Using Laplacian Eigenvalues and Eigenvectors
in the Analysis of Frequency Assignment Problems[*]

Jan van den Heuvel   and   Snežana Pejić

Department of Mathematics
London School of Economics
Houghton Street, London WC2A 2AE, U.K.

Abstract

A Frequency Assignment Problem (FAP) is the problem that arises when to a given set of transmitters frequencies have to be assigned such that spectrum is used efficiently and the interference between the transmitters is minimal. In this paper we see the frequency assignment problem as a generalised graph colouring problem, where transmitters are presented by vertices and interaction between two transmitters by a weighted edge. We generalise some properties of Laplacian matrices that hold for simple graphs. We investigate the use of Laplacian eigenvalues and eigenvectors as tools in the analysis of properties of a FAP and its generalised chromatic number (the so-called span).

Keywords: Frequency assignment problem, weighted graph, Laplacian matrix, eigenvalues, eigenvectors, span.

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1 Introduction

Many methods for solving Frequency Assignment Problems (FAPs) rely on knowledge about certain structural properties about the problem to facilitate the algorithms. But if no such structural information is given explicitly, it may be very difficult to obtain this information from just the numerical data.

In this note we discuss the possibilities to use algebraic techniques to investigate the structure of Frequency Assignment Problems, using numerical data such as a constraint matrix only. Algebraic objects have several advantages over combinatorial properties. For instance, calculation with high numerical precision is fast and stable, even for large problems. And those properties usually do not depend on the actual representation of a problem, such as order of the nodes, etc.

In order to explain our approach a little further, we first describe some of the underlying assumptions. Frequency Assignment Problems can be described in different ways, using different collections of information. We will always assume that we are given a collection of N transmitters, numbered $1, \ldots, N$. But what is further given may vary from problem to problem. For instance, we may be given information about the geometric position of the set of transmitters, information about the power of the transmitters, and maybe some information about the terrain properties.

For our purposes we assume that all we have available is the *constraint matrix*:

Definition 1.1 *A constraint matrix $W = [w_{ij}]$ is the $N \times N$ matrix such that if f_i denotes the frequency assigned to transmitter i, for $i = 1, \ldots, N$, then in order to limit interference it is required that $|f_i - f_j| \geq w_{ij}$, for all i, j.*

We use the term "transmitter" in a more general sense as a unit to which one channel needs to be assigned. In a system where certain "actual transmitters" need to be assigned more than one channel, we will consider each channel as a separate "transmitter". This makes sense because a transmitter with multiple channels is indistinguishable from a system where there is a number of transmitters of the same power all in more or less the same position.

Definition 1.2 *A feasible assignment of a FAP with N transmitters, given the constraint matrix W, is a labelling $f : \{1, \ldots, N\} \longrightarrow \{ x \in \mathbb{R} \mid x \geq 0 \}$ such that $|f_i - f_j| \geq w_{ij}$, for all i, j, $i \neq j$.*
The span *of a FAP is the minimum over all feasible assignments of the largest label used. By* solving a FAP *we will mean determining its span and a feasible labelling that will give that span.*

Accepting the hypotheses in the definitions above, the constraint matrix is the unique object needed to "solve the FAP". Nevertheless, additional information, such as geometric position of transmitters, may be available, and may be useful for a method to "solve the FAP".

The main aim of the methodology discussed in this note is to obtain some further information about the FAP which may be helpful for algorithms for solving the FAP.

For several reasons, including historical developments of solution methods for FAPs, we will usually discuss FAPs in the context of *weighted graphs*.

Definition 1.3 *A weighted graph (G, w) is a triple $(V(G), E(G), w)$, where $V(G)$ and $E(G)$ form the vertex set and edge set, respectively, of a simple graph G, and w is a weight-function defined on $E(G)$. We will assume that all weights $w(e)$, $e \in E(G)$, are non-negative real numbers.*

If no confusion about the weight can arise, we often just use G to denote a weighted graph.

Also to make notation easier, we identify the situation that there is no edge between two vertices u and v and the situation where there is an edge uv with $w(uv) = 0$.

If a FAP with constraint matrix $W = [w_{ij}]$ is given, then we can easily form a weighted graph G by setting $V(G) = \{1, \ldots, N\}$, joining two vertices i, j whenever $w_{ij} > 0$, and setting the weight of an edge $e = ij$ equal to $w(e) = w_{ij}$. Similarly, given a weighted graph, we can formulate a FAP on the vertex set in the reverse manner.

From now on we will mix both FAPs and weighted graphs, and hence assume we can talk about the *span of a weighted graph G*, notation $sp(G)$, as the span of the related FAP. We also freely use a mixture of FAP-terms and graph theoretical terms. So a vertex and a transmitter should be considered as being the same object.

The *chromatic number $\chi(G)$* of a graph G is the smallest number of labels needed for a labelling in which adjacent vertices must receive different labels. The span of a weighted graph (G, w) is one less than the chromatic number of G if $w \equiv 1$, but the two parameters can be completely different for general w.

As indicated before, we always assume that a weighted graph G has an associated matrix W. We define the *(weighted) degree of a vertex v* as $d(v) = \sum\limits_{u \neq v} w(uv)$.

Definition 1.4 *Given a weighted graph G with associated matrix W, denote by D the diagonal matrix with the degrees of the vertices on the diagonal. Then we define the* Laplacian L of G *as the matrix $L = D - W$; hence $L_{uv} = \begin{cases} -w(uv), & \text{if } u \neq v, \\ d(v), & \text{if } u = v \end{cases}$ (here, and in the definition of degree above, we follow our convention that $w(uv) = 0$ if $uv \notin E(G)$).*

If we want to emphasise the weighted graph determining the Laplacian we denote the Laplacian as $L(G)$.

The matrix L can be seen as a generalisation of the *Laplacian matrix* from algebraic graph theory, in a similar way as W is a generalisation of the *adjacency matrix* of a graph.

Notice that the information contained in the Laplacian is the same as that in the constraint matrix. In that sense, there seems to be no good reason to study algebraic properties of the Laplacian and not the constraint matrix. But it seems that the Laplacian is more useful when one wants to obtain structural information about the underlying graph (or FAP). Also, the Laplacian has some additional algebraic properties (for instance, it is non-negative definite) which give a head start in the algebraic analysis.

Definition 1.5 *Given a weighted graph or a FAP with Laplacian L, a* Laplacian eigenvalue *and a* Laplacian eigenvector *is an eigenvalue and eigenvector of L. So $\lambda \in \mathbb{R}$ is a Laplacian eigenvalue with corresponding Laplacian eigenvector $x \in \mathbb{R}^N$ if $x \neq 0$ and $L x = \lambda x$.*

Since L is a symmetric matrix ($L_{uv} = L_{vu}$) we know that it has N real eigenvalues $\lambda_1, \ldots, \lambda_N$, and a collection of corresponding eigenvectors $x_1, \ldots, x_N$ that form a basis of $\mathbb{R}^N$. Because

of the fairly nice structure of the Laplacian, determining the eigenvalues and eigenvectors is computationally fairly easy, even for larger values of N. Experiments with $N \approx 10,000$ have been performed on a standard PC, where running times were in the order of minutes.

The guiding question in the remainder of this note is:

> *What are the properties of the Laplacian eigenvalues and eigenvectors of a FAP, and what (structural) information regarding the FAP can be obtained from the eigenvalues and eigenvectors?*

Different aspects of this question will be considered in the different sections that follow.

2 Properties of the Laplacian eigenvalues of weighted graphs

2.1 Bounds for the eigenvalues in terms of number of vertices and components of a graph

In this section we obtain bounds for the Laplacian eigenvalues in terms of the number of vertices and the number of components of the weighted graph. These results are inspired by theorems known for simple graphs (see, e.g., [1], [3] and [6]).

Definition 2.1 *The* weighted complete graph $K_N^{(c)}$, *where $c \in \mathbb{R}$, $c > 0$, is the graph on N vertices with all vertices adjacent and all edges have constant weight c.*

Lemma 2.2 *The eigenvalues of the Laplacian L of a weighted complete graph $K_N^{(c)}$ are 0 with multiplicity 1, and cN with multiplicity $N-1$.*

Proof It is easy to see that $L\mathbf{1} = \mathbf{0}$, so we conclude that 0 is a Laplacian eigenvalue of the graph.

Let x be any vector orthogonal to $\mathbf{1}$. Then we easily obtain $Lx = (cN)x$. Since there are exactly $N-1$ linearly independent vectors orthogonal to $\mathbf{1}$, cN is a Laplacian eigenvalue with multiplicity $N-1$. ∎

Definition 2.3 *The* complement $(G, \overline{w})$ *(or just $\overline{G}$) of a weighted graph G is the weighted graph with the same vertex and edge set, and with weights $\overline{w}(e) = C - w(e)$ for an edge $e \in E(G)$, where $C = \max\limits_{e \in E(G)} w(e)$.*

Theorem 2.4 *If λ is a Laplacian eigenvalue of a weighted graph G on N vertices and maximal edge-weight C, then $0 \le \lambda \le CN$.*

A vector orthogonal to $\mathbf{1}$ is an eigenvector of the Laplacian $L(G)$ corresponding to the eigenvalue λ, if and only if it is an eigenvector of the Laplacian $L(\overline{G})$ of the complement $\overline{G}$ corresponding to $CN - \lambda$.

Proof Choose an orientation on the edges of the considered weighted graph G, and suppose G has M edges. The vertex-edge incidence matrix Q of G is the $N \times M$ matrix Q such that

$$
Q_{ve} = \begin{cases} \sqrt{w(e)}, & \text{if edge } e \text{ points toward vertex } v, \\ -\sqrt{w(e)}, & \text{if edge } e \text{ points away from vertex } v, \\ 0, & \text{otherwise.} \end{cases}
$$

Note that this means that for the Laplacian we have $L = Q\,Q^T$.

For an eigenvalue λ of the Laplacian L there exist a vector x, with $\|x\| = 1$, such that $L\,x = \lambda\,x$. Thus $\lambda = \langle \lambda\,x, x \rangle = \langle L\,x, x \rangle = \langle Q\,Q^T x, x \rangle = \langle Q^T x, Q^T x \rangle = \|Q^T x\|^2$. Therefore λ is real and non-negative.

Further, by the definition of the complement of the weighted graph G we have $L(G) + L(\overline{G}) = L(K_n^{(C)})$. If $L(G)\,u = \lambda\,u$ for some vector u orthogonal to $\mathbf{1}$ then by using Lemma 2.2 we find

$$L(\overline{G})\,u \;=\; L(K_n^{(C)})\,u - L(G)\,u \;=\; (CN - \lambda)\,u. \tag{1}$$

Since the Laplacian eigenvalues of $\overline{G}$ are also non-negative, we get $\lambda \leq CN$.

Expression (1) also proves the last part of the theorem. $\blacksquare$

Theorem 2.5 *Let G be a weighted graph with N vertices and maximal edge-weight C. The multiplicity of the eigenvalue 0 equals the number of components of the graph and the multiplicity of CN is equal to one less than the number of components of the complementary graph $\overline{G}$.*

Proof Define the incidence matrix Q as in the previous proof, and let $v_1, v_2, \dots, v_k$ be the vertices of a connected component of the graph G. The sum of the corresponding rows of the incidence matrix Q is $\mathbf{0}^T$ and any $k - 1$ of these rows are independent. We conclude that the rank of the matrix formed from these k rows is $k - 1$ and its nullity is 1. Therefore the nullity of Q, and hence of $L(G) = Q\,Q^T$, is equal to the number of components of the graph, i.e., the multiplicity of the eigenvalue 0 equals the number of components of the graph.

Furthermore, by Theorem 2.4 the value CN is an eigenvalue of $L(G)$ with multiplicity m if and only if 0 is an eigenvalue of $L(\overline{G})$ of multiplicity $m + 1$. The latter is true if and only if $\overline{G}$ has $m + 1$ components. Thus the result follows. $\blacksquare$

Definition 2.6 *The complete $p^{(c)}$-partite graph $K_{N_1, N_2, \dots, N_p}^{(c)}$ is the weighted graph isomorphic to $K_{N_1, N_2, \dots, N_p}$ in which all edges have weight c. In other words, the vertices of $K_{N_1, \dots, N_p}^{(c)}$ can be partitioned into p disjoint sets $V_1, V_2, \dots, V_p$, where each V_i contains N_i vertices, and two vertices are joined by an edge of weight c if and only if the vertices belong to different parts.*

Theorem 2.7 *Let G be a weighted graph on N vertices and maximal edge weight C. Then the multiplicity of CN as an eigenvalue of the Laplacian matrix L is $k - 1$ if and only if G contains a complete $k^{(C)}$-partite graph $K_{N_1, \dots, N_k}^{(C)}$ as a spanning subgraph of G.*

Proof By Theorem 2.5 the multiplicity of CN is equal to one less then k, the number of components of the complementary graph $\overline{G}$. On the other hand the number k of components of $\overline{G}$ gives us the information on the existence of a complete $k^{(C)}$-partite graph $K_{N_1, \dots, N_k}^{(C)}$ on $V(G)$ as a subgraph of G. $\blacksquare$

2.2 Bounds for the eigenvalues in terms of edge weights

In this subsection we state two results concerning upper bounds for the smallest non-zero and the largest Laplacian eigenvalue of a weighted graph. The proofs of the theorems are presented after all terminology needed and theorems are introduced.

Definition 2.8 *The* degree $d(v)$ *of a vertex v of a weighted graph G is the sum of all weights of the edges incident to that vertex, i.e., $d(v) = \sum\limits_{u \neq v} w(uv)$.*

Theorem 2.9 *Let G be a weighted graph on N vertices and suppose its Laplacian L has eigenvalues $0 = \lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_1$. Then we have*

$$\lambda_{N-1} \leq w(uv) + \frac{1}{2}\left(d(u) + d(v)\right),$$

for any two vertices u, v, $u \neq v$.

Definition 2.10 *The* 2-degree $d_2(v)$ *of a vertex v of a weighted graph G is the sum of the degrees of its neighbours times the weight of the edge with which v is joined to that neighbour. So $d_2(v) = \sum\limits_{u \neq v} w(uv)\, d(u)$.*

Theorem 2.11 *Let λ_1 be the largest Laplacian eigenvalue of a connected weighted graph G. Then $\lambda_1 \leq \max\limits_{v \in V(G)} \left\{ d(v) + \dfrac{d_2(v)}{d(v)} \right\}$.*

Before we proceed we shortly recall some of the matrix theory regarding eigenvalues of matrices. Assume that the sequence $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_N(A)$ represent the eigenvalues of a symmetric matrix A in decreasing order, and let the coordinates of the matrix and corresponding vectors be indexed by a set V, $|V| = N$. A useful characterisations of the eigenvalues are given by the Rayleigh's and Courant-Fisher's formulae (see [7]):

$$\lambda_N(A) = \min\{\, \langle A\,x, x \rangle \mid x \in \mathbb{R}^V,\ \|x\| = 1 \,\},$$
$$\lambda_1(A) = \max\{\, \langle A\,x, x \rangle \mid x \in \mathbb{R}^V,\ \|x\| = 1 \,\},$$
$$\lambda_{N-k+1}(A) = \min\{\, \max\{\, \langle A\,x, x \rangle \mid x \in X,\ \|x\| = 1 \,\} \mid X \text{ a } k\text{-dimensional subspace of } \mathbb{R}^V \,\},$$

for $k = 1, \dots, N$.

Also,

$$\lambda_{N-k+1}(A) = \min\{\, \langle A\,x, x \rangle \mid \|x\| = 1,\ x \perp x_i \text{ for } i = N - k + 1, \dots, N \,\},$$

where $x_N, x_{N-1}, \dots, x_{N-k+2}$ are pairwise orthogonal eigenvectors of $\lambda_N, \lambda_{N-1}, \dots, \lambda_{N-k+2}$, respectively.

For a weighted graph G with Laplacian matrix L we have

$$\lambda_{N-1}(L) = \min\{\, \langle L\,x, x \rangle \mid \|x\| = 1,\ x \perp \mathbf{1} \,\},$$

since $\mathbf{1}$ is the eigenvector corresponding to $\lambda_N(L) = 0$.

Now, we can give the proof of Theorem 2.9.

Proof of Theorem 2.9 Choose two vertices u, v, $u \neq v$, and define $x \in \mathbb{R}^N$ by

$$x_w = \begin{cases} 1, & \text{if } w = u, \\ -1, & \text{if } w = v, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, x is orthogonal to $\mathbf{1}$ and therefore we have

$$\lambda_{N-1} \leq \frac{\langle L\,x, x\rangle}{\langle x, x\rangle} = \frac{(L\,x)^T x}{x^T x} = \frac{d(u) + d(v) + 2\,w(uv)}{2},$$

which proves the theorem. ∎

From this theorem it can be concluded that if a weighted graph has at least one pair of non-adjacent vertices u, v, then $\lambda_{N-1} \leq \frac{1}{2}(d(u) + d(v)) \leq \frac{1}{2}(d_{max} + d_{max}) = d_{max}$, where d_{max} denotes the maximum degree.

Note that the value of λ_{N-1} can be greater than d_{max} in the case of a weighted graph in which any two vertices are adjacent. For example $\lambda_{N-1}(K_N^{(c)}) = c\,N > c\,(N-1) = d_{max}(K_N^{(c)})$.

To prove Theorem 2.11, we use Geršgorin Theorem and its corollaries. These are classical results regarding the location of eigenvalues of general matrices, and can be found in, e.g., [5].

Theorem 2.12 (Geršgorin Theorem) *Let $A = [a_{ij}]$ be an $N \times N$ matrix, and let*

$$R_i'(A) = \sum_{j=1,\,j \neq i}^{N} |a_{ij}|, \qquad i \in \{1, \ldots, N\},$$

denote the deleted absolute row sums of A. Then all the (complex) eigenvalues of A are located in the union of N discs

$$\bigcup_{i=1}^{N} \{\, z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i'(A) \,\} = G(A).$$

Furthermore, if a union of k of these N discs forms a connected region in the complex plan that is disjoint from all other $N - k$ discs, then there are precisely k eigenvalues of A in this region.

Corollary 2.13 *Let $A = [a_{ij}]$ be an $N \times N$ matrix and let*

$$C_j'(A) = \sum_{i=1,\,i \neq j}^{N} |a_{ij}|, \qquad j \in \{1, \ldots, N\},$$

denote the deleted absolute column sums of A. Then all the (complex) eigenvalues of A are located in the union of N discs

$$\bigcup_{j=1}^{N} \{\, z \in \mathbb{C} \mid |z - a_{jj}| \leq C_j'(A) \,\} = G(A^T).$$

Furthermore, if a union of k of these N discs forms a connected region that is disjoint from all other $N - k$ discs, then there are precisely k eigenvalues of A in this region.

Theorem 2.12 and Corollary 2.13 together show that all the eigenvalues of A lie in the intersection of the two regions, that is, in $G(A) \cap G(A^T)$.

Corollary 2.14 *If $A = [a_{ij}]$ is an $N \times N$ matrix, then*

$$\max\{\,|\lambda| \mid \lambda \text{ an eigenvalue of } A\,\} \leq \min\Big\{ \max_i \sum_{j=1}^{N} |a_{ij}|, \ \max_j \sum_{i=1}^{N} |a_{ij}| \Big\}.$$

We are now ready to give a proof of Theorem 2.11.

Proof of Theorem 2.11 In a graph with no edges both sides of the inequality are 0. Furthermore, it is enough to consider connected weighted graphs only. Denote by D the diagonal matrix $D = \operatorname{diag}(d(1), d(2), \ldots, d(N))$. Then the matrix $D^{-1}L\,D$ is the matrix given by

$$(D^{-1}L\,D)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -w(uv)\,d(v)/d(u), & \text{if there is an edge joining } u \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$

Since D is invertible, $D^{-1}L\,D$ has the same eigenvalues as L. Therefore, application of Corollary 2.14 to the rows of $D^{-1}L\,D$ gives us

$$\lambda_1 \leq \max_u \Big\{ d(u) + \frac{\sum_v w(uv)\,d(v)}{d(u)} \Big\},$$

and hence the result. ■

2.3 Continuity of the Laplacian eigenvalues

The following properties from the perturbation theory of matrices appear to be important for our research. They support our idea of a possibility to classify FAP problems according to their similarities, i.e., slightly changed initial information in the influence matrix should not lead to a major change in the approach for solving a given problem. This becomes extremely important when we recognise that a new problem can be compared to an "almost similar problem" for which we have developed a way to solve it. But in order to recognise similar problems with our approach, it is essential that the Laplacian eigenvalues from similar FAPs should be comparable as well.

Definition 2.15 *The* eigenvalue vector *of a symmetric matrix is the vector whose entries are the eigenvalues of the matrix in increasing order.*

Definition 2.16 *The Euclidean norm, or ℓ_2-norm, for a vector $x \in \mathbb{C}^N$ is*

$$\|x\|_2 = \Big(\sum_{i=1}^{N} |x_i|^2 \Big)^{1/2}.$$

And the Euclidean norm or ℓ_2-norm of an $N \times N$ matrix A is

$$\|A\|_2 = \Big(\sum_{i,j=1}^{N} |a_{ij}|^2 \Big)^{1/2}.$$

Our aim in the remainder of this section is to prove that small changes in a Laplacian matrix bring small changes to its eigenvalue vector. In other words we want to establish:

Theorem 2.17 *Let T be the operator that maps the set of symmetric matrices into the set of vectors, such that the image of a matrix of size N is its eigenvalue vector. Then T is a continuous operator.*

Before we continue we recall the definition of a normal matrix.

Definition 2.18 *A matrix A is said to be* normal *if $A^*A = AA^*$, where A^* denotes the Hermitian adjoint of A. That is, A is normal if it commutes with its Hermitian adjoint.*

The following theorem is the crucial for us. Both the theorem and its proof can be found in [5].

Theorem 2.19 (Hoffman and Wielandt) *Let A and E be $N \times N$ matrices, such that A and $A + E$ are both normal. Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of A in some given order, and let $\lambda'_1, \dots, \lambda'_N$ be the eigenvalues of $A + E$ in some order. Then there exists a permutation σ of the integers $1, 2, \dots, N$ such that*

$$\left[\sum_{i=1}^{N} |\lambda'_{\sigma(i)} - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2.$$

This result has a corollary regarding symmetric matrices which more or less directly answers our query and therefore proves Theorem 2.17.

Corollary 2.20 *Let A and E be $N \times N$ matrices, such that A and $A + E$ are real symmetric. Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of A arranged in increasing order ($\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$), and let $\lambda'_1, \dots, \lambda'_N$ be the eigenvalues of $A + E$ also ordered increasingly ($\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_N$). Then we have*

$$\left[\sum_{i=1}^{N} |\lambda'_i - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2.$$

Proof By Theorem 2.19, there is some permutation σ of the integers $1, 2, \dots, N$ for which $\left[\sum_{i=1}^{N} |\lambda'_{\sigma(i)} - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2$. If the eigenvalues of $A + E$ in the list $\lambda'_{\sigma(1)}, \dots, \lambda'_{\sigma(N)}$ are already in increasing order, then there is nothing to prove. Otherwise, there are two successive eigenvalues in the list that are not ordered in this way, say

$$\lambda'_{\sigma(k)} > \lambda'_{\sigma(k+1)}, \quad \text{for some } k \in \{1, \dots, N-1\}.$$

But since

$$|\lambda'_{\sigma(k)} - \lambda_k|^2 + |\lambda'_{\sigma(k+1)} - \lambda_{k+1}|^2$$
$$= |\lambda'_{\sigma(k+1)} - \lambda_k|^2 + |\lambda'_{\sigma(k)} - \lambda_{k+1}|^2 + 2(\lambda_k - \lambda_{k+1}) \cdot (\lambda'_{\sigma(k+1)} - \lambda'_{\sigma(k)}),$$

and since $\lambda_k - \lambda_{k+1} \leq 0$ by assumption, we see that

$$|\lambda'_{\sigma(k)} - \lambda_k|^2 + |\lambda'_{\sigma(k+1)} - \lambda_{k+1}|^2 \geq |\lambda'_{\sigma(k+1)} - \lambda_k|^2 + |\lambda'_{\sigma(k)} - \lambda_{k+1}|^2.$$

Thus, the two eigenvalues $\lambda'_{\sigma(k)}$ and $\lambda'_{\sigma(k+1)}$ can be interchanged without increasing the sum of squared differences. By a finite sequence of such interchanges, the list of eigenvalues $\lambda'_{\sigma(1)}, \ldots, \lambda'_{\sigma(N)}$ can be transformed into the list $\lambda'_1, \lambda'_2, \ldots, \lambda'_N$ in which the real parts are increasing and the asserted bound holds. $\blacksquare$

For real symmetric matrices A and B and their eigenvalues arranged in increasing order we have

$$\left[\sum_{i=1}^{N} |\lambda_i(A) - \lambda_i(B)|^2 \right]^{1/2} \leq \|A - B\|_2$$

and this is exactly the relation that proves Theorem 2.17.

3 Laplacian eigenvalues and eigenvectors and their connection to structural properties of weighted graphs

Since a matrix is not uniquely characterised by its collection of eigenvalues, one cannot expect all important information of a FAP to be obtainable from the set of Laplacian eigenvalues. But when we combine the eigenvalues with a full collection of eigenvectors, then we can reconstruct the matrix. So we can expect to obtain more detailed information about a FAP by considering both the eigenvalues and certain eigenvectors simultaneously.

The following are some results on possible structural properties of a FAP that can be recognised using the combination of eigenvalues and eigenvectors.

We mainly write the theorems using graph theory terminology, but sometimes rephrase them in radio terms so that their relation to frequency assignment problems is clear.

The first result is useful to check whether a given FAP can be divided into independent FAPs of smaller size.

Theorem 3.1 *Let G be a weighted graph on N vertices. The multiplicity k of the Laplacian eigenvalue 0 is equal to the number of components of G.*

If each components has N_i vertices, $i = 1, \ldots, k$, then the values of N_i and the vertices in each component can be recognised as follows: The $N \times k$ matrix M whose columns are k independent eigenvectors that correspond to the eigenvalue 0 contains k different rows $r^{(i)}$, $i = 1, \ldots, k$, where row $r^{(i)}$ occurs N_i times. Moreover, indices of rows that are identical correspond to vertices belonging to the same component of G.

Proof Theorem 2.5 already explains that the multiplicity of the eigenvalue 0 represents the number of components of the corresponding weighted graph.

Let us, therefore, concentrate on the second part of the theorem about the connection between coordinates of the eigenvectors and the components of the graph. Consider the matrix M, described in the statement of the theorem; where we claim that identical rows of M correspond to the vertices from the same component.

The fact that G has k connected components of sizes $N_1, \ldots, N_k$ helps to reconstruct the Laplacian matrix L of G. Without loss of generality we can assume that the first N_1 rows of L

correspond to the first component, the second N_2 rows correspond to the second component, etc. The Laplacian matrix is then of the form:

$$
L = \begin{bmatrix}
L_{11} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & L_{22} & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & L_{kk}
\end{bmatrix},
$$

where each L_{ii} is an $N_i \times N_i$ submatrix that represents the Laplacian matrix of the i-th component of G.

Define the vector x_i as the vector with coordinates 1 for all coordinates that correspond to the component i and 0 everywhere else, for $i = 1, \ldots, k$. And let y be any vector of the form

$$ y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k, $$

where the α_i's are real numbers.

It is easy to prove that $L y = 0 y$, and since there are exactly k linearly independent choices for y, we can conclude that any such choice gives a basis of the Laplacian eigenspace that corresponds to the eigenvalue 0. This also means that a matrix having these k eigenvectors as its columns, for any choice of k independent eigenvectors, is such that it contains at most k different rows of which the first N_1 are identical (to $r^{(1)}$, say), the second N_2 are identical, (to $r^{(2)}$), etc. In fact, this matrix has exactly k distinct rows, for the rank of the matrix is k. This proves the existence of a bijection between the rows of the matrix M and the vertices of G so that indices of the identical rows are the indices of the vertices that belong to the same component of the graph G. ■

Next, we describe a method that can identify channels belonging to a multiple-channel transmitter in a given set of channels.

Theorem 3.2 *A weighted graph G on N vertices contains a collection S of $n > 1$ vertices with the following properties:*

- $w(uv) = w$, *for some constant w, for all $u, v \in S$, $u \neq v$;*
- $w(uv) = w_v$ *(i.e., w_v is only depending on v) for all $u \in S$ and $v \in V(G) \setminus S$;*

if and only if there is a Laplacian eigenvalue of multiplicity $n-1$ such that the $n-1$ eigenvectors have the following form:

- *they are orthogonal to $\mathbf{1}$;*
- *there exist $N - n$ coordinates for which the $n - 1$ eigenvectors are all 0.*

Moreover, the n coordinates for which not all $n-1$ eigenvectors are 0 represent the vertices that belong to the set S.

Proof First suppose that a collection S as described in the theorem exist. Without loss of generality we can assume that the first n rows of the Laplacian matrix L of G correspond to

the n vertices in S. Therefore we have the following form of L:

$$L \; = \; \left[\begin{array}{cccc|ccc} d & -w & \cdots & -w & \cdots & -w_v & \cdots \\ -w & d & \ddots & \vdots & & \vdots & \\ \vdots & \ddots & \ddots & -w & & \vdots & \\ -w & \cdots & -w & d & \cdots & -w_v & \cdots \\ \hline \vdots & & & \vdots & & & \\ -w_v & \cdots & \cdots & -w_v & & & \\ \vdots & & & \vdots & & & \end{array}\right],$$

where d is the degree of a channel in S (which is constant), and v is any transmitter outside S.

It is not difficult to prove that any vector of the form $x = [\alpha_1, \alpha_2, \cdots, \alpha_n, 0, \cdots, 0]^T$, where $\sum_i \alpha_i = 0$ (i.e., x is orthogonal to $\mathbf{1}$), is an eigenvector of L with eigenvalue $d + w$. And there are $n - 1$ linearly independent vertices of this form. This proves the "only if" part of the theorem.

Next suppose that there is an eigenvalue λ of multiplicity $n - 1$, and denote the $n - 1$ eigenvectors from the statement of the theorem by $x^{(1)}, x^{(2)}, \cdots, x^{(n-1)}$. Again, we can assume that the coordinates for which not all eigenvectors are equal to 0 are the first n. So each $x^{(k)}$ looks like

$$x^{(k)} \; = \; [\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots, \alpha_n^{(k)}, 0, \cdots, 0]^T.$$

Writing out the relevant coordinates involved in the equations $L\, x^{(k)} = \lambda\, x^{(k)}$ gives the following:

$$\left[\begin{array}{cccc|cc} x_{11} & x_{12} & \cdots & x_{1n} & \cdots & \cdots \\ x_{21} & x_{22} & \cdots & x_{2n} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} & \cdots & \cdots \\ \hline \vdots & \vdots & & \vdots & \cdots & \cdots \\ y_{v1} & y_{v2} & \cdots & y_{vn} & & \\ \vdots & \vdots & & \vdots & \cdots & \cdots \end{array}\right] \left[\begin{array}{c} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \\ 0 \\ \vdots \\ 0 \end{array}\right] \; = \; \lambda \left[\begin{array}{c} \alpha_1^{(k)} \\ \alpha_2^{(k)} \\ \vdots \\ \alpha_n^{(k)} \\ 0 \\ \vdots \\ 0 \end{array}\right], \qquad (2)$$

where v is some vector outside the first n. In order to prove this part of the theorem we need to show that

- for every vector v outside the first n, the vector $[y_{v1}, y_{v2}, \cdots, y_{vn}]^T$ is a multiple of $\mathbf{1}$;
- there exist constants d and w such that $x_{ii} = d$ for all $i = 1, \ldots, n$, and $x_{ij} = -w$ for $i \neq j$.

The first part we can do by considering the $n - 1$ linear equations that we obtain for $y_{v1}, y_{v2}, \cdots, y_{vn}$ using the matrix equation in (2) for each eigenvector. This gives:

$$y_{v1}\, \alpha_1^{(1)} + y_{v2}\, \alpha_2^{(1)} + \cdots + y_{vn}\, \alpha_n^{(1)} \; = \; 0,$$
$$y_{v1}\, \alpha_1^{(2)} + y_{v2}\, \alpha_2^{(2)} + \cdots + y_{vn}\, \alpha_n^{(2)} \; = \; 0,$$
$$\cdots$$
$$y_{v1}\, \alpha_1^{(n-1)} + y_{v2}\, \alpha_2^{(n-1)} + \cdots + y_{vn}\, \alpha_n^{(n-1)} \; = \; 0.$$

Since the $n-1$ eigenvectors are independent, the system above is a homogeneous system of $n-1$ linearly independent equations with n unknowns. Using the condition of the theorem that the sum of all entries in each eigenvector is 0, we get that the a solution is: $y_{v1} = y_{v2} = \cdots = y_{vn} = 1$, i.e, $[y_{v1}, \dots, y_{vn}]^T = \mathbf{1}$. But since the solution space has dimension 1, all solutions must be a multiple of $\mathbf{1}$, as required.

The approach to solve the second part is similar. Writing out the top line of (2) for each eigenvector gives the system:

$$
\begin{aligned}
x_{11}\,\alpha_1^{(1)} + x_{12}\,\alpha_2^{(1)} + \cdots + x_{1n}\,\alpha_n^{(1)} &= \lambda\,\alpha_1^{(1)}, \\
x_{11}\,\alpha_1^{(2)} + x_{12}\,\alpha_2^{(2)} + \cdots + x_{1n}\,\alpha_n^{(2)} &= \lambda\,\alpha_1^{(2)}, \\
&\cdots \\
x_{11}\,\alpha_1^{(n-1)} + x_{12}\,\alpha_2^{(n-1)} + \cdots + x_{1n}\,\alpha_n^{(n-1)} &= \lambda\,\alpha_1^{(n-1)}.
\end{aligned}
$$

Since the eigenvectors are linearly independent, we find that the solution space must be 1-dimensional, which means that every solution must be of the form $x_{11} = \lambda + k_1$, $x_{12} = \cdots = x_{1n} = k_1$, for some $k_1 \in \mathbb{R}$ (use that $\sum_j \alpha_j^{(k)} = 0$ for all k). Repeating this for the other lines in the matrix equation gives that we must have $x_{ii} = \lambda + k_i$, and $x_{i1} = \cdots = x_{i(i-1)} = x_{i(i+1)} = \cdots = x_{in} = k_i$, for some $k_i \in \mathbb{R}$, for all $i = 1, \dots, n$. Because of the symmetry of the Laplacian we get that all k_i must be equal, say to a real number w. So we obtain that there exist real number w such that

$$
x_{ij} = \begin{cases} \lambda + w, & \text{if } i = j, \\ w & \text{if } i \neq j. \end{cases}
$$

Setting $d = \lambda + w$ completes the proof of the theorem. ∎

Corollary 3.3 *Let L be the Laplacian matrix of a FAP with N channels. This system contains a collection S of $n > 1$ channels with the following properties:*

- *the n channels all belong to the same transmitter, with mutual influence w, for some constant w;*

- *the influence between any of the channels in S and a channel outside S does not depend on the channel in S;*

if and only if there is a Laplacian eigenvalue of multiplicity $n-1$ such that the $n-1$ eigenvectors have the following form:

- *they are orthogonal to $\mathbf{1}$;*

- *there exist $N - n$ coordinates for which the $n - 1$ eigenvectors are all 0.*

Moreover, the n coordinates for which not all $n-1$ eigenvectors are 0 represent the channels that belong to the set S.

The following theorem explores whether a given set of transmitters can be separated into subsets so that vertices from different subsets influence one another maximally. Recall the definition of the complete $p^{(c)}$-partite graph $K_{N_1,N_2,\dots,N_p}^{(c)}$ in Definition 2.6.

Theorem 3.4 *Let G be a weighted graph on N vertices and maximal edge weight C. The graph G contains a complete $k^{(C)}$-partite graph $K_{N_1,\ldots,N_k}^{(C)}$ as a spanning subgraph if and only if the multiplicity of CN as an eigenvalue of the Laplacian matrix L is $k-1$.*

Further, the $N \times (k-1)$ matrix M whose columns are $k-1$ independent eigenvectors that correspond to the eigenvalue CN contains k different rows $r^{(i)}$, $i = 1,\ldots,k$ where each row $r^{(i)}$ occurs N_i times. Moreover, indices of rows that are identical correspond to vertices belonging to the same part V_i of the $k^{(C)}$-partition of G.

Proof In Theorem 2.7 we have already proved that multiplicity of CN is equal to $k-1$ if and only if G contains a complete $k^{(C)}$-partite graph $K_{N_1,\ldots,N_k}^{(c)}$ as a spanning subgraph. The main point for discussion here is the remaining part of the theorem, describing the relation between entries of the eigenvectors and the partition of vectors.

In order to realize which vertices of G belong to which part we consider $k-1$ independent eigenvectors that correspond to CN. More precisely we look at the matrix M, described in the statement of the theorem. Our claim is that identical rows of M correspond to vertices in the same part.

In order to this, consider the complementary weighted graph $\overline{G}$. Following Theorem 2.4, we see that $\overline{G}$ has a Laplacian eigenvalue 0 of dimension k, and all eigenvectors of $L(G)$ for the eigenvalue CN are also eigenvectors of $L(\overline{G})$ for the eigenvalue 0. Additionally, the all-1 vector $\mathbf{1}$ is an eigenvector of $L(\overline{G})$ corresponding to the eigenvalue 0. Using this information in Theorem 3.1 immediately reveals that $\overline{G}$ has a structure consisting of k components, where the vertices of each component can be found by looking at identical rows in the matrix of eigenvectors (the extra vector $\mathbf{1}$ makes no difference here). Going back to the original graph G, this means a structure for G as described in the theorem. ∎

Corollary 3.5 *Let L be the Laplacian of a FAP with N channels, where the maximal interference between any pair is C. This system contains a partition $S_1, S_2, \ldots, S_k$ of the channels into k parts with size N_i $i = 1,\ldots,k$, with the following properties:*

- *the interference between any pair of vertices from different pairs is C;*

if and only if there is a Laplacian eigenvalue CN of multiplicity $k-1$ such that the $k-1$ corresponding eigenvectors have the following form:

- *they are orthogonal to $\mathbf{1}$;*

- *he $N \times (k-1)$ matrix M whose columns are the $k-1$ eigenvectors contains k different rows $r^{(i)}$, $i = 1,\ldots,k$ where each row $r^{(i)}$ occurs N_i times.*

Moreover, indices of rows in M that are identical correspond to channels belonging to the same part S_i.

In Theorem 4.6 in the next section we will show that knowledge about the structure of a FAP along the lines of the corollary above, can be very useful when trying to find the span of the FAP.

4   Determining the span using the information provided by algebraic methods

Let us recall the definition of the span from the introduction.

Definition 4.1 *The span of a weighted graph G is the minimum over all feasible assignments of the largest label used.*

We have already mentioned that we don't see it as our task to compute the span for given FAPs, but our plan is to investigate how much the eigenvalues and eigenvectors can help in discovering useful structures of the FAP that can give better approximations for the values of the span. Some first results in that direction follow.

4.1   The chromatic number of a weighted graph

Definition 4.2 *The* chromatic number $\chi(G)$ *of a graph G (weighted or not) is the smallest number of labels needed for a labelling in which adjacent vertices must receive different labels.*
An alternative definition is to say that the chromatic number of a weighted graph with all weights positive is the smallest number of labels needed in a feasible assignment.

Note that the chromatic number carries less information for frequency assignment then the span does, but it is certainly a useful parameter to get some ideas about the order of the span.
The following theorem gives bounds on the chromatic number of weighted graphs.

Theorem 4.3 *Let G be a connected weighted graph on N vertices, with at least one pair of non-adjacent vertices. Let $\lambda_1, \dots, \lambda_N$ be the Laplacian eigenvalues of G in decreasing order and denote by d_{max} the maximal vertex degree of G. Then we have:*

- *If $\lambda_{N-1} < d_{max}$, then the chromatic number $\chi(G)$ satisfies:*

$$\chi(G) \geq \frac{\lambda_1 - 2\,\lambda_{N-1}}{d_{max} - \lambda_{N-1}}.$$

- *If $\lambda_{N-1} = \lambda_{N-2} = \dots = \lambda_{N-k+1} = d_{max}$ and $\lambda_{N-k} > d_{max}$, then*

$$\chi(G) \leq \frac{(k+1)\,\lambda_{N-k} - \lambda_1 - (k-1)\,d_{max}}{\lambda_{N-k} - d_{max}}.$$

Recall from the paragraph following the proof of Theorem 2.9 that for a weighted graph with at least one pair of non-adjacent vertices, we have that the second smallest Laplacian eigenvalue λ_{N-1} is smaller than or equal to the maximum vertex degree. Note that the condition that G must have at least one non-adjacent pair of vertices is not really a restriction. Any graph that fails that restriction has all vertices adjacent, which means that the chromatic number is equal to the number of vertices.
The following result from matrix analysis is used in the proof of Theorem 4.3. Its proof can be found in [4].

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Lemma 4.4 *Let A be a real symmetric matrix of order N, and let $S_1, \ldots, S_t$, $t \geq 2$, be a partition of $\{1, \ldots, N\}$ into non-empty subsets. Denote by A_{kk} the submatrix of A with row and column indices from S_k. If $0 \leq i_k \leq |S_k|$, $k = 1, \ldots, t$, then*

$$\lambda_{i_1 + i_2 + \ldots + i_t + 1}(A) + \sum_{i=1}^{t-1} \lambda_{n-i+1}(A) \leq \sum_{k=1}^{t} \lambda_{i_k+1}(A_{kk}),$$

where $\lambda_i(X)$, $i = 1, 2, \ldots$ are the eigenvalues of the matrix X in decreasing order.

Proof of Theorem 4.3. Set $t = \chi(G)$. There exists a partition $S_1, \ldots, S_t$ of the vertices of G such that each of the subgraphs of G induced by S_i contains no edges. With $i_1 = i_2 = \cdots = i_t = 0$, Lemma 4.4 gives:

$$\lambda_1 + \sum_{i=1}^{t-1} \lambda_{N-i+1} \leq \sum_{k=1}^{t} \lambda_1(L_{kk}). \tag{3}$$

Furthermore, since $\lambda_N = 0$, we get:

$$\sum_{i=1}^{t-1} \lambda_{N-i+1} = 0 + \lambda_{N-1} + \sum_{i=3}^{t-1} \lambda_{N-i+1} \geq (t-2)\lambda_{N-1}. \tag{4}$$

Since the sum of the eigenvalues of a symmetric matrix is equal to the sum of its diagonal elements, and the diagonal elements of L are the degrees of the vertices, we get for each k that

$$\sum_{i=1}^{|S_k|} \lambda_i(L_{kk}) = \sum_{v \in S_k} d(v) \leq |S_k| d_{max}.$$

Since the L_{kk} are diagonal matrices, their eigenvalues are just the elements on the diagonal. So we have for all k, $\lambda_1(L_{kk}) = \max v \in S_k\{d(v)\} \leq d_{max}$, hence

$$\sum_{k=1}^{t} \lambda_1(L_{kk}) \leq t\, d_{max}. \tag{5}$$

The last two inequalities (4) and (5) together with (3) give:

$$\lambda_1 + (t-2)\lambda_{N-1} \leq t\, d_{max}. \tag{6}$$

If $\lambda_{N-1} < d_{max}$, then after simplifying expression (6) we get:

$$\chi(G) = t \geq \frac{\lambda_1 - 2\lambda_{N-1}}{d_{max} - \lambda_{N-1}}.$$

In the second case if $\lambda_{N-1} = d_{max}$, then there is a k such that $\lambda_{N-2} = \cdots = \lambda_{N-k+1} = d_{max}$ and $\lambda_{N-k} > d_{max}$. Then we can write

$$\sum_{i=1}^{t-1} \lambda_{N-i+1} = 0 + \lambda_{N-1} + \cdots + \lambda_{N-k+1} + \sum_{i=k+1}^{t-1} \lambda_{N-i+1}$$

$$\geq (k-1)\, d_{max} + (t-k-1)\, \lambda_{N-k}. \tag{7}$$

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From equations (5) and (7) we obtain

$$\lambda_1 + (k-1)\,d_{max} + (t-k-1)\,\lambda_{N-k} \;\leq\; t\,d_{max},$$

which is equivalent to

$$\chi(G) \;=\; t \;\leq\; \frac{(k+1)\,\lambda_{N-k} - \lambda_1 - (k-1)\,d_{max}}{\lambda_{N-k} - d_{max}}.$$

as required. ∎

The following theorem gives another way of finding an upper bound for the chromatic number.

Theorem 4.5 *Let G be a connected weighted graph on N vertices. Let $\lambda_1, \ldots, \lambda_N$ be the Laplacian eigenvalues of G in decreasing order and denote by d_{total} the sum of the degrees of all vertices of G. Then we have:*

$$\chi(G) \;\leq\; 2 + \frac{d_{total} - \lambda_1}{\lambda_{N-1}}.$$

Proof We follow the proof of Theorem 4.3 until equation (4). Then we continue with the approximation $\sum\limits_{k=1}^{t} \lambda_1(L_{kk}) \leq d_{total}$, and we get $t\,\lambda_{N-1} \leq d_{total} - \lambda_1 + 2\,\lambda_{N-1}$. The result follows, since λ_{N-1} is always a positive number for connected graphs. ∎

4.2 The span of weighted graphs with a specific structure

In the introduction we mentioned the search for useful structures, where by useful we mean structural information that can assists us in determining the span of a FAP. Many existing algorithms use properties such as the existence of (near-)cliques to assist them in the search for good solutions. But finding these structural properties may be equally hard to solving the FAP. Algebraic methods have at least the property that they are fairly easy to compute.

We can actually give an example of a structural property that can be determined by algebraic means and provides information about the span of a weighted graph. The following is a consequence of Theorem 3.4.

Theorem 4.6 *Let G be a weighted graph on N vertices and maximal edge weight C. If the graph has a Laplacian eigenvalue CN with multiplicity $k-1$, then G contains a complete $k^{(C)}$-partite graph $K_{N_1,\ldots,N_k}^{(C)}$ as a spanning subgraph.*

For $i = 1, \ldots, k$, let S_i denote the weighted subgraph of G spanned by the vertices belonging to part i of $K_{N_1,\ldots,N_k}^{(C)}$ and let s_i denotes the span of S_i. Then the span $sp(G)$ of the whole graph G is equal to $\sum\limits_{i=1}^{k} s_i + (k-1)\,C$.

Proof We first show that we can find a feasible labelling for G using labels from 0 to $\sum\limits_{i=1}^{k} s_i + (k-1)\,C$. Namely, we find an optimal assignment, using labels in $[0, s_i]$ for each of the subgraphs S_i, independently. Then we combine these "independent" assignments in the following way:

Keep the labels of S_1 unchanged. Increase each label of S_2 by $s_1 + C$; increase each label of S_3 by $s_1 + s_2 + 2C$, etc. In other words, any label assigned to a vertex in S_j, for some j, will be increased by $\sum_{i=1}^{j-1} s_i + (j-1)C$. It is easy to see that the resulting labelling is feasible and that the maximum label used is $\sum_{i=1}^{k} s_i + (k-1)C$, which implies

$$sp(G) \leq \sum_{i=1}^{k} s_i + (k-1)C. \tag{8}$$

We still have to prove that it is not possible to do any better, so that the span is at least the sum in the right hand side of (8). We will prove this by induction on k.

If there is only one part ($k = 1$), then there is nothing to prove.

So suppose that the theorem is true for $k-1$, $k \geq 2$, i.e., if a weighted graph with maximum weight C contains a complete $(k-1)^{(C)}$-partite graph $K_{N_1,\cdots,N_{k-1}}^{(C)}$ as a spanning subgraph, then the span is at least $\sum_{i=1}^{k-1} s_i + (k-2)C$. And we will prove that under this assumption the statement is true for k.

So consider a feasible labelling of the weighted graph G with maximum weight C for which the maximum label is as small as possible. We will refer to this particular labelling as $\mathcal{L}$, and to its .

Now consider the subgraph G' spanned by vertices of the parts $S_1, S_2, \ldots, S_{k-1}$. Suppose that the labelling $\mathcal{L}$ labels the vertices of G' using labels from $[p, p']$, for some $p, p' \geq 0$, where at least one vertex gets label p and another one has label p'. Since G' contains $K_{N_1,N_2,\cdots,N_{k-1}}^{(c)}$ as a spanning subgraph, by induction this means that the span of G' is at least $\sum_{i=1}^{k-1} s_i + (k-2)C$. So we certainly must have $p' - p \geq \sum_{i=1}^{k-1} s_i + (k-2)C$.

Next look at the labels that $\mathcal{L}$ assigns to the vertices of S_k. Suppose these labels are in the in the range $[q, q']$, for some $q, q' \geq 0$, where again the labels q and q' actually occur. If $p' \leq q$, then we must have in fact $p' + C \leq q$. Since the span of the graph spanned by S_k is s_k, we get that $q' - q \geq s_k$. This gives for the largest label in $\mathcal{L}$, which is q' in this case:

$$q' \geq q + s_k \geq p' + C + s_k \geq \sum_{i=1}^{k-1} s_i + (k-2)C + p + C + s_k = \sum_{i=1}^{k} s_i + (k-1)C + p,$$

which would complete the proof. If $q' \leq p$, then we are done using a similar argument.

So we are left to consider the cases $p < q < p'$ or $q < p < q'$. Since these two cases are more or less symmetric, we only consider the first case. This means that the labelling $\mathcal{L}$ uses some labels t and t' for G' such that in the interval (t, t') no other label is used for G', but there are vertices in S_k that get labels from (t, t'). Let a be the smallest and a' be the largest label in (t, t') which is assigned to some vertex in S_k. Then we have in fact $t + C \leq a \leq a' \leq t' - C$. Suppose the largest label in $\mathcal{L}$ is m.

Now we transform the labelling $\mathcal{L}$ as follows:

- All labels in the range $[0, t]$ stay the same.

- Every vertex which has a label b in the range (t, t') (all these vertices are in S_k) will be relabelled to $b + (m + C - t')$.

- Vertices that have a label b larger than or equal to t' are relabelled with $b - (t' - a)$.

It is straightforward to check that this transformation gives a new feasible labelling $\overline{\mathcal{L}}$ of G. Moreover, the largest label $\overline{m}$ of $\overline{\mathcal{L}}$ will be $\overline{m} = a' + (m + C - t') \leq m$, since $a' \leq t' - C$. Since $\mathcal{L}$ was a feasible labelling with minimal largest label, we in fact must have $\overline{m} = m$.

The largest label that a vertex in G' receives in $\overline{\mathcal{L}}$ is $\overline{p}' = p' - (t' - a) \leq p' - C$. And the smallest label that in vertex in S_k receives in $\mathcal{L}'$ either remains q (if $q < t$) or it is increased to $\overline{q} = q + (m + C - t') \geq q + C$ ($q = t$). We conclude that the described transformation gives a new feasible labelling $\overline{\mathcal{L}}$ with the same smallest and largest label and in which the numbers of labels of S_k occurring between labels of G' is smaller than in $\mathcal{L}$.

Repeating the transformation, if necessary, gives as an end result a feasible labelling of the type $p' \leq q$ with the same smallest and largest label. And we have shown earlier in the proof that the existence of such a labelling proves the theorem. ∎

5 Eigenvalues of Laplacian matrices for specific FAPs

In this section we will determine eigenvalues and eigenvectors of the Laplacian matrix for the following FAPs:

- A system of one transmitter with multiple channels.

- The FAP formed by transmitters regularly placed on a square grid, with influence based on a collection of constraints that are of the same type for each transmitter.

- The FAP formed by transmitters regularly placed on a triangular grid, with influence based on a collection of constraints that are of the same type for each transmitter.

5.1 FAP with one multiple-channel transmitter

In the introduction we described that a multiple-channel transmitter with N channels would be considered as a set of several N transmitters whose influence matrix is an $N \times N$ matrix with zeros on the diagonal and a constant C on all other positions, where C is the channel separation between any two channels. The spectrum for the Laplacian matrix L of such system has an eigenvalue 0 with multiplicity 1, and an eigenvalue CN with multiplicity $N - 1$. In particular, apart from one eigenvalue 0, all other eigenvalues are concentrated in one large number.

5.2 Infinite grids

The approach in this section is taken from [2]. In order to discuss Laplacian eigenvalues of FAPs defined on infinite grids, we represent them as weighted graphs defined on two-dimensional integer lattices. The vertices of the graphs are the points in $\mathbb{Z}^2$, i.e., points of the form $v_x = (x_1, x_2)$, where x_1 and x_2 are integers. The edges are determine by an 'edge-defining

multi-set' E' such that two vertices v_x and v_y are joined if and only if the ordered pair $v_y - v_x = (y_1 - x_1, y_2 - x_2)$ belongs to E'.

Since we are now working with infinite graphs, we must extend our algebraic methods. We will not give full details, but only give an informal description. For an infinite graph with finite degrees it is possible to define the *Laplacian operator* L. If f is a function defined on the set of vertices V, then the function $L f$ is defined by

$$(L f)(v) \;=\; \sum_{uv \in E} [f(v) - f(u)],$$

where E denotes the set of edges of the graph. Generalising the finite case, we call a number λ an eigenvalue of L if there exists a non-zero function f such that $L f = \lambda f$, i.e., such that for all $v \in V$ we have

$$\lambda f(v) \;=\; \sum_{uv \in E} [f(v) - f(u)].$$

For the grids defined as above, this becomes equivalent to

$$\lambda f(x_1, x_2) \;=\; \sum_{(y_i, y_2) \in \mathbb{Z}^2, \, (y_1 - x_1, y_2 - x_2) \in E'} [f(x_1, x_2) - f(y_1, y_2)], \qquad \text{for all } (x_1, x_2) \in \mathbb{Z}^2.$$

In order to overcome the problem that a suitable function f must satisfy an infinite number of conditions, we use the translational symmetry of the lattice graphs. Specifically we define a function f by assigning a value f_0 to the vertex $(0, 0)$: $f(0, 0) = f_0$. Then we assign values to the remaining vertices (x_1, x_2) as follows:

$$f(x_1, x_2) \;=\; h_1^{x_1} \, h_2^{x_2} \, f_0, \qquad \text{for all } (x_1, x_2) \in \mathbb{Z}^2,$$

for certain h_1, h_2. It can be shown that this construction gives enough eigenvalues and eigenvectors, enabling us to give detailed information on the algebraic properties of infinite grid FAPs.

The infinite square grid

As stated before, we assume the vertices of the two-dimensional square grid we set vertices to be the elements of $\mathbb{Z}^2$. The edge-defining multi-set E' is $\{(0, 1), (0, -1), (1, 0), (-1, 0)\}$, and we assume that all edges have weight one, i.e., there is a constraint one between any two "adjacent" transmitters.

As described above, in order to find the eigenvectors we consider function f on the vertex set with $f((0, 0)) = f_0$. By the given information on the edge-defining multi-set E' and constraints, it follows that the neighbouring vertices to $(0, 0)$ are $(0, 1)$, $(-1, 0)$, $(1, 0)$ and $(0, -1)$ and therefore the $\lambda f = L f$ condition at $(0, 0)$ is:

$$\lambda f(0, 0) \;=\; [f(0, 0) - f(1, 0)] + [f(0, 0) - f(-1, 0)] + [f(0, 0) - f(0, 1)] + [f(0, 0) - f(0, -1)].$$

Substituting $f(x_1, x_2) = h_1^{x_1} \, h_2^{x_2} \, f_0$ for these points gives the equation

$$\lambda f_0 \;=\; \left([1 - h_1] + [1 - h_1^{-1}] + [1 - h_2] + [1 - h_2^{-1}] \right) f_0,$$

which can be simplified to

$$\lambda = 4 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2}.$$

In particular we see that every choice for h_1, h_2 different from 0 gives an eigenvalue λ and corresponding eigenfunction f

The infinite triangular grid

Again, in this case the set of vertices is the set of all two-dimensional integer pairs. The edge-defining multi-set is this time $E' = \{(0,1), (0,-1), (1,0), (-1,0), (1,1), (-1,-1)\}$. All edges are supposed to have weight one.

Following similar calculations as for the square grid, we get the eigenvalues:

$$\lambda = 6 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2} - h_1 h_2 - \frac{1}{h_1 h_2},$$

for each choice for h_1, h_2 different from 0.

5.3 Laplacian eigenvalues of finite parts of grids

The calculations in the previous subsection can give us an idea about eigenvalues of regular infinite grids. In this subsection we want to show what happens if we are only looking at a finite part of the grids under consideration.

A first approach, which allows analytical analysis, is to consider the final part as being a finite grid embedded on a torus. In this way we avoid boundary problems. In other words, the gives us a finite problem with 'infinite-like' properties.

Following the arguments above, we think of the grids as lying on a torus curved in two directions. And we assume that after going for m steps in one direction we return to the original position, and after n steps in the other direction we do the same.

To compute the eigenvalues in this case we make a clever choice of the multipliers h_1 and h_2. In each coordinate direction we must make sure that the graph does not repeat indefinitely, but folds to itself after m repetitions with respect to the first coordinate and n repetitions with respect to the second coordinate. Hence we can no longer use any values for h_1 and h_2, but we must have $h_1^m = h_1^0 = 1$ and $h_2^n = h_2^0 = 1$. These equations have the solutions:

$$h_1 = \exp\left(2\pi i \frac{r}{m}\right), \qquad \text{for some } r, \ 0 \le r \le m-1;$$
$$h_2 = \exp\left(2\pi i \frac{s}{n}\right), \qquad \text{for some } s, \ 0 \le s \le n-1.$$

For each possible value of h_1, h_2, hence for each possible value of r, s, we get a Laplacian eigenvalue according to the formulae for the infinite grids.

- For the square grid we have $\lambda = 4 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2}$, which, after simplifying, gives the following eigenvalues:

$$\lambda_{rs} = 4 - 2\cos\left(2\pi \frac{r}{m}\right) - 2\cos\left(2\pi \frac{s}{n}\right), \qquad r = 0, 1, \dots, m-1, \ s = 0, 1, \dots, n-1.$$

A graphical representation of these eigenvalues, for $m = n = 30$, can be found below.

- And for the triangular grid we derived $\lambda = 6 - h_1 - \dfrac{1}{h_1} - h_2 - \dfrac{1}{h_2} - h_1 h_2 - \dfrac{1}{h_1 h_2}$, which results in the following $m\,n$ eigenvalues:

$$\lambda_{rs} = 6 - 2\cos\left(2\pi\,\frac{r}{m}\right) - 2\cos\left(2\pi\,\frac{s}{n}\right) - 2\cos\left(2\pi\left(\frac{r}{m} + \frac{s}{n}\right)\right), \qquad \begin{aligned} &r = 0, 1, \dots, m-1, \\ &s = 0, 1, \dots, n-1. \end{aligned}$$

Although we can give the eigenvalues of grids embedded on a torus explicitly, it is not clear how close they are to the eigenvalues of 'real' finite parts, lying in a plane. The problem is that it seems very hard to give exact analytical results for eigenvalues for these cases. Numerical analysis can provide an outcome here. Using the program Maple 6, we've calculated the eigenvalues of the Laplacian associated to finite parts of the infinite square and triangular grid with 900 points.

For the square grid we took a 30×30 square part of the infinite square grid embedded in the plane. The distribution of the eigenvalues of this system is given in Figure 1 on the right. For comparison, the distribution for the 30×30 square grid on the torus is shown in the same picture on the left. Note that the symmetry from the part on the torus is broken in the right side because of edge effects.
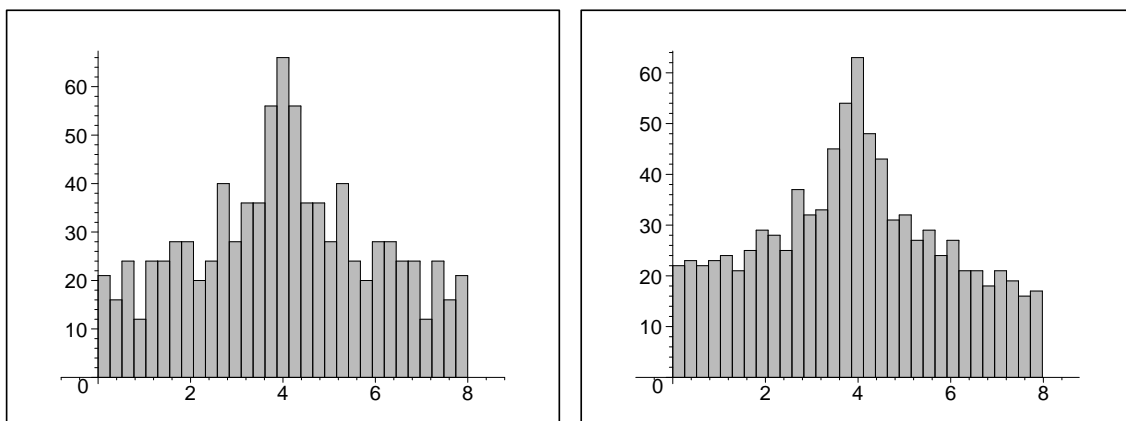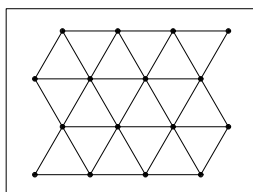

Figure 1 Distribution of Laplacian eigenvalues for 30×30 finite parts of the square lattice. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

A similar calculation was done for a part of the triangular lattice with 900 points embedded in the plane. Since this lattice is not of the same form in the x and y-directions, we took the points lying in a rectangular box similar to the shape of the 4×4 part below.

The distribution of the eigenvalues for both the case of the part of the triangular grid with 900 points embedded on the torus and in the plane are given in Figure 2.
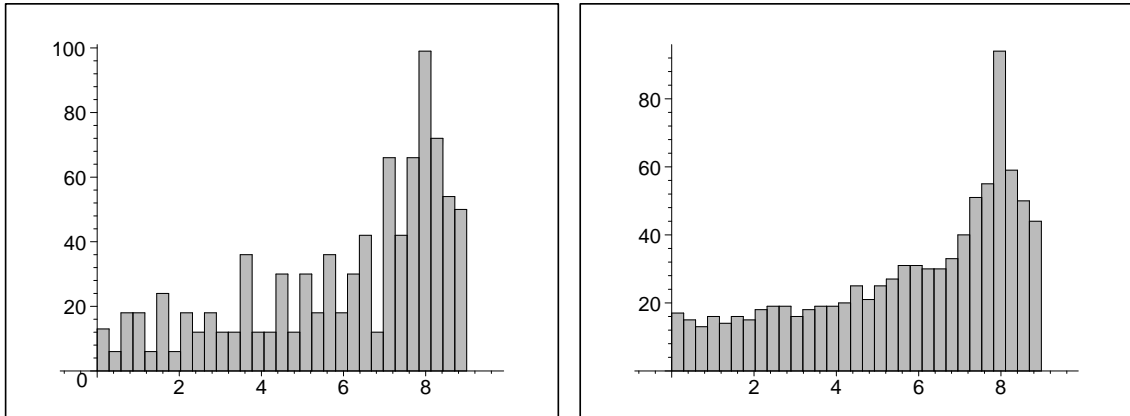
Figure 2 Distribution of Laplacian eigenvalues for 30×30 finite parts of the triangular lattice. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

5.4 Square and triangular grids with different constraints

The calculations above (both analytical and numerical) can also be done for more involved situations. As an example of this we consider both the square and triangular grid with constraints given in Figure 3.
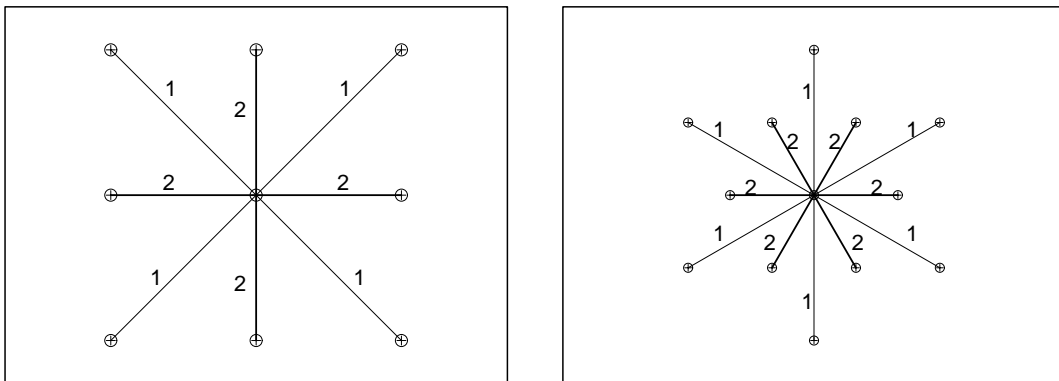

Figure 3 More complicated constraint patters for grids. The pictures indicate the constraints experienced by one point related to the position of its neighbouring points, for the square grid (left) and the triangular grid (right).

For the analytical analysis, we use the same technique as before. In order to take into account of edges with weight more than one, we just increase their multiplicity in the edge-defining multi-set. So for the square grid we again use the set of all two-dimensional integer pairs as the set of vertices. But the edge-defining multi-set giving the constraints as illustrated in the left of Figure 3 we use

$$
\begin{aligned}
E' = \ & \{(1,0), (1,0), (0,1), (0,1), (-1,0), (-1,0), (0,-1), (0,-1), \\
& (1,1), (-1,1), (-1,-1), (1,-1)\}.
\end{aligned}
$$

When we consider the grid embedded on the torus, then using a similar procedure as before gives the form of the eigenvalues:

$$\lambda_{rs} \; = \; 12 - 4\cos\left(2\pi\,\frac{r}{m}\right) - 4\cos\left(2\pi\,\frac{s}{n}\right) - 2\cos\left(2\pi\,(\frac{r}{m} + \frac{s}{n})\right) - 2\cos\left(2\pi\,(\frac{r}{m} - \frac{s}{n})\right),$$
$$r = 0, 1, \ldots, m-1, \; s = 0, 1, \ldots, n-1.$$

The distribution of the eigenvalues for the case $m = n = 30$, hence with 900 vertices, is given on the left side of Figure 4. Again, for comparison, we numerically calculated the Laplacian eigenvalues of a 30×30 part of the square grid in the plane, with similar constraints. This distribution can be found in Figure 4 as well.
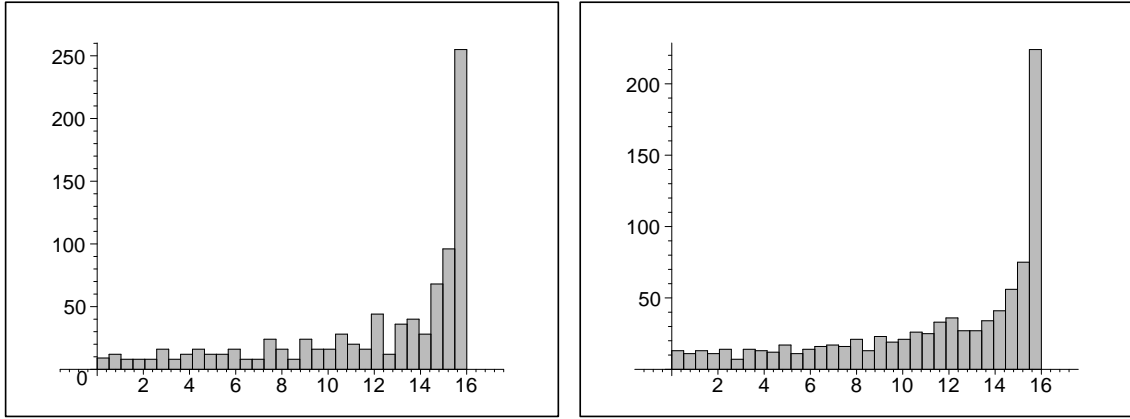

Figure 4   Distribution of Laplacian eigenvalues for 30×30 finite parts of the square lattice, with constraints determined by the left picture in Figure 3.. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

Finally, for the triangular lattice with constraints according to Figure 3, we can analytically determine the Laplacian eigenvalues for the part embedded on the torus as follows. Take as the edge-defining multi-set:

$$E' \; = \; \{(1,1), (1,1), (0,1), (0,1), (-1,0), (-1,0), (-1,-1), (-1,-1), (0,-1), (0,-1),$$
$$(1,0), (1,0), (1,2), (-1,1), (-2,-1), (-1,-2), (1,-1), (2,1)\},$$

where multiple entries correspond to edges with weight 2. This gives the following form of the eigenvalues:

$$\lambda_{rs} \; = 18 - 4\cos\left(2\pi\,\frac{r}{m}\right) - 4\cos\left(2\pi\,\frac{s}{n}\right) - 4\cos\left(2\pi\,(\frac{r}{m} + \frac{s}{n})\right) - 2\cos\left(2\pi\,(\frac{r}{m} - \frac{s}{n})\right)$$
$$- 2\cos\left(2\pi\,(\frac{2r}{m} + \frac{s}{n})\right) - 2\cos\left(2\pi\,(\frac{r}{m} + \frac{2s}{n})\right), \qquad r = 0, 1, \ldots, m-1,$$
$$s = 0, 1, \ldots, n-1.$$

And, finally, also for this case we calculated these eigenvalues for the case $m = n = 30$. And we did numerical calculations for this part of the triangular grid with the same constraints for a similar finite part, but now embedded in the plane. A graphical representation of these outcomes are given in Figure 5.
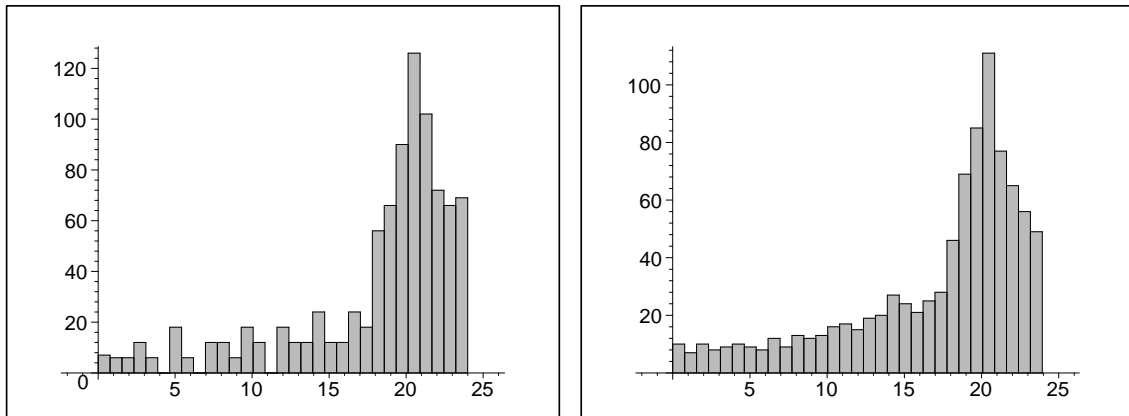
Figure 5 Distribution of Laplacian eigenvalues for 30×30 finite parts of the triangular lattice, with constraints determined by the right picture in Figure 3. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

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