Binet matrices, an extension of network matrices[1]

Gautam Appa, Balázs Kotnyek

Department of Operational Research
London School of Economics
Houghton Street, London WC2A 2AE, U.K.
e-mail: {g.appa, b.kotnyek}@lse.ac.uk

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Abstract

We define a new class of matrices called *binet matrices* – $\mathcal{B}$, closely related to network matrices – $\mathcal{N}$. We show that if a rational matrix A belongs to class $\mathcal{B}$, then the polyhedron $P(A,b) = \{\,x : Ax \leq b,\, x \geq 0\,\}$ is integral for all integral b vectors whose common divisor is 2. Starting with the node-edge incidence matrices of *bidirected graphs* having elements 0, ± 1 and ± 2, binet matrices with elements 0, ± 1, ± 2 and $\pm 1/2$ are derived as their generalization, in parallel with the derivation of network matrices as generalised node-edge incidence matrices of directed graphs. It is shown that the two well known totally unimodular matrices which are not network matrices are binet matrices, but the transpose of a binet matrix is not necessarily a binet matrix. The most striking result obtained is that class $\mathcal{B}$ is closed under pivoting and taking submatrices. Some further theorems on binet matrices are developed, leaving open the problem of an efficient recognition algorithm.

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1 Introduction

Totally unimodular matrices play a basic role in combinatorial optimisation. The classical theorem of Hoffman and Kruskal [7] states that the polyhedron $P(A, b) = \{x : x \geq 0, Ax \leq b\}$, where A is an integral matrix, is an integral polyhedron (i.e., has only integral extreme points) for every integral vector b if and only if A is totally unimodular. A natural way of extending this result is to consider which class of rational matrices leads to integral polyhedra. A further extension can be to relax the conditions of the theorem by requiring the integrality of $P(A, b)$ for only a special subset of integral vectors, namely for those in which every element of b has a common divisor k. Matrices guaranteeing integral polyhedra for this restricted subset of b vectors are called *k-regular*.

The most important class of totally unimodular matrices is that of network matrices (Tutte [11]). Its importance lies both in its theoretical properties and practical advantage. On the theoretical side it is worth mentioning that by Seymour's decomposition theorem [10] any totally unimodular matrix can be constructed from network matrices and two exceptional non-network but totally unimodular matrices (see also Bixby [2]) by applying simple operations. The practical advantage of network matrices is twofold. First, many of the most familiar and practically interesting totally unimodular matrices can be shown to be network matrices. Secondly, to solve the linear programming problem we can use the network simplex method (introduced by Dantzig in [3], for more references see Schrijver [8]), which is much more efficient than a general linear programming algorithm. This efficiency is a consequence of the strong connection between network matrices and directed graphs.

In this paper we generalise the well-known method of deriving network matrices from directed graphs to *bidirected graphs*. Bidirected graphs can be viewed as a common extension of directed and undirected graphs, involving directed and bidirected edges and single and double loops. Schrijver [9] also defined bidirected graphs starting with an integral matrix in which the sum of absolute values in each row is less than or equal to 2. This characterisation is obviously the transpose of what is used in Section 4 of this paper. In other words, Schrijver gave the edge-node incidence matrix of a bidirected graph, while we use the node-edge incidence matrix. The node-edge incidence matrices of bidirected graphs lead to the definition of *binet matrices* (binets for short) - the extension of network matrices. We show that in parallel with the directed case, there exists a simple method to calculate the columns of a binet matrix using the underlying bidirected graph.

Having introduced binets we prove that they are 2-regular. Next, we look for connections with (1-regular) totally unimodular matrices. All network matrices are easily shown to be a special class of binet matrices. The two exceptional 5×5 totally unimodular matrices, shown by Bixby [2] to be neither graphic nor co-graphic (i.e., neither the matrices nor their transposes can be represented as network matrices) also turn out to be binet matrices. We show their underlying bidirected graphs. This leads to a modification of Seymour's theorem which showed that any totally unimodular matrix can be decomposed into network matrices, transposes of network matrices and the the two exeptional matrices. We can claim that any totally unimodular matrix can be decomposed into 1-regular binet matrices and their transposes.

The paper is organised as follows. The second section highlights ways in which rational matrices behave differently from integral ones and introduces k-regularity. The third section connects k-regularity to the integrality of the polyhedra $P(A,b)$. We define bidirected graphs in Section 4. In Section 5 binet matrices are defined and their basic properties are established. Here we also provide an algorithm to calculate the columns of binet matrices and provide examples. Section 6 develops basic operations that maintain the binet property, leading to the main result that any submatrix of a binet matrix is also binet. Some further theorems about binet matrices are proved in Section 7. The last section concludes with a pointer towards connections with signed matrices and areas of further research.

2 Rational matrices

Let M be a matrix of rational elements and $P(M,b)$ be the polyhedron $\{\, x : Mx \leq b,\, x \geq 0 \,\}$. A non-empty polyhedron is said to be *integral*, if it has only integral extreme points. The integrality of the polyhedron $P(M,b)$ is strongly related to the constraint matrix M. If M is an integral matrix, then the polyhedron is integral for all integral vectors b if and only if the matrix is *totally unimodular*, i.e., all of its square submatrices have determinant 0 or ± 1. For integral matrices total unimodularity is equivalent to the integrality of the inverse of every non-singular submatrix of M. The next definition gives the generalisation of this property for rational matrices.

Definition 1. A matrix M of rational elements is said to be *k-regular* if for every non-singular square submatrix R of M the elements of (kR^{-1}) are integral.

The classical theory of unimodularity [7] is about 1-regular integral matrices. In fact, as mentioned above, 1-regularity of an *integral* matrix is equivalent to its total unimodularity. Appa [1] extended this to k-regular integral matrices, pointing out the special case of node-edge incidence matrices of non-bipartite undirected graphs giving rise to 2-regular matrices. In this paper we deal with 2-regular rational matrices with elements $0, \pm 1/2, \pm 1$ and ± 2. We begin, therefore, by developing theorems connecting rational k-regular matrices M to the integrality of polyhedra $P(M,b)$ for special vectors b. All the results from the classical theory of unimodularity (see e.g. Schrijver [8]) and extensions in Appa [1] can be derived as special cases of Theorem 2 below.

In order to emphasise the distinction between results about rational matrices as opposed to integral matrices, consider the rational matrix M below.

$$M = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}, \text{ with the inverse } M^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It is obvious that M is 1-regular. However, M^{-1} is not 1-regular. In contrast, if M were an integral 1-regular matrix (i.e., totally unimodular), then its inverse (and that of any non-singular square submatrix of it) would be 1-regular (and totally unimodular). Moreover, no totally unimodular matrix can be derived from M above by scaling its rows and columns with non-zero numbers, so 1-regular rational matrices are more than scaled versions of totally unimodular matrices.

Another difference between integral and rational matrices is related to unimodularity. For any full row rank $m \times n$ matrix A, let a *basis* be a non-singular square submatrix of order m. An integral full row rank matrix A is called *unimodular* if each basis of A has determinant ± 1. It is easy to see that matrix A is totally unimodular if and only if the matrix $[A, I]$ (where I denotes the unit matrix of appropriate size) is unimodular. This equivalence does not remain valid for 1-regular rational matrices, as the next example illustrates.

$$M = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad [M, I] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{bmatrix}$$

where M is 1-regular, since all the 1×1 submatrices of M have integral inverses and

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}. \text{ But } \quad \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix},$$

so $[M, I]$ is not 1-regular because it has a basis the inverse of which is not integral.

3 Integral polyhedra and k-regular matrices

Let M be a rational matrix of size $m \times n$. Let $P(M, b)$ be the polyhedron $\{\, x : Mx \leq b,\, x \geq 0 \,\}$, where b is a vector of appropiate size. The next theorem connects 1-regularity of M to the integrality of polyhedra $P(M, b)$. The proof follows the ideas of Veinott and Dantzig [12].

Theorem 1. *The polyhedron $P(M, b)$ is integral for each integral vector b if and only if M is 1-regular.*

Proof: First we prove that $P(M, b)$ is integral if M is 1-regular. Note that if x is an extreme point of $P(M, b)$, there exists a non-singular square submatrix R of M such that (after permuting the columns of M) $x = (R^{-1}b^R, 0)^T$ where b^R consists of the components of b corresponding to the rows of R. As M is 1-regular, R^{-1} is integral by definition. So x is integral.

To see the converse it is enough to show that $R^{-1}e_i$ is integral for each unit vector e_i. Since R is a rational matrix, there exists an integral y such that Ry is integral and $z = y + R^{-1}e_i \geq 0$. Then $b^R = Rz$ is integral. Let z' arise from

z by adding zero-components corresponding to the colums that are not in R and let us extend b^R to the integral vector b so that $Mz' \leq b$. Then z' is an extreme point of $P(M, b)$ (since it satisfies n linearly independent constraints with equality), so it is integral. Therefore z and $R^{-1}e_i = z - y$ are integral, too. $\square$

The well-known theorem of Hoffman and Kruskal [7] about the integrality of $P(A, b)$ where A is an integral matrix is a direct corollary of the above theorem in view of the fact that an integral matrix is totally unimodular if and only if the inverse of its every non-singular submatrix is integral.

If we require the integrality of $P(M, b)$ not for all but only for a subset of integral vectors b, we get the following, more general theorem.

Theorem 2. *For a given positive integer k, the polyhedron $P(M, kb)$ is integral for each integral vector b, if and only if M is k-regular.*

Proof: It is obvious that $P(M, kb) = \{\, x : Mx \leq kb,\ x \geq 0 \,\}$ is integral for each integral b if and only if $P(M/k, b) = \{\, x : (M/k)x \leq b,\ x \geq 0 \,\}$ is integral for each integral vector b, where $(M/k)_{i,j} = [m_{ij}/k]$. By Theorem 1 this second polyhedron is integral if and only if for each non-singular square submatrix (R/k) of (M/k) the elements of $(R/k)^{-1} = kR^{-1}$ are integral. $\square$

An important special case of Theorem 2 occurs when $k = 2$, leading to half-integral solutions, i. e. solutions of the form $a/2$ where $a \in \mathbb{Z}^n$.

Theorem 3. *If M is 2-regular then for each integral vector b the extreme points of polyhedron $P(M, b)$ are half-integral.*

The next theorem deals with problems where the variables in $P(M, b)$ have upper bounds.

Theorem 4. *If an integral matrix A is k-regular, then the polyhedron $\{\, x : Ax \leq kb,\ 0 \leq x \leq ku \,\}$ is integral for each integral vectors b and u.*

Proof: Let $\begin{bmatrix} S & R \\ I & 0 \end{bmatrix}$ be a non-singular square submatrix of $\begin{bmatrix} A \\ I \end{bmatrix}$. Then

$$\begin{bmatrix} S & R \\ I & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ R^{-1} & -R^{-1}S \end{bmatrix},$$

so $\begin{bmatrix} A \\ I \end{bmatrix}$ is k-regular if A is such. $\square$

Note that in general the theorem is not true for a rational matrix M replacing A. For example, if $k = 2$ and

$$M = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

then M is 2-regular, but $\begin{bmatrix} M \\ I \end{bmatrix}$ is not, as $x = [2, 0.5]$ is a non-integral extreme point of $\{ x : Mx \leq 2b,\, 0 \leq x \leq 2u \}$. However, we show later that in the special case of binet matrices, which are rational, Theorem 4 does apply for $k = 2$.

If $k = 1$ in Theorem 4, then $ku = u$, and requiring general integer upper bounds does not affect the integrality of a polyhedron. Note, however, that for $k > 1$, if an integral matrix A is k-regular and the upper bounds on the variables in $\{ x : Ax \leq kb,\, 0 \leq x \leq u \}$ are general integers (and not multiples of k), then integer solution cannot be guaranteed. For example, if

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then A is 2-regular, but $x = [1, 1, 1, 0.5, 0.5, 0.5]$ is an extreme point of $\{ x : Ax \leq kb,\, 0 \leq x \leq u \}$.

4 Bidirected graphs

Network matrices form a special and important subset of totally unimodular matrices. We would like to find their counterparts among 2-regular matrices. Network matrices arise from the node-edge incidence matrices of directed graphs. If each column of a matrix contains exactly one $+1$, one -1 and no other non-zero elements, then it is the node-edge incidence matrix of a directed graph. It is well-known that such matrices are totally unimodular. The next theorem gives their generalization to 2-regular matrices.

Theorem 5. *Let A be an $m \times n$ integral matrix satisfying*

$$\sum_{i=1}^{m} |a_{ij}| \leq 2 \qquad \text{for } j = 1, \ldots, n.$$

Then A is 2-regular.

Proof: Let R be a non-singular square submatrix of A. We need to show that $2R^{-1}$ is integral. If R can be decomposed into t blocks R_1 to R_t, each block must also be a non-singular square submatrix because otherwise R is singular.

$$\text{Let } R = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_t \end{bmatrix}, \text{ then} \quad R^{-1} = \begin{bmatrix} R_1^{-1} & & & \\ & R_2^{-1} & & \\ & & \ddots & \\ & & & R_t^{-1} \end{bmatrix}.$$

Note that all entries in R and R^{-1} outside the blocks are zeros.

So it is enough to show that any non-decomposable R is 2-regular. We show this by proving that for every such R, $det(R) = \pm 1$ or ± 2. This together with the fact, that the co-factor of any element R_{ij}, denoted by $cof(R_{ij})$, is integer (since the matrix is integral) implies that any element of the inverse, given by $R^{-1}_{ji} = cof(R_{ij})/det(R)$, is half-integral.

Let R be a non-singular and non-decomposable square submatrix of A of size l. As R is a sub-matrix of A, total number of non-zero elements in R cannot exceed $2l$ because each column of A has at most two. As R is non-decomposable and non-singular, total number of non-zero elements in R must be at least $2l-1$. To see this consider the intersection graph of R, denoted $G(R)$, in which there are $2l$ nodes - one for each row and column - and an edge joining a row node i to a column node j if and only if $R_{ij} \neq 0$. For R to be non-decomposable, its intersection graph must be connected, i.e., it must have at least $2l - 1$ edges.

Consider the case where R has exactly $2l - 1$ non-zero elements. Its intersection graph is a tree graph. So the determinant of R can be obtained by expansion through a row or a column containing only one non-zero entry. It is fairly obvious that the absolute value of the determinant is the absolute value of the product of l non-zero elements of R. Since there is only one column that has less than two non-zeros, R contains at most one ± 2 element. Thus the product of any subset of the non-zero elements is ± 1 or ± 2. Hence $|det(R)| = 1$ or 2 and the theorem follows.

Consider now the case where the total number of non-zero elements in R is $2l$. Then each column of R has exactly two non-zero elements each of which is ± 1. By permuting the rows and columns we can ensure that the two non-zero elements in column j are in rows j and $j + 1$ (mod m). Then

$$det(R) = R_{11}cof(R_{11}) - R_{21}cof(R_{21})$$

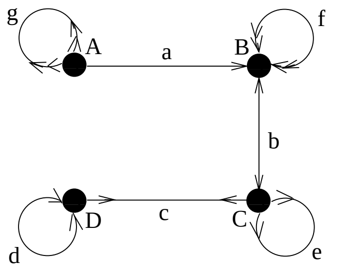
where $cof(R_{11})$ and $cof(R_{21})$ are the determinants of submatrices of R such that they contain one column with one non-zero (being ± 1) and exactly two non-zeros in all other columns. Following the argument of the previous paragraph, $cof(R_{11}) = \pm 1$ and $cof(R_{21}) = \pm 1$ and R is non-singular by assumption, hence $det(R) = \pm 2$. □

The condition on the elements of A in Theorem 5 implies that if a column of A contains two non-zeros, they are ± 1 each. If we restrict the condition to the special case where $|a_{ij}| \leq 1$ ($i = 1, \dots, m$ and $j = 1, \dots, n$), the resulting 2-regular $0, \pm 1$ matrices can be viewed as generalised node-edge incidence matrices of a mixed graph with self-loops. The reason why we have chosen to allow columns with a singleton ± 2 has to do with the fact that binet matrices and their properties developed in later sections require them.

The next definition gives a generalisation of mixed graphs with self-loops so that the resulting graphs are in one to one correspondance with 2-regular matrices of Theorem 5.

Definition 2. The graph $G = (V, E)$ with node-set V and edge-set E is a *bidirected graph* if each edge $e = (u, v)$ belongs to one of the following seven categories:

(a) e is directed from u to v, u is its out-node, v is its in-node,



	a	b	c	d	e	f	g
A	-1	0	0	0	0	0	-2
B	1	1	0	0	0	2	0
C	0	1	-1	0	-1	0	0
D	0	0	-1	1	0	0	0

Figure 1: A bidirected graph, and its incidence matrix

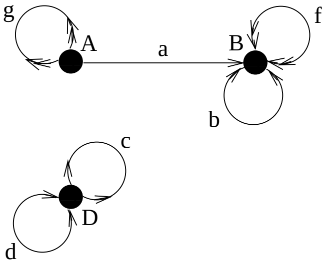
(b) e is an in-bidirected edge, both u and v are in-nodes of e,
(c) e is an out-bidirected edge, both u and v are out-nodes of e,
(d) e is an in-1-loop, that is $u = v$ and e is directed to it,
(e) e is an out-1-loop, that is $u = v$ and e is directed from it,
(f) e is an in-2-loop, that is $u = v$ and e is directed to it,
(g) e is an in-2-loop, that is $u = v$ and e is directed from it.

Edges in categories (b)-(g) are called *bidirected.* Thus all edges other than directed edges between two distinct nodes are called bidirected by definition. Categories (d) and (e) define *1-loops*, while (f) and (g) define *2-loops*, a new construct to represent a column with a single non-zero element ± 2. (In Figure 1 edges a to g illustrate categories (a) to (g) above.)

Clearly, bidirected graphs form a common generalization of both directed and undirected graphs, as edges in category (a) are the normal directed edges and category (b) edges are normally represented as undirected edges.

Bidirected graphs were constructed to represent the matrices of Theorem 5. The representation is quite straightforward. Let the rows and columns of the matrix correspond to the nodes and edges of the graph, respectively. If an edge is not a 2-loop, then the corresponding column contains $+1$ in the row(s) of its in-node(s), -1 in the row(s) of its out-node(s), and 0 in every other rows. If the edge is a 2-loop, then the corresponding column has $+2$ or -2 in the node-row where the loop is located depending on whether the loop is directed to or from it, and 0 in other rows. We will call the representation matrix of a bidirected graph its *(node-edge) incidence matrix.* So the 2-regular matrices defined in Theorem 5 are all incidence matrices of bidirected graphs. Figure 1 shows a bidirected graph and its incidence matrix. Columns b and c with two unit elements of the same sign are represented in the graph by bidirected edges. All columns with a single non-zero element are represented by self-loops. If the singleton non-zero is ± 2, we draw a 2-loop distinguished by double arrows. As to the signs of non-zero elements, the in-arrows represent plus and out-arrows minus.

In what follows we develop some properties of bidirected graphs and their incidence matrices.



	a	b	c	d	f	g
A	-1	0	0	0	0	-2
B	1	1	0	0	2	0
D	0	0	-1	1	0	0

Figure 2: Deleting node C from the graph of Figure 1 and the corresponding operation on the matrix

An incidence matrix A of a bidirected graph G is called *connected* if G is connected. Connectedness of G implies that A is non-decomposable, i.e., $A \neq \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1, A_2 \neq 0$. Deleting a row of A (and then deleting the columns that have become all-zero) means deleting the corresponding node and its loops from G. The loops located at the deleted node disappear, while the non-loop edges incident to it are 'turned back' i.e., they become loops. For example, deleting row C from the matrix in Figure 1 corresponds to deleting node C from the graph of Figure 1, leading to Figure 2.

Since after deleting rows, the matrix is still an incidence matrix of a bidirected graph we can assume, without loss of generality, that A is of full row rank. We can also assume A and G to be non-decomposable or equivalently, connected. If they consist of more than one connected components, then the components can be viewed separately, as individual matrices and graphs, and the whole theory developed hereafter can be applied to them. Therefore, for the sake of simpicity in what follows we will assume the incidence matrices of bidirected graphs to be connected and of full row rank.

Submatrices of A correspond to subgraphs of G. Let $G(R)$ denote the subgraph related to submatrix R. The next theorem relates non-singularity of R to $G(R)$.

Theorem 6. *Any square submatrix R of the incidence matrix A of a bidirected graph G is non-singular if and only if each connected component of $G(R)$ contains exactly one cycle and this cycle has an odd number of bidirected edges.*

Proof: If R is a 1×1 matrix, then the theorem is trivial as a loop is a cycle and contains only one bidirected edge. Let us assume that R is of size m, and the theorem is true for submatrices smaller than m. If R is not connected, then its connected components are smaller non-singular square submatrices, so the theorem follows by induction. Thus, let us assume that R is connected.

Since R is a square submatrix, $G(R)$ has as many edges as nodes, so it contains exactly one cycle. Let R_1 be the submatrix of R corresponding to the cycle (i.e., its rows correspond to the nodes, and columns to the edges of the cycle). If the cycle is a loop, then R_1 consists of an only element and it is non-singular. If the cycle is not a loop, then by permuting the rows and columns of R and scaling them by ± 1 it can be transformed to the following form:

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}, \quad \text{where} \quad R_1 = \begin{bmatrix} 1 & & & & \pm 1 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

Deleting the nodes and edges of the cycle from $G(R)$ the remaining subgraph $G(R_3)$ consists of components containing a loop as an only cycle, because R_1 was the only cycle in $G(R)$. The size of any component in $G(R_3)$ is at most $m-1$. Thus, by induction, R_3 is non-singular, and non-singularity of R depends on the non-singularity of R_1.

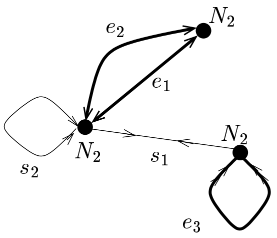
One can easily show by cofactor expansion, that in case of a non-loop cycle R_1 is non-singular if and only if the element standing in its upper right hand corner is $+1$, i.e., if $G(R_1)$ contains one bidirected edge. We now show that row and column scaling operations could lead to a R_1 such that $G(R_1)$ has only one bidirected edge only if the original cycle in $G(R)$ had an odd number of bidirected edges. Note first that scaling columns by -1 only changes direction of arrows and has no effect on whether the edge is directed or bidirected. Scaling a row by -1 changes every non-loop edge incident to the corresponding node from bidirected to directed and vice versa. As the cycle graph $G(R_1)$ has exactly two edges incident to any node, scaling of a row by -1 changes the number of bidirected edges in the cycle by an even number. Since scaling has no influence on singularity and the scaled cycle is non-singular if and only if it contains an odd number $(=1)$ of bidirected edges, the theorem follows. □

Remark 1. Let R' be a submatrix of A such that its columns are linearly independent. Since A is of full row rank, R' can be extended to a non-sigular square submatrix R by adding column vectors. This implies that any component of $G(R')$ (since it comes from $G(R)$ by deletion of edges)is either an isolated node or forms a tree or contains exatly one cycle with odd number of bidirected edges.

Note that the converse is also true: if any component of $G(R')$ either forms a tree or contains exactly one cycle with odd number of bidirected edges, then the columns of R' are linearly independent.

5 Binet matrices

We are now ready to define binet matrices from the incidence matrices of bidirected graphs in the same way as network matrices are defined from incidence matrices of directed graphs. Recall that the rows of a network matrix correspond to the edges of a tree basis and the columns to the non-basic edges. We generalise this to the bidirected case.



$$B = R^{-1}S = \begin{array}{c|c|c|} & w_1 & w_2 \\ \hline e_1 & -1/2 & 1/2 \\ \hline e_2 & 1/2 & -1/2 \\ \hline e_3 & -1/2 & 0 \\ \hline \end{array}$$

Figure 3: Binet representation of binet matrix B.

Let us recall that a basis from an $m \times n$ full row rank matrix is a non-singular square submatrix of size m.

Definition 3. Let A be the incidence matrix of a bidirected graph G, R be a basis of it and $A = [R, S]$. The matrix $B = R^{-1}S$ is called a *binet matrix*.

For the sake of simplicity, let us introduce some terminology. If R is a basis of the incidence matrix A of a bidirected graph G, then $G(R)$ is called the *basis of* G, and its edges are called the *basic edges*. The edges of G that are not in the basis $G(R)$ (i.e., those of $G(S)$) are the *non-basic edges*. Let us call a cycle an *odd cycle* if the number of bidirected edges in it is odd, and an *even cycle* otherwise. Theorem 6 claims that a subgraph of G is a basis if and only if its every component contains exactly one cycle, which is odd. The rows and columns of a binet matrix are related to basic and non-basic edges, respectively. Binet matrices may have different graphical representations, that is the same binet matrix can arise from different bidirected graphs. We will call a graphical representation of a binet matrix its *binet representation*, and a bidirected graph that represents it, a *binet graph*. Figure 3 shows a possible binet representation of binet matrix B for which R, S and R^{-1} are as follows.

$$R = \begin{array}{c|c|c|c|} & e_1 & e_2 & e_3 \\ \hline N_1 & 1 & -1 & 0 \\ \hline N_2 & 1 & 1 & 0 \\ \hline N_3 & 0 & 0 & 2 \\ \hline \end{array}, \quad S = \begin{array}{|c|c|} s_1 & s_2 \\ \hline -1 & 1 \\ \hline 0 & 0 \\ \hline -1 & 0 \\ \hline \end{array} \quad \text{and} \quad R^{-1} = \begin{array}{|c|c|c|} \hline 1/2 & 1/2 & 0 \\ \hline -1/2 & 1/2 & 0 \\ \hline 0 & 0 & 1/2 \\ \hline \end{array}$$

The basis comprises of edges e_1, e_2 and e_3. The columns of B corresponding to non-basic edges s_1 and s_2 are denoted by w_1 and w_2.

Elements of a network matrix can be determined with the help of its graph representation (see in [8]). We now develop a similar method of calculating the elements of a binet matrix from its binet representation, without reference to R^{-1} or S.

Let s be a non-basic edge, i.e., the related column of the node-edge incidence matrix is in S. Let the corresponding column of B be w, i.e. $w = R^{-1}s$. The m-dimensional column vectors of our non-singular matrix R are linearly independent, so they span an m-dimensional Eucledian vectorspace $\mathbb{R}^m$. As

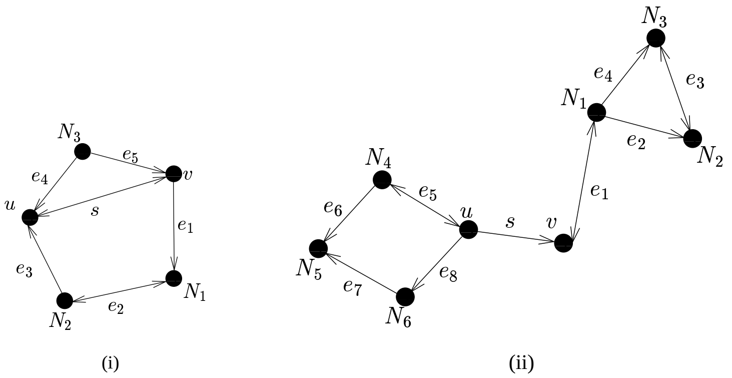


Figure 4: Examples of minimal dependent subgraphs

$Rw = s$, column w represents the unique coordinates of vector s in $\mathbb{R}^m$. Let $\{ e_1, e_2, \dots , e_t \}$ be the subset of columns (edges) of R $(G(R))$ where the coordinates are non-zero and let $s = e_0$. Then s is equal to a linear expression made up of e_i's with non-zero coefficients $w(e_i)$'s.

$$s = w(e_1)e_1 + w(e_2)e_2 + \cdots + w(e_t)e_t, \qquad w(e_i) \neq 0, \quad i = 1, \dots , t \tag{1}$$

So by defining $w(s) = w(e_0) = -1$, we get $\sum_{i=0}^{n} w(e_i)e_i = 0$. Thus the columns of the matrix $R' = [\, e_0, e_1, e_2, \dots , e_t \,]$ form a minimal dependent set in $\mathbb{R}^m$, i.e., deleting any member, the resulting vectors are linearly independent. Therefore, deleting any edge from subgraph $G^S(R')$ spanned by edges $s, e_1, e_2, \dots , e_t$ we are left with an 'independent' subgraph, i.e., a subgraph which can be extended to a basis. So by Remark 1 any of its components which is not an isolated node is either a tree or contains exactly one cycle which is an odd cycle. On the other hand $G^S(R')$ is connected since $w(e_i)$'s are non-zero and unique coordinates of s. (If it is not connected, R' is decomposable, and s is spanned by the columns that are in the same submatrix as s. The other columns have zero coordinates, which contradicts the fact that all $w(e_i)$'s are non-zero.) It can be easily seen that this implies that $G^S(R')$ must consist of an even-cycle or two edge-disjoint odd-cycles connected with a (maybe zero-length) path. Figure 4 illustrates the two types of minimal dependent sets. In the first graph edge-set $G^S(R') = \{ s, e_1, e_2, e_3 \}$ forming an even-cycle is a minimal dependent set. In the second example the entire graph forms a minimal dependent set with two odd cycles. We will call $G^S(R')$ the *fundamental circuit* of s in G. This is in parallel with each non-basic edge giving rise to a unique cycle in a network matrix where the basic edges make up a tree graph. A fundamental circuit is a closed walk in the graph, starting and finishing at the same node. Let s connect nodes u and v, and let us traverse its fundamental circuit starting from u along s. Let the order in which edges and nodes are travelled be denoted as below by

$q+1$ edges f_0 to f_q and q nodes n_0 to n_{q-1}:

$$u, s = f_0, v = n_0, f_1, n_1, f_2, n_2, \ldots, n_{q-1}, f_q, u \tag{2}$$

In the first example of Figure 4, q equals 3 and $f_1 = e_1, n_1 = N_1, f_2 = e_2, n_2 = N_2, f_3 = e_3$. In the second example $q = 10$ and $f_1 = e_1, n_1 = N_1, f_2 = e_2, n_2 = N_2, f_3 = e_3, n_3 = N_3, f_4 = e_4, n_4 = N_1, f_5 = e_1, n_5 = v, f_6 = s, n_6 = u, f_7 = e_5, n_7 = N_4, f_8 = e_6, n_8 = N_5, f_9 = e_7, n_9 = N_6, f_{10} = e_8$.

In general, a particular node or edge of the subgraph $G^S(R')$ may be traversed twice in the walk so that two different node (or edge) numbers n_i (or f_i) in the walk may refer to the same node (or edge) of the graph. For example, nodes u, v and N_1 and edges e_1 and s in the second example are traversed twice. A node of the walk is called an *even-node* if it is an in-node of both or neither of the edges that stand next to it in the list. Otherwise it is called an *odd-node*. In the second example, v is an even-node while it is an odd-node in the first example.

Coefficients $w(e_i)$ (i=0,1, ... ,t) are non-zero weights corresponding to columns of R' so that their weighted sum is zero (as shown in (1)). As rows of the incidence matrix are related to the nodes of G and columns to the edges, $w(e_i)$'s can be viewed as weights assigned to the edges of the fundamental circuit so that at every node the sum of the weights is zero. Knowing that $w(e_0) = -1$ and that exactly two f_i edges are incident to every node of the fundamental circuit, this zero sum property provides us with a quick method for tracing the other coefficients. The rules below summarize this method by walking along the fundamental circuit. By applying these rules we get weights that satisfy (1). Since we know that w is unique, our rules provide us with the correct coefficients.

First we assign the values $b(f_i)$ to every basic edge of the walk as intermediate weights:

<u>Assigning values $b(f_i)$</u>

1. $b(f_0) = -1$

2. Let $i = 1$ and while $i \leq q-1$ let

$$b(f_i) = \begin{cases} -b(f_{i-1}) & \text{if } n_{i-1} \text{ is an even-node} \\ b(f_{i-1}) & \text{if } n_{i-1} \text{ is an odd-node} \end{cases}$$

   and increase i by 1.

Instead of determining $b(f_i)$'s succesively, they can be directly determined with the help of the node-values $c(n_j)$ $(j = 0, 1, \ldots, q-1)$ defined below:

$$c(n_j) = \begin{cases} 0 & \text{if } n_j \text{ is an odd-node} \\ +2 & \text{if } n_j \text{ is an in-node of both } f_i \text{ and } f_{i+1} \\ -2 & \text{if } n_j \text{ is an out-node of both } f_i \text{ and } f_{i+1} \end{cases}$$

Then for $i = 1, 2, \ldots, q$

$$b(f_i) = \begin{cases} 1 & \text{if } \sum_{j=0}^{i-1} c(n_j) \equiv 2 \pmod 4 \\ -1 & \text{if } \sum_{j=0}^{i-1} c(n_j) \equiv 0 \pmod 4 \end{cases}$$

Since different f_i's and n_i's could be equal to the same edge or node of the graph (for example if f_j is a loop, then $n_{j-1} = n_j$) $b(f_i)$'s must be adjusted to obtain the correct coefficients.

<ins>Calculating coefficients $w(e_i)$</ins>

1. If s is a 1-loop, then divide every $b(f_i)$ by 2: $b(f_i) := b(f_i)/2$ for $i = 1, \ldots, q$
2. If f_i is a 1-loop and $f_i \neq s$, then multiply $b(f_i)$ by 2: $b(f_i) := 2b(f_i)$
3. If s appears twice in the list (so it lies on the connecting path), then divide every $b(f_i)$ by 2: $b(f_i) := b(f_i)/2$ for $i = 1, \ldots, q$
4. If basic edge e_i appears twice in the list, then let $w(e_i)$ be the sum of the (modified) values of its two appearences:

   $$w(e_i) := b(f_{j_1}) + b(f_{j_2}) \ \text{ if } f_{j_1} = e_i \text{ and } f_{j_2} = e_i$$

   Otherwise let $w(e_i)$ equal the value of its only appearence:

   $$w(e_i) := b(f_j) \ \text{ if } f_j = e_i$$

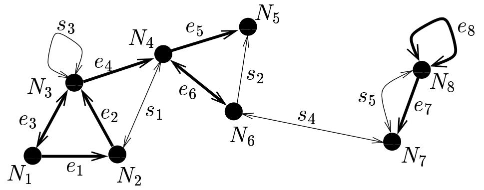
We illustrate our procedure with reference to Figure 5 which shows an example of a binet graph and its binet matrix. The rows and columns of the binet matrix are denoted by the signs of the corresponding edges of the binet graph. The basis comprises of edges $e_1, e_2, \ldots, e_8$.

The fundamental circuit of s_1 is $\{N_4, f_0 = s_1, n_0 = N_2, f_1 = e_1, n_1 = N_1, f_2 = e_3, n_2 = N_3, f_3 = e_4, N_4\}$. Nodes N_2 and N_4 are even nodes, nodes N_1 and N_3 are odd-nodes. The non-zero elements of column s_1 in the binet matrix are calculated as:

$$w(e_1) = b(f_1) = 1, \ w(e_3) = b(f_2) = 1, \ w(e_4) = b(f_3) = 1.$$

The minimal dependent set including s_2 is $\{s_2, e_6, e_4, e_2, e_1, e_3, e_5\}$. The fundamental circuit of s_2 is $\{N_5, f_0 = s_2, N_6, f_1 = e_6, N_4, f_2 = e_4, N_3, f_3 = e_2, N_2, f_4 = e_1, N_1, f_5 = e_3, N_3, f_6 = e_4, N_4, f_7 = e_5, N_5\}$. The non-zero elements of column s_2 in the binet matrix are calculated as:

$$w(e_6) = b(f_1) = -1, \ w(e_4) = b(f_2) + b(f_6) = 2, \ w(e_2) = b(f_3) = 1,$$
$$w(e_1) = b(f_4) = 1, \ w(e_3) = b(f_5) = 1, \ w(e_5) = b(f_7) = 1.$$



	s_1	s_2	s_3	s_4	s_5
e_1	1	1	1/2	-1/2	0
e_2	0	1	1/2	-1/2	0
e_3	1	1	1/2	-1/2	0
e_4	1	2	0	-1	0
e_5	0	1	0	0	0
e_6	0	-1	0	1	0
e_7	0	0	0	1	1
e_8	0	0	0	1	2

Figure 5: Example of a binet graph, and its binet matrix. The basic edges $e_1, e_2, \ldots, e_8$ are in bold.

The fundamental circuit of s_3 is $\{N_3, f_0 = s_3, N_3, f_1 = e_2, N_2, f_2 = e_1, N_1, f_3 = e_3, N_3\}$. For s_3 is a 1-loop, the non-zero elements of column s_3 in the binet matrix are calculated as:

$$w(e_2) = \frac{1}{2}b(f_1) = \frac{1}{2}, \quad w(e_1) = \frac{1}{2}b(f_2) = \frac{1}{2}, \quad w(e_3) = \frac{1}{2}b(f_3) = \frac{1}{2}.$$

The fundamental circuit of s_4 is $\{N_7, f_0 = s_4, N_6, f_1 = e_6, N_4, f_2 = e_4, N_3, f_3 = e_3, N_1, f_4 = e_1, N_2, f_5 = e_2, N_3, f_6 = e_4, N_4, f_7 = e_6, N_6, f_8 = s_4, N_7, f_9 = e_7, N_8, f_{10} = e_8, N_8, f_{11} = e_7, N_7\}$. As s_4 appears twice in the fundamental circuit and e_8 is a 1-loop, the non-zero elements of column s_4 in the binet matrix are calculated as:

$$w(e_6) = \frac{1}{2}(b(f_1) + b(f_7)) = 1, \ w(e_4) = \frac{1}{2}(b(f_2) + b(f_6)) = -1,$$
$$w(e_3) = \frac{1}{2}b(f_3) = -\frac{1}{2}, \ w(e_1) = \frac{1}{2}b(f_4) = -\frac{1}{2}, \ w(e_2) = \frac{1}{2}b(f_5) = -\frac{1}{2},$$
$$w(e_7) = \frac{1}{2}(b(f_9) + b(f_{11})) = 1, \ w(e_8) = \frac{1}{2}2b(f_{10}) = 1.$$

The fundamental circuit of s_5 is $\{N_8, f_0 = s_5, N_7, f_1 = e_7, N_8, f_2 = e_8, N_8\}$. The non-zero elements of column s_5 in the binet matrix are calculated as:

$$w(e_7) = b(f_1) = 1, \quad w(e_8) = 2b(f_2) = 2.$$

This method of calculating the elements of binet matrices provides us with a method of calculating the inverse of basis matrix R. Let us add a 1-loop to each node of the graph so that the incidence matrix A contains a unit matrix I, corresponding to the 1-loops. The binet matrix $B = R^{-1}S$ contains R^{-1} since S contains I. A column of R^{-1} can be calculated by determining the fundamental circuit of a 1-loop. The following lemma makes use of this representation of the columns.

Lemma 7. *If R is a basis of the node-edge incidence matrix of a bidirected graph, then each row of the inverse of its every component is of the following two types:*

(a) contains only $\pm 1/2$ entries,

(b) contains only $0, \pm 1$ entries.

Further, if the only cycle in a component is a 1-loop, then its inverse is integral (has only rows of type (b)).

Proof: Let us suppose that the basis is non-decomposable (connected) as each component can be viewed separately with respect to taking inverse. Then $G(R)$ contains exactly one cycle, which is an odd-cycle. As we mentioned above columns of R^{-1} are related to 1-loop non-basic edges. Each cycle edge of $G(R)$ appears once in the fundamental circuit of each non-basic 1-loop. A non-cycle basic edge, if it is contained in the fundamental circuit, appears twice. Following the rules of calculating coefficients $w(e_i)$, the $b(f_i)$ values are divided by 2. Therefore, if the odd-cycle of $G(R)$ is not a 1-loop, then coefficients $w(e_i)$ are $\pm 1/2$ on the cycle edges, and ± 1 on the non-cycle basic edges for every non-basic 1-loop. As each column of R^{-1} refers to a 1-loop non-basic edge, the main part of the theorem (statements (a) and (b)) is proven. In the special case where the unique cycle of the basis graph $G(R)$ is a 1-loop, following the rules for calculating $w(e_i)$'s the only cycle edge row is multiplied by 2 so that all rows contain only $0, \pm 1$ entries. $\square$

As we can see from the examples and from Lemma 7, binet matrices are not necessarily integral matrices. They may contain $\pm 1/2$. The next lemma gives the conditions for integral binet matrices. Before stating the lemma we recall that the bidirected graph G is connected.

Lemma 8. *A binet matrix is integral if and only if it has a binet representation in which*

(a) every cycle in each component of its basis is a 1-loop, or

(b) the basis is connected and there are no 1-loop non-basic edges.

A stronger version of (b) is equivalent:

(b') the basis is connected and the binet graph has no 1-loops.

Proof: The lemma follows easily from Lemma 7, or can be proved directly using the rules of calculating the elements of the matrix and the fact that G is connected. $\square$

The next theorem establishes an important property of binet matrices.

Theorem 9. *Binet matrices are 2-regular.*

Proof: Let $B = R^{-1}S$ be a binet matrix, where $A = [S, R]$ is the incidence matrix of a bidirected graph. Then $[B, I] = R^{-1}A$. Let C be a non-singular square submatrix of B. To show that $2C^{-1}$ is integral let $[B, I]$ be transformed so that

$$[B, I] = \begin{bmatrix} D_2 & C & 0 & I_2 \\ D_3 & D_1 & I_1 & 0 \end{bmatrix} \quad \text{and let} \quad E = \begin{bmatrix} C & 0 \\ D_1 & I_1 \end{bmatrix}$$

where I_1 and I_2 are unit matrices of appropriate size. There is a basis T of A such that $E = R^{-1}T$. By Lemma 7, $2T^{-1}$ is integral, hence $2E^{-1} = 2T^{-1}R$ and then $2C^{-1}$ is integral too. $\square$

Remark 2. If we add a 1-loop to every node of a bidirected graph, and take the corresponding unit matrix as a basis, we get $B = I^{-1}A = A$. So the incidence matrix of a bidirected graph is a binet matrix, which is an alternative proof of theorem 5.

The next theorem lists special characteristics of bidirected graphs that guarantee totally unimodular binet matrices.

Theorem 10. *If $B = R^{-1}S$ is a binet matrix represented by G, and*

(a) all the bidirected edges in G are 1-loops, or

(b) G does not contain 1-loops, and all the possible bases of G are connected,

then B is totally unimodular.

Proof: We will use the notations introduced in the proof of Theorem 9, and show that if the conditions of the theorem are satisfied, then the inverse of each non-singular square submatrix C of B is integral, so that B is 1-regular. Since by Lemma 8 B is integral, this is equivalent to its total unimodularity.

If there are no bidirected edges in G except 1-loops, then all the cycles of any basis T are 1-loops, so T^{-1} is integral by Lemma 7. In case (b), T is connected, so a row of T^{-1} contains only $\pm 1/2$ entries, or $0, \pm 1$ entries. There are no 1-loops in $G(R)$, so $\sum_i |r_{ij}| = 2$ for all j where r_{ij}'s are the elements of R. Thus $T^{-1}R$ is an integral matrix. $\square$

A binet matrix B is 2-regular, so the polyhedron $P(B, b) = \{ x : Bx \leq b, x \geq 0 \}$ is integral for each even vector b. A bit more can be claimed.

Theorem 11. *Let $B = R^{-1}S$ be a binet matrix. $P(B, b)$ is integral if b satisfies the following conditions:*

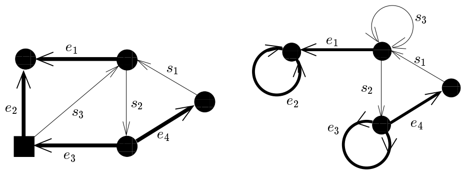
(a) the elements of b that correspond to non-cycle basic edges are even,

(b) if the cycle of a component is not a 1-loop, then the elements of b that correspond to its edges are all odd or all even.

(c) if the cycle of a component is a 1-loop, then the element of b that corresponds to it is even.

Proof: For the integrality of $P(B, b)$ it is enough to prove that the polyhedron $\{ z : [B, I]z = b, z \geq 0 \}$ is integral. Since $[B, I] = R^{-1}A$, this polyhedron is the same as $\{ z : Az = Rb, z \geq 0 \}$.

First we show that for every vector b that satisfies the conditions of the theorem Rb is even. Consider the i^{th} row of R. Its non-zeros are ± 1 corresponding to the non-2-loop edges incident to node i in $G(R)$, or ± 2 corresponding to the 2-loops located at i. At any node of $G(R)$ the number of incident cycle edges can be 0, or 1 if the cycle is a loop, or 2 if the cycle is not a loop. If the number of incident cycle edges at a node i is zero, $\sum_j r_{ij} b_j$ is even anyway by condition (a). If not, and if the cycle is not a loop, then only cycle edges may have odd b



	s_1	s_2	s_3
e_1	-1	1	-1
e_2	1	-1	1
e_3	1	-1	0
e_4	-1	0	0

Figure 6: The creation of a network matrix. The basic edges are in bold.

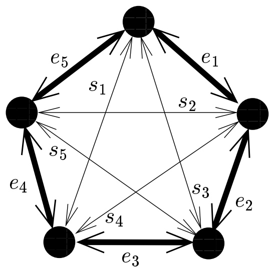
values. As all of them are odd or even, the sum of any two (with weights ± 1) is even. If the cycle is a 1-loop, then its weight is even, while if it is a 2-loop, then its weight is multiplied with 2. So $\sum_j r_{ij} b_j$ is even.

Since A is the incidence matrix of a bidirected graph and Rb is an even vector, by Theorem 5 the polyhedron is integral. $\square$

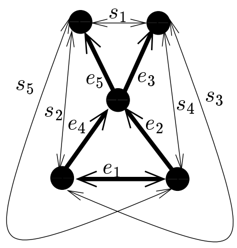
Let us now consider some examples of binet matrices. First of all, every network matrix is a binet matrix. Network matrices arise from the incidence matrices of connected directed graphs by deleting an arbitrary row, choosing a basis (which corresponds to a spanning tree), and taking $R^{-1}S$ just as we did for binet matrices. To translate this to bidirected graphs, observe that row deletion means deleting a node and creating 1-loops out of its incident edges (see Figure 6). A directed graph becomes bidirected after this operation. So network matrices form a very special set of binet matrices, where the only bidirected edges are 1-loops. Hence Theorem 10 provides us with a proof of the total unimodularity of network matrices.

Next consider matrices B_1 and B_2 below. These are the two well-known totally unimodular matrices that play an important role in the decomposation theorem of totally unimodular matrices (see Seymour [10]). Neither they nor their transposes are network matrices. The graphs drawn next to them show that B_1 and B_2 are binet matrices. Theorem 10 gives a proof of their total unimodularity.

$$B_1 = \begin{array}{c|c|c|c|c|c|} & s_1 & s_2 & s_3 & s_4 & s_5 \\ \hline e_1 & 1 & 0 & 0 & 1 & -1 \\ \hline e_2 & -1 & 1 & 0 & 0 & 1 \\ \hline e_3 & 1 & -1 & 1 & 0 & 0 \\ \hline e_4 & 0 & 1 & -1 & 1 & 0 \\ \hline e_5 & 0 & 0 & 1 & -1 & 1 \\ \hline \end{array}$$



$$B_2 = \begin{array}{c|c|c|c|c|c|} & s_1 & s_2 & s_3 & s_4 & s_5 \\ \hline e_1 & 1 & 1 & 1 & 1 & 1 \\ \hline e_2 & 1 & 1 & 1 & 0 & 0 \\ \hline e_3 & 1 & 0 & 1 & 1 & 0 \\ \hline e_4 & 1 & 0 & 0 & 1 & 1 \\ \hline e_5 & 1 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$



6 Operations on binet matrices

Let us start with some simple operations under which a binet matrix remains such.

Theorem 12. *A binet matrix remains a binet matrix under the following operations:*

(a) permuting rows and columns
(b) scaling a row or a column by -1
(c) pivoting, i.e. transforming $\begin{bmatrix} \alpha & c \\ b & D \end{bmatrix}$ *to* $\begin{bmatrix} 1/\alpha & c/\alpha \\ -b/\alpha & D - \frac{1}{\alpha}bc \end{bmatrix}$
(d) repeating a row or a column
(e) deleting a column
(f) adding a row or a column with one non-zero element ± 1.

Proof: We give the equivalent operations on binet graphs.

Let $e_1, e_2, \ldots, e_n$ be the basic edges related to the rows, and $s_1, s_2, \ldots, s_k$ be the non-basic edges related to the columns of the matrix.

(a) Trivial.

(b) Scaling a row or a column by -1 means changing the direction of the related edge, that is making out-nodes into in-nodes and vice versa.

(c) Pivoting on the non-zero element b_{ij} of B is equal to changing the basis, replacing e_i by s_j.

(d) Repeating a column is equivalent to including an identical copy of the non-basic edge related to the column. Repeating a row is equivalent to dividing a basic edge into two new basic edges by inserting a new odd-node, in the following way.

If e is a non-loop basic edge connecting nodes u and v (see Figure 7 (i)), then insert a new node t into the graph, connect e to t instead of v with the same direction, connect nodes t and v with a new directed edge r and direct r so that t becomes an odd node.

If e is a 2-loop basic edge at node u (see Figure 7 (ii)) , then insert a new node t into the graph, connect e to t so that it goes between u and t and remains bidirected with the same direction at u, connect t and u with another, directed edge r and direct r so that t becomes an odd-node.

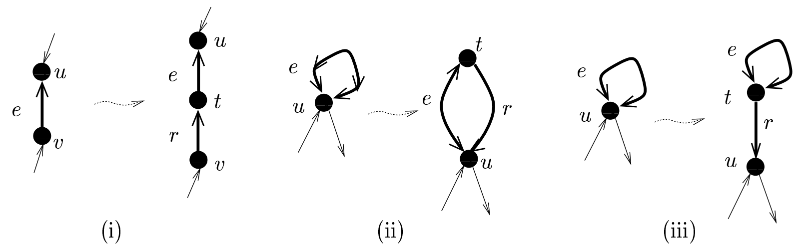


Figure 7: The graphical equivalent of repeating row e

If e is a 1-loop basic edge at node u (see Figure 7 (iii)), then insert a new node t, remove 1-loop e from u and locate it at t, and connect t and u with a new directed edge r so that t becomes an odd-node.

Each fundamental circuit that goes through e will go through r as well, and one checks easily with reference to calculating coefficients $w(e_i)$ in Section 5 that the value of both $w(e)$ and $w(r)$ will be equal to $w(e)$ calculated in the old graph. Since t is an odd node, $c(t) = 0$ so that all the other values in each fundamental circuit remain unchanged. Therefore new row r of the binet matrix of the transformed graph equals row e of the original matrix and other rows are unchanged.

(e) Deleting a column is simply the same as deleting the non-basic edge that is related to it.

(f) Adding a column with one $+1$ in row i simply translates to including an identical copy of e_i as a non-basic edge.

As for rows, let us add a row with a $+1$ in the column of a non-basic edge s. If we do the same operation with s as we did in *(d)* with e, then r will be a new basic edge, which is only used in the fundamental circuit of s, and it gets the value $+1$ in it. So row r of the binet matrix of the transformed graph is a unit row, which contains $+1$ in the column of s and zeros in other columns.

As to adding a row or a column with the only non-zero being -1, we refer to operation *(b)* introduced above. $\square$

Note that the operation of deleting a row is left out in Theorem 12. This is much more complex and dealt with separately in Theorem 16.

Theorem 12 can be used to prove easily Theorem 4 for binet matrices and $k = 2$.

Theorem 13. *Let B be a binet matrix. Then all the extreme points of the polyhedron $\{ x : Bx \le b,\ 0 \le x \le u \}$ are half-integral for all integral vectors b and u.*

Proof: Because of part *(f)* of Theorem 12, $\begin{bmatrix} B \\ I \end{bmatrix}$ is a binet matrix, so it is 2-regular, and half-integrality follows from Theorem 3. $\square$

In Lemma 8 we gave the necessary and sufficient conditions for the integrality of a binet matrix. The next lemma embellishes that result with reference to pivoting.

Lemma 14. *By finite number of pivot operations any binet matrix can be converted to an integral binet matrix*

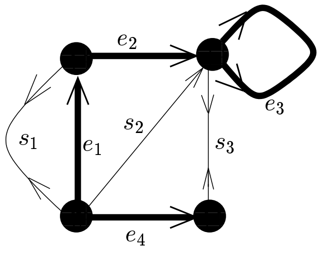
Proof: Let B be a binet matrix and b_{ij} a non-integral element of it. Then $b_{ij} = \pm 1/2$ and that can occur if and only if e_i is an edge of a cycle which is not a 1-loop and s_j is a 1-loop or lies on the connecting path between two components. By part *(c)* of Theorem 12 pivoting on entry b_{ij} means changing the basis, and exchanging basic edge e_i with non-basic edge s_j. After pivoting the basis contains more 1-loops or fewer components, therefore after finite number of pivots the basis satisfies the conditions of Lemma 8, and B is integral. □

It is very easy to see that deletion of a column of a binet matrix preserves its binet property. Deleting a row, however, causes problems. For network matrices, deletion of a row simply translates to the contraction of the related basic (tree) edge, i.e., deletion of the edge and unification of its end-nodes. This operation cannot be applied directly to binet matrices, as during the calculation of the matrix elements the parity of the nodes and cycles (whether they are even or odd) can change after contracting an edge. If we contract a bidirected basic edge, then the number of bidirected edges in a basic cycle may become even, in spite of the fact that a basis must contain odd cycles. Furthermore, contracting a bidirected edge of a fundamental circuit changes the position of odd-nodes in the circuit, which determines values $b(f_i)$. One can easily see, however, that none of these problems occur when contracting a directed basic edge. Hence a representation of the matrix with as few bidirected edges as possible may help us in deleting a row. Clearly, at least one bidirected edge must occur in every component, since each contains an odd-cycle. The next theorem claims that one is enough.

Theorem 15. *There always exists a representation of a binet matrix where each component of the basis contains only one bidirected edge.*

Proof: Let us suppose that the basis R is connected. We have shown in the proof of Theorem 6 that by scaling the rows the submatrix of R that corresponds to the cycle can be transformed so that it contains only one bidirected edge (note that column scaling is not needed here). By further scaling of the rows of R, the non-cycle basic edges can also be transformed to directed edges. So there exists a diagonal $0, \pm 1$ matrix E such that if $R' = ER$, then $G(R')$ contains only one bidirected edge. Then $B = R^{-1}S = R^{-1}E^{-1}ES = R'^{-1}S'$, where $S' = ES$ and R' is a basis of $[R', S'] = EA$. Thus B has a binet representation where the basis (R') contains only one bidirected edge. □

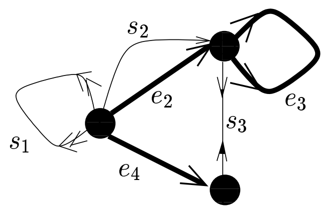
Remark 3. The transformation described above changes only the directions of the edges in the binet graph, leaving its structure unchanged. Futhermore, the only bidirected edge in each component must be in its cycle, and we are free to choose which cycle-edge remains bidirected.



B=

	s_1	s_2	s_3
e_1	1	1	1
e_2	2	1	1
e_3	2	0	2
e_4	0	0	-1

Figure 8: The binet graph representing binet matrix B



$B_1=$

	s_1	s_2	s_3
e_2	2	1	1
e_3	2	0	2
e_4	0	0	-1

Figure 9: The binet graph and matrix after deleting e_1

Theorem 16. *If we delete a row from a binet matrix, then the remaining matrix is still a binet matrix.*

Proof: Let B be a binet matrix and e_1 be one of its rows. Consider a graph representation $G(B)$ of B. If e_1 is not a loop in the graph, then by the previous theorem and remark we can find a representation of B where e_1 is not bidirected. (Such a representation is shown in Figure 8.) Let us contract this edge as shown in Figure 9 by replacing the end-nodes of e_1 with a new node and connecting edges that were incident to either of them to the new node. If a bidirected edge is parallel to e_1, then we replace it with a new 2-loop. If a directed edge is parallel to e_1, then we delete it from the graph to avoid creating all-zero columns in the matrix. (As e_1 is a directed edge, the fundamental circuit created by an s edge which is a directed edge parallel to it is s and e_1, leading to zeros in the column of s in B in every row other than e_1 which is being deleted.) The binet matrix B_1 of the transformed graph is equal to the matrix we get deleting row e_1 from B. The sums $\sum_{j=0}^{i-1} c(n_j)$ do not change, so the $b(f_i)$ values $(i = 2, 3, \ldots, m)$ of the non-deleted edges in every fundamental circuit remain unchanged. As no new 1-loops, components or basic edges were created, coefficients $w(e_i)$'s do not change either.

If e_1 is a loop in $G(B)$, then we delete it from the graph together with the node it is incident to as defined earlier in Section 4 (see Figure 10). Let $G(B_1)$ and R_1 denote the remaining graph and basis respectively, where B_1 denotes the matrix we get after deleting row e_1 from B. The following formulae show,

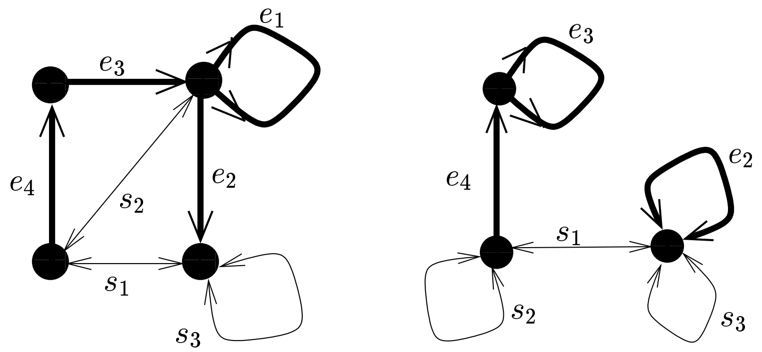


Figure 10: Deletion of loop e_1

that B_1 is equal to the binet matrix represented by $G(B_1)$ with the basis R_1.

$$A = [R, S] = \begin{bmatrix} 0 & R_1 & S_1 \\ \alpha & * & * \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} * & 1/\alpha \\ R_1^{-1} & 0 \end{bmatrix}, \quad B = R^{-1}S = \begin{bmatrix} * \\ R_1^{-1}S \end{bmatrix}$$

where the first column of A and the first row of B represent e_1, $\alpha = \pm 1$ or ± 2 depending on whether e_1 is a 1-loop or a 2-loop, and $*$ denotes elements the actual values of which are irrelevant.

□

As a consequence of the previous theorem and Theorem 12 (e), we conclude:

Corollary 17. *Let $\mathcal{B}$ denote the set of binet matrices in a general sense, where all-zero columns are allowed. Then $\mathcal{B}$ is closed under taking submatrices.*

Remark 4. This is the first point where 2-loops are necessary. So far the whole theory could have been developed without 2-loops. But if we define bidirected graphs and binet matrices without 2-loops, then Theorem 16 and Corollary 17 do not remain true. The matrix of Figure 9 serves as an example. Deleting e_1 from the matrix of Figure 8 we are left with a column (s_1) with only 0 and ± 2 entries. But if we have no 2-loops, then a binet matrix cannot have a column with only 0 and ± 2 elements.

7 Further theorems about binet matrices

We provide new results on binet matrices, some of which were discovered in our attempt at finding a recognition algorithm.

Theorem 18. *If B is an integral binet matrix, then there exist network matrices N_1 and N_2 such that*
(a) $B = N_1 + N_2$
(b) both $[N_1, N_2]$ and $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ are network matrices.

Proof: Let binet graph $G = (V, R, S)$ with node set V, basic edges R, and non-basic edges S, represent B. By Theorem 15 and Lemma 8, G is one of the following two types. We deal with them in turn.

Case I. All the basic cycles of G are 1-loops, and they are the only bidirected edges in the basis. (See Figure 11, for example.)

We introduce a way of constructing a spanning tree and two connected directed graphs giving network matrices that satisfy the conditions of the theorem.

Let us introduce a new node v_0 into G and replace each basic cycle loop with an edge connected to v_0. If the cycle was an in-loop located at v, then the new edge is directed from v_0 to v, while if the cycle was an out-loop, then the edge is directed to v_0 from V. (See Figure 12.) So the basis R becomes a spanning tree T on the node set $V \cup v_0$. We shall construct connected directed graphs D_1 and D_2 on this node set so that they both contain T. Based on this tree in graphs D_1 and D_2 we shall construct network matrices N_1 and N_2 , respectively, which satisfy the conditions of the theorem.

(a) If, as for s_1 in Figure 11, $s \in S$ is a directed edge, then let it remain the same in D_1. (Let s_1 denote this new edge too.) Column s of B is equal to column s of N_1. Let us suppose that N_2 contains an all-zero column corresponding to s and an 'empty' edge corresponds to it in D_2.

(b) If, as for s_2 in Figure 11, $s \in S$ is a 1-loop, then connect it to v_0 in D_1 and direct it as described above. An 'empty' edge represents it in D_2.

(c) If $s \in S$ is a bidirected edge which is not a 1-loop (as is the case for s_3 and s_4 in Figure 11), and $u, v \in V$ are its end-nodes, then *imagine* two new 1-loops l_u and l_v located at u and v and directed so that u (or v) is an out-node of l_u (l_v) if and only if it is an out-node of s. If (as for s_4) s is a 2-loop (i.e., $u = v$), then let l_u and l_v be two identical loops. It is clear that the column of the incidence matrix of G that corresponds to s is the sum of the columns of the matrix that correspond to the related loops, and this remains true for binet matrix B. These imagined, artifical 1-loops will help us to find edges in D_1 and D_2 that represent s.

Connect l_u and l_v to v_0 in D_1 and D_2, respectively, i.e., let us suppose that D_1 contains edge $s = (u, v_0)$, and D_2 contains edge $s = (v, v_0)$ with the correct, formerly described directions. Figures 12 and 13 illustrates the operations.

Column s of B now equals the sum of columns s of N_1 and N_2 for any non-basic edge. We have derived every column of the binet matrix as the sum of two corresponding columns of network matrices. So $B = N_1 + N_2$.

$[N_1, N_2]$ is trivially a network matrix, as both N_1 and N_1 are based on T. As for $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, consider the following construction. Change the direction of every edge in D_2 (getting D_2'), unify the v_0 node in D_1 and D_2', as shown in Figure 14. As to non-tree edges, if $(u, v_0) \in D_1$ and $(v, v_0) \in D_2'$ are two non-tree directed edges corresponding to the same non-basic edge of B, then we substitute them by a new edge (u, v) in the unified graph, and keep the directions at u and v. The network matrix of the so defined unified directed graph is $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$.

Case II. There are no 1-loops in G, the basis is connected and it contains

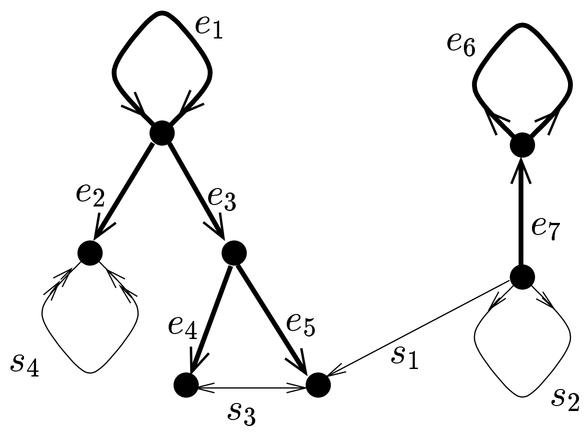


Figure 11: Binet graph B with only 1-loop basic cycle edges

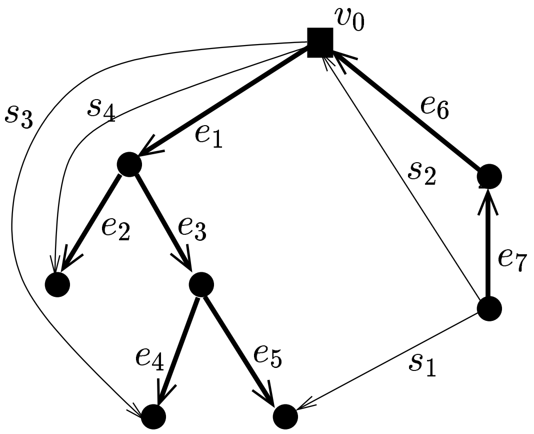


Figure 12: Directed graph D_1 made from binet graph B of Figure 11

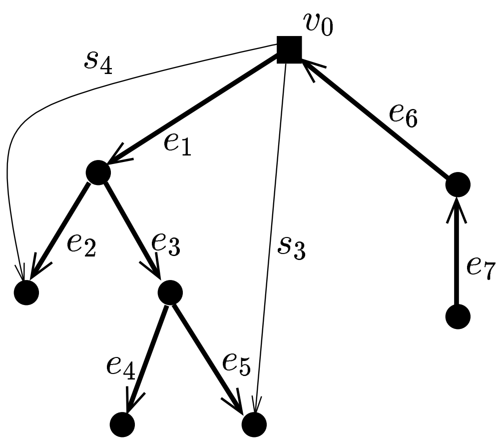


Figure 13: Directed graph D_2 made from binet graph B of Figure 11

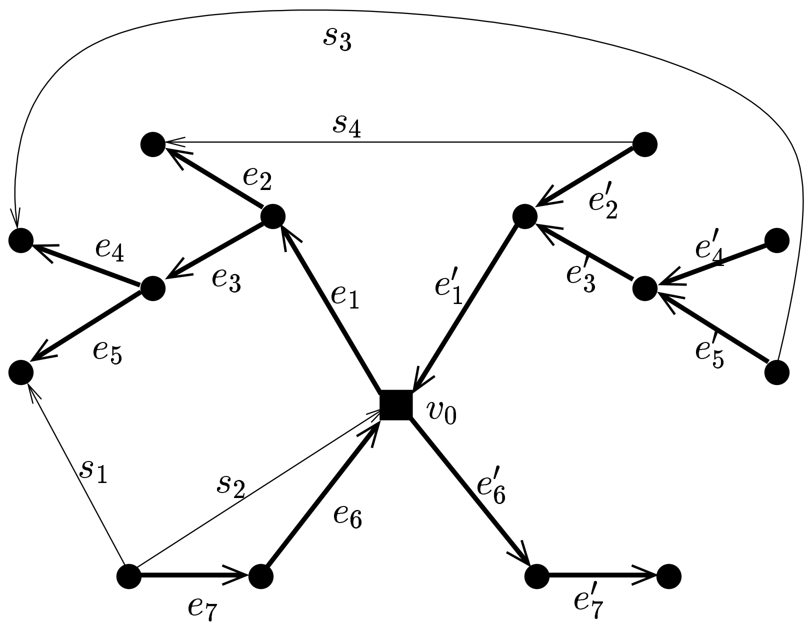


Figure 14: Unification of D_1 and the modified D_2

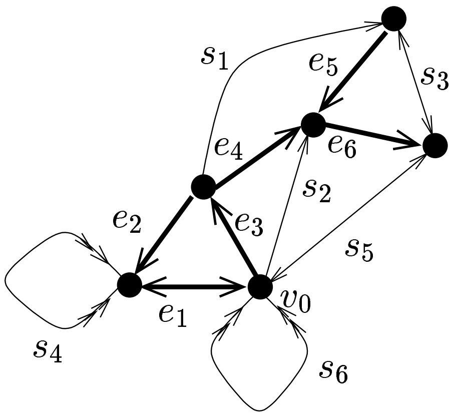


Figure 15: Binet graph B with a connected basis

only one bidireced edge e_1. (Figure 15 illustrates this case.)

Let v_0 be an end-node of e_1. We 'open' the cycle at v_0, by splitting v_0 into two separate nodes v_0' and v_0'' so that e_1 is incident to v_0' and other basic edges incident to v_0 are incident to v_0''. Furthermore, we delete the direction of e_1 at v_0' so that the basis becomes a directed tree T. Figure 16 shows this operation.

We will construct directed graphs D_1 and D_2 such that they both contain T as spanning tree. Network matrices N_1 and N_2, defined respectively on D_1 and D_2 satisfiy the conditions of the theorem. Figure 15 illustrates.

If $s \in S$ is a directed edge and v_0 is not its end-node (s_1 in Fig. 15), then it remains the same in D_1 and becomes an 'empty' edge in D_2. If v_0 is its end-node (as s_2 in Fig. 15), then v_0'' is its end-node in D_1 and it is represented by an 'empty' edge in D_2. If $s \in S$ is a bidirected edge connecting u and v where $u, v \neq v_0$ (but u might be equal to v) as in case of s_4 in Fig. 15, then let $s = (u, v_0')$ and $s = (v_0'', v)$ be directed edges of D_1 and D_2, resp., such that u (v) is an out-node of s in D_1 (D_2) if it is an out-node of s in G. If $v = v_0$ but s is not a loop (s_5), so $u \neq v_0$ then let $s = (u, v_0')$ be a directed edge in D_1 with the direction at u as described above and let s be an 'empty' edge in D_2. If s is a 2-loop at v_0 so $u = v = v_0$ (as s_6), then let $s = (v_0', v_0'')$ be a directed edge in D_1 with the appropiate direction at v_0' and let s be an 'empty' edge in D_2.

We have listed all the possible cases, and in each case column s of B is equal to the sum of columns s of N_1 and N_2. So $B = N_1 + N_2$.

The network representations of $[N_1, N_2]$ and $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ are made in the same way as before, unifying v_0' of D_1 with v_0'' of D_2' in the latter case. $\square$

Remark 5. The converse of Theorem 18 is not true as shown by the following counterexample in which $[N_1, N_2]$ and $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ are network matrices with $B = N_1 + N_2$ and B is integral, but not a binet matrix, as it is not even 2-regular.

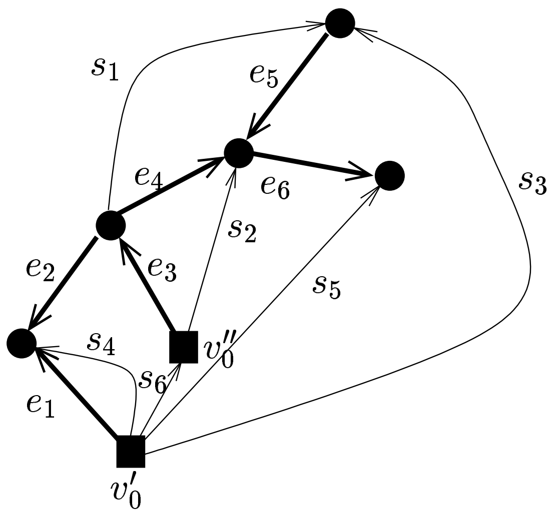


Figure 16: Directed graph D_1 made from binet graph B of Figure 15 by opening the cycle at v_0

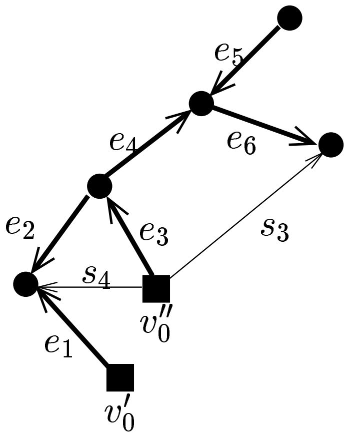


Figure 17: Directed graph D_2 made from binet graph B of Figure 15 by opening the cycle at v_0

$$B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3}$$

It is well-known (Ghouila-Houri [4]) that for each collection of columns of a totally unimodular matrix there exists a scaling of the selected columns by ± 1 such that the sum of the scaled columns is a vector of $0, \pm 1$ elements, and this is true for rows as well. Obviously, this does not remain true for 2-regular matrices, as matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ shows. However, for binet matrices we can say something similar.

Theorem 19. *For each collection of columns or rows of an integral binet matrix there exists a scaling of the selected columns or rows by ± 1 such that the sum of the scaled columns or rows is a vector with elements $\{0, \pm 1, \pm 2\}$.*

Proof: Since column (row) deletion preserves binet property of binet matrix B, we can assume that every column (row) is selected.

1. Columns. By Theorem 18 there exist network matrices N_1 and N_2 such that $B = N_1 + N_2$ and $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ is also a network matrix. By the Ghouila-Houri property of totally unimodular matrices, there exists a ± 1 vector d such that Nd is a vector of $\{0, \pm 1\}$ elements, hence $Bd = (N_1 + N_2)d = N_1 d + N_2 d$ is a vector of $\{0, \pm 1, \pm 2\}$ elements. Thus vector d gives the required scaling.

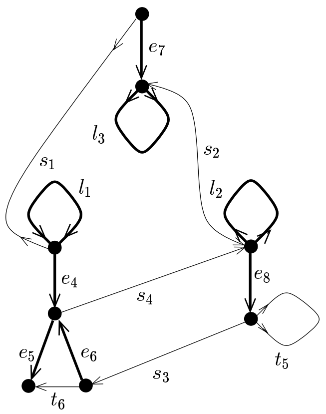
2. Rows. Very similar using the fact that $[N_1, N_2]$ is a network matrix. □

Remark 6. For totally unimodular matrices the converse of the related theorem is true in the sense that if an integral matrix has the scaling property, then it is totally unimodular. However, for binet matrices the conserve of Theorem 19 is not true. Matrices of (3) in Remark 5 serve as a counterexample.

Theorem 20. *A totally unimodular binet matrix B is also a network matrix if and only if it has a binet representation in which each basic cycle is a 1-loop.*

Proof: 1. Necessity–By the operation mentioned earlier and illustrated in Figure 6 all the network matrices can be represented with a binet graph in which the basic cycle of each component is a 1-loop.

2. Sufficiency–Let G be the binet representation of B in which each basic cycle is a 1-loop, and the non-loop basic edges are directed. Such a representation exists by Theorem 15. Let $C_1, C_2, \ldots, C_k$ be the components of G. Since B is totally unimodular, it does not contain ± 2 entries so any non-basic edge of G can be a 1-loop, a directed edge or a bidirected edge connecting two components. (See Figure 18.) Furthermore, if two components are connected with a bidirected edge, then all the other edges that connect these two components are also bidirected. The same holds for the directed case. This is because otherwise the matrix is not totally unimodular as it contains a $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ submatrix (corresponding to the rows of two 1-loops of the components in question). We



	s_1	s_2	s_3	s_4	t_5	t_6
l_1	-1	0	1	-1	0	0
l_2	0	1	1	-1	1	0
l_3	1	1	0	0	0	0
e_4	0	0	1	-1	0	0
e_5	0	0	0	0	0	1
e_6	0	0	-1	0	0	1
e_7	-1	0	0	0	0	0
e_8	0	0	-1	0	-1	0

Figure 18: Example binet graph G and matrix B for Theorem 20 with basic edges in bold

will show that there exists a representation of B with only 1-loop and directed non-basic edges leading to a network representation of B.

Let $l_1, l_2, \ldots, l_k$ be the basic cycles all of which are 1-loops, and $s_1, s_2, \ldots, s_l$ be the non-basic edges that connect two components. The submatrix of B made up of rows $l_1, l_2, \ldots, l_k$ and columns $s_1, s_2, \ldots, s_l$ is a totally unimodular matrix with exactly two nonzero entries in each column. Such a matrix is a network matrix, with a 'star' representation, i.e., based on a tree T that consists of a node u and edges $l_1, l_2, \ldots, l_k$ all incident to u. (See Figure 19.) The existence of such a representation follows easily from Ghouila-Houri's characterization [4] of totally unimodular matrices (see [8, pp 283]).

Let us insert a new node v_0 into graph G, and connect the basic 1-loops to it so that if the loop is an in-loop, then the new edge is directed from v_0, and vice versa. In the new graph G', v_0 and edges $l_1, l_2, \ldots, l_k$ form a star, which may differ from T only in the directions of the edges. If the direction of l_i is different in G' and T, then let us change the direction of all the edges that are in C_i or incident to it: in-nodes will be out-nodes and vice versa. (C_3 was such in G, so we change the direction of l_3, e_7, s_1 and s_2.) With this modification all the connecting edges will become directed, the other non-basic edges remain 1-loops or directed edges. If we connect the 1-loops to v_0 as described above, then G' is a directed graph and it represents B. (Figure 19 shows the network matrix G' made of graph G of Figure 18.) □

The following corollary is an easy consequence of Lemma 8 and the previous theorem.

Corollary 21. *If a binet matrix B is totally unimodular, but not a network matrix, then in its every binet representation the basis is connected, and there are no 1-loops in the binet graph.*

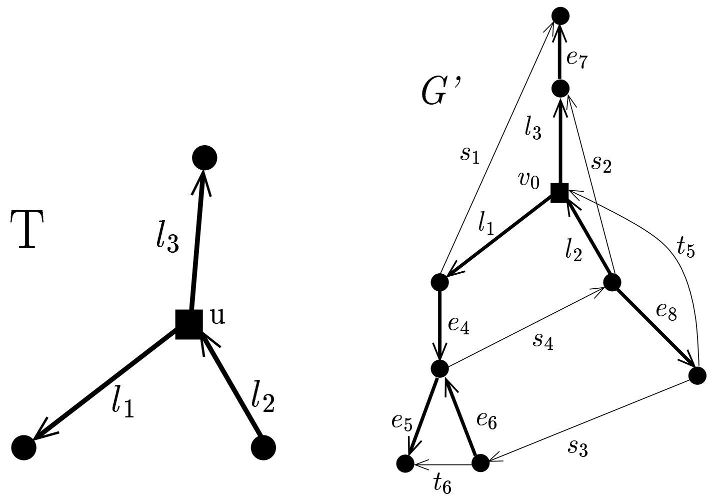


Figure 19: The tree T and the network matrix G' that represents B of Fig. 18

We have shown that network matrices have a binet representation in which the basic cycle in each component of the basis graph is a 1-loop, and Theorem 20 stated that this property is a characterisc of totally unimodular network matrices. It does not mean that this is the only kind of binet representation of network matrices. In fact, next theorem claims the opposite.

Theorem 22. *There allways exists a binet representation of any network matrix with at least two rows, where the basis is connected, and the basic cycle is not a loop.*

Proof: Let a directed graph with a spanning tree T represent the network matrix. We show how to construct a binet graph with a non-loop cycle to represent the network matrix. To this end first insert a new bidirected basic edge f into the graph so that it connects two nodes that are not connected with an edge of the tree. (Figure 20 illustrates the operation on the graph and matrix of Figure 6.) The tree extended with this new edge is a binet basis graph in which the basic cycle contains at least three edges. Moreover, no fundamental circuit uses the new bidirected edge. So the binet matrix of this graph is equal to the original network matrix plus an all-zero row. If we delete the all-zero row we get the original matrix. In the graph row deletion translates to the redirection of the edges so that f is directed, and then to the contraction of f. Since the basic cycle contained at least three edges, the cycle of the new basis is not a loop, and it is obvious that the basis is connected. □

We have shown that every network matrix is binet. Unfortunately this is not true of every totally unimodular matrix. Below we give an example of a matrix which is the transpose of a node-edge incidence matrix of a directed graph but has no binet representation.

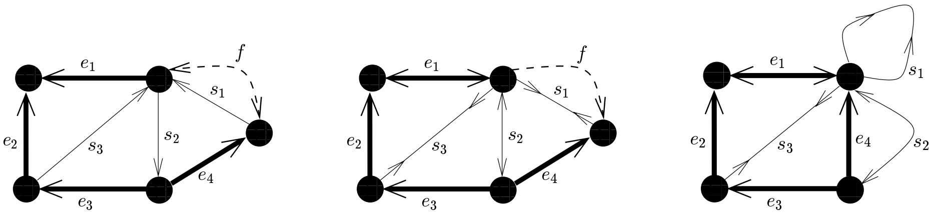


Figure 20: Inserting a bidirected edge f to the network matrix; redirecting the edges and contracting f

Let K_6 be the complete graph on six nodes, and let us direct its edges arbitrarily. Let A be its node-edge incidence matrix having 6 rows and 15 columns. Let B be the transpose of this matrix. We claim that B is not a binet matrix.

Let us suppose that B is binet. Then there exists a bidirected graph G that represents it. Since B has 15 rows and 6 columns, G has 15 nodes and 21 edges. Let us assign a degree $d(n)$ to each node n measuring the number of the incident edges and counting the loops twice, so that $\sum d(n) = 42$. This implies that G must have a node with a degree at most 2. Let n_1 is such. If $d(n_1) = 1$, then there is only one incident (non-loop) edge to n_1, which must be a basic edge. The row corresponding to this edge is all-zero, as no fundamental circuit can go through the edge. Since B does not contain all-zero rows, this case is impossible.

If $d(n_1) = 2$ and a loop is incident to n_1, then this is the only incident edge, it must be in the basis, and the corresponding row is all-zero again. If two non-loop edges (say e and f) are incident to n_1, then at least one of them is in the basis. If one of the edges (say e) is basic, the other is non-basic then there is only one fundamental circuit goes through e, the basic edge, the one corresponding to f. This means that B has a row which contains only one non-zero, contradicting the fact that all of its rows have two non-zeros. If both of the incident edges e and f are basic, then the rows related to them can differ only in the sign of the non-zero elements as any fundamental circuit that goes through either of them goes through the other too. This is again a contradiction, since there are no parallel edges in K_6. We have examined all the possible cases and all of them led to a contradiction, so B cannot be a binet matrix.

In fact, we proved the following lemma.

Lemma 23. *Let B be a binet matrix of size $m \times n$. If $m > 2n$, then*

(a) B has an all-zero row, or

(b) B has a row with exactly one non-zero, or

(c) B has two rows the absolute values of which are equal.

8 Connections with other research areas

Results similar to some of ours have appeared in many guises. One particularly worth mentioning is signed graphs. They were introduced by Harary in [5] and are more or less the same as bidirected graphs. The way signed graphs are defined by Zaslavsky in [13] is even closer, the only difference from bidirected graphs being that he allows directed loops (positive loops in his terminology) but that has no consequence in the main theorems. He develops the theory of signed graphic matroids proving that such a matriod can be defined based on the notion of balance. His balanced circles represent the same phenomenon as our odd-cycles, thus independence in the signed graphic matroid means the same as in bidirected graphs. Zaslavsky introduces the incidence matrix of a signed graph as a representation of the signed graphic matroid, and proves its 2-regularity. This latter result also appears in Hochbaum et al. [6] as a property of the generalised 2-SAT problem.

The main difference from Zaslavsky's results is that the approach in our paper is matrix driven instead of a matroid driven one. The main new points here are binet matrices and their 2-regularity. While incidence matrices of bidirected graphs, since they contain at most two nonzeros in every column, cover only a very restricted part of rational matrices, binet matrices provide us with a broader applicapability.

Acknowledgements

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References

[1] G.M. Appa, k-integrality, an extension of total unimodularity, *Operations Research Letters*, 13 (1993), 159-163.

[2] R. E. Bixby, Kuratowski's and Wagner's theorems for matroids, *Journal of Combinatorial Theory (B)*, 22 (1977), 31-53.

[3] G. B. Dantzig, Application of the simplex method to a transportation problem, in: *Activity Analysis of Production and Allocation* (ed: Tj. C. Koopmans), Wiley, New York, 1951, pp. 339-347.

[4] A. Ghouila-Houri, Caractérisation des matrices totalement unimodulaires, *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris)*, 254 (1962) 1192-1194.

[5] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2 (1953-1954), 143-146.

[6] B.S. Hochbaum, N. Megiddo, J. Naor and A. Tamir, Tight bounds and 2-approximation algorithms for integer programs with two variables per inequality, *Mathematical Programming*, 62 (1993), 69-83.

[7] A. Hoffman and J. Kruskal, Integral boundary points of convex polyhedra, in: K. Kuhn and A. Tucker (eds.), *Linear Inequalities and Related Systems*, Princeton University Press, Princeton, NJ, 1956, 223-246.

[8] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986.

[9] A. Schrijver, Disjoint Homotopic Paths and Trees in a Planar Graph, *Discrete & Computational Geometry*, 6 (1991), 527-574.

[10] P.D. Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory (B)*, 28 (1980), 305-359.

[11] W. T. Tutte, Lectures on matroids, *Journal of Research of the National Bureau of Standards (B)*, 69 (1965), 1-47.

[12] A. Veinott and G. Dantzig, Integral extreme points, *SIAM Review*, 10 (1968), 371-372.

[13] Th. Zaslavsky, Signed graphs, *Discrete Applied Mathematics*, 4 (1982), 47-74.