_0, l_0)$ and $s = (i_0, j_1, k_1, l_1)$. The node set of the clique is $Q(c,s) = \{(i_0, j_0, k_0, l_0), (i_0, j_0, k_1, l_1), (i_0, j_1, k_0, l_1), (i_0, j_1, k_1, l_0)\}$, the induced inequality being

$$x_{i_0 j_0 k_0 l_0} + x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_0 l_1} + x_{i_0 j_1 k_1 l_0} \leq 1 \tag{4.57}$$

If the inequality (4.57) is of Chvátal rank 1, then there exists $0 < \epsilon < 1$, such that every solution to the LP-relaxation of *OLS*, i.e. $Ax = e$, $x \geq 0$, satisfies:

$$x_{i_0 j_0 k_0 l_0} + x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_0 l_1} + x_{i_0 j_1 k_1 l_0} \leq 2 - \epsilon \tag{4.58}$$

However, any solution having $x_{i_0 j_0 k_0 l_0} = x_{i_0 j_0 k_1 l_1} = x_{i_0 j_1 k_0 l_1} = x_{i_0 j_1 k_1 l_0} = \frac{1}{2}$ violates (4.58). We will show that such a solution always exists for $n \geq 3$, distinguishing between the cases where n is even and n is odd, thus proving that the rank of the inequality (4.57) is at least 2.

If n is even then for each $(i_v, j_u) \in I \times J$ define two possible value pairs: either $\{k(i_v, j_u), l(i_v, j_u)\} = \{y_1, z_1\}$ or $\{k(i_v, j_u), l(i_v, j_u)\} = \{y_2, z_2\}$ where

$$y_1 = (2 \cdot \left\lceil \frac{v}{2} \right\rceil + 2 \cdot \left\lfloor \frac{u}{2} \right\rfloor) \bmod n,\ z_1 = (y_1 + 2 \cdot \left\lfloor \frac{v}{2} \right\rfloor + u - 2 \cdot \left\lfloor \frac{u}{2} \right\rfloor) \bmod n,$$

and

$$y_2 = (2 \cdot \left\lceil \frac{v}{2} \right\rceil + 2 \cdot \left\lfloor \frac{u}{2} \right\rfloor + 1) \bmod n,\ z_2 = (y_2 + 2 \cdot \left\lfloor \frac{v}{2} \right\rfloor - u + 2 \cdot \left\lfloor \frac{u}{2} \right\rfloor) \bmod n.$$

The proposed solution is:

$$x_{i_v j_u k_y l_z} = \begin{cases} \frac{1}{2},\ \text{if } y = y_1, z = z_1 \text{ or } y = y_2, z = z_2 \\ 0,\ \text{otherwise} \end{cases}$$

for all $(i_v, j_u) \in I \times J$. It is easy to see that $v, u, z, y \in \{0, 1, 2, ..., n-1\}$.

To illustrate this solution with respect to an *OLS* structure, assume that $I = J = K = L = \{0, 1, ..., n-1\}$, and n even. I is the row set, J is the column set and K, L are the value sets of the first and the second latin square, respectively. For each cell $(i, j) \in I \times J$, we define:

$$k_1 = (2 \cdot \left\lceil \frac{i}{2} \right\rceil + 2 \cdot \left\lfloor \frac{j}{2} \right\rfloor) \bmod n,\ l_1 = (k_1 + 2 \cdot \left\lfloor \frac{i}{2} \right\rfloor + j - 2 \cdot \left\lfloor \frac{j}{2} \right\rfloor) \bmod n,$$

and

$$k_2 = (2 \cdot \left\lceil \frac{i}{2} \right\rceil + 2 \cdot \left\lfloor \frac{j}{2} \right\rfloor + 1) \bmod n,\ l_2 = (k_2 + 2 \cdot \left\lfloor \frac{i}{2} \right\rfloor - j + 2 \cdot \left\lfloor \frac{j}{2} \right\rfloor) \bmod n.$$

The solution must satisfy:

$$x_{ijkl} = \begin{cases} \frac{1}{2}, \text{ if } k = k_1, l = l_1 \text{ or } k = k_2, l = l_2 \\ 0, \text{ otherwise} \end{cases}$$

The non zero variables for $n = 6$ are depicted at Table 78. Pair (k, l) placed at cell (i, j) implies that $x_{ijkl} = \frac{1}{2}$. The

Table 78: A solution violating (4.58) for $n = 6$

	0	1	2	3	4	5
0	00,11	01,10	22,33	23,32	44,55	45,54
1	22,33	23,32	44,55	45,54	00,11	01,10
2	24,35	25,34	40,51,	41,50	02,13	03,12
3	40,51	41,50	02,13	03,12	24,35	25,34
4	42,53	43,52	04,15	05,14	20,31	21,30
5	04,15	05,14	20,31	21,30	42,53	43,52

marginal row values are the values of index i, whereas the marginal column values are the ones of index j. It is easy to verify that exactly two variables at the left-hand side of each constraint are set to $\frac{1}{2}$ by checking that each value of index k (or l) appears exactly twice in each row/column, each pair (k, l) appears at exactly two cells and each cell has exactly two non-zero variables.

If n is odd, the solution does not follow a pattern as concrete as above, therefore it can be better described in the format used in Table 78. Assume again w.l.o.g. that $I = J = K = L = \{0, 1, ..., n\}$, n odd and construct a square matrix, with the marginal values on rows and columns corresponding to elements of sets I and J, respectively. The matrix is symmetric, with respect to the main bottom-left to top-right diagonal (i.e. the diagonal $(n-1, 0), (n-2, 1), ..., (0, n-1)$) and is illustrated in Table 79.

The non-zero x_{0jkl} variables (i.e. the ones in the first row) are fixed exactly as in the case of n being even, except for

Table 79: A solution violating (4.58) for n odd

	0	1			n-3	n-2	n-1
0	00,11	01,10	...	...	(n-3)(n-3),(n-2)(n-2)	(n-3)(n-2),(n-2)(n-3)	(n-1)(n-1)
1	(n-5)(n-3),(n-4)(n-2)		...	...	(n-1)0,(n-1)1		(n-3)(n-2),(n-2)(n-3))
2							(n-3)(n-3),(n-2)(n-2)
...	...	...	...	...	...	...	...
n-2	(n-1)2,(n-1)3		...	...			01,10
n-1						(n-5)(n-3),(n-4)(n-2)	00,11

variable $x_{0(n-1)(n-1)(n-1)}$, the single variable set to 1. The non-zero $x_{i(n-1)kl}$ variables (i.e. the ones in the last column) are fixed in the manner posed by the above mentioned symmetry. Excluding the first row and the last column, we are left with a $(n-1) \times (n-1)$ square submatrix (of even size), which can be split into $\frac{(n-1)^2}{4}$ 2×2 submatrices.

Each such 2×2 submatrix, which is "above" the main diagonal, involves two pairs of consecutive indices from each of the sets K, L according to the pattern illustrated at Table 80 (remember all four sets contain elements $\{0, 1, ..., n-1\}$),

Table 80:

$yz, (y+1)(z+1)$	$zy, (z+1)(y+1)$
$(z+1)y, z(y+1)$	$y(z+1), (y+1)z$

where $y, z = 0, 2, ..., n-3$.

It is therefore enough to define only the upper left element of each such submatrix, appearing at rows $1, 3, ..., n-4$ and columns $0, 2, ..., n-5$. At row i and column j ($i \leq n-4$ and odd, $j \leq n-5$ and even, $i + j \leq n-4$), there must be $y = n - (i + j) - 4$ and $z = y + n - i$. If $z > n-3$, then z is replaced by $z \bmod n-3$.

The remaining 2×2 submatrices are the last to be filled according to the pattern illustrated at Table 81, where $y =$

Table 81:

$(n-1)y, (n-1)(y+1)$	$y(n-1), (y+1)(n-1)$
$y(n-1), (y+1)(n-1)$	$(n-1)y, (n-1)(y+1)$

$0, 2, ..., n-3$. Note that the top-right and bottom-left cells are on the diagonal induced by the cells $(n-1, 0), (n-2, 1), ..., (0, n-1)$. The value of y must be the one appearing only once in that column, after all other cells have been filled. A solution violating inequality (4.58) for $n = 7$ is illustrated in Table 82. It is easy to check that exactly two variables at the left-hand side of each constraint are set to $\frac{1}{2}$ or exactly one variable (i.e. x_{0666}) is set to 1.

We have shown that the rank of (4.57) is at least two. Now we will show that the rank is at most 2, by deriving (4.57) as a linear combination of rank 1 inequalities. Adding the rows $(i_0, j_0), (i_0, k_0), (i_0, l_1)$, each one weighted by $\frac{1}{2}$, gives an inequality, where variables $x_{i_0 j_0 k_0 l_0}, x_{i_0 j_0 k_1 l_1}$ and $x_{i_0 j_1 k_0 l_1}$ appear with coefficient 1, variable $x_{i_0 j_0 k_0 l_1}$ appears with coefficient $\frac{3}{2}$ and all other variables appear with coefficient $\frac{1}{2}$. The r.h.s. is $\frac{3}{2}$. Rounding down both sides results in the inequality:

Table 82: A solution violating (4.58) for $n = 7$.

	0	1	2	3	4	5	6
0	00,11	01,10	22,33	23,32	44,55	45,54	66
1	24,35	42,53	02,13	20,31	60,61	06,16	45,54
2	43,52	25,34	21,30	03,12	06,16	60,61	44,55
3	04,15	40,51	64,65	46,56	03,12	20,31	23,32
4	50,41	05,14	46,56	64,65	21,30	02,13	22,33
5	62,63	26,36	05,14	40,51	25,34	42,53	01,10
6	26,36	62,63	50,41	04,15	43,52	24,35	00,11

$$x_{i_0 j_0 k_0 l_0} + x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_0 l_1} + x_{i_0 j_0 k_0 l_1} \leq 1 \tag{4.59}$$

Applying the same procedure to rows $(i_0, j_0), (i_0, k_1), (i_0, l_0)$ results in the inequality:

$$x_{i_0 j_0 k_0 l_0} + x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_1 l_0} + x_{i_0 j_0 k_1 l_0} \leq 1 \tag{4.60}$$

Applying the same procedure to rows $(i_0, j_1), (i_0, k_0), (i_0, l_0)$ results in the inequality:

$$x_{i_0 j_0 k_0 l_0} + x_{i_0 j_1 k_0 l_1} + x_{i_0 j_1 k_1 l_0} + x_{i_0 j_1 k_0 l_0} \leq 1 \tag{4.61}$$

Applying the same procedure to rows $(i_0, j_1), (i_0, k_1), (i_0, l_1)$ results in the inequality:

$$x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_0 l_1} + x_{i_0 j_1 k_1 l_0} + x_{i_0 j_1 k_1 l_1} \leq 1 \tag{4.62}$$

Adding inequalities (4.59)-(4.62) gives the following inequality:

$$3(x_{i_0 j_0 k_0 l_0} + x_{i_0 j_0 k_1 l_1} + x_{i_0 j_1 k_1 l_0} + x_{i_0 j_1 k_0 l_1}) + (x_{i_0 j_0 k_0 l_1} + x_{i_0 j_0 k_1 l_0} + x_{i_0 j_1 k_0 l_0} + x_{i_0 j_1 k_1 l_1}) \leq 4 \tag{4.63}$$

Dividing inequality (4.63) by 3 and rounding down both sides gives inequality (4.57). This implies that inequality (4.57) is of rank at most 2 and the proof is complete.

$\square$

5 Violated clique inequalities

Facet-defining inequalities are of great importance since they describe the convex hull of integer solutions for a problem. Therefore, if we knew all facet-defining inequalities of an integer polytope, we would be able to solve the integer problem by incorporating them into the constraint matrix and then solving the linear programming relaxation. In practice, however, this is not easy, since for most problems a) not all the facets of the underlying convex hull of integer points are known, and b) the number of facets is not polynomially bounded on the size of the problem, thus yielding a constraint matrix of exponential size. For these reasons most algorithms consider the known facet inequalities only when they are violated by some points of the linear relaxation polytope. Although determining whether an arbitrary non-integer solution violates a facet-defining inequality of the convex hull of integer solutions for an $NP-$ hard problem is generally also $NP-$ hard, it is sometimes possible to do that efficiently for certain classes of facets. In particular, with respect to the *OLS*, we propose two polynomial procedures, one for each class of facet defining cliques, that deal with the problem of detecting a violated clique inequality.

First we give an algorithm that detects a violated facet-defining inequality induced by cliques of type II.

Algorithm 1 Separation of Cliques of type II.

Let $x \in P_L$ and $v \in \mathbb{N}$ such that $v \geq 5$.

Step 1: Set $d_c = 0$ for all $c \in C$.

Step 2: For all $s \in C$ check x_s. If $x_s \geq \frac{1}{vn}$ then set $d_c = d_c + x_s$ for all $c \in Q(s)$. If $d_c > 1$ stop: the inequality $\sum\{x_q : q \in Q(c)\} \leq 1$ is violated. Otherwise continue.

Step 3: For all $c \in C$ if $d_c > \frac{v-4}{v}$ then check whether the inequality $\sum\{x_q : q \in Q(c)\} \leq 1$ is violated. If so stop; otherwise continue.

In order to prove the correctness and complexity of the algorithm we need some intermediate results.

Lemma 5.1. *For a point $x \in P_L$ and a positive integer v, the number of components of x with value $\geq v$ is $\leq \frac{n^2}{v}$.*

Proof. The value of the linear program $L = \max\{ex : x \in P_L\}$ can be easily shown to be n^2 since the vectors $x \in \mathbb{R}^{n^4}$ and $u \in \mathbb{R}^{6n^2}$ defined by $x_c = \frac{1}{n^2}, \forall c \in C$ and $u_r = \frac{1}{6}, \forall r \in R$ are feasible solutions to L and its dual. Therefore they are optimal. Thus if more than $\frac{n^2}{v}$ components of x have values greater than or equal to v then the value of ex would be greater than n^2 contradicting the above.

□

Lemma 5.2. *For any $x \in P_L$ and any positive integer v, the number of $c \in C$, such that $\sum\{x_q : q \in Q(c)\} \geq v$ is $\leq \frac{n^2(4n-3)}{v}$.*

Proof. Consider

$$\sum\{\sum\{x_q : q \in Q(c)\} : c \in C\} \tag{5.1}$$

We know that $\mid Q(c) \mid = 4n-3$ for all $c \in C$ which implies that each x_c appears $4n-3$ times in (5.1). Hence

$$\sum\{\sum\{x_q : q \in Q(c)\} : c \in C\} = (4n-3)\sum\{x_c : c \in C\} \leq (4n-3)n^2$$

If there were more than $\frac{n^2(4n-3)}{v}$ $\sum\{x_q : q \in Q(c)\}$ with a value greater than or equal to v then it would be $\sum\{\sum\{x_q : q \in Q(c)\} : c \in C\} > n^2(4n-3)$ contradicting the above.

□

Theorem 5.3. *Algorithm 1 determines in $O(n^4)$ steps whether a given $x \in P_L$ violates a facet-defining inequality of type II.*

Proof. Let us first prove that the algorithm is correct. Assume that the inequality $\sum\{x_q : q \in Q(c)\} \leq 1$ is violated for some $c \in C$. Then

$$\begin{aligned} &d_c = \sum\{x_q : q \in Q(c), x_q \geq \frac{1}{vn}\} > 1 - \sum\{x_q : q \in Q(c), x_q < \frac{1}{vn}\} \\ &\geq 1 - \frac{4n-3}{vn} \geq \frac{v-4}{v} \end{aligned}$$

Hence, violation is detected at *Step* 3 of the algorithm. Therefore the algorithm is correct.

Let us now examine the complexity of the algorithm. At *Step* 1 we initialize n^4 counters. At *Step* 2 there can be at most vn^3 components of a fractional point x which are examined. For each of these, $4n-3$ counters are updated since there are $4n-3$ nodes in the node set of a clique of type II. So, in the worst case the complexity of *Step* 2 is $vn^3(4n-3)$. At *Step* 3 the number of $c \in C$ for which $\sum\{x_q : q \in Q(c)\} > \frac{v-4}{v}$ is at most $\frac{vn^2(4n-3)}{v-4}$ (Lemma 5.2). For each such c we need $4n-3$ extra steps to check whether the corresponding inequality is indeed violated. Hence, the complexity of *Step* 3 is $\frac{vn^2(4n-3)^2}{v-4}$. Thus, the overall complexity of the algorithm is

$$f(v,n) = n^4 + vn^3(4n-3) + \frac{vn^2(4n-3)^2}{v-4} \tag{5.2}$$

which is $O(n^4)$.

The value of v that minimizes $f(v,n)$ is found by setting the first derivative with respect to v to zero:

$$\frac{\partial f(v,n)}{\partial v} = n(v-4)^4 - 4(4n-3) = 0 \Rightarrow v = 4 + \sqrt{16 - \frac{12}{n}}$$

which for large n produces $v = 8$. For this value of v (5.2) becomes $f(n) = n^4 + 8n^3(4n-3) + 2n^2(4n-3)^2$.

Note that the complexity of the above algorithm remains linear with respect to the number of variables, therefore it is the lowest possible.

□

Now we will give a separation algorithm for cliques of type III.

Algorithm 2 Separation of Cliques of type III

Let $x \in P_L$.

Step 1:For all $c \in C$ if $1 > x_c > \frac{1}{4}$

Step 2: Then for all $t \in C$ with $\mid c \cap t \mid = 2$ if $x_t > \frac{1-x_c}{3}$

Step 3: Then for all $s \in C$, such that $\mid c \cap s \mid = 1$ and $\mid s \cap t \mid = 3$ if $\sum\{x_q : q \in Q(c,s)\} > 1$ stop; otherwise continue.

Theorem 5.4. *Algorithm 2 determines in $O(n^4)$ steps whether a given $x \in P_L$ violates a facet defining inequality of type III.*

Proof. Let us first consider the correctness of the algorithm. $Q(c,s)$ is a node set of a 4-clique thus if a non-integer point x violates $\sum\{x_q : q \in Q(c,s)\} \leq 1$ at least one component of x must be $> \frac{1}{4}$. W.l.o.g. assume that $x_c > \frac{1}{4}$ for $c = (i_0, j_0, k_0, l_0)$. Since $x \in P_L$ then it belongs to the constraints

$$\sum\{x_{ijk_0l_0} : i \in I, j \in j\} = 1 \tag{5.3}$$

$$\sum\{x_{i_0jkl_0} : j \in J, k \in K\} = 1 \tag{5.4}$$

$$\sum\{x_{ij_0kl_0} : i \in I, k \in K\} = 1 \tag{5.5}$$

$$\sum\{x_{i_0j_0kl} : k \in K, l \in L\} = 1 \tag{5.6}$$

$$\sum\{x_{ij_0k_0l} : i \in I, l \in L\} = 1 \tag{5.7}$$

$$\sum\{x_{i_0jk_0l} : j \in J, l \in L\} = 1 \tag{5.8}$$

If $x_c = 1$ then for all $t \in C$ such that $\mid c \cap t \mid = 2$ we have $x_t = 0$. Hence, the inequality $\sum\{x_q : q \in Q(c,s)\} \leq 1$ is satisfied as equality for all $s \in C$ such that $\mid c \cap s \mid = 1$. Therefore if x violates such an inequality the $x_c < 1$ and the range $1 > x_c > \frac{1}{4}$ are correct. Since $\mid Q(c,s) \mid = 4$ the condition $x_t > \frac{1-x_c}{3}$ must hold for at least one $t \in Q(c,s)$. Consequently, algorithm 2 is correct.

Concerning the complexity of the algorithm, we note that the comparison in *Step* 1 is executed in the worst case n^4 times, once for each variable. The number of variables with value $> \frac{1}{4}$ is at most $4n^2$ (Lemma 5.1). For each such variable there are $6(n-1)^2$ $t \in C$ such that $\mid c \cap t \mid = 2$ as indicated by constraints (5.3),...,(5.8). Hence, we need $24n^2(n-1)^2$

comparisons to identify all such ordered pairs (c,t), yielding complexity of $O(n^4)$. For each such c the number of t cannot be more than 3 in each of (5.3),...,(5.8) since otherwise one of these inequalities would be violated. Thus the total number of t given c for which $x_t > \frac{1-x_c}{3}$ is satisfied is 18. For c,t given there are at most $n-1$ $s \in C$ such that $\mid c \cap s \mid = 1$. Thus, *Step* 3 will be executed $4n^2 \times 18 \times (n-1)$ times, i.e. its complexity is of the order $O(n^3)$. Thus, the total complexity is of $O(n^4)$.

□

Corollary 5.5. *Whether there exists a violated clique inequality can be detected in linear time with respect to the number of variables, i.e. in $O(n^4)$ steps.*

□

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