Chromatic Polynomials and Representations of the Symmetric Group

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Abstract

The chromatic polynomials considered in this paper are associated with graphs constructed in the following way. Take n copies of a complete graph K_b and, for $i = 1, 2, \ldots, n$, join each vertex in the ith copy to the same vertex in the $(i+1)$th copy, taking $n + 1 = 1$ by convention. Previous calculations for $b = 2$ and $b = 3$ suggest that the chromatic polynomial contains terms that occur in 'levels'. In the present paper the levels are explained by using a version of the sieve principle, and it is shown that the terms at level ℓ correspond to the irreducible representations of the symmetric group Sym_ℓ. In the case of the two linear representations the terms can be calculated explicitly by methods based on the theory of distance-regular graphs. For the nonlinear representations the calculations are more complicated. An illustration is given in Section 10, where the complete chromatic polynomial for the case $b = 4$ is obtained.

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Chromatic polynomials and representations of the symmetric group

1. Introduction

The chromatic polynomials considered in this paper are associated with graphs, which we call *bracelets*, constructed in the following way. Take n copies of a *base graph*, and join certain vertices in the ith copy to certain vertices in the $(i+1)$th copy, the joins being the same for each i, and $n+1=1$ by convention. Here we take the base graph to be the complete graph K_b, and the joins to be the matching in which each vertex in one copy of K_b is joined to the same vertex in the next copy. This gives a bracelet that we denote by $B_n(b)$. More generally, bracelets can be constructed by allowing the base graph to be incomplete, and by allowing the copies to be joined in a more complicated way [**5**,**7**].

The chromatic polynomials for $B_n(2)$ and $B_n(3)$ are as follows [**3**, **4**, **6**, **7**]:

$$P(B_n(2);k) \;=\; (k^2-3k+3)^n + (k-1)\big((1-k)^n + (3-k)^n\big) + (k^2-3k+1).$$

$$\begin{aligned} P(B_n(3);k) \;=\; & (k^3-6k^2+14k-13)^n \\ & + (k-1)\Big((-k^2+7k-13)^n + 2(-k^2+4k-4)^n\Big) \\ & + k(k-3)/2\Big((k-5)^n + 2(k-2)^n\Big) \\ & + (k-1)(k-2)/2\Big((k-1)^n + 2(k-4)^n\Big) \\ & + (k^3-6k^2+8k-1)(-1)^n. \end{aligned}$$

A partial explanation of the form of these results is provided by the transfer-matrix method, which leads to a formula of the kind

$$P(B_n(b);k) \;=\; \sum_r m_r(k)\lambda_r(k)^n,$$

where $\lambda_r(k)$ and $m_r(k)$ are the eigenvalues and multiplicities of suitable matrices $T(k)$, k being a positive integer. However, the results quoted above suggest that more is true: the terms occur in 'levels', the terms at level ℓ being of the form

$$\big(\text{Polynomial of degree } \ell\big) \times \big(\text{Integer}\big)\big(\text{Polynomial of degree } b-\ell\big)^n.$$

The levels can be explained, in general terms, by using the sieve method introduced in [**5**]. In the present paper the method is applied to the graphs $B_n(b)$, where it turns out that all the required polynomials of degree $b-\ell$ can be derived (in principle) from the representation theory of the symmetric group Sym_ℓ.

There are two reasons why this result is significant. The first is that the polynomials that are raised to the nth power determine the behaviour of the chromatic roots of $B_n(b)$ as $n \to \infty$, for all b. Indeed, a theorem of Beraha, Kahane and Weiss [**2**] can be used to determine the curves that comprise the limit points of the chromatic roots. The second reason is that the same general structure is observed to apply to many other families of bracelets, with different base graphs and different links. Since these graphs correspond to the lattice-like models studied in theoretical physics, information about the limiting behaviour of their chromatic roots is potentially of great interest [**10**,**11**,**12**].

The main result proved here (Theorem 4) is that, for every partition π of ℓ ($\ell \leq b$), the corresponding representation R^π of Sym_ℓ gives rise to a 'collapsed' matrix N^π whose eigenvalues are also eigenvalues of the transfer matrix $T(k)$; and conversely, every eigenvalue of $T(k)$ is an eigenvalue of N^π for some π and ℓ. It follows that the chromatic polynomial of $B_n(b)$ can be written in the form

$$P(B_n(b);k) \;=\; \sum_{\ell=0}^{b} \sum_{|\pi|=\ell} m_\pi(k)\ \mathrm{tr}(N^\pi)^n,$$

for certain polynomials $m_\pi(k)$.

In the case of the two linear representations of Sym_ℓ, the matrices N^π and their spectra can be calculated by methods based on the theory of distance-regular graphs, using the Johnson graphs $J(b,\ell)$. This calculation produces explicitly $d+1$ polynomials associated with the principal representation (where $d = \min(\ell, b-\ell)$), and two polynomials associated with the alternating representation (Theorems 5 and 6). For nonlinear representations the calculations are more complicated. As an illustration, the case $b = 4$, $\ell = 3$ is studied in Section 10, and the complete chromatic polynomial of $B_n(4)$ is obtained. The result agrees with that obtained by Chang [**8**], using a different method.

2. The sieve formula

We take $V = \{1, 2, \ldots, b\}$ as the vertex-set of a complete graph K_b, and regard the set $\{1, 2, \ldots, k\}$, where $k \geq b$, as a set of colours. The set of all proper k-colourings of K_b will be denoted by P_k, and $\mathcal{V}_k$ will denote the vector space of real-valued functions defined on P_k. A basis for $\mathcal{V}_k$ is given by the set of functions $[\gamma]$, $\gamma \in P_k$, such that $[\gamma](\alpha) = 0$ unless $\alpha = \gamma$, in which case the value is 1.

Let X be a non-empty subset of V and let θ be a k-colouring of the subgraph of K_b induced by X. For any $\gamma \in P_k$, denote by γ_X the restriction of γ to X, and define $[X|\theta]$ to be the element of $\mathcal{V}_k$ given by

$$[X|\theta] = \sum_{\gamma_X = \theta} [\gamma].$$

In other words, $[X|\theta]$ is the function that takes the value 1 on the colourings that agree with θ on X, and 0 otherwise.

To cover the case when $X = \emptyset$ and ϕ is the empty function it makes sense to define $[\emptyset|\phi] = \sum_\gamma [\gamma]$, where the sum is taken over all $\gamma \in P_k$. So $[\emptyset|\phi]$ is the function which takes the constant value 1 on every element of P_k. We also denote this function by u.

We say that a pair (α, β) of members of P_k is *compatible* if $\alpha(v) \neq \beta(v)$ for all $v \in V$. Suppose this condition holds. Then, if two disjoint copies of K_b are linked by edges, each vertex v in the first copy being joined to the vertex v in the second, and the colourings

α and β are given to the first and second copy respectively, these colourings combine to form a (proper) colouring of the linked graph.

For each $k \geq b$ we define a *compatibility matrix* $T = T(k)$ (and an associated *compatibility operator* on $\mathcal{V}_k$) as follows. The rows and columns of T correspond to the elements of P_k, and the entries are given by:

$$(T)_{\alpha\beta} \;=\; \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are compatible;} \\ 0 & \text{otherwise.} \end{cases}$$

The sieve formula [5] Suppose that k is given and let $T = T(k)$ be the associated compatibility operator. For any given $\alpha \in P_k$ let α_X denote the restriction of α to X. Then

$$T[\alpha] \;=\; \sum_{X \subseteq V} (-1)^{|X|} [X|\alpha_X].$$

3. Invariant subspaces

In this section we shall prove the following result.

Theorem 1 Let S be a subset of the colours $\{1, 2, \ldots, k\}$, and let $\mathcal{U}(S)$ be the subspace of $\mathcal{V}_k$ spanned by all functions $[Y|\beta]$ with $\beta(Y) \subseteq S$. Then $\mathcal{U}(S)$ is invariant under the action of T.

The proof is in several stages. We begin with the following simple calculation:

$$\begin{aligned} T[Y|\beta] \;&=\; T\Big(\sum_{\alpha_Y = \beta} [\alpha]\Big) \\ &=\; \sum_{\alpha_Y = \beta} T[\alpha] \\ &=\; \sum_{\alpha_Y = \beta} \sum_{X \subseteq V} (-1)^{|X|}\; [X|\alpha_X] \\ &=\; \sum_{X \subseteq V} (-1)^{|X|}\; \sigma(X, Y, \beta), \end{aligned}$$

where

$$\sigma(X, Y, \beta) = \sum_{\alpha_Y = \beta} [X|\alpha_X].$$

In order to prove Theorem 1 we require an expression for $\sigma(X, Y, \beta)$ which, for all X, involves only functions of the form $[Z|\gamma]$, with $\gamma(Z) \subseteq \beta(Y)$.

If δ is an injection from X to $\{1, 2, \ldots, k\}$ such that δ and β agree on $X \cap Y$, then there is a function $\delta * \beta$ from $X \cup Y$ to $\{1, 2, \ldots, k\}$ defined by

$$(\delta * \beta)(v) = \begin{cases} \delta(v) & \text{if } v \in X; \\ \beta(v) & \text{if } v \in Y. \end{cases}$$

Note that $\delta * \beta$ does not exist unless $\delta_{X\cap Y} = \beta_{X\cap Y}$, and if it does exist, it may not be an injection (because it is possible that, for some $x \in X \setminus (X \cap Y)$ and $y \in Y \setminus (X \cap Y)$, $\delta(x) = \beta(y)$). We define

$$G(\beta) = \{\theta \in P_k \mid \theta_X * \beta \text{ exists and is an injection}\}.$$

Recall that for any complex number z and non-negative integer n the 'falling factorial' $(z)_n$ is defined by

$$(z)_0 = 1, \qquad (z)_n = z(z-1)_{n-1} \quad (n \geq 1).$$

In other words, $(z)_n = z(z-1)\cdots(z-n+1)$.

Lemma 1 Suppose that X and Y are subsets of V, where $|X \cup Y| = p$ and $|V| = b$, and let β be an injection from Y into $\{1, 2, \ldots, k\}$. Then

$$\sigma(X, Y, \beta) = (k-p)_{b-p} \sum_{\theta \in G(\beta)} [\theta].$$

Proof Using the definitions of $\sigma(X, Y, \beta)$ and $[X|\alpha_X]$ we have

$$\begin{aligned}\phi(X, Y, \beta) &= \sum_{\alpha_Y = \beta} [X|\alpha_X] \\ &= \sum_{\alpha_Y = \beta} \sum_{\theta_X = \alpha_X} [\theta].\end{aligned}$$

In order to reverse the order of summation it is convenient to write this as a sum taken over pairs (α, θ):

$$\sigma(X, Y, \beta) = \sum_{(\alpha,\theta) \in A(\beta)} [\theta],$$

where

$$A(\beta) = \{(\alpha, \theta) \mid \alpha_X = \theta_X \text{ and } \alpha_Y = \beta\}.$$

Treating this a repeated sum we have

$$\sum_{(\alpha,\theta) \in A(\beta)} [\theta] = \sum_{\theta} [\theta] \sum_{(\alpha,\theta) \in A(\beta)} 1 = \sum_{\theta} [\theta]\, m(\theta, \beta),$$

where $m(\theta, \beta)$ is the number of α such that $\alpha_X = \theta_X$ and $\alpha_Y = \beta$.

Suppose first that θ is in the set $G(\beta)$ defined above. Then $\theta_X * \beta$ exists and is an injection on $X \cup Y$, and $m(\theta, \beta)$ is the number of ways of extending this injection to an injection defined on the whole of V. This number is just

$$(k-p)(k-p-1)\ldots(k-b+1) = (k-p)_{b-p}.$$

On the other hand, if θ is not in the set $G(\beta)$, there can be no injection α satisfying the conditions, and $m(\theta,\beta)=0$. $\square$

If Z is a subset of X containing $X \cap Y$ define

$$\Pi(Z) = \{\pi \mid \pi : Z \to Y \text{ is an injection and } \pi_{X\cap Y} = id\}.$$

Note that $\Pi(X\cap Y)$ contains only the inclusion function $X \cap Y \to Y$.

Lemma 2 With $\Pi(Z)$ as above, and the other notation as in Lemma 1,

$$\sigma(X,Y,\beta) \ = t(X,Y) \sum_{X\cap Y \subseteq Z \subseteq X} (-1)^{|Z|}\tau(Z),$$

where

$$t(X,Y) = (-1)^{|X\cap Y|}(k-p)_{b-p}, \qquad \tau(Z) = \sum_{\pi\in\Pi(Z)} [Z \mid \beta\pi].$$

Proof Given the result of Lemma 1, it remains to evaluate $\sum[\theta]$ taken over the set $G(\beta)$. Recall that $G(\beta)$ consists of those θ in P_k for which (i) $\theta_X * \beta$ is defined, and (ii) $\theta_X * \beta$ is an injection on $X \cup Y$. The set of all θ satisfying condition (i) consists of those for which $\theta_{X\cap Y} = \beta_{X\cap Y}$, and the sum of $[\theta]$ taken over this set is, by definition,

$$[X\cap Y|\beta_{X\cap Y}].$$

We apply the sieve principle to obtain the relationship between this sum and the sum taken over all θ for which both conditions hold. It is convenient to put $Z = (X\cap Y)\cup D$, so that summing over Z such that $X \cap Y \subseteq Z \subseteq X$ corresponds to summing over all subsets D of $X \setminus (X\cap Y)$.

So we write the sum over $G(\beta)$ as

$$\sum_{D\subseteq X\setminus(X\cap Y)} (-1)^{|D|}F(D,\beta),$$

where $F(D,\beta)$ is the sum taken over θ that satisfy condition (i), but fail to satisfy condition (ii) on the set D. The second statement means that for each $v \in D$ there is some $\pi(v)$ in Y such that $\theta(v) = \beta(\pi(v))$. Clearly, this defines an injection π from D to $Y \setminus (X\cap Y)$. We can extend π to an injection from $(X \cap Y)\cup D$ to Y, by defining it to be the identity on $X \cap Y$, and since condition (i) holds this means that $\theta_{(X\cap Y)\cup D} = \beta\pi$. In other words

$$F(D,\beta) = \sum_{\pi\in\Pi(X\cap Y)\cup D} [(X\cap Y)\cup D \mid \beta\pi] = \tau((X\cap Y)\cup D).$$

Putting $Z = (X \cap Y)\cup D$ we have the result. $\square$

The results obtained in Lemmas 1 and 2 provide the required formula for $T[Y|\beta]$.

Lemma 3 The action of T is given by

$$T[Y|\beta] \;=\; \sum_{X\subseteq V}(-1)^{|X\setminus(X\cap Y)|}(k-|X\cup Y|)_{b-|X\cup Y|} \sum_{X\cap Y\subseteq Z\subseteq X}(-1)^{|Z|}\tau(Z),$$

where

$$\tau(Z) \;=\; \sum_{\pi\in\Pi(Z)}[Z|\beta\pi].$$

□

We can now complete the proof of Theorem 1.

Proof of Theorem 1 Let Y,β be such that $\beta(Y)\subseteq S$. According to Lemma 3, $T[Y|\beta]$ is a linear combination of terms of the form $[Z|\beta\pi]$. Here π is an injection $Z\to Y$, and so $\beta\pi(Z)\subseteq\beta(Y)\subseteq S$, as claimed. □

4. Explicit form of the coefficients

In this section we shall obtain a formula (Theorem 2) that gives the coefficients of the terms $[A|\alpha]$ that appear in $T[Y|\beta]$. This formula is remarkably simple, despite the complicated arguments that are needed to justify it.

Lemma 4 Suppose that Y and β are given, and that A and α are such that

$$\alpha(A)\subseteq\beta(Y), \qquad \alpha_{A\cap Y}=\beta_{A\cap Y}.$$

Then $[A|\alpha]$ appears as a term in $\sigma(X,Y,\beta)$ if and only if X is a subset of V satisfying

$$A\subseteq X\subseteq V\setminus Y',$$

where $Y'=Y\setminus(A\cap Y)$.

Proof It follows from Lemma 2 that the terms $[A|\alpha]$ appearing in $\sigma(X,Y,\beta)$ are precisely those for which

$$[A|\alpha]=[Z|\beta\pi], \quad \text{where } X\cup Y\subseteq Z\subseteq X, \text{ and } \pi\in\Pi(Z).$$

The given conditions on α ensure that $\alpha=\beta\pi$ for some $\pi\in\Pi(A)$, so the effective condition is that $X\cup Y\subseteq A\subseteq X$. Rearranging this as a condition on X we obtain the stated result. □

Now that we have established which terms occur in $T[Y|\beta]$, it remains only to find the coefficients.

For each integer $n\geq 0$ and any complex number z, the following formula defines a monic polynomial of degree n:

$$C(n,z) \;=\; \sum_{r=0}^{n}(-1)^r\binom{n}{r}(z-r)_{n-r}.$$

In particular, the polynomials $C(n,z)$ for $0 \le i \le 4$ are:

$$C(0,z)=1, \quad C(1,z)=z-1, \quad C(2,z)=z^2-3z+3,$$

$$C(3,z)=z^3-6z^2+14z-13, \quad C(4,z)=z^4-10z^3+41z^2-84z+73.$$

The relevance of these polynomials stems ultimately from the fact that $C(b,k)$ is the number of k-colourings τ of K_b such that τ is compatible with any given k-colouring σ. In other words, it is the number of τ that are 'derangements' of σ: $\sigma(v) \neq \tau(v)$ for all $v \in \{1,2,\ldots,b\}$.

For simplicity of notation, we fix the integers b and $k \ge b$, and define

$$c_i = C(b-i, k-i) \quad (0 \le i \le b).$$

Theorem 2 Let A and Y be subsets of $V=\{1,2,\ldots,b\}$, and let α, β be injections from A and Y respectively to $\{1,2,\ldots,k\}$, such that $\alpha(A) \subseteq \beta(Y)$ and $\alpha_{A\cap Y}=\beta_{A\cap Y}$. Then the coefficient of $[A|\alpha]$ in $T[Y|\beta]$ is

$$(-1)^{|A\cap Y|}\, c_{|A\cup Y|}.$$

Proof According to Lemma 2, the coefficient of a term $[A|\alpha]$ that occurs in $\sigma(X,Y,\beta)$ is $t(X,Y)(-1)^{|A|}$. If α satisfies the given conditions then, by Lemma 3, $[A|\alpha]$ occurs in $\sigma(X,Y,\beta)$ if and only if X is such that $A \subseteq X \subseteq V \setminus Y'$. Since

$$T[Y|\beta] = \sum_{X\subseteq V}(-1)^{|X|}\sigma(X,Y,\beta),$$

it follows that the coefficient of $[A|\alpha]$ in $T[Y|\beta]$ is

$$\sum_{A\subseteq X\subseteq V\setminus Y'}(-1)^{|X|}t(X,Y)(-1)^{|A|}.$$

Writing $X = A\cup R$, with $A\cap R=\emptyset$, the condition $A\subseteq X\subseteq V\setminus Y'$ is equivalent to the condition that R is a subset of $V\setminus(A\cup Y)$. Furthermore,

$$|X| = |A|+|R|, \quad |X\cap Y| = |A\cap Y|, \quad |X\cup Y| = |A\cup Y|+|R|.$$

Hence

$$\begin{aligned} t(X,Y) &= (-1)^{|X\cap Y|}(k-|X\cup Y|)_{b-|X\cup Y|} \\ &= (-1)^{|A\cap Y|}(k-|A\cup Y|-|R|)_{b-|A\cup Y|-|R|}, \end{aligned}$$

and the required coefficient is

$$\sum_{R\subseteq V\setminus(A\cup Y)} (-1)^{|A|+|R|} t(X,Y)(-1)^{|A|}$$

$$\begin{aligned}
&= (-1)^{|A\cap Y|} \sum_{R\subseteq V\setminus(A\cup Y)} (-1)^{|R|}(k-|A\cup Y|-|R|)_{b-|A\cup Y|-|R|}\\
&= (-1)^{|A\cap Y|} \sum_{r=0}^{b-|A\cup Y|} (-1)^r \binom{b-|A\cup Y|}{r}(k-|A\cup Y|-r)_{b-|A\cup Y|-r}\\
&= (-1)^{|A\cap Y|}\, C(b-|A\cup Y|,\ k-|A\cup Y|).
\end{aligned}$$

□

5. Levels 0 and 1

The action of T on the invariant subspaces $\mathcal{U}(S)$ with $|S| = 0, 1$ can be described fairly easily. When $S = \emptyset$ the only relevant function is $u = [\emptyset|\phi]$, and the equation $Tu = c_0 u$ follows from Theorem 2. So the invariant subspace $\mathcal{U}(\emptyset)$ is just $\langle u\rangle$. Another proof of the equation $Tu = c_0u$ follows from the observation, already noted above, that $c_0 = C(b, k)$ is the number of k-colourings β of K_b such that (α, β) is a compatible pair, for any given α. In other words, c_0 is the number of 1's in each row of the matrix T. Standard arguments lead to the conclusion that the eigenvalue c_0 has multiplicity 1.

Suppose that $S = \{h\}$, where h is any colour ($1 \leq h \leq k$). Denote by $[y|h]$ the function $[Y|\beta]$ such that $Y = \{y\}$ and $\beta(y) = h$. The terms that appear in $T[y|h]$ are the $[A|\alpha]$ for which $A = \emptyset$, $A = \{y\}$ and $A = \{x\}$, for all $x \neq y$. Clearly, for each A, there is a unique α, and Theorem 2 tells us that the complete formula is

$$T[y|h] = c_1\Big(u - [y|h]\Big) + c_2\Big(\sum_{x\neq y}[w|h]\Big).$$

Since $Tu = c_0u$, it follows that

$$\mathcal{U}(h) = \langle u, [1|h], [2|h], \ldots, [b|h]\rangle.$$

When $k = b$, the set of functions $u, [1|h], [2|h], \ldots, [b|h]$ is linearly dependent (u is the sum of the other functions, since every b-colouring of K_b assigns colour h to exactly one vertex). However, if $k > b$ there are k-colourings of K_b that do not use the colour h, so u is linearly independent of the other functions, and the set is a basis for $\mathcal{U}(h)$. We shall assume henceforth that this condition holds.

We have shown that the action of T on $\mathcal{U}(h)$, with respect to the given basis, is represented by the matrix

$$\begin{pmatrix}
c_0 & c_1 & c_1 & . & . & . & c_1\\
0 & -c_1 & c_2 & . & . & . & c_2\\
0 & c_2 & -c_1 & . & . & . & c_2\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
0 & c_2 & c_2 & . & . & . & -c_1
\end{pmatrix}.$$

We write this matrix as

$$\begin{pmatrix} M_0 & * \\ O & M_1 \end{pmatrix}.$$

Here M_0 is the 1×1 matrix (c_0), and M_1 is the $b \times b$ matrix in the lower right-hand corner; O and $*$ stand for zero and non-zero matrices respectively. In terms of the identity matrix I and the all-1 matrix J we have

$$M_0 = c_0 I; \qquad M_1 = -c_1 I + c_2(J - I).$$

In the following sections we shall develop a general theory from which these two equations follow as special cases.

For the time being, the eigenvalues and eigenfunctions of M_0 and M_1 can be obtained by elementary means. The trivial case M_0 has already been discussed. For M_1 there are just two distinct eigenvalues, and they are:

$$\begin{aligned} -c_1 + (b-1)c_2 & \quad \text{(multiplicity 1);} \\ -c_1 - c_2 & \quad \text{(multiplicity } b-1\text{).} \end{aligned}$$

The corresponding eigenfunctions can be obtained directly from the formula for the action of T. First, let

$$\phi_h = [1|h] + [2|h] + \cdots + [b|h] - (b/k)u.$$

It is easy to check that $T\phi_h = ((b-1)c_2 - c_1)\phi_h$, so that ϕ_h spans a 1-dimensional space of eigenfunctions.

Next, for any vertices $v \neq w$ let

$$\psi_h^{vw} = [v|h] - [w|h].$$

Again, it is easy to check that $T\psi_h^{vw} = (-c_2 - c_1)\psi_h^{vw}$. So we have $b(b-1)/2$ eigenfunctions, one for each edge vw of K_b. These eigenfunctions span a space of dimension $b-1$, a basis for which is obtained by taking the functions corresponding to the edges of any spanning tree of K_b. For example, we may fix $w = b$, and take $v = 1, 2, \ldots, b-1$.

Up to this point we have been considering the action of T on the subspace $\mathcal{U}(h)$, for a given colour h. In fact, for each one of the k colours there is an eigenfunction ϕ_h, and the only linear relationship between them is that their sum is identically zero. Thus, in the action of T on the whole space $\mathcal{V}_k$, the multiplicity of the eigenvalue $(b-1)c_2 - c_1$ is $(k-1)$. Similarly, the multiplicity of $-c_2 - c_1$ is $(b-1)(k-1)$.

This completes the analysis of the terms at levels 0 and 1. We can now exhibit the leading terms of the chromatic polynomial of $B_n(b)$, for all $b \geq 2$:

$$P(B_n(b);k) \;=\; c_0^n + (k-1)\Big((-c_1 + (b-1)c_2)^n + (b-1)(-c_1 - c_2)^n\Big) + \ldots\;,$$

where $c_i = C(b-i, k-i)$, $i = 0, 1, 2$.

6. The general case

In this section we consider the action of T on an invariant subspace $\mathcal{U}(S)$ with $|S| = \ell \geq 2$. As in the previous section, it is often convenient to use a more explicit form of the notation, so we write

$$[y_1, y_2, \ldots, y_r | h_1, h_2, \ldots, h_r]$$

for the function $[Y|\beta]$ when $Y = \{y_1, y_2, \ldots, y_r\}$ and $\beta(y_i) = h_i$ $(1 \leq i \leq r)$.

We begin with the case $\ell = 2$. Theorem 2 provides the following basic formula for the action of T on the space $\mathcal{U}(S)$ with $S = \{h_1, h_2\}$:

$$\begin{aligned} T[v, v'|h_1, h_2] \ &= c_2\Big(u - [v|h_1] - [v'|h_2] + [v, v'|h_1, h_2]\Big) \\ &\quad + \ c_3\Big(\sum_{w \neq v, v'} [w|h_1] + [w|h_2] - [v, w|h_1, h_2] - [w, v'|h_1, h_2]\Big) \\ &\quad + \ c_4\Big(\sum_{w, w' \neq v, v'} [w, w'|h_1, h_2] + [w', w|h_1, h_2]\Big). \end{aligned}$$

Note that for each set A of two vertices $\{w, w'\}$ there are two injections from A to $\{h_1, h_2\}$, and thus two terms in the last sum.

Clearly the space $\mathcal{U}(h_1, h_2)$ contains as subspaces $\mathcal{U}(\emptyset)$, $\mathcal{U}(h_1)$ and $\mathcal{U}(h_2)$. It is easy to check that

$$\mathcal{U}(h_1) \cap \mathcal{U}(h_2) = \mathcal{U}(\emptyset),$$

so the subspace $\mathcal{U}(h_1) + \mathcal{U}(h_2)$ has dimension $2(b+1) - 1 = 2b+1$. A basis for this subspace is

$$u, [1|h_1], [2|h_1], \ldots, [b|h_1], [1|h_2], [2|h_2], \ldots, [b|h_2],$$

which can be extended to a basis of $\mathcal{U}(h_1, h_2)$ by adding the functions $[x, y|h_1, h_2]$ and $[y, x|h_1, h_2]$ for all pairs $\{x, y\}$. Then the action of T on $\mathcal{U}(h_1, h_2)$ with respect to this basis is represented by a matrix of the form

$$\begin{pmatrix} M_0 & * & * & * \\ O & M_1 & O & * \\ O & O & M_1 & * \\ O & O & O & M_2 \end{pmatrix}.$$

Here M_0 and M_1 are the matrices discussed in Section 5, O denotes a matrix of zeros, and $*$ denotes a non-zero matrix. The matrix M_2 is a $b(b-1) \times b(b-1)$ matrix whose rows and columns correspond to the functions $[P|\gamma]$, where $|P| = 2$ and $\gamma : P \to \{h_1, h_2\}$ is a bijection.

The entry in row $[P|\gamma]$ and column $[Q|\delta]$ of M_2 is the coefficient of $[P|\gamma]$ in $T[Q|\delta]$, so using the formula for $T[v, v'|h_1, h_2]$ displayed above, it follows that the entries of M_2 are given by:

$$(M_2)_{[P|\gamma],[Q|\delta]} = \begin{cases} c_2 & \text{if } P = Q \text{ and } \gamma = \delta; \\ -c_3 & \text{if } P \cap Q = \{x\} \text{ and } \gamma(x) = \delta(x); \\ c_4 & \text{if } P \cap Q = \emptyset \ ; \\ 0 & \text{otherwise.} \end{cases}$$

We now turn to the general situation when S is a fixed set of ℓ colours, $\{h_1, h_2, \dots, h_\ell\}$, $\ell \geq 2$. In general, the formula for $T[Q|\delta]$ with $\delta(Q) = S$ contains terms $[P|\gamma]$ with $\gamma(P) = S'$ for each subset S' of S. Thus the action of T on $\mathcal{U}(S)$ can be represented by a matrix $\hat{T}$ of a form analogous to that given above for the case $|S| = 2$. Specifically, for $r = 0, 1, 2, \dots, \ell$, $\hat{T}$ has a 'diagonal' consisting of $\binom{\ell}{r}$ submatrices M_r, one for each r-subset of S. The entries in the lower triangle of $\hat{T}$ are all zero, so the eigenvalues of $\hat{T}$ are the eigenvalues of the diagonal submatrices. We shall suppose that these eigenvalues have been determined for $r < \ell$, and concentrate on the matrix M_ℓ. Henceforth ℓ will be fixed and we write $M = M_\ell$.

For each $P \subseteq V$ of size ℓ there are $\ell!$ injections $\gamma : P \to S$ and $\ell!$ corresponding functions $[P|\gamma]$. Thus the number of rows and columns of M is $\binom{b}{\ell} \times \ell! = (b)_\ell$, and M can be partitioned into square blocks M^{PQ} of size $\ell!$. The block M^{PQ} corresponds to the intersection of the rows labelled $[P|\gamma]$ and the columns labelled $[Q|\delta]$, for a given pair of ℓ-subsets (P, Q). According to Theorem 2, in each block the entries are $(-1)^{|P\cap Q|} c_{|P\cup Q|}$ if $\gamma_{P\cap Q} = \delta_{P\cap Q}$, and 0 otherwise.

In order to analyse M^{PQ} we need some notation. Write $P = \{p_1, p_2, \dots, p_\ell\}$, where $p_1 < p_2 < \cdots < p_\ell$. Given a permutation σ in Sym_ℓ, the symmetric group on $\{1, 2, \dots, \ell\}$, define the function

$$[P, \sigma] = [p_{\sigma(1)}, p_{\sigma(2)}, \dots, p_{\sigma(\ell)} \mid h_1, h_2, \dots, h_\ell].$$

We consider M^{PQ} as a matrix whose rows and columns correspond to the members of Sym_ℓ, the entries being

$$(M^{PQ})_{\sigma\tau} = M_{[P,\sigma]\ [Q,\tau]}.$$

Suppose $|P \cap Q| = i$. Then, using our standard notation for P and Q, there are subscripts α_j, β_j such that $p_{\alpha_j} = q_{\beta_j}$, $j = 1, 2, \dots, i$. Let

$$F_{PQ} = \{\phi \in Sym_\ell \mid \phi(\alpha_j) = \beta_j,\ 1 \leq j \leq i\}.$$

Note that, given any $\phi \in F_{PQ}$, the set F_{PQ} is a left coset ϕH in Sym_ℓ, where H is the subgroup fixing $\beta_1, \beta_2, \dots, \beta_i$ pointwise. Thus $|F_{PQ}| = (\ell - i)!$.

Lemma 5 Let $X(\rho)$ be the permutation matrix representing ρ in the regular representation of Sym_ℓ on itself; that is, $X_{\sigma\tau}(\rho)$ is 1 if $\tau = \rho\sigma$ and 0 otherwise. Also, let $g_{PQ} = (-1)^{|P\cap Q|} c_{|P\cup Q|}$.

Then the block M^{PQ} of M can be written as

$$M^{PQ} \ = \ g_{PQ} \sum_{\rho \in F_{PQ}} X(\rho).$$

Proof The functions $[P,\sigma]$ and $[Q,\tau]$ correspond to the colourings σ^* of P and τ^* of Q defined by

$$\sigma^*(p_{\sigma(r)}) = h_r, \quad \tau^*(q_{\tau(r)}) = h_r, \quad r = 1, 2, \ldots, \ell.$$

The term $(M^{PQ})_{\sigma\tau}$ is non-zero if and only if $\sigma^*_{P\cap Q} = \tau^*_{P\cap Q}$, and then it is equal to g_{PQ}. Thus it is sufficient to prove that $\sigma^*_{P\cap Q} = \tau^*_{P\cap Q}$ if and only if $X_{\sigma\tau}(\rho) = 1$ for some $\rho \in F_{PQ}$, or equivalently, $\tau\sigma^{-1} \in F_{PQ}$.

Given $p_{\alpha_j} = q_{\beta_j} \in P \cap Q$, let $\alpha_j = \sigma(x)$. Then

$$\tau^*(q_{\tau(x)}) = h_x = \sigma^*(p_{\sigma(x)}) = \sigma^*(p_{\alpha_j}).$$

Suppose $\tau\sigma^{-1} \in F_{PQ}$, so that $\tau(x) = \tau\sigma^{-1}(\alpha_j) = \beta_j$. It follows that $\tau^*(q_{\beta_j}) = \sigma^*(p_{\alpha_j})$, so $\sigma^*_{P\cap Q} = \tau^*_{P\cap Q}$, as required.

Conversely, suppose $\sigma^*_{P\cap Q} = \tau^*_{P\cap Q}$. Then $\sigma^*(p_{\alpha_j}) = \tau^*(q_{\beta_j})$ and so $\beta_j = \tau(x)$. Since $x = \sigma^{-1}(\alpha_j)$, it follows that $\tau\sigma^{-1}$ is in F_{PQ}. $\square$

7. The collapsed matrix associated with a representation

In this section b and ℓ are fixed positive integers with $b \geq \ell$. If R is an r-dimensional matrix representation of Sym_ℓ, let N_R be the matrix defined as follows. In the formula for the block M^{PQ} of M obtained in Lemma 5, replace each permutation matrix $X(\rho)$ by the corresponding $R(\rho)^t$, where the superscript denotes the transpose. Thus N_R is a matrix with blocks

$$(N_R)^{PQ} = g_{PQ} \sum_{\rho \in F_{PQ}} R(\rho)^t.$$

Note that the blocks are now of size r, instead of $\ell!$. Given P, designate the row of N_R given by the ith rows of the blocks $(N_R)^{PQ}$ by (P,i), and similarly for the columns. If f is an eigenvector of N_R with eigenvalue λ, so that f is defined on pairs (P,i), then for all such pairs we have

$$\sum_{(Q,j)} (N_R)_{(P,i),(Q,j)} f(Q,j) \;=\; \lambda f(P,i).$$

Since

$$(N_R)_{(P,i),(Q,j)} \;=\; g_{PQ} \sum_{\rho \in F_{PQ}} R_{ji}(\rho).$$

this can be rewritten as

$$\sum_{Q} g_{PQ} \sum_{j=1}^{h} f(Q,j) \sum_{\rho \in F_{PQ}} R_{ji}(\rho) \;=\; \lambda f(P,i).$$

Theorem 3 Let R be a matrix representation of Sym_ℓ, of degree r, and N_R the matrix defined above. Then each eigenvector f of N_R with eigenvalue λ can be lifted to r linearly independent eigenvectors of M with the same eigenvalue.

Proof We shall show that, for $i = 1, 2, \ldots, r$, the vector f_R^i defined by

$$f_R^i[P,\sigma] = \sum_{j=1}^{r} R_{ji}(\sigma)\ f(P,j)$$

is an eigenvector of M with eigenvalue λ.

We have

$$\begin{aligned}
(Mf_R^i)[P,\sigma] &= \sum_{[Q,\tau]} M_{[P,\sigma],[Q,\tau]} f_R^i[Q,\tau] \\
&= \sum_Q \sum_\tau g_{PQ} \sum_{\rho\in F_{PQ}} X_{\sigma\tau}(\rho)\ f_R^i[Q,\tau], \\
&= \sum_Q\ g_{PQ} \sum_{\rho\in F_{PQ}} \sum_\tau X_{\sigma\tau}(\rho)\ f_R^i[Q,\tau],
\end{aligned}$$

The last sum is

$$\begin{aligned}
\sum_\tau X_{\sigma\tau}(\rho) f_R^i[Q,\tau] &= f_R^i[Q,\rho\sigma] \\
&= \sum_{j=1}^{r} R_{ji}(\rho\sigma)\ f(Q,j) \\
&= \sum_{j=1}^{r} f(Q,j) \sum_{h=1}^{r} R_{jh}(\rho) R_{hi}(\sigma).
\end{aligned}$$

Thus, since $N_R f = \lambda f$,

$$\begin{aligned}
(Mf_R^i)[P,\sigma] &= \sum_Q\ g_{PQ} \sum_{\rho\in F_{PQ}} \sum_{j=1}^{r} f(Q,j) \sum_{h=1}^{r} R_{jh}(\rho) R_{hi}(\sigma) \\
&= \sum_{h=1}^{r} R_{hi}(\sigma) \left(\sum_Q\ g_{PQ} \sum_{j=1}^{r} f(Q,j) \sum_{\rho\in F_{PQ}} R_{jh}(\rho) \right) \\
&= \sum_{h=1}^{r} R_{hi}(\sigma) \lambda f(P,h) \\
&= \lambda f_R^i[P,\sigma].
\end{aligned}$$

The linear independence of the r vectors f_R^i follows from the fact that each matrix $R(\rho)$ is invertible, and hence has rank r. $\square$

The irreducible representations of Sym_ℓ correspond to the partitions π of ℓ. If R is the representation R^π, denote N_R by N^π. The degree n_π of R^π is given by a well-known formula of Frobenius, and the size of N^π is $n_\pi\binom{b}{\ell}$.

Suppose that the eigenvalues of N^π are

$$\nu_\pi^i \quad \text{with multiplicities} \quad f_\pi^i \quad (i = 1, 2, \ldots, i_\pi),$$

so that

$$\sum_i f_\pi^i = n_\pi \binom{b}{\ell}.$$

According to Theorem 3, ν_π^i is an eigenvalue of M with multiplicity $n_\pi f_\pi^i$ so, as eigenvalues of M, the total multiplicity of all the ν_π^i is

$$\sum_{\pi,i} n_\pi f_\pi^i = \sum_\pi n_\pi^2 \binom{b}{\ell}.$$

It is a standard result that $\sum_\pi n_\pi^2 = \ell!$, the order of Sym_ℓ. Thus the total multiplicity is $\ell!\binom{b}{\ell} = (b)_\ell$. Since this is equal to the size of M we have the following result.

Theorem 4 Every eigenvalue of M is an eigenvalue of N^π for some partition π of ℓ, and conversely. $\square$

The symmetric group Sym_ℓ has two representations of degree 1: the principal representation defined by $R^{pri}(\sigma) = 1$ and the alternating representation defined by $R^{alt}(\sigma) = sign(\sigma)$. These representations correspond to the partitions $[\ell]$ and $[1^\ell]$. Each has a corresponding matrix whose PQ block is the scalar

$$g_{PQ} \sum_{\rho \in F_{PQ}} \chi(\rho),$$

where χ is R^{pri} or R^{alt} (considered as a character). These matrices will be denoted by N^{pri} and N^{alt}. In the following sections we shall obtain explicit formulae for the eigenvalues of these two matrices.

8. The principal terms

The *Johnson graph* $J(b, \ell)$ is defined to be the graph whose vertices are the ℓ-subsets of a b-set, two of them being joined by an edge whenever they intersect in a set of size $\ell - 1$ [**1**]. This is a connected graph of diameter $d = \min(\ell, b - \ell)$, and the distance between two vertices P and Q is $\ell - |P \cap Q|$. The *distance matrices* $A_0, A_1, \ldots, A_d$ are therefore defined by

$$(A_j)_{PQ} = \begin{cases} 1 & \text{if } |P \cap Q| = \ell - j; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6 The matrix $N = N^{pri}$ can be expressed in terms of the the distance matrices of the Johnson graph $J(b,\ell)$, as follows:

$$N = \sum_{j=0}^{d} (-1)^{\ell-j}\, j!\, c_{\ell+j}\, A_j.$$

Proof According to the formula given at the end of the previous section, N_{PQ} is equal to $g_{PQ}|F_{PQ}|$. If $|P \cap Q| = i$ then $g_{PQ} = (-1)^i c_{2\ell-i}$ and $|F_{PQ}| = (\ell - i)!$, so

$$N_{PQ} = (-1)^i c_{2\ell-i}(\ell - i)!\ .$$

The result follows by putting $i = \ell - j$ and using the definition of the distance matrices.

□

Note that when $\ell = 0, 1$ the formula gives $N = c_0A_0$ and $N = -c_1A_0 + c_2A_1$, respectively. The matrix A_0 is the identity, and when $\ell = 1$ the graph $J(b, 1)$ is a complete graph, so $A_1 = J - I$. In these cases we have $M = N$ and so we recover the formulae for M_0 and M_1 obtained by direct means in Section 5.

Since $J(b,\ell)$ is a distance-regular graph, each matrix A_j is a polynomial function of the *adjacency matrix* $A = A_1$, that is $A_j = v_j(A)$. In fact, the polynomials v_j can be obtained explicitly from the *Eberlein polynomials*, defined formally by the rule

$$E_j(u) = \sum_{t=0}^{j} (-1)^t \binom{u}{t}\binom{\ell-u}{j-t}\binom{b-\ell-u}{j-t}.$$

Although $E_j(u)$ is ostensibly a polynomial in u, of degree $2j$, it turns out [**1**] that it is also a polynomial function v_j of $\lambda_u = (\ell - u)(b - \ell - u) - u$, of degree j. Since the eigenvalues of $A = A_1$ are

$$\lambda_i = (\ell - i)(b - \ell - i) - i \qquad (0 \le i \le \ell),$$

it follows that the eigenvalues of $A_j = v_j(A)$ are $v_j(\lambda_i) = E_j(i)$.

Theorem 5 Let $T(k)$ be the compatibility operator for k-colourings of the untwisted bracelet on K_b, where $k > b$. Given ℓ such that $0 \le \ell \le b$ put $d = \min(\ell, b-\ell)$. Then $T(k)$ has $d+1$ eigenvalues $\nu_{\ell i}$, $0 \le i \le d$, given by the formula

$$(-1)^{\ell}\nu_{\ell i} \;=\; \sum_{j=0}^{d} (-1)^j\, j!\, c_{\ell+j}\, E_j(i),$$

where $c_{\ell+j} = C(b-\ell-j, k-\ell-j)$ and E_j is the Eberlein polynomial defined above. Each $(-1)^{\ell}\nu_{\ell i}$ is a monic polynomial in k of degree $b - \ell$.

Proof The set of eigenvalues of $T(k)$ contains the set of eigenvalues of the matrix $\hat{T}$ representing its action on a subspace $\mathcal{U}(S)$, where $|S| = \ell$. Among these eigenvalues are the eigenvalues of M, and by Theorem 4, these in turn include the eigenvalues of N.

The result follows from the formula for N given in Lemma 6, and the expression for the eigenvalues of A_j in terms of the Eberlein polynomials. $\square$

The multiplicities of the eigenvalues of N^{pri} can be deduced from the general theory of distance-regular graphs. It is known [**1**] that the multiplicity of the eigenvalue $\lambda_i = (\ell - i)(b - \ell - i) - i$ of the adjacency matrix A_1 of $J(b, \ell)$ is

$$m_i = \binom{b}{i} - \binom{b}{i-1}.$$

Since N^{pri} is a polynomial function of A, the eigenvalue $\nu_{\ell i}$ also has multiplicity m_i.

It has already been mentioned that, when $\ell = 0$ and $\ell = 1$, M is equal to N^{pri}, so all the eigenvalues of M are given by Theorem 5. When $\ell = 2$ and $b \geq 4$ we have $N = c_2 A_0 - c_3 A_1 + 2c_4 A_2$, and there are three distinct eigenvalues of N, which are also eigenvalues of M. They can be obtained as follows. The relevant Eberlein polynomials are

$$\begin{aligned} E_0(u) &= 1; \\ E_1(u) &= \lambda_u = (2-u)(b-2-u) - u; \\ E_2(u) &= \frac{1}{4}\Big(\lambda_u^2 - (b-2)\lambda_u - (2b-4)\Big). \end{aligned}$$

It follows that the eigenvalues of N are:

$$\begin{aligned} c_2 - (2b-4)c_3 + (b-2)(b-3)c_4, &\quad \text{multiplicity } 1; \\ c_2 - (b-4)c_3 - 2(b-3)c_4, &\quad \text{multiplicity } b-1; \\ c_2 + 2c_3 + 2c_4, &\quad \text{multiplicity } b(b-3)/2. \end{aligned}$$

9. The alternating terms

The matrix N^{alt} is simpler than N^{pri}, for the following reason.

Lemma 7 If $|P \cap Q| \leq \ell - 2$ then $(N^{alt})_{PQ} = 0$.

Proof If $|P \cap Q| = \ell - j$, the set F_{PQ} is a coset of the stabilizer of a symmetric group of order $j!$. When $j \geq 2$ this coset contains equal numbers of odd and even permutations, so

$$\sum_{\rho \in F_{PQ}} sign(\rho) = 0.$$

The result follows from the definition of N^{alt}. $\square$

When $P = Q$ the set F_{PQ} contains only the identity permutation, and when $|P \cap Q| = \ell - 1$ it contains just one, non-identity, permutation ρ_{PQ}. It follows that

$$N^{alt} = (-1)^\ell (c_\ell I - c_{\ell+1} D),$$

where

$$D_{PQ} = \begin{cases} sign(\rho_{PQ}) & \text{if } |P \cap Q| = \ell - 1; \\ 0 & \text{otherwise.} \end{cases}$$

It is helpful to think of D as the adjacency matrix of a signed graph $J^{\pm}(b, \ell)$, that is, $J(b, \ell)$ with the edge PQ given the sign of ρ_{PQ}. It will be convenient to write

$$sign(\rho_{PQ}) = (-1)^{\epsilon(P,Q)},$$

and consider $\epsilon(P, Q)$ as an integer modulo 2.

The next lemma provides a useful method of determining $\epsilon(P, Q)$. Given a subset A of $\{1, 2, \ldots, b\}$ and an element $a \in A$ define $\pi(a, A)$, the *place* of a in A, to be the value of s such that a is the sth member of A in the standard (increasing) order.

Lemma 8 Suppose that P and Q are ℓ-subsets of $\{1, 2, \ldots, b\}$ such that $|P \cap Q| = \ell - 1$, and let $P \setminus (P \cap Q) = \{p\}$, $Q \setminus (P \cap Q) = \{q\}$. Then

$$\epsilon(P, Q) = \pi(p, P) + \pi(q, Q).$$

Proof Let $s = \pi(p, P), t = \pi(q, Q)$, and suppose that $s > t$. Then

$$p_i = q_i \ (i = 1, \ldots, t - 1,\ i = s + 1, \ldots, \ell), \quad p_i = q_{i+1} \ (i = t, \ldots, s - 1).$$

Thus ρ_{PQ} must be the cycle $(t, t + 1, \ldots, s)$, which has sign $(-1)^{s-t} = (-1)^{s+t}$. It is easy to check that the result holds for $s \leq t$ also. □

Lemma 9 If D is the adjacency matrix of the signed graph $J^{\pm}(b, \ell)$ then

$$D^2 = \ell(b - \ell)I + (b - 2\ell)D.$$

Proof Since $(D^2)_{PQ} = \sum_Z D_{PZ} D_{ZQ}$, it is clear that $(D^2)_{PQ} = 0$ whenever the distance between the vertices P and Q of $J^{\pm}(b, \ell)$ is greater than 2; that is, whenever $|P \cap Q| < \ell - 2$.

If $P = Q$, we have $(D^2)_{PP} = \sum_Z D_{PZ}^2$. Here the non-zero summands correspond to the vertices Z that are adjacent to P in $J^{\pm}(b, \ell)$, and each non-zero summand is 1. Since there are $\ell(b - \ell)$ such vertices, $(D^2)_{PP} = \ell(b - \ell)$.

If $|P \cap Q| = \ell - 1$ or $\ell - 2$ the non-zero summands $D_{PZ} D_{ZQ}$ correspond to the vertices Z that are adjacent to both P and Q.

Suppose first that P and Q are adjacent, and let $R = P \cap Q$ where $|R| = \ell - 1$ and $P = R \cup p$, $Q = R \cup q$. Then there are $b - 2$ vertices adjacent to both P and Q, of two types:

$$\begin{array}{rl} b - \ell - 1 \text{ vertices } R \cup x, & (x \notin R \cup p \cup q); \\ \ell - 1 \text{ vertices } R \setminus y \cup p \cup q, & (y \in R). \end{array}$$

For a vertex $X = R \cup x$ of the first type we have

$$\epsilon(P, X) = \pi(p, R \cup p) + \pi(x, R \cup x), \quad \epsilon(X, Q) = \pi(x, R \cup x) + \pi(q, R \cup q).$$

Hence $\epsilon(P,X)+\epsilon(X,Q)=\epsilon(P,Q)$, and $D_{PX}D_{XQ}=D_{PQ}$ for all $b-\ell-1$ vertices of this type.

For a vertex $Y=R\setminus y\cup p\cup q$ of the second type, we have

$$\epsilon(P,Y)=\pi(y,R\cup p)+\pi(q,R\setminus y\cup p\cup q),\quad \epsilon(Y,Q)=\pi(p,R\setminus y\cup p\cup q)+\pi(y,R\cup q).$$

Here it turns out that $\epsilon(P,Y)+\epsilon(Y,Q)=\epsilon(P,Q)+1$. For example, suppose that $p<y<q$; then

$$\pi(p,R\setminus y\cup p\cup q)=\pi(p,R\cup p),\quad \pi(q,R\setminus y\cup p\cup q)=\pi(q,R\cup q),$$

$$\pi(y,R\cup p)=\pi(y,R\cup q)+1,$$

which gives the result. Hence, for all $\ell-1$ vertices Y of this type, $D_{PY}D_{YQ}=-D_{PQ}$.

So the value of $(D_{PQ})^2$ when $|P\cap Q|=\ell-1$ is therefore

$$(b-\ell-1)D_{PQ}+(\ell-1)(-D_{PQ})=(b-2\ell)D_{PQ}.$$

If $|P\cap Q|=\ell-2$, suppose $P=S\cup a\cup b$, $Q=S\cup c\cup d$, where $|S|=\ell-2$. For brevity, write $P=Sab$, and so on. Then there are just four vertices adjacent to both P and Q: Sac, Sad, Sbc, Sbd.

Suppose $a<b<c<d$. Then it is easy to check that

$$\begin{aligned}\epsilon(Sab,Sac)+\epsilon(Sac,Scd)&=1+\epsilon(Sab,Sbc)+\epsilon(Sbc,Scd)\\ \epsilon(Sab,Sad)+\epsilon(Sad,Scd)&=1+\epsilon(Sab,Sbd)+\epsilon(Sbd,Scd).\end{aligned}$$

It follows that $(D_{PQ})^2=0$ in this case, as required. The other cases can be verified in a similar way. □

Theorem 6 Let $T(k)$ be the compatibility operator for k-colourings of the untwisted bracelet on K_b, where $k>b$. Then for each ℓ in the range $2\le\ell\le b-1$, $T(k)$ has two eigenvalues $\nu^*_{\ell i}$, $i=0,1$, given by the formulae

$$\nu^*_{\ell 0}=(-1)^\ell(c_\ell-(b-\ell)c_{\ell+1}),\quad \nu^*_{\ell 1}=(-1)^\ell(c_\ell+\ell c_{\ell+1}).$$

(Note that when $\ell=b$ the eigenvalue $\nu^*_{b0}=(-1)^b$ can also be regarded as belonging to this family - see Section 10).

Proof The set of eigenvalues of $T(k)$ contains the set of eigenvalues of the matrix $\hat{T}$ representing its action on a subspace $\mathcal{U}(S)$, where $|S|=\ell$. Among these eigenvalues are the eigenvalues of M, and by Theorem 4, these in turn include the eigenvalues of N^{alt}.

It follows from Lemma 9 that the eigenvalues λ_1,λ_2 of D are the roots of the quadratic equation

$$\lambda^2-(b-2\ell)\lambda-\ell(b-\ell)=0,\quad \text{that is } \lambda_1=b-\ell,\ \lambda_2=-\ell.$$

Consequently the eigenvalues of $N^{alt} = (-1)^\ell(c_\ell I - c_{\ell+1}D)$ are $(-1)^\ell(c_\ell - (b-\ell)c_{\ell+1})$ and $(-1)^\ell(c_\ell + \ell c_{\ell+1})$. $\square$

The multiplicities of the eigenvalues $\nu^*_{\ell i}$ can be computed directly. Since the trace of D is zero, the multiplicities f_0 and f_1 of its eigenvalues $b-\ell$ and $-\ell$ satisfy

$$f_0(b-\ell) + f_1(-\ell) = 0, \quad f_0 + f_1 = \binom{b}{\ell}.$$

Hence the multiplicities of the eigenvalues $\nu^*_{\ell 0}$ and $\nu^*_{\ell 1}$ are

$$f_0 = \binom{b-1}{\ell-1}, \quad f_1 = \binom{b-1}{\ell}.$$

Since the principal and alternating representations are the only representations of Sym_ℓ when $\ell = 2$, in this case the eigenvalues of N^{pri} and N^{alt} account for all the eigenvalues of M. For $b \geq 4$ there are three distinct eigenvalues of N^{pri} (Section 8) and for $b \geq 3$ there are two distinct eigenvalues of N^{alt}, which are

$$\begin{aligned} c_2 - (b-2)c_3, &\quad \text{multiplicity } b-1; \\ c_2 + 2c_3, &\quad \text{multiplicity } (b-1)(b-2)/2. \end{aligned}$$

10. Conclusion: the cases b=2,3,4

In the Introduction it was pointed out that the terms in the chromatic polynomial of $B_n(b)$ appear to occur in levels, the terms at level ℓ being of the form

$$\big(\text{Polynomial of degree } \ell\big) \times \big(\text{Integer}\big)\big(\text{Polynomial of degree } b-\ell\big)^n.$$

The foregoing theory reveals that each 'Polynomial of degree $b-\ell$' is an eigenvalue of a matrix $M = M_\ell$, and the corresponding 'Integer' is its multiplicity as a eigenvalue of M. Furthermore all these expressions can be obtained from collapsed matrices N_R associated with representations R of Sym_ℓ. In this section we shall study the cases $b = 2, 3, 4$, obtaining all the eigenvalues and multiplicities explicitly. In these cases a little extra work will produce the full chromatic polynomial, specifically the 'Polynomial of degree ℓ' involved in each term. This polynomial arises because the matrix M is defined for a fixed ℓ-subset of the k colours available, and so M occurs many times as a block of the compatibility matrix $T(k)$. This means that each eigenvalue has a *global multiplicity* as an eigenvalue of $T(k)$, as well as its *local multiplicity* as an eigenvalue of M.

All the eigenvalues that we have obtained can be expressed in terms of the functions $c_i = C(b-i, k-i)$, where C was defined in Section 4. For a given b, c_i is a monic polynomial in k of degree $b-i$. Each eigenvalue λ at level ℓ is a linear combination of the c_i's, with coefficients that depend on b, not k, and it is easy to check that $(-1)^\ell\lambda$ is a monic polynomial of degree $b-\ell$. It is also worth remarking that there are many relationships among the λ's, arising from the recursive form of the definition of the c_i's.

In the following exposition it will appear that the cases $b = 2, 3, 4$ are mainly covered by the general theory of the principal and alternating representations, as developed in Sections 8 and 9. In addition there is the simple observation that, when $\ell = b$, the matrix M is $(-1)^b I$. It remains only to deal with the case $b = 4$, $\ell = 3$, which involves a nonlinear representation.

The case $b = 2$

In this case the relevant polynomials are

$$c_0 = k^2 - 3k + 3, \quad c_1 = k - 2, \quad c_2 = 1.$$

At level 0 we have the eigenvalue c_0, and at level 1 two principal eigenvalues

$$-c_1 + c_2 = -(k-3), \quad -c_1 - c_2 = -(k-1),$$

both with local multiplicity 1 and global multiplicity $k-1$ (Section 5). At level 2, we have the eigenvalue $c_2 = 1$, which is both a principal eigenvalue and an alternating one. It can be shown [**3**] that the global multiplicities are $k(k-3)/2$ and $(k-1)(k-2)/2$ respectively, so the total global multiplicity of 1 is $k^2 - 3k + 1$. The chromatic polynomial of $B_n(2)$ is thus as given in Section 1.

The case $b = 3$

In this case the relevant polynomials are

$$c_0 = k^3 - 6k^2 + 14k - 13, \quad c_1 = k^2 - 5k + 7, \quad c_2 = k - 3, \quad c_3 = 1.$$

At level 0 we have the eigenvalue c_0, and at level 1 two principal eigenvalues

$$-c_1 + 2c_2 = -(k^2 - 7k + 13), \quad -c_1 - c_2 = -(k^2 - 4k + 4),$$

with local multiplicities 1 and 2 respectively, and global multiplicity $k-1$. At level 2 we have $d = \min(\ell, b - \ell) = 1$, so there $d + 1 = 2$ principal eigenvalues, and two alternating eigenvalues:

$$c_2 - 2c_3 = k - 5, \quad c_2 + c_3 = k - 2, \quad c_2 - c_3 = k - 4, \quad c_2 + 2c_3 = k - 1,$$

with local multiplicities 1,2,2,1 respectively. It turns out that the principal eigenvalues have global multiplicity $k(k-3)/2$, and the alternating eigenvalues have global multiplicity $(k-1)(k-2)/2$ (compare the case $b = 2$, $\ell = 2$).

Finally, at level 3 we have the eigenvalue $-c_3 = -1$. Its global multiplicity is $k^3 - 6k^2 + 8k - 1$, so this completes the derivation of the chromatic polynomial of $B_n(3)$, as given in Section 1.

The case $b = 4$

In this case the relevant polynomials are

$$c_0 = k^4 - 10k^3 + 41k^2 - 84k + 73, \quad c_1 = k^3 - 9k^2 + 29k - 34,$$

$$c_2 = k^2 - 7k + 13, \quad c_3 = k - 4, \quad c_4 = 1.$$

At level 0 we have the eigenvalue c_0, and at level 1 two principal eigenvalues

$$-c_1 + 3c_2 = -(k^3 - 12k^2 + 50k - 73), \quad -c_1 - c_2 = -(k^3 - 8k^2 + 22k - 21),$$

with multiplicities 1 and 3 respectively. At level 2, general formulae for the three principal eigenvalues were obtained in Section 8, and for the two alternating eigenvalues in Section 9. Putting $b = 4$ in these formulae we obtain the eigenvalues

$$c_2 - 4c_3 + 2c_4 = k^2 - 11k + 31, \quad c_2 - 2c_4 = k^2 - 7k + 11, \quad c_2 + 2c_3 + 2c_4 = k^2 - 5k + 7,$$

$$c_2 - 2c_3 = k^2 - 9k + 21, \quad c_2 + 2c_3 = k^2 - 5k + 5,$$

with multiplicities 1,3,2,3,3 respectively.

The calculation at level 3 presents a new feature, because the nonlinear irreducible representation of Sym_3 is involved. Such representations can be constructed using the methods given by James and Kerber [**9**], for example. In this case the representation is associated with the partition [21] and is generated by the following matrices

$$R^{[21]}(12) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad R^{[21]}(123) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Returning to our theory, we proceed as follows. Enumerate the 3-subsets P of $\{1, 2, 3, 4\}$ in the order 234, 134, 124, 123, and label the rows and columns of M accordingly. This means that M can be written as a 4×4 matrix composed of blocks M^{PQ}, each block being a 6×6 matrix given by the formula in Lemma 5. Here the set F_{PQ} contains only one permutation in each case, and since $c_3 = k - 4$, $c_4 = 1$, we can write $M = Z - (k - 4)I$ where

$$Z = \begin{pmatrix} O & I & X(12) & X(123) \\ I & O & I & X(23) \\ X(12) & I & O & I \\ X(132) & X(23) & I & O \end{pmatrix}.$$

According to Theorem 4, every eigenvalue of M is an eigenvalue of one of the collapsed matrices N^{pri}, N^{alt}, $N^{[21]}$, corresponding to the irreducible representations of Sym_3. For N^{pri} and N^{alt} the eigenvalues are given explicitly by theory developed in Sections 8 and 9.

The matrix $N^{[21]}$ is the 8×8 matrix obtained by replacing the 6×6 matrices $X(\rho)$ by the appropriate 2×2 matrices $R^{[21]}(\rho)^t$. Writing $N^{[21]} = Z^{[21]} - (k - 4)I$, we find that $Z^{[21]}$ has eigenvalues $2, 0, -2$ with multiplicities $3, 2, 3$ respectively, so the eigenvalues of $N^{[21]}$ are $-(k - 6)$, $-(k - 4)$, $-(k - 2)$, with the same multiplicities. Recalling that, as eigenvalues of M, the multiplicities are doubled (because the degree of $R^{[21]}$ is 2), we have the complete set of eigenvalues and multiplicities of M:

$$\begin{array}{rllllll} \text{principal}: & -k+7 & (1) & -k+3 & (3) & & \\ \text{alternating}: & -k+1 & (1) & -k+5 & (3) & & \\ [21]: & -k+6 & (6) & -k+4 & (4) & -k+2 & (6). \end{array}$$

The global multiplicities are as follows: $\frac{1}{6}k(k-1)(k-5)$ for the principal eigenvalues, $\frac{1}{6}(k-1)(k-2)(k-3)$ for the alternating eigenvalues, and $\frac{1}{6}k(k-2)(k-4)$ for the $R^{[21]}$ eigenvalues. (Note that the last expression is generally a half-integer, but everything comes out right because the local multiplicities are double-integers.)

The multiplicity of the level 4 eigenvalue $c_4 = 1$ can now be found by *ad hoc* methods, giving the following result, in agreement with Chang [**8**]:

$$\begin{aligned} P(B_n(4);k) \;=\; & (k^4-10k^3+41k^2-84k+73)^n \\ & + (k-1)\left((-k^3+12k^2-50k+73)^n + 3(-k^3+8k^2-22k+21)^n\right) \\ & + k(k-3)/2\left((k^2-11k+31)^n + 3(k^2-7k+11)^n + 2(k^2-5k+7)^n\right) \\ & + (k-1)(k-2)/2\left(3(k^2-9k+21)^n + 3(k^2-5k+5)^n\right) \\ & + k(k-1)(k-5)/6\left((-k+7)^n + 3(-k+3)^n\right) \\ & + (k-1)(k-2)(k-3)/6\left((-k+1)^n + 3(-k+5)^n\right) \\ & + k(k-2)(k-4)/6\left(6(-k+6)^n + 4(-k+4)^n + 6(-k+2)^n\right) \\ & + k^4-10k^3+29k^2-24k+1. \end{aligned}$$

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