Equimodular curves

Norman Biggs

Centre for Discrete and Applicable Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
U.K.

CDAM Research Report LSE-CDAM-2000-17

September 2000

Abstract

This paper is motivated by a problem that arises in the study of partition functions of antiferromagnetic Potts models, including as a special case the chromatic polynomial. It relies on a theorem of Beraha, Kahane and Weiss, which asserts that the zeros of certain sequences of polynomials approach the curves on which a matrix has two eigenvalues with equal modulus. It is shown that (in general) the equimodular curves comprise a number of segments, the end-points of which are the roots of a polynomial equation, representing the vanishing of a discriminant. The segments are in bijective correspondence with the double roots of another polynomial equation, which is significantly simpler than the discriminant equation. Singularities of the segments can occur, corresponding to the vanishing of a Jacobian. These results are illustrated by explicit calculations in a specific case.

MSC 2000: 05C15, 05C50, 05E99

Equimodular curves

1. Introduction

This paper is motivated by a problem that arises in the study of partition functions of antiferromagnetic Potts models, including as a special case the chromatic polynomial. The critical behaviour of these models leads to the study of the complex zeros of the partition function, the classic result in this field being the Lee-Yang theorem [7,8].

In 1972 it was observed [3] that the complex zeros of certain chromatic polynomials exhibit interesting behaviour, although the reason for it was not understood at that time. Subsequently, a theorem of Beraha, Kahane and Weiss [1] provided a general explanation for such behaviour. (Specific examples are given in [2,9].) The theorem asserts that, as $n \to \infty$, the zeros of certain sequences of polynomials $f_n(z)$ approach the curves on which a 'transfer matrix' $T(z)$ has two eigenvalues with equal modulus.

In the theoretical physics literature, the Beraha-Kahane-Weiss theorem has been exploited in papers by Shrock and his colleagues [5,11,12], and by Salas and Sokal [10]. It is clear that it requires intensive computational resources, and consequently some work on the method itself is desirable. That is the subject of the present paper. It is shown that (in general) the equimodular curves comprise a number of segments, the end-points of which are the roots of a polynomial equation, representing the vanishing of a discriminant. The segments are in bijective correspondence with the double roots of another polynomial equation, which is significantly simpler than the discriminant equation. Singularities of the segments can occur, corresponding to the vanishing of a Jacobian. These results are illustrated by explicit calculations in a specific case.

The algebraic and computational techniques employed here are part of the current revival of interest in the techniques of classical algebraic geometry, in particular the use of resultants [6]. The computations in Section 6 were done with the aid of Maple, version 6, and I am grateful to Philipp Reinfeld for help with them.

2. The polynomial criterion

In the intended application of these results we are given an $m \times m$ matrix $T(z)$, each of whose entries is a polynomial function of the complex variable z, with integer coefficients, and it is required to find the points z for which $T(z)$ has two eigenvalues of equal modulus. Roughly speaking, these points lie on curves in the complex plane, the 'equimodular curves' of the title.

The key idea stems from the observation that if $|\lambda_A| = |\lambda_B|$, then there is a complex number s with $|s| = 1$ such that $\lambda_A = s\lambda_B$. So if λ_A and λ_B are roots of the polynomial equation $p(\lambda) = 0$, where $p(\lambda) = \det(\lambda I - T(z))$, then it follows that λ_B is a common root of $p(\lambda) = 0$ and $p_s(\lambda) = 0$, where $p_s(\lambda) = p(s\lambda)$.

Let $a_i(z)$ be the coefficient of λ^{m-i} in $p(\lambda)$, that is,

$$\det(\lambda I - T(z)) = \lambda^m + a_1(z)\lambda^{m-1} + a_2(z)\lambda^{m-2} + \cdots + a_m(z).$$

The $a_i(z)$ are the sums of principal minors of $T(z)$, and so they are polynomials with integer coefficients, and we have

$$p_s(\lambda) = p(s\lambda) = s^m\lambda^m + s^{m-1}a_1(z)\lambda^{m-1} + s^{m-2}a_2(z)\lambda^{m-2} + \cdots + a_m(z).$$

It is convenient to begin by regarding s and the a_i as indeterminates, so that we consider p_s and p as elements of the ring of polynomials with coefficients in $\mathbf{Z}[s, a_1, a_2, \ldots, a_m]$. It is a classical result that a necessary and sufficient condition for p_s and p to have a non-constant common factor is that the *resultant* $\det R$ vanishes, where $R = R(p_s, p)$ is the following $2m \times 2m$ matrix:

$$\begin{pmatrix}
s^m & s^{m-1}a_1 & \ldots & sa_{m-1} & a_m & 0 & \ldots & 0 & 0 \\
0 & s^m & \ldots & s^2a_{m-2} & sa_{m-1} & a_m & \ldots & 0 & 0 \\
0 & 0 & . & s^3a_{m-3} & s^2a_{m-2} & sa_{m-1} & \ldots & 0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
0 & 0 & \ldots & s^m & s^{m-1}a_1 & s^{m-2}a_2 & \ldots & sa_{m-1} & a_m \\
1 & a_1 & \ldots & a_{m-1} & a_m & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & a_{m-2} & a_{m-1} & a_m & \ldots & 0 & 0 \\
0 & 0 & \ldots & a_{m-3} & a_{m-2} & a_{m-1} & \ldots & 0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
0 & 0 & \ldots & 1 & a_1 & a_2 & \ldots & a_{m-1} & a_m
\end{pmatrix}.$$

For convenience, let $a_0 = 1$. Then we can describe non-zero entries of R as follows:

$$r_{ij} \;=\; \begin{cases} s^{j-i+m}a_{j-i} & \text{if } 1 \le i \le m; \\ a_{j-i+m} & \text{if } m+1 \le i \le 2m. \end{cases}$$

By definition, $\det R$ is the sum over all permutations π of $\{1, 2, \ldots, 2m\}$ of terms

$$sign(\pi)\; r_{1,\pi(1)}r_{2,\pi(2)} \cdots r_{2m,\pi(2m)}.$$

For $1 \le i \le m$ the non-zero entries of R in rows i and $i+m$ are in columns $i, i+1, \ldots i+m$. It follows that the non-zero terms in $\det R$ arise only from permutations that satisfy the condition

$$\{\pi(i), \pi(i+m)\} \;\subset \{i, i+1, \ldots, i+m\}.$$

We shall denote the set of such permutations by Π_m. It is clear that, for each $\pi \in \Pi_m$, there are non-negative integers $n_1(\pi), n_2(\pi), \ldots, n_m(\pi)$ and a non-negative integer $e(\pi)$ such that

$$\det R = \sum_{\pi \in \Pi_m} sign(\pi)\; a_1^{n_1(\pi)} a_2^{n_2(\pi)} \;\cdots\; a_m^{n_m(\pi)}\; s^{e(\pi)}.$$

In this formulation, $\det R$ is a polynomial in s with coefficients in $\mathbf{Z}[a_1, a_2, \ldots, a_m]$. We shall investigate the form of the coefficients.

Given $\pi \in \Pi_m$ define functions

$$t_\pi, b_\pi : \{1,2,\ldots,m\} \to \{0,1,\ldots,m\}$$

by the rules

$$t_\pi(i) = \pi(i) - i, \quad b_\pi(i) = \pi(i+m) - i.$$

Consider the term in $\det R$ arising from a given $\pi \in \Pi_m$. For $1 \le i \le m$ the element $r_{i,\pi(i)}$ involves a_k if and only if $k = \pi(i) - i$, and the element $r_{i+m,\pi(i+m)}$ involves a_k if and only if $k = \pi(i+m) - i$. Thus

$$n_k(\pi) \;=\; \# t_\pi^{-1}(k) \;+\; \# b_\pi^{-1}(k).$$

Lemma 1 For all $\pi \in \Pi_m$ we have

$$n_1(\pi) + 2n_2(\pi) + \cdots + mn_m(\pi) = m^2.$$

Proof It follows from the formula for $n_k(\pi)$ given above that

$$\begin{aligned}\sum_{k=0}^{m} kn_k(\pi) &= \sum_{i=1}^{m}\Big(t_\pi(i) + b_\pi(i)\Big)\\ &= \sum_{i=1}^{m}(\pi(i) - i) \;+\; \sum_{i=1}^{m}(\pi(i+m) - i)\\ &= \sum_{i=1}^{2m}\pi(i) \;-\; 2\sum_{i=1}^{m} i.\end{aligned}$$

Since π is a permutation, this is equal to

$$\sum_{i=1}^{2m} i - 2\sum_{i=1}^{m} i \;=\; (2m^2 + m) \;-\; 2(m(m+1)/2) = m^2.$$

□

The lemma says that each monomial $a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$ that occurs as a coefficient corresponds to a partition of m^2. Furthermore, it is clear from the definition that no part is greater than m, and not more than m parts are equal.

Lemma 2 Given $\pi \in \Pi_m$, define π^* as follows

$$\pi^*(i) = \begin{cases} \pi(i+m) & \text{if } 1 \le i \le m, \\ \pi(i-m) & \text{if } m+1 \le i \le 2m. \end{cases}$$

Then π^* is in Π_m and

$$sign(\pi^*) = (-1)^m sign(\pi), \quad n_k(\pi^*) = n_k(\pi), \quad e(\pi^*) = m^2 - e(\pi).$$

Proof Let τ_i $(1 \leq i \leq m)$ denote the transposition that switches $\pi(i)$ and $\pi(i+m)$. Then $\pi^* = \tau_1\tau_2 \dots \tau_m\pi$, so $sign(\pi) = (-1)^m sign(\pi^*)$.

It follows from the definitions that $t_{\pi^*} = b_\pi$ and $b_{\pi^*} = t_\pi$. Hence $n_k(\pi^*) = n_k(\pi)$.

Finally, s^k occurs in a term $r_{i,\pi(i)}$ if and only if $1 \leq i \leq m$ and $k = m - (\pi(i) - i)$. Hence

$$e(\pi) = \sum_{i=1}^{m}(m - t_\pi(i)) = m^2 - \sum_{i=1}^{m} t_\pi(i),$$

and

$$e(\pi^*) = \sum_{i=1}^{m}(m - b_\pi(i)) = m^2 - \sum_{i=1}^{m} b_\pi(i).$$

Hence, using the formula obtained in the proof of Lemma 1, $e(\pi) + e(\pi^*) = m^2$. □

Considering $\det R(p_s, p)$ as an element of $\mathbf{Z}[a_1, a_2, \dots, a_m][s]$, the lemma says that each monomial in $a_1, a_2, \dots, a_m$ that contributes to the coefficient of s^j also contributes to s^{m^2-j}, with the same sign if m is even but opposite sign if m is odd. When the a_i's are polynomial functions of $z \in \mathbf{C}$, the coefficients are also polynomial functions of z, and we can summarise the results so far in the following way.

Theorem 1 Let T be an $m \times m$ matrix over the ring $\mathbf{Z}[x]$ of polynomials with integer coefficients, and let $T(z)$ be the matrix of complex numbers obtained by substituting the complex number z for x. Then there is a polynomial ρ in $\mathbf{Z}[s, x]$ with the following properties

(i) ρ has degree m^2 in s and $s^{m^2}\rho(s^{-1}, x) = (-1)^m \rho(s, x)$;

(ii) the polynomial $\rho_z \in \mathbf{C}[s]$, obtained by substituting z for x, has a zero on the unit circle $|s| = 1$ if and only if $T(z)$ has two eigenvalues with equal moduli. □

3. Simplification of the condition

In this section we shall obtain a reduction of the polynomial condition stated in Theorem 1. In Section 4 the case $m = 3$ will be worked out in full, and the result for $m = 4$ will be stated. For the time being, we shall revert to considering $a_1, a_2, \dots, a_m$ as indeterminates, rather than functions of a complex variable.

Lemma 3 Let $\rho(s) = \det R(p_s, p)$. Then we have

$$\rho(s) = a_m(s-1)^m \Delta_m(s),$$

where $\Delta_m(s)$ is a reciprocal polynomial of degree $m(m-1)$.

Proof The factors follow immediately from the explicit form of $R(p_s, p)$. Subtract row $m + i$ is from row i, for each i in the range $1 \leq i \leq m$. The resulting non-zero entries are, in row i and column j,

$$(s^{m-j+i} - 1)a_{j-i} = (s-1)(1 + s + \dots + s^{m-j+i-1})a_{j-i} \quad (i \leq j \leq m + i - 1).$$

The other entries in row i are zero. In particular, the new entry in row m and column $2m$ is zero, so there is only one nonzero entry in that column, which is a_m in the last row. Expanding in terms of the last column and removing the factor $(s-1)$ from each of the terms in rows 1 to m, we get

$$\rho(s) = a_m (s-1)^m \Delta_m(s),$$

where $\Delta_m(s)$ is a determinant of size $2m-1$.

In the previous section we showed that $s^{m^2}\rho(s^{-1}) = (-1)^m \rho(s)$. Thus

$$s^{m^2} a_m (s^{-1}-1)^m \Delta_m(s^{-1}) = (-1)^m a_m (s-1)^m \Delta_m(s).$$

This implies that

$$s^{m(m-1)}\Delta_m(s^{-1}) = \Delta_m(s),$$

which means that $\Delta_m(s)$ is a reciprocal polynomial. □

Corollary $\Delta_m(1)$ is an integer multiple of the discriminant of p.

Proof In each of the rows 1 to m of $\Delta(1)$ the non-zero entries are as follows.

$$m \quad (m-1)a_1 \quad (m-2)a_2 \quad \dots \quad 2a_{m-2} \quad a_{m-1}$$

These are the coefficients of p', so $\Delta(1)$ is the resultant of p' and p, which is a multiple of the discriminant of p. □

The corollary is no surprise, because when $s=1$ the equation $p(\lambda)=0$ has a double root $\lambda_A = \lambda_B$.

The reciprocal polynomial $\Delta_m(s)$ can be written as

$$s^{m(m-1)/2}\left(b_0 + \sum_{i=1}^{m(m-1)/2} b_i(s^i + s^{-i})\right),$$

where the coefficients b_i are elements of $\mathbf{Z}[a_1, a_2, \dots, a_m]$. Further simplification occurs when the variable s is replaced by

$$t = s + s^{-1} + 2 = s^{-1}(s+1)^2.$$

Then the powers of t are given by

$$t^k = s^{-k}(s-1)^{2k} = \sum_{i=0}^{k-1}\binom{2k}{i}(s^i + s^{-i}) \ + \ \binom{2k}{k},$$

and this system of equations can be inverted so that, for each $i > 0$, $s^i + s^{-i}$ is expressed as a polynomial $\phi_i(t)$ with integer coefficients. For example,

$$\phi_1(t) = t - 2, \quad \phi_2(t) = t^2 - 4t + 2, \quad \phi_3(t) = t^3 - 6t^2 + 9t - 2.$$

For convenience we define $\phi_0(t) = 1$. Then $\Delta_m(s) = s^{m(m-1)/2} r_m(t)$, where

$$r_m(t) = \sum_{i=0}^{m(m-1)/2} b_i \phi_i(t) \qquad (t = s + s^{-1} + 2).$$

Formally $r_m(t)$ is an element of $\mathbf{Z}[t, a_1, a_2, \ldots, a_m]$; we may think of it either as a polynomial in t with coefficients in $\mathbf{Z}[a_1, a_2, \ldots, a_m]$ or as a polynomial in $a_1, a_2, \ldots, a_m$ with coefficients in $\mathbf{Z}[t]$.

We can now reformulate Theorem 1. Given the matrix T with entries in $\mathbf{Z}[x]$, the coefficients $a_1, a_2, \ldots, a_m$ of its characteristic polynomial are also elements of $\mathbf{Z}[x]$, so we have a polynomial $v(t, x) \in \mathbf{Z}[t, x]$ defined by

$$v(t, x) = r_m(t, a_1(x), a_2(x), \ldots, a_m(x)).$$

The condition $|s| = 1$ implies that $s = \exp(i\theta)$, where θ is real, and so $t = 4\cos^2(\theta/2)$. Hence t is real and lies in the range $0 \le t \le 4$.

Theorem 2 Let T be an $m \times m$ matrix over the ring $\mathbf{Z}[x]$, and let $T(z)$ be the matrix of complex numbers obtained by substituting the complex number z for x. Then there is a polynomial v in $\mathbf{Z}[t, x]$ with the following properties

(i) v has degree $m(m-1)/2$ in t;

(ii) the polynomial $v(t, z)$ obtained by substituting a given complex number z for x has a zero in the interval $0 \le t \le 4$ if and only if $T(z)$ has two eigenvalues with equal moduli.

□

4. The cases m=3 and m=4

When $m = 3$ we have

$$\det R(p_s, p) = \det \begin{pmatrix} s^3 & s^2 a_1 & s a_2 & a_3 & 0 & 0 \\ 0 & s^3 & s^2 a_1 & s a_2 & a_3 & 0 \\ 0 & 0 & s^3 & s^2 a_1 & s a_2 & a_3 \\ 1 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 1 & a_1 & a_2 & a_3 \end{pmatrix}.$$

Expanding, we obtain a polynomial of degree 9 in s:

$$\begin{aligned} &a_3^3(s^9 - 1) - a_3^2 a_2 a_1 (s^8 - s) \\ &\quad + (-2a_3^2 a_2 a_1 + a_3^2 a_1^3 + a_3 a_2^3)(s^7 - s^2) \\ &\quad + (-3a_3^3 + 6a_3^2 a_2 a_1 - a_3^2 a_1^3 - a_3 a_2^3 - a_3 a_2^2 a_1^2)(s^6 - s^3) \\ &\quad + (-a_3^2 a_2 a_1 - 2a_3^2 a_1^3 - 2a_3 a_2^3 + 3a_3 a_2^2 a_1^2)(s^5 - s^4). \end{aligned}$$

Note that there are five partitions of 3^2 in which no part exceeds 3 and no part is repeated more than three times. In this case all five corresponding monomials occur as coefficients. Removing the factors a_3 and $(s-1)^3$ yields the reciprocal polynomial $\Delta_3(s)$, where

$$\begin{aligned} s^{-3}\Delta_3(s) = a_3^2(s^3+s^{-3}) &+ (3a_3^2 - a_1a_2a_3)(s^2+s^{-2}) \\ &+ (6a_3^2 - 5a_1a_2a_3 + a_2^3 + a_1^3a_3)(s+s^{-1}) \\ &+ (7a_3^2 - 6a_1a_2a_3 + 2a_2^3 + 2a_1^3a_3 - a_1^2a_2^2). \end{aligned}$$

Changing the variable to $t = s + s^{-1} + 2$ means that we replace $s^i + s^{-i}$ by $\phi_i(t)$, where the polynomials $\phi_i(t)$ are displayed in the previous section. This gives $r_3(t)$, which can be written in a fairly compact form as an element of $\mathbf{Z}[t][a_1, a_2, a_3]$:

$$r_3(t) = (t-1)^3a_3^2 - (t-1)(t+2)a_1a_2a_3 + ta_2^3 + ta_1^3a_3 - a_1^2a_2^2.$$

Putting $t = 4$ gives

$$r_3(4) = 27a_3^2 - 18a_1a_2a_3 + 4a_2^3 + 4a_1^3a_3 - a_1^2a_2^2.$$

Since $t = 4$ corresponds to $s = 1$, this is the case $m = 3$ of the corollary to Lemma 3: $r_3(4)$ is (apart from an integer factor) the discriminant of the cubic polynomial p. The case $t = 0$ is also worth noting: $r_3(0) = -(a_3 - a_1a_2)^2$. This is part of a general pattern, which will be explained in the next section.

The corresponding results for $m = 4$ will be stated without proof. There are several noteworthy features of the expression for $r_4(t)$ given below. Of the 20 possible monomials, only 15 actually occur, and 9 of them occur in sets of two or three with the same coefficient. Also, the five terms that do not involve a_4 are equal to $-a_3^2r_3(t)$.

$$\begin{aligned} r_4(t) \ &= t^2(t-4)^4a_4^3 - t(t-2)^2(t^2-2t+4)a_1a_3a_4^2 - 2t^2(t-2)^2a_2^2a_4^2 \\ &+ t(t-1)(t^2-4)(a_1^2a_2a_4^2 + a_2a_3^2a_4) - (t-1)^3(a_1^4a_4^2 + a_3^4) - (t-1)(t-2)a_1^2a_3^2a_4 \\ &- t(t^2+2t-4)a_1a_2^2a_3a_4 + (t+2)(t-1)(a_1^3a_2a_3a_4 + a_1a_2a_3^3) \\ &+ t^2a_2^4a_4 - t(a_1^3a_3^3 + a_2^3a_3^2) + a_1^2a_2^2a_3^2. \end{aligned}$$

$$\begin{aligned} r_4(4) \ &= 256a_4^3 - 192a_1a_3a_4^2 - 128a_2^2a_4^2 \\ &+ 144(a_1^2a_2a_4^2 + a_2a_3^2a_4) - 27(a_1^4a_4^2 + a_3^4) - 6a_1^2a_3^2a_4 \\ &- 80a_1a_2^2a_3a_4 + 18(a_1^3a_2a_3a_4 + a_1a_2a_3^3) \\ &+ 16a_2^4a_4 - 4(a_1^3a_3^3 + a_2^3a_3^2) + a_1^2a_2^2a_3^2. \end{aligned}$$

$$r_4(0) = (a_1^2a_4 + a_3^2 - a_1a_2a_3)^2.$$

5. The square property

The values $t = 0$ and $t = 4$ are obviously special. It has been shown that for all m, $r_m(4)$ is a multiple of the discriminant. Here it will be shown that there exists $y_m \in \mathbf{Z}[a_1, a_2, \ldots, a_m]$ such that $r_m(0) = \pm y_m^2$, and a determinantal formula for y_m will be obtained.

The value $t = 0$ corresponds to $s = -1$. In this case the equation $p(\lambda) = 0$ has roots λ_A, λ_B with $\lambda_A = -\lambda_B$. Trivially, it also true that $\lambda_B = -\lambda_A$, so we should expect the resultant of p and p_s to have double roots when $s = -1$.

The algebraic form of this observation is as follows. Let

$$f(\lambda) = \frac{1}{2}\big(p(\lambda) + p(-\lambda)\big), \quad g(\lambda) = \frac{1}{2\lambda}\big(p(\lambda) - p(-\lambda)\big).$$

Then if m is even

$$f(\lambda) = \lambda^m + a_2\lambda^{m-2} + \cdots + a_{m-2}\lambda^2 + a_m,$$

$$g(\lambda) = a_1\lambda^{m-2} + a_3\lambda^{m-4} + \cdots + a_{m-3}\lambda^2 + a_{m-1},$$

and if m is odd

$$f(\lambda) = a_1\lambda^{m-1} + a_3\lambda^{m-3} + \cdots + a_{m-1}\lambda^2 + a_m,$$
$$g(\lambda) = \lambda^{m-1} + \cdots + a_{m-3}\lambda^2 + a_{m-1}.$$

In both cases we can consider f and g as polynomials in $\mu = \lambda^2$. They have a common root $\mu_A = \lambda_A^2$, and so the resultant of f and g vanishes. The resultant is the determinant of an $(m-1) \times (m-1)$ matrix S_m, which, in the even case, is

$$S_m = \begin{pmatrix} 1 & a_2 & a_4 & \dots & a_m & 0 & \dots & 0 \\ 0 & 1 & a_2 & \dots & a_{m-2} & a_m & \dots & 0 \\ . & . & . & \dots & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_4 & a_6 & \dots & a_m \\ a_1 & a_3 & a_5 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & a_{m-1} & 0 & \dots & 0 \\ . & . & . & \dots & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_3 & a_5 & \dots & a_{m-1} \end{pmatrix}.$$

There is a similar form when m is odd. For example, when $m = 3, 4, 5$, the matrices are

$$S_3 = \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 1 & a_2 & a_4 \\ a_1 & a_3 & 0 \\ 0 & a_1 & a_3 \end{pmatrix}, \qquad S_5 = \begin{pmatrix} 1 & a_2 & a_4 & 0 \\ 0 & 1 & a_2 & a_4 \\ a_1 & a_3 & a_5 & 0 \\ 0 & a_1 & a_3 & a_5 \end{pmatrix}.$$

The argument given above shows that a necessary and sufficient condition for p_{-1} and p to have a common root is that $\det S_m = 0$. This condition is related to our general framework by the following result.

Lemma 4 If S_m is the $(m-1) \times (m-1)$ matrix described above, and $r_m(t)$ is the polynomial of degree $m(m-1)/2$ defined in Section 3, then

$$r_m(0) \;=\; (-1)^{m(m+1)/2}(\det S_m)^2.$$

Proof Let $R = R(p_{-1}, p)$ and

$$X = \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix},$$

where the submatrices are all of size $m \times m$. Then XR has one non-zero entry (1) in the first column, and one non-zero entry (a_m) in the last column. Expanding $\det XR$ in terms of these two columns gives

$$\det XR = a_m \det(S_m \otimes I_2),$$

where $\otimes$ denotes the Knonecker product, and I_2 is the 2×2 identity matrix. Since $\det X = 2^{-m}$ and $\det(S_m \otimes I_2) = (\det S_m)^2$, we have

$$\det R = 2^m a_m (\det S_m)^2.$$

Putting $s = -1$ in Lemma 3, we have $\det R = a_m(-2)^m \Delta_m(-1)$, and from the definition of r_m, $\Delta_m(-1) = (-1)^{m(m-1)/2} r_m(0)$. The result follows. □

6. The equimodular curves as unions of arcs

We now return to the situation where $T(z)$ is an $m \times m$ matrix, each entry of which is an integral polynomial function of a complex number z. Then the coefficients of its characteristic polynomial are also integral polynomial functions, $a_i(z)$. According to Theorem 2, there is a function $v : \mathbf{R} \times \mathbf{C} \to \mathbf{C}$, defined by

$$v(t, z) \;=\; r_m(t, a_1(z), a_2(z), \ldots, a_m(z)),$$

such that the set $E \subseteq \mathbf{C}$ of points z where $T(z)$ has two eigenvalues of equal modulus is

$$\{z \in \mathbf{C} \mid v(t, z) = 0 \text{ for some } t,\ 0 \le t \le 4\}.$$

For a given t_0, $v(t_0, z) = 0$ is a polynomial equation in z, of degree k, say. Let z_0 be any one of its k roots. By the implicit function theorem, if the Jacobian of the mapping $z \mapsto v(t_0, z)$ is not zero at z_0, then there is a neighbourhood N of t_0 and a continuously differentiable function $z(t)$ defined on N, such that $z(t_0) = z_0$ and $v(t, z(t))$ is identically zero for $t \in N$. This means that E is a union of k images of the interval $[0, 4]$, and these images are differentiable arcs, except possibly at the end-points of the interval and points where the Jacobian is zero.

The points corresponding to $t = 0$ are particularly important. In the previous section we showed that $\pm r_m(0)$ is a perfect square, so the roots of $v(0, z) = 0$ are double roots. Consequently the images of $[0, 4]$ occur in pairs, the end-points corresponding to $t = 0$ of two paired arcs being coincident. In terms of the parameter θ (recall that $t = 4\cos^2(\theta/2)$), the double-arc is the image of the interval $[0, 2\pi]$. It is not however a homeomorphic image of the circle $|s| = 1$: since the transformation $t = s + s^{-1} + 2$ is not regular at $s = 0$ the image of $\theta = 0$ is (in general) different from the image of $\theta = 2\pi$.

In summary, the set E consists of a number of isolated points (degenerate arcs), together with segments of curves. The end-points of each segment are two distinct roots of $v(4, z) = 0$, and each segment contains a double root of $v(0, z) = 0$ as an interior point. The segments are smooth, except at points where the Jacobian vanishes.

For each t the Jacobian of the mapping $z \mapsto v(t, z)$ is $|v'(t, z)|^2$, where v' denotes the derivative with respect to z. So the Jacobian vanishes if and only if $v'(t, z) = 0$. Thus the condition that there is a point which lies on one of the equimodular curves, and where the Jacobian vanishes, is that there exist t^* and z^* such that $v(t^*, z^*) = 0$ and $v'(t^*, z^*) = 0$. In other words, there is a value t^* such that the equations $v(t^*, z) = 0$ and $v'(t^*, z) = 0$ have a common root; that is, $v(t^*, z)$ has a multiple root.

To determine the values of t^* for which the Jacobian vanishes we write

$$v(t, z) = c_0(t)z^k + c_1(t)z^{k-1} + \ \dots \ + c_k(t),$$

where the coefficients $c_i(t)$ $(1 \leq i \leq k)$ are polynomials in t. Then the condition for this polynomial in z to have a multiple root is that its discriminant is zero. The discriminant is (a multiple of) the resultant of $v(t, z)$ and $v'(t, z)$, and it is a polynomial $D(t)$ in $\mathbf{Z}[t]$, the determinant of a $(2k - 1) \times (2k - 1)$ matrix.

In fact, $D(t)$ has a factor $(t - t^*)^\mu$ if and only if $v(t^*, z) = 0$ has μ double roots. Since all the roots of $v(0, z) = 0$ are double roots, this means that $D(t)$ has a factor t^ℓ, where 2ℓ is the degree of $v(0, z)$.

Example In the analysis of the family of generalised dodecahedron graphs [4] the following matrix was obtained:

$$T(z) = \begin{pmatrix} z^4 + 2z^3 + 3z^2 + z + 1 & z^3 + z & z^3 + z^2 + 2z \\ -1 & z^2 & -z \\ -(z^2 + z + 1) & -z & -z \end{pmatrix}.$$

The coefficients of the characteristic polynomial are

$$\begin{aligned} a_1(z) &= z^4 + 2z^3 + 4z^2 + 1 \\ a_2(z) &= z(z + 1)(z^4 + z^3 + 2z^2 + 2) \\ a_3(z) &= z^2(z + 1)^2. \end{aligned}$$

As shown in Section 3,

$$r_3(t) = (t - 1)^3 a_3^2 - (t - 1)(t + 2)a_1 a_2 a_3 + t a_2^3 + t a_1^3 a_3 - a_1^2 a_2^2,$$

so in this case $v(t, z)$ can be written as $-z^2(z+1)^2 u(t, z)$, where $u(t, z)$ is a monic polynomial of degree 16 in z, all its coefficients being integer polynomials of degree not exceeding 3 in t:

$$u(t, z) \ = \ z^{16} + 6z^{15} + (25 - t)z^{14} + \cdots + (2t^2 - 6t - 4)z + (4 - t).$$

The factors $z^2(z + 1)^2$ imply that the points 0 and -1 are (double) roots of $v(t, z) = 0$ for all t; we can think of them as degenerate arcs.

For the values $t = 0$ and $t = 4$ we have

$$u(0, z) = (z^8 + 3z^7 + 8z^6 + 8z^5 + 11z^4 + 5z^3 + 9z^2 - z + 2)^2,$$
$$u(4, z) = z(z + 1)(z^3 + z - 1)^2(z^8 + 5z^7 + 14z^6 + 24z^5 + 31z^4 + 19z^3 + 3z^2 - 27z + 4).$$

The points where $u(t, z) = 0$ for some $t \in [0, 4]$ comprise eight segments of curves. Each segment has one of the eight double roots of $u(0, z) = 0$ as an interior point, and if these roots are denoted by β_j then we can define Γ_j to be the corresponding segment ($j = 1, 2, \dots, 8$). We know that Γ_j has two roots α_{j1}, α_{j2} of $u(4, z) = 0$ as its end-points; and in order to determine which segment contains which roots, we can calculate the roots of $u(t, z) = 0$ for suitable values of t in the range $0 < t < 4$. This leads to the following table, and the configuration illustrated in Fig. 1.

j	α_{j1}	α_{j2}	β_j
1	$-1.87 + 1.13i$	$-0.34 + 1.16i$	$-1.08 + 1.73i$
2	$-0.34 + 1.16i$	0.61	$0.16 + 0.47i$
3	$-1.87 - 1.13i$	$-0.34 - 1.16i$	$-1.08 - 1.73i$
4	$-0.34 - 1.16i$	0.15	$0.16 - 0.47i$
5	$0.14 + 1.64i$	0.68	$0.25 + 0.93i$
6	0.68	$0.14 - 1.64i$	$0.25 - 0.93i$
7	$-1.02 + 1.51i$	0	$-0.84 + 1.17i$
8	$-1.02 - 1.51i$	-1	$-0.84 - 1.17i$

Fig.1: The eight segments. The eight double roots of $u(0, z)$ are denoted by $\circ$. Double roots of $u(4, z) = 0$ are denoted by $\bullet$, and double roots of $u(t, z) = 0$ for other values of t by $\diamond$.

For further clarification we calculate the discriminant $D(t)$ of $u(t,z)$ which, in this case, is a polynomial of degree 59 in t. Since there are eight double roots when $t=0$, we know that t^8 is a factor, and in fact $D(t)$ factorises over $\mathbf{Z}[t]$ as follows:

$$D(t) \;=\; Ct^8(t-4)^3 f_2(t)^2 f_3(t)^2 f_5(t)^2 f_7(t)^2 f_{14}(t),$$

where C is an integer and $f_d(t)$ is a polynomial of degree d.

The factor $(t-4)^3$ corresponds to the fact that there are three double roots of $u(4,z)=0$, arising from the factor $(z^3+z-1)^2$. These roots are $\alpha_{12}=\alpha_{21}$, $\alpha_{32}=\alpha_{41}$ and $\alpha_{52}=\alpha_{61}$, so the three pairs of segments (Γ_1,Γ_2), (Γ_3,Γ_4), (Γ_5,Γ_6) join up at the respective end-points.

It turns out that there are only two roots of $D(t)=0$ in the open interval $(0,4)$:

$$t_1 = 2.35869\ldots \quad \text{and} \quad t_2 = 3.08501\ldots.$$

The corresponding double roots of $u(t_1,z)=0$ and $u(t_2,z)=0$ are $\gamma_1 = 0.337\ldots$ and $\gamma_2 = -0.534\ldots$ respectively. These are the points on the real axis where the pairs of segments (Γ_2,Γ_4) and (Γ_7,Γ_8) intersect.

The behaviour at γ_k $(k=1,2)$ can be visualised as follows. When $t=t_k-\epsilon$ there are roots of $v(t_k,z)=0$ just above and just below γ_k. When $t=t_k$, these roots collide. When $t=t_k+\epsilon$ the roots are on the real axis, one either side of γ_k. □

7. Further properties

In order to apply the Beraha-Kahane-Weiss theorem it is important to know which of the equimodular curves are dominant: that is, which of them have the property that the modulus of the pair of eigenvalues that defines the curve is larger than the moduli of the other eigenvalues.

We have seen that each point where $r_m(0)=0$ determines a segment of an equimodular curve. It follows (by a simple continuity argument) that the points that have the dominance property determine the dominant equimodular curves. Thus we are led to the following situation. Suppose that the roots of

$$x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m = 0$$

are $\gamma, -\gamma, \lambda_1, \lambda_2, \ldots, \lambda_{m-2}$; then we require that each eigenvalue λ_i satisfies $|\lambda_i| < |\gamma|$.

In general, the question of which curves are dominant can be answered by direct calculation of the eigenvalues. But it is interesting to note that in the cases $m=3,4,5$ the polynomial can be factorised 'generically', that is, in the ring $\mathbf{Z}[a_1,a_2,\ldots,a_m][x]$, and this leads to a condition involving the coefficients. For example, when $m=3$ the 'existence condition', which guarantees that the roots of $x^3+a_1x^2+a_2x+a_3=0$ are of the form $\gamma,-\gamma,\lambda$, is $a_1a_2=a_3$, so we have

$$a_2(x^3+a_1x^2+a_2x+a_3) = (x^2+a_2)(a_2x+a_3).$$

It follows that the condition that $|\lambda| < |\gamma|$ is equivalent to

$$|a_3/a_2| < \sqrt{|a_2|}, \quad \text{or equivalently} \quad |a_1|^2 < |a_2|.$$

In the example discussed in Section 6, this means that the equimodular curve Γ_j is dominant if and only if $|a_1(\beta_j)|^2 < |a_2(\beta_j)|$, where a_1 and a_2 are the relevant polynomials. Thus the dominant curves are $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$.

References

1. S. Beraha, J. Kahane, N.J. Weiss. Limits of zeros of recursively defined families of polynomials. *Studies in Foundations and Combinatorics, Advances in Math., Supplementary Studies* 1 (1978) 213-232.

2. N.L. Biggs. A matrix method for chromatic polynomials -II. *CDAM Research Report Series*, LSE-CDAM-2000-4, April 2000.

3. N.L. Biggs, R.M. Damerell, A.D. Sands. Recursive families of graphs. *J. Combinatorial Theory (B)* 12 (1972) 123-131.

4. N.L. Biggs, P. Reinfeld. The chromatic roots of generalised dodecahedra. *CDAM Research Report Series*, LSE-CDAM-2000-7, June 2000.

5. S-C. Chang, R. Shrock. Ground state entropy of the Potts antiferromagnet on strips of the square lattice. http://uk.arXiv.org/ps/cond-mat/0004161

6. D. Cox, J. Little, D. O'Shea. *Using Algebraic Geometry.* Springer, New York, 1998.

7. T.D.Lee, C.N.Yang. Statistical theory of equations of state and phase transitions II. Lattice gas and Ising model. *Phys. Rev.* 87 (1952) 410-419.

8. E.H. Lieb, A.D. Sokal. A general Lee-Yang theorem for one-component and multicomponent ferromagnets. *Commun. Math. Phys.* 80 (1981) 153-179.

9. R. Read, G.F. Royle. Chromatic roots of families of graphs. In: *Graph Theory, Combinatorics and Applications* (ed. Y.Alavi, G. Chartrand, O.R. Ollerman, A.J.Schwenk) Wiley 1991.

10. J. Salas, A.D. Sokal. Transfer matrices and partition-function zeros for antiferromagnetic Potts models. http://uk.arXiv.org/ps/cond-mat/0004330

11. R. Shrock, S-H. Tsai. Asymptotic limits and zeros of chromatic polynomials and ground state entropy of Potts antiferromagnets. *Phys. Rev.* E55 (1997) 5165-5179.

12. R. Shrock, S-H. Tsai. Ground-state degeneracy of Potts antiferromagnets: homeomorphism classes with non-compact W boundaries. *Physica A* 265 (1999) 186-223.