Abstract

There is an extensive theory regarding optimal continuous path search for a mobile or immobile 'target'. The traditional theory assume that the target is one of three types: (i) an object with a known distribution of paths, (ii) a mobile or immobile hider who wants to avoid or delay capture, or (iii) a rendezvouser who wants to find the searcher. This paper introduces a new type of search problem by assuming that aims of the target are not known to the searcher. The target may be either a type (iii) cooperator (with a known cooperation probability c) or a type (ii) evader. This formulation models search problems like that for a lost teenager who may be a 'runaway', or a lost intelligence agent who may be a defector. In any given search context, it produces a continuum of search problems $\Gamma(c)$, $0 \leq c \leq 1$, linking a zero-sum search game (with $c = 0$) to a rendezvous problem (with $c = 1$). These models thus provide a theoretical bridge between two previously distinct parts of Search Theory, namely Search Games and Rendezvous Search.
keywords:search, rendezvous, game

Searching For An Agent Who May Or May Not Want To Be Found

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Revised Version

1 Introduction

A large portion of the traditional theory of optimal search deals with the problem of finding an optimal search path for a continuously moving target. A typical scenario is for the searcher and the target (or 'agent') to be placed in a known region X according to a known distribution, and for the two of them to move at unit speed until the first time T that they meet (e.g., come within a given detection range). The searcher wishes to minimize the expected value $\hat{T}$ of this meeting time. The nature of the search problem is determined by the aims (if any) of the agent, and is placed into one of three branches of the theory as follows:

1. **One-Sided Search for a Moving (or Immobile) Target**: Here the distribution of the target paths in known, and this is a stochastic optimization problem. (See for example [10], [9], [22].)

2. **Search Games with a Mobile (or Immobile) Hider**: This is a zero-sum game formulation, with payoff function T, and the target is a maximizer of capture time T. (See for example [13], [16], [18],[19].)

3. **Rendezvous Search**: In this formulation, both Searcher and Agent are minimizers of T. In the symmetric form, both players must use the same mixed strategy; in the asymmetric form (which we use here) they may use distinct strategies, and there is no need to mix (randomize). (See for example [2], [6], [21].)

The novel element which we introduce in this article is an uncertainty regarding the motives of the lost Agent: he may be a mobile or immobile hider (Evader) as in problem 2, or he may be a cooperating rendezvouser who shares the same aim as the Searcher, as in problem 3. We assume that the probability c of cooperation is known to the Searcher and to the Agent. In any given search context (search space and player motions) G we obtain a continuous family $\Gamma_G(c)$ of search problems, $0 \leq c \leq 1$, where $\Gamma_G(0)$ is a search game with mobile or immobile Hider and $\Gamma_G(1)$ is an asymmetric rendezvous problem. We obtain a unique continuous Value function $V_G(c)$ which gives the least expected value of T for the given (cooperation) probability c that the target wants to be found. As we shall demonstrate, this uncertainty regarding *a priori* agent motives affects both the paths (strategies) chosen by the Searcher and the paths chosen by the (cooperating or evading) Agent. One might say that this article introduces the game theoretic notion of incomplete information into the theory of search. Our formalization of the cooperative portion of the problem will be that of asymmetric rendezvous search, where the Searcher and Agent may agree on the roles (paths) they will take in the event that the agent gets lost and wants to be found. For example a mother (the Searcher) may tell her child (the Agent) what to do in this event, knowing however that this instruction may be disregarded if the child does not want to be found. The child will follow these (rendezvous) instructions if he wants to be found, and may use the knowledge of these instructions in deciding on an evasion strategy if he doesn't.

This uncertainty as to the Agent's aims is common for example when a teenager is reported missing to the police by the parents. In such cases the police usually ascribe a probability ($1-c$ in our notation) that the teenage is not lost or abducted but rather a runaway who does not want to be found. Parents usually complain that the police overestimate this probability, and indeed this is a subject of some controversy, as in the following passage from [11, p, 467].

> "Bannister had gone to a local police station to make a report in person...From a police detective ... he'd heard 'Look, its only been a few weeks...She's probably alive and healthy somewhere, and ninety nine out of a hundred of these cases turn out to be a girl who just wanted to spread her wings [an Evader]'. Not his Mary, Bannister had replied."

Another example of such incomplete information search is found in the novel 'Hunt for Red October' [12], by Tom Clancy, where a Russian submarine of that name becomes lost to Soviet command. The difficulties faced by the Russian search effort are exactly those formalized in this article, as they are uncertain whether the sub is indeed lost (their first assumption) or is defecting to the West (as they gradually come to believe). The current SETI (Search for Extraterrestrial Intelligence) Project is based on an implicit assumption regarding not only the existence of EI but of a sufficiently large value of c. In the SETI context, T would be the first time when transmitter and listener are on the same frequency and the message is understood to be nonrandom. Search and Rescue operations (seeking lost hikers, for example) also make judgements about c in determining where to look first.

After formalizing this 'rendezvous-evasion' problem in section 2, we devote the next two sections to illustrative examples. Section 3 considers the problem where the Searcher is placed at a given and known node of a metric tree, and the immobile Agent can choose any other node of the tree to wait at. The Searcher then moves at unit speed until the Agent is found. This is a problem where the rendezvous extreme $c = 1$ is trivial (Agent hides at nearest node to Searcher and Searcher goes there), while the search game extreme $c = 0$ is complicated and has been the subject of several papers. Section 4 considers the problem where the Searcher and a mobile Agent are placed a unit distance apart on a line, and faced in random directions. The Agent is assumed to have a speed $m < 1$ while the Searcher has speed 1. (Speed $m = 1$ is also allowed when $c = 1$.) This, on the contrary, is a problem which is simple at the search game extreme $c = 0$, but rather subtle at the rendezvous extreme $c = 1$, where it has been the subject of several papers (for $m = 1$). We conclude the paper with a final section 5 which gives a summary and suggests future directions of research.

We call the problem introduced in this paper the 'rendezvous-evasion' problem because this term is so descriptive, but the reader is warned that this name also appears in the literature for a different problem (*e.g.* [17]). We should also mention that much of the literature on search does not concentrate on optimal search paths, but rather optimal distribution of effort (see [20]).

2 General Analysis

Let S be the set of search strategies and let $\mathcal{A}$ be the set of agent strategies (for cooperator or evader-hider). Let $\mathcal{S}^*$ and $\mathcal{A}^*$ denote the corresponding sets of mixed strategies. The cooperation probability c is a given parameter of the problem. Let T denote the 'capture' time when the Searcher first meets the Agent, and $\hat{T}$ its expected value. Prior to the start of the search problem the Searcher and the Agent agree a rendezvous strategy $r \in \mathcal{A}$ for the (cooperating) Agent. This strategy will be followed with probability c when search begins. When the search begins the Searcher follows some mixed path strategy $s \in \mathcal{S}^*$ and the Agent, who is by now either a Cooperator or an Evader, follows respectively either the agreed path r or some mixed path strategy $h \in \mathcal{A}^*$. When choosing h, the Evader is aware of the agreed rendezvous (Cooperator) strategy r. (The child would remember what Mother told him to do; the defecting submarine would have the protocol book for lost or disabled submarines.) Hence the formalization of the story we have in mind is as follows.

$$v\left(c\right) = \min_{r \in \mathcal{A}} \left[\min_{s \in \mathcal{S}^*} \max_{h \in \mathcal{A}^*} \left(c\hat{T}\left(s, r\right) + \left(1 - c\right) \hat{T}\left(s, h\right) \right) \right]. \tag{1}$$

Here r denotes the pure rendezvous strategy (of the Cooperator), s the (possibly mixed) strategy of the Searcher, and h the (possibly mixed) strategy of the Evader. (We call the two types of Agent the Cooperator and the Evader.) Note that after r is chosen, a minimax theorem ensures that the following min and max are interchangeable: If the path spaces $\mathcal{S}$ and $\mathcal{A}$ consist of Lipshitz continuous paths on a metric space X and are endowed with the topology of uniform convergence on compact time intervals, then the meeting time T is lower semicontinuous in each variable and consequently the existence of a value follows from the results in Appendix 1 of [13] or the more general result of [3]. Actually the Lipshitz condition on the paths in $\mathcal{A}$ is not required by the latter result - continuity is sufficient. Note that when $c = 0$, the rendezvous strategy r of the Cooperator is irrelevant and the problem reduces to a zero-sum 'search game' with $v\left(0\right)$ reducing to the minimax value of that game; when $c = 1$, the Evader strategy h is irrelevant and the problem reduces to an asymmetric rendezvous problem, with $v\left(1\right)$ equal to the 'asymmetric rendezvous value' R^a of [2]. Observe that necessary conditions for a triple r', s', h' to constitute optimal values for some c are that h' is an optimal response to s', r' is an optimal response to s', and s' is an optimal response to the $(c, 1-c)$ mixture of r' and h'.

3 Immobile Agent on a Tree

We begin with a model of the following search problem. A Mother drives a teenager son to a roadside drop-off point N. From there he will hike in a large park. If it rains, he will go to one of a group of huts (none of which are at N) , where he will wait, while the Mother drives back to N and, covered with proper raingear, begins a search. The Mother tells the child which hut he should go to if it rains, knowing full well that if he is enjoying himself at that time he will disregard her instructions. The huts and the drop-off point N are all connected by a network of paths that forms a tree.

In this model the search domain will be a tree Q with a distinguished node called N where the Searcher must start his search. (In keeping with the search and rendezvous literature, we will refer to the agents as 'he' even when they adopt the 'wait for mummy' strategy, or when the motivating example includes females.) Each edge of Q has a certain length, and the sum of these is called the length of Q, denoted $\mu(Q)$. For simplicity we will normalize the length of the given tree so that $\mu(Q) = 1$. The Agent in this problem is immobile, he simply picks a node of the tree other than N, and stays there. The Searcher moves along the tree at unit speed until he reaches the node chosen by the Agent, aiming to minimize the expected time $\hat{T}$ to reach this node. With probability c, the Agent is a Cooperator (or rendezvouser) who also wishes to minimize $\hat{T}$; and with probability $1-c$ the Agent is an Evader who wishes to maximize $\hat{T}$. Simple domination arguments are sufficient to show that a Cooperator will always chose a node adjacent to the starting node N, while the Evader will always chose a terminal node of Q. We will denote by $v(Q, N, c)$ the minimal time $v(c)$ defined more generally in equation (1) for this specific problem. The problem dealt with here generalizes the $c = 0$ 'search game with immobile hider on a tree' which was solved by the second author in [13] and generalized by Reijnierse and Potters in [18]. The corresponding rendezvous problem ($c = 1$) is trivial – the Searcher goes to the node nearest to N (or to a predetermined one such node if there are several nearest nodes), where the Cooperator waits.

3.1 A Simple Example

Before considering the case of a general tree, we look at a very simple example of the (pre-normalized) tree based on the closed interval $[-1, +1]$ with an additional Searcher starting node N place at 0. Since the Agent pure strategy space $\mathcal{A}$ is simply the symmetric set $\{-1, +1\}$, we may assume without loss of generality that the agreed rendezvous strategy for the Cooperator has been announced as $r = -1$. That is, the Cooperator part of the Agent has agreed to wait at the node -1 if he gets lost. This fact is known to the Evader part of the Agent. Assume that the probability c that the Agent cooperates is close to 1. In the subsequent zero-sum game between the Searcher and the Evader, the optimal strategies are clearly for the Searcher to go first to -1 and then to $+1$ and for the Evader to wait at $+1$. (Our subsequent general analysis will show that this is optimal for $c \geq 1/2$, and it is easy enough to check directly.) The value $v(c)$ (least expected time to find the Agent) is therefore given by $(c)(1) + (1-c)(3) = 3 - 2c$. In particular, the expected time to find the Evader is 3. This simple example avoids all the graph theoretic material we will need for the general case, but nevertheless illustrates the important point that *the Evader needs to know the agreed rendezvous strategy r in order to determine his optimal mixed strategy.* For any evader mixed strategy h that does not depend on r, the largest time to capture the Evader can guarantee is $(1/2)(1) + (1/2)(3) = 3/2$, which is achieved only for the symmetric strategy of hiding equiprobably at either end.

3.2 The Search Game $c = 0$

The case where the Agent is certainly an Evader (hider might be a better term, since he is immobile) has been solved by the second author in [13], Theorem 3, page 29. We quote it without proof below, in our more general notation. This zero-sum game is a special case of what is known as a search game with an immobile hider, the special case being the tree. The recursive derivation of the optimal hider strategy $h^*(Q, B)$ can be found in [13].(An extension of this result to more general settings is given in [18].)

Theorem 1 *Let Q be a tree with distinguished node B and length $\mu(Q)$. Then the value of the search game with immobile hider on Q, with the Searcher starting at B, is given by $v(Q, B, 0) = \mu(Q)$. An optimal mixed strategy for the Searcher is to select a minimal tour of Q which starts and ends at B (such a tour takes time $2\mu(Q)$) and traverse it equiprobably in either direction. An optimal mixed strategy for the Evader is a certain probability distribution $h^*(Q, B)$ over the terminal nodes of Q.*

We note that the stated Searcher strategy ensures that the value $v(Q, B, 0)$ is no more than $\mu(Q)$, since if one direction of the tour reaches a node at time t, the other direction reaches it at time $2\mu(Q) - t$. The difficult part of this result is the determination of the evader strategy $h^*(Q, B)$ which ensures that the Evader will not be found on average before time $\mu(Q)$.

3.3 Solution for fixed node A

We return now to our setting of a normalized tree Q, $\mu(Q) = 1$, with a distinguished node N which is the starting point for the Searcher. We now fix any node A of Q which is adjacent (but not equal) to N. (Note that any nonadjacent node B is dominated by the node adjacent to N on the path to B.) We consider the game determined by the rendezvous strategy $r = A$ of staying at the node A. We determine the value $v_A(c)$ of this game in terms of certain parameters of the node A, and in the next subsection we use this formula for the value to find the best rendezvous node A as a function of the cooperation probability c. The parameters we will need are the distance $d = d_A$ from N to A , the length of the edge e between N and A, and the total length $w = w_A = \mu(Q_A)$ of the subtree of Q_A of the tree Q (with the root N) starting at A. (When the open edge e is deleted from the tree, the resulting graph $Q - e$ has two components; the one containing A but not N is denoted Q_A, while the one containing N but not A is denoted Q_{-A}.) We will now describe two search strategies, one of which will be optimal. Which one of these is optimal will depend on whether c is less than or greater than the cutoff value of $d_X / (1 - w_Z)$, where $A = X$ minimizes d_A

and $A = Z$ maximizes w_A. (See Figure 1 and Theorem 4.) Note that when N is a terminal node, $d_X = 1 - w_z$, so that the cutoff value is 1.

When c is large we might expect that the best search strategy would begin by going directly to A. In the (unlikely) case that the Agent is not found there, the Searcher might play the optimal strategy for the resulting search game on Q with A as the distinguished starting node. Since this subgame starts at time d (when the Searcher reaches A) its value as determined by Gal's Theorem above is $d + \mu(Q) = d + 1$. Hence the strategy of going first to A and then traversing a tour of Q starting and ending at A equiprobably in either direction gives the following bound,

$$v_A(c) \leq cd + (1-c)(d+1) = 1 + d - c. \tag{2}$$

When c is small we might expect the Searcher to use the strategy given in the previous subsection for $c = 0$, with expected meeting time to meet the Evader of $\mu(Q) = 1$, and an expected time of d to meet the Cooperator (at A) if the search begins by going to A, and $2(1-d-w) + d$ otherwise. Here $1 - d - w$ is the length $\mu(Q_{-A})$ of the subtree of Q given by $Q - e - Q_A$, where e is the edge between N and A. It follows that another upper bound on the rendezvous-evasion value is given by

$$v_A \leq (c/2)(d) + (c/2)(2(1-d-w)+d) + (1-c)(1) = 1 - cw. \tag{3}$$

Observe that the two estimates are equal when $1 + d - c = 1 - cw$, or $c = \frac{d}{1-w}$. Hence we have shown.

Lemma 2 *Let Q be a unit-length tree with Searcher starting node N. For any other node A of Q adjacent to N, let d_A be the length of the edge NA and let $w_A = \mu(Q_A)$ denote the total length of the subtree determined by A. If the rendezvous strategy r is to wait at the node A, then an upper bound for the value v_A of the resulting subgame is given by*

$$v_A(c) \leq \begin{cases} 1 - cw_A & \text{if } c \leq \frac{d_A}{1-w_A}, \\ 1 - c + d_A & \text{if } c \geq \frac{d_A}{1-w_A}. \end{cases}$$

We claim that this upper bound is the actual rendezvous-evasion value v_A for the subgame with $r = A$. To prove this, we have to provide evader strategies which give the opposite inequalities.

The evader strategies which we propose are mixtures of the two mixed strategies $h_{-A} = h^*(Q_{-A}, N)$ and $h_A = h^*(Q_A, A)$, which Gal's Theorem says are optimal in the respective subtrees for the pure search games $(c = 0)$. In particular, define the mixture $h(p) = p(h_{-A}) + (1-p)(h_A)$. Our analysis below implicitly uses Gal's observation (proved in section 2.3 of [13]) that an optimal search which enters the subtree Q_A or Q_{-A} will complete the search of that tree before leaving it. The analysis which follows is independent of Lemma 2 and in fact derives optimal Search and Evader strategies simultaneously, as functions of the cooperation probability c (and of course for the given rendezvous strategy $r = A$ for the Cooperator).

Once a particular node A has been chosen as a rendezvous strategy r for the Cooperator, it is convenient to renormalize the tree so that the distance from the starting node N to A is 1. In this normalization the total length of Q will be some number L (greater than 1). The length of Q_A will be called a and the length of Q_{-A} will be called b, so that $b + 1 + a = L$. The analysis given by the second author in section 2.3 of [13] shows that if an optimal search strategy enters the subtree Q_A or Q_{-A} we can assume it will complete this search before leaving the subtree. Since any search strategy begins either by going to A or entering Q_{-A}, we need only consider the following three search strategies s_1, s_2, s_3: the first go to A, continues to search Q_A using a minimal tour with equiprobable direction, and goes back to search Q_{-A} in a similar manner; the second searches Q_{-A} in this manner, then goes to A and continues to search Q_A in this manner; the third strategy begins by going to A, then goes back to search Q_{-A}, and finally goes back to A to search Q_A in this way. The evader strategies which we propose are mixtures of the two mixed strategies $h_{-A} = h^*(Q_{-A}, N)$ and $h_A = h^*(Q_A, A)$, which Gal's Theorem says are optimal in the respective subtrees for the pure search games $(c = 0)$. In particular, define the mixture $h(p) = p(h_{-A}) + (1-p)(h_A)$. When the agent is a Cooperator (hides at A) the matrix consisting of the expected capture time $\hat{T}$ for each of the strategies (rows) s_1, s_2, s_3 for the minimizing Searcher is given by,

$$\begin{pmatrix} 1 \\ 2b+1 \\ 1 \end{pmatrix}.$$

Similarly when the evader uses h^*_{-A} (left column) or h^*_A (right column) we have expected capture time given by the matrix.

$$\begin{pmatrix} 2+2a+b & 1+a \\ b & 2b+1+a \\ 2+b & 3+2b+a \end{pmatrix}.$$

So the game matrix is given by

$$\begin{aligned} & c\begin{pmatrix} 1 & 1 \\ 2b+1 & 2b+1 \\ 1 & 1 \end{pmatrix} + (1-c)\begin{pmatrix} 2+2a+b & 1+a \\ b & 2b+1+a \\ 2+b & 3+2b+a \end{pmatrix} \\ = & \begin{pmatrix} -c+2+2a+b-2ca-cb & 1+a-ca \\ cb+c+b & 2b+1+a-ca \\ -c+2+b-cb & -2c+3+2b+a-2cb-ca \end{pmatrix}. \end{aligned}$$

If the evader chooses left column with probability p, the payoffs for the three search strategies are given by

$$\left(\begin{array}{ll} -c+2+2a+b-2ca-cb & 1+a-ca \\ cb+c+b & 2b+1+a-ca \\ -c+2+b-cb & -2c+3+2b+a-2cb-ca \end{array}\right)\left(\begin{array}{c} p \\ 1-p \end{array}\right)=$$

$$\left(\begin{array}{l} -p\left(cb+c-b-1-a+ca\right)+1+a-ca \\ p\left(cb+c-b-1-a+ca\right)+2b+1-ca+a \\ p\left(cb+c-b-1-a+ca\right)-2c+3+2b+a-2cb-ca \end{array}\right). \tag{4}$$

Since the payoffs corresponding to the last two strategies are parallel, we see that the middle strategy (s_2) dominates the bottom strategy (s_3) when

$$2b+1-ca+a<-2c+3+2b+a-2cb-ca$$

or equivalently, when $c<\frac{1}{1+b}$ (while if $c>\frac{1}{1+b}$, then s_3 dominates s_2).

Case I: $c<\frac{1}{1+b}$

We must equate the top two rows of the final matrix in (4) to find the optimal value of $p\left(c\right)$. This gives $p\left(c\right)=\frac{b}{L(1-c)}$. Observe that the optimal Evader strategy depends in a continuous manner on the cooperation probability c. Note in particular that in the pure search game case $c=0$ this gives $p=\frac{b}{L}$. That is, the Evader hides in the subtree Q_{-A} rooted at N with a probability equal to the ratio $\mu\left(Q_{-A}\right)/\mu\left(Q\right)$. This agrees with the analysis given in [13]. Setting $p\left(c\right)=\frac{b}{L(1-c)}$ in the undominated (top two) rows of the matrix in (4) gives $b+1+a-ca=L-ca$ in both cases. To find the optimal Searcher strategy, note that the coefficients of p in the two expressions have opposite sign. Hence the weighted average of these two lines which has zero slope is the equiprobable simple average. So the searcher in this case should optimally choose the undominated rows s_1 and s_2 with equal probabilities. This is the same as Gal's solution for $c=0$. If the expected time $L-ca$ is renormalized into our original system L becomes 1, a becomes $w_A=\mu\left(Q_A\right)$, and $L-ca$ becomes $1-cw_A$ so that the Evader strategy $h\left(p\left(c\right)\right)$ gives the estimate

$$v_A\left(c\right)\geq 1-cw_A. \tag{5}$$

If we wish to write b/L in our original notation, it is simply $\mu\left(Q_{-A}\right)/\mu\left(Q\right)=\mu\left(Q_{-A}\right)$, so $p\left(c\right)=\mu\left(Q_{-A}\right)/\left(1-c\right)=\left(1-d_A-w_A\right)/\left(1-c\right)$.

Case II: $c>\frac{1}{1+b}$.

Here we equate the first and third rows of (4) getting:

$$-p\left(L(c-1)\right)+1+a-ca=p\left(L(c-1)\right)-2c+3+2b+a-2cb-ca$$
with solution $p=\frac{b+1}{L};1-p=\frac{a}{L}$.

Unlike the previous case, the optimal value of p does not depend on c. Substituting this value for p into the top or bottom row of the final matrix in (4) gives the expected time

$$-cb + 2 - c + b + a - ca = 1 + L\,(1 - c)\,.$$

Again, the fact that the coefficients of p have opposite signs shows that the Searcher chooses the undominated rows (s_1 and s_3) with equal probability. This is equivalent to saying that the Searcher goes to A and then plays optimally (equiprobable direction tour) in the pure search game ($c = 0$) that results when the Agent is not found at A. To get the expected time $1 + L\,(1 - c)$ back into our original normalization, the distance 1 becomes d_A and the total length L become 1. The expected time therefore becomes $d_A + 1 - c$. Hence in our original normalization, we have that the fixed strategy $h\left(\frac{b+1}{L}\right)$ guarantees the estimate

$$v_A\,(c) \geq 1 - c + d_A.$$

Combined Cases

If we combine the analyses given for the two above cases, we obtain a stronger form of Lemma 2.

Lemma 3 *Let Q be a unit-length tree with Searcher starting node N. For any other node A of Q adjacent to N, let d_A be the length of the edge NA and let $w_A = \mu\,(Q_A)$ denote the total length of the subtree determined by A. If the rendezvous strategy r is to wait at the node A, the value v_A of the resulting subgame is given by*

$$v_A\,(c) = \begin{cases} 1 - cw_A & \text{if } c \leq \frac{d_A}{1-w_A}, \\ 1 - c + d_A & \text{if } c \geq \frac{d_A}{1-w_A}. \end{cases}$$

Furthermore, for the case $c \leq \frac{d_A}{1-w_A}$, the optimal searcher strategy is the same as for the pure search game ($c = 0$) – namely select a minimal tour of Q starting at N and traverse it equiprobably in either direction. The optimal strategy for the Evader is $h(p(c))$,where $p(c) = \mu(Q_{-A})/(1-c)$. This results in a total Agent strategy of $h^*\,(Q, N)$, the optimal hiding strategy in the search game for an immobile hider on the tree. For the case $c \geq \frac{d_A}{1-w_A}$, the optimal strategy for the Searcher is to go directly to A, and then to follow a minimal tour of Q starting and ending at A, taking it equiprobably in either direction.

3.4 General Solution

In the previous subsection we found the solution to the subgame that results from the announcement of the rendezvous strategy $r = A$, where A is a node adjacent to the starting node N in the tree Q. Figure 1 below depicts a tree $\bar{Q}$ with the three possible values for the rendezvous node A labeled as X, Z and Y. To achieve the required normalization, all numbers in the figure measure 1/18'ths, so the total length is one.

Figure 1: The Tree $\bar{Q}$

It is worth observing that the closest node is X, with $d_X = 1/18$; while the node which generates the largest subtree is Z, with $w_Z = 6/18$. For c very small, all three value functions v_A $(A = X, Y, Z)$ will be of the form $1 - cw_A$, so clearly the least will be v_Z. On the other hand, for $c = 1$ we have $v_A = d_A$, so that the least is v_X. These two facts are obviously true for all trees. By plotting the three curves v_X, v_Z, v_Y schematically in Figure 2 below, we can see that the two curves which are least at the extremes are best (one of them) everywhere. For the particular tree $\bar{Q}$ shown in Figure 1, we see that for small c the node Z is optimal for the Cooperator, while for large c the node X is optimal. The node Y, which has intermediate values of d and w, is never optimal.

Figure 2: Curves v_A for the tree $\bar{Q}$

The three curves each consist of two straight line segments, the one to the left of slope $-w_A$ the one to the right of slope -1. At $c = 0$ the curves all have height 1, while at $c = 1$ they have height d_A. The minimum curve begins with the segment with least slope $-w_Z$ and ends with the segment which has minimum height d_X at $c = 1$. Theorem 4 below says that this is the general situation not only for the particular tree $\bar{Q}$, but for any tree.

Theorem 4 *Let Q be a unit-length tree with Searcher starting node N. Let X be any node at minimum distance d_X to N and let Z be any node which determines a subtree Q_Z of maximum total length $\mu(Q_Z) = w_Z$. Then the value $v(Q, N, c)$ of the rendezvous-evasion problem on (Q, N) with cooperation probability c is given by*

$$v(Q, N, c) = \begin{cases} 1 - cw_Z & \text{if } c \leq \frac{d_X}{1 - w_Z}, \\ 1 - c + d_X & \text{if } c \geq \frac{d_X}{1 - w_Z}. \end{cases}$$

For $c \leq \frac{d_X}{1-w_Z}$, Z is an optimal rendezvous strategy for the Cooperator, a traversal of a minimal tour of Q from N (equiprobably in either direction) is optimal *for the Searcher, and an optimal strategy for the Evader is to adopt $h^*(Q_{-Z}, N)$ with probability $\mu(Q_{-Z}) / (1-c)$ and $h^*(Q_Z, N)$ with the complementary probability. For $c \geq \frac{d_X}{1-w_Z}$, X is an optimal rendezvous strategy for the Cooperator, a move to X followed by a minimal tour of Q from X (equiprobably in either direction) is optimal of the Searcher, and the strategy $h^*(Q, X)$ is optimal for the Evader.*

P roof. First observe that the formula claimed for $v(Q, N, c)$ is simply the minimum of $v_Z(c)$ and $v_X(c)$, since these lines intersect at $c = \frac{d_X}{1-w_Z}$. It is thus a concave function. The result will follow immediately from Lemma 3 if we can establish the fact that

$$v_Y(c) \geq \min[v_X(c), v_Z(c)], \text{ for all } c \in [0,1], \text{ all nodes } Y \neq N \text{ of } Q. \quad (6)$$

To establish this fact, fix any node $Y \neq N$. For $c \leq d_Y / (1-w_Y)$ we have by Theorem 1 that $v_Y(c) = 1 - cw_Y \geq 1 - cw_Z \geq \min[v_X(c), v_Z(c)]$, since the line $1 - cw_Z$ is a support line of the concave function $\min[v_X(c), v_Z(c)]$. Similarly for $c \geq d_Y / (1-w_Y)$ we have $v_Y(c) = 1 - c + d_Y \geq 1 - c + d_X \geq \min[v_X(c), v_Z(c)]$, since the line $1 - c + d_X$ is a support line to the concave function $\min[v_X(c), v_Z(c)]$. This establishes the claim (6) and shows that for any c either X or Z is optimal. The rest follows from Lemma 3. ■

4 Mobile Agent on the Line

In this section we consider the rendezvous-evasion problem on the line, when the Searcher and a mobile Agent are initially placed a unit distance apart and faced in random directions. Neither knows the direction of the other, and they have no common notion of a positive direction on the line. This problem has been extensively studied in the $c = 1$ rendezvous context. The symmetric rendezvous version was proposed in [2], but the upper bound on the symmetric rendezvous value given there has been successively improved by Anderson and Essegaier [5] and Baston [8]. The asymmetric version is more relevant for our current purposes. This was solved (see solution below) by the authors [4]. A similar but distinct problem for the unit interval (rather than the infinite line) has been solved by Howard [15]. These versions all assume that both players have unit speed. However this is not a suitable assumption for rendezvous-evasion, as the evader strategy of going forward at unit speed achieves infinite T (no capture) with positive probability (of .5) against any searcher strategy, and hence infinite expected time $\hat{T}$. For this reason, we assume a slower agent speed of m, $0 \leq m < 1$. We allow m to be 1 only in the pure rendezvous case of $c = 1$, to include the problem studied in [4]. Of course this problem is fairly trivial in the pure search game ($c = 0$) context. In this section we will give a complete solution to the rendezvous-evasion problem on the line for all parameter values (m, c) in $[0,1) \times [0,1]$. For $c = 1$ and $m < 1$ this analysis can also be viewed as an extension of the asymmetric rendezvous problem to the case of differing speeds. The analysis concentrates on the optimal strategies for the Searcher and the cooperating Agent, as the Evader has only one strategy which meets the necessary condition (see last sentence of the previous section) of optimizing against the Search strategies which we consider: namely, pick a direction and go at maximum speed m until caught. We will first review the optimal rendezvous strategy found by the authors in [4] (that is, for $m = c = 1$) and we then consider two strategy types for the general case; those in which the Searcher turns twice, and those in which the Searcher turns once. It will turn out that there are three regions of the $m \times c$ square with qualitatively distinct optimal strategies. Those readers we would like to see at this point a qualitative description of the solution to the problem should look at Theorem 5 which is located just before the Conclusions section.

In all these analyses, we consider that the Searcher begins at location 0 facing in the positive direction, and that the Agent is initially placed at -1 or $+1$ and faced left or right. This gives four equiprobable starting situations for the Agent. Since the Agent has two strategies (r with probability c if a cooperating rendezvouser, and h with probability $1-c$ if an evader) it follows that there are 8 possible trajectories of the Agent: the four Cooperator trajectories $\pm 1 \pm r\,(t)$, each with probability $c/4$; and the four Evader trajectories $\pm 1 \pm h\,(t)$, each with probability $(1-c)\,/4$. We will say figuratively that the Searcher must find all four Cooperators and all four Evaders. It follows that given strategies of the Searcher, the Cooperator and the Evader, there are eight (not necessarily distinct) capture times t_i, $i = 1, \ldots, 8$, and that the expected meeting time $\hat{T}$ is a weighted average (with respect to c) of these. By contrast, note that in the simpler pure rendezvous context, there are only four such capture times. To make this more explicit, note that if the Agent's (pure) strategy (path relative to his starting point and initial direction) is $g(t)$, where $g(0) = 0$, his actual path relative to the Searcher's coordinate system is $\pm 1 \pm g\,(t)$, with each of the four combined choices of sign chosen with probability $1/4$. If the Searcher chooses path $f\,(t)$, with $f\,(0) = 0$, then this is also his actual path. The four meeting times for f and g are given by the four values (corresponding to choice of signs) of

$$t_i = \min \{t : f(t) = \pm 1 \pm g\,(t)\} \,.$$

In the rendezvous-evasion version, there will be a strategy g for the cooperator (called r) and another strategy g for the evader (called h for hider), giving a eight paths and consequently eight meeting times t_i.

4.1 Solution for $m = c = 1$

The optimal solution case of two unit speed rendezvouser initially placed a unit distance apart was found by the authors (1995) to be as follows. One player (call him the Searcher) moves forward for time 1, then reverses direction for a further 2 time units. This path is drawn in a thick solid line in the Figure 3 below. The other player (call him the Agent) moves forward for time .5, then backwards for time .5, then forwards for time 1, and finally backwards for time

1. This path (relative to his starting point) is drawn in dotted line below. All of these motions are taken at maximum speed (that is, one). By time 3 they have definitely met. Depending on the initial placement of the Agent, the meeting time will be .5, 1, 2, or 3, an average of $6.5/4 = 13/8 = V(m = 1, c = 1)$. All of these motions are taken at maximum speed (that is, 1). In Figure 3, the Search strategy is given by the thick line, and the four possible Agent (Cooperator) paths (starting at ± 1, in direction ± 1) are drawn as two thin and two medium lines. The intersections of the thick line with each of the other lines occur at the four meeting times mentioned above.

Figure 3: Pure Rendezvous Solution, $m = c = 1$.

4.2 General Solution with Two Searcher Turns

In this and the following subsection, we derive the optimal search pair (s, r) in the classes of strategies where the Searcher respectively makes two turns or one turn. The two-turn strategies have the Searcher moving forward at unit speed for time at least one, and reversing direction when both Cooperators starting at $+1$ have been found. The Searcher reverses again when he intersects the path $-1 - mt$ (and hence has found all Agents starting at -1), until he reaches the path $+1 + mt$. The one-turn strategy is unique, going at unit speed unil $+1 + mt$ is reached, and then reversing until $-1 - mt$ is reached. It is easy to see the following two properties: Searcher strategies with more than two turns cannot be optimal. Against a Search strategy s which is either a two-turn strategy or the one-turn strategy, as described above, the only Evader strategy $h(t)$ which maximizes $\hat{T}(s, h)$ is the simple 'no-turn' strategy $h_0(t) = mt$ (going at maximum speed without turning) which gives the four paths $\pm 1 \pm mt$. By the fact (see section 2) that a necessary condition for a triple (s', r', h') to be optimal is that $\hat{T}(s', h)$ is maximized by $h = h'$, we need only consider the 'no-turn' strategy $h_0(t) = mt$ for the Evader. Our method of analysis is therefore different from that of section 3. We evaluate pairs (s, r) against h_0, first (in this subsection) considering pairs where s has two turns and then (in the next subsection) where it is the one-turn strategy.

In the rendezvous solution given in previous section, the Searcher reverses direction (at time 1) as soon as he is sure he has met the Cooperator if he was in the forward direction (started at +1). It seems plausible that if the cooperation probability c is sufficiently close to 1, then this should remain true. In this case, when the Searcher meets the last of the four Cooperators he will have Evaders on either side of him. The subgame optimal strategy for the searcher is to meet the closer Evader first (one who started at -1), and then reverse to eventually meet the remaining Evader who started at +1. Such a Searcher strategy involves two changes of direction. We will show later that indeed when c is close to 1 the optimal Search strategy will be of this type, involving two turns.

We now analyze optimal strategies of this two-turn type. We consider that the Searcher must find all four Cooperators and all four Evaders. (The evaders are simply the four functions $\pm 1 \pm mt$.) After going to the right to find both cooperators in this direction, he finds all cooperators and evaders in the other direction (who started at -1), before turning once again to find the remaining evader who started at +1. We determine the optimal Searcher and Cooperator paths of this type, as a function of the given parameters m and c.

Without loss of generality, we may assume that the first meeting of the Searcher with a Cooperator will take place at time and location a, for some value $1/(1+m) \le a \le 1$. Note that for the case $m = c = 1$ discussed above, we have $a = 1/2$. A lemma of the authors [4] shows that the Searcher and Cooperator should go to location a at maximum speed. So we have

$$t_1 = a. \tag{7}$$

At time t_1, the other Cooperator who started at +1 will be at location $1 + (1-a)$, so if he reverses at time t_1 and moves at maximum speed m, he can meet the Searcher in additional time

$$t_2 - t_1 = \frac{2(1-a)}{1+m}. \tag{8}$$

In this class of strategies the Searcher and Cooperator both turn at time t_2, so that the Cooperator starting at -1 who is closer to the Searcher next meets him. Equating the positions of this Cooperator and the Searcher at time t_3 gives the equation

$$\begin{aligned} -1 + (1-a) + m\left(-(t_2 - t_1) + (t_3 - t_2)\right) &= 2t_2 - t_3 & (9)\\ \text{at } x_3 &= -(t_3 - 2t_2) = 2t_2 - t_3. & (10) \end{aligned}$$

At time t_3 the distance between the Searcher and the remaining Cooperator is $2(x_3 - (-1))$, since the midpoint of the Cooperators who start at -1 is always

-1. Hence the Searcher meets the remaining cooperator (who turns at time t_3) at time t_4, in additional time

$$t_4 - t_3 = \frac{2\,(2t_2 - t_3 + 1)}{1+m} \text{ at } x_4 = -\,(t_4 - 2t_2)\,. \tag{11}$$

At time t_4 the Searcher still has to meet the remaining left Evader $-1-mt$ and the remaining right Evader $+1+mt$. The Searcher's path after t_4 is given by $x_4 - (t - t_4)$. So he meets the remaining left Evader at time t_5 given by

$$x_4 - (t_5 - t_4) = -1 - m * t_5 \text{ at } x_5 = -\,(t_5 - 2t_2) \tag{12}$$

At time t_5 the Searcher has found the Agent if he started at -1, so he reverses direction with path $x_5 + (t - t_5)$ to find the remaining right evader with path $+1+mt$ at time t_6,

$$x_5 + (t_6 - t_5) = 1 + m * t_6. \tag{13}$$

The game is over at time t_6. However we must calculate two more earlier times, t_7 and t_8, when the Searcher met the evaders $1-mt$ and $-1+mt$ respectively. Clearly

$$\begin{aligned} t_7 &= 1/\,(1+m) && \text{(14a)} \\ -1 + m * t_8 &= -\,(t_8 - 2t_2)\,. && \text{(14b)} \end{aligned}$$

Starting with $t_1 = a$, we may successively solve the numbered equations for the times t_2 through t_8, obtaining,

$$\begin{aligned} t_1 &= a \\ t_2 &= \frac{-a + am + 2}{1+m} \\ t_3 &= \frac{am - a + 4}{1+m} \\ t_4 &= \frac{-3a + am^2 + 6m + 6 + 2am}{(1+m)^2} \\ t_5 &= -\frac{-2a + 2am + 5 + m}{(1+m)\,(-1+m)} \\ t_6 &= \frac{2am - m - 2a + 7}{(-1+m)^2} \\ t_7 &= 1/\,(1+m) \\ t_8 &= \frac{-2a + 2am + 5 + m}{(1+m)^2}. \end{aligned}$$

The expected meeting time $\hat{T}$ is given by

$$\begin{aligned} \hat{T} &= c\,(t_1 + t_2 + t_3 + t_4)\,/4 + (1-c)\,(t_5 + t_6 + t_7 + t_8)\,/4 && (15) \\ &= \frac{(m-1)\left(2cm^3 + m^2 - cm^2 - 4cm + 3 - c\right)a - 3c - 8cm - 7cm^2 + 6cm^3 + 2m + m^2 + 9}{2\,(1+m)^2\,(-1+m)^2}. \end{aligned}$$

Since the parameter a appears linearly, and $m-1$ is negative, we minimize $\hat{T}$ by setting a equal to $1/(1+m)$ or 1, according as the expression

$$\phi = 2cm^3 + m^2 - cm^2 - 4cm + 3 - c \tag{16}$$

is respectively negative or positive, or equivalently as the cooperation probability c is larger or smaller than the function $\tau(m) = \frac{m^2+3}{-2m^3+m^2+4m+1}$. Note that setting $a = 1/(1+m)$ corresponds to the Agent going to meet the approaching Searcher at maximum speed m, while setting $a = 1$ corresponds to the Agent staying still for the first unit of time. Observe that in the pure rendezvous case $c = 1$ (but with distinct speeds), the agent waits when his speed m is such that the expression $2m^3 + m^2 - m^2 - 4m + 3 - 1 = 2(m-1)\left(m^2+m-1\right)$ is positive, or equivalently when m is less than the golden mean $\frac{\sqrt{5}-1}{2} \doteq .618\,03$. Thus the solution obtained by the authors for equal speeds ($m = 1$) hold qualitatively for sufficiently large m, but fails for m below the cutoff value. As mentioned above, the optimal Searcher-Cooperator strategies when the Searcher uses two turns depends on whether the parameter pair (m, c) lies above or below the curve $c = \tau(m)$ which is drawn on $m \times c$ unit square in Figure 2 below. The curve intersects the line $c = 1$ at the golden mean $m = \frac{\sqrt{5}-1}{2} \doteq .618\,03$ and again at $m = 1$.

Figure 4: Below curve $c = \tau(m)$, Agent waits for time 1.

Substituting the values $a = 1/(1+m)$ and $a = 1$ into the formula 15 for $\hat{T}$, we obtain two expressions for the value function $v_2(m,c)$ using the strategy family of this section,

$$v_2(m,c) = \begin{cases} \dfrac{4cm^4 + m^3 - 2cm^3 - 9cm^2 + 7m - 4cm + m^2 + 3 - c}{(m+1)^3 (m-1)^2}, & \text{if } c \geq \tau(m), \\ \dfrac{2cm^4 + m^3 + 3cm^3 - 10cm^2 + 5m - 5cm + 6 - 2c}{2(m+1)^2 (m-1)^2}, & \text{if } c \leq \tau(m). \end{cases} \tag{17}$$

The formula $v_2(m,c)$ gives the least expected time obtainable by 2-turn search strategies of the type described in this subsection.

When $c \geq \tau(m)$, the optimal Searcher and Cooperator strategies obtained by setting $a = 1/(1+m)$ do not involve any waiting, and will be denoted respectively by s_2 and r_2. The subscript 2 denotes the fact that the Searcher is turning twice. These strategies (which depend on m as well) are depicted, together with the Evader paths $\pm 1 \pm mt = \pm 1 \pm h_0(t)$, in Figure 5 below.

Figure 5: Two-turn strategies for $c \geq \tau(m)$.

The thick solid line is the Searcher strategy s_2. The four Hider paths are drawn in the light, or dashed, line. The four Cooperator paths are drawn in medium lines which are discontinued when they meet the Searcher. To save space we have not drawn the portion of s_2 after it meets the Cooperator $-1 - r_2$. It then goes to meet the Evader $-1 - h_0$ and then the Evader $+1 + h_0$. Observe that this strategy takes the Searcher past the point $+1$ (which is as far as he gets in the case $m = c = 1$ considered in the previous subsection) - to show this we have drawn in the line at $+1$. Aside from the path s_2 which has slope ± 1, all the other lines have slope $\pm m$. As m approaches 1, the Searcher and Cooperator paths approach those drawn in Figure 3.

As we have seen above, when $c \leq \tau(m)$ we set $a = 1$. This means that the Cooperator stays still until time $t = 1$. So we denote this Cooperator strategy by $r_{2,w}$ and call it the 'waiting' strategy (hence the w). We denote the corresponding Searcher strategy obtained by taking $a = 1$ by $s_{2,w}$. The paths resulting from these strategies, together with the Evader paths $\pm 1 \pm h_0(t) = \pm 1 \pm mt$, are shown below in Figure 6.

Figure 6: Two-turn strategies for $c \leq \tau(m)$.

Observe that since the Cooperator waits until time $t = 1$, the Searcher does not have to go past $+1$ in his search. This makes the Searcher's expected time to find the Evader slightly shorter, and hence is an improvement overall when c is small.

4.3 One-Turn Search Strategy

In this subsection, we evaluate the expected time $\hat{T}$ when the Searcher does not turn until he is sure that the Agent is not in his forward direction. At this point he turns and moves at maximum speed until he is sure to have found any Agent. Such a search procedure seems reasonable when the likelihood of an Evader $(1 - c)$ is high. With such a searcher, it is always optimal for the Cooperator to move a maximum speed m for the initial time $z_1 = z_0 = 1/(1+m)$. We call the eight possible capture times $z_0 = z_1 \leq z_2 \leq \cdots \leq z_7$ (unless $m = 0$, the inequalities will be strict). The Searcher always moves at speed 1; going forward until time z_3, when he turns around. At time z_1,the Searcher meets a Cooperator and an Evader. The Cooperators turn at time z_1. At time z_2, he meets the remaining forward Cooperator, and the Cooperators change direction. At time z_3 the Searcher meets the remaining forward Evader. At this time he knows the Agent started at -1, so he turns around. At time z_4 he meets the Evader with motion $-1+mt$, and at time z_5 he meets a Cooperator who started at -1. At this time the remaining Cooperator changes direction. He meets this cooperator at time z_6 and meets the Evader with motion $-1-mt$ at time z_7, when the game ends. We shall not include the derivation of the formulae for the z_i but merely list them as follows:

$$
\begin{aligned}
z_1 &= \frac{1}{1+m}\\
z_2 &= \frac{1+3m}{(1+m)^2}\\
z_3 &= \frac{1}{1-m}\\
z_4 &= \frac{m-3}{(m+1)(m-1)}\\
z_5 &= \frac{-3-5m-5m^2+5m^3}{(1+m)^3(-1+m)}\\
z_6 &= \frac{-3-14m-12m^2+7m^4-10m^3}{(1+m)^4(-1+m)}\\
z_7 &= \frac{3-m}{(m-1)^2}
\end{aligned}
$$

It is worth noting that at the two times z_3, z_7 when an outward moving evader is met, an additional factor $(m-1)$ comes into the denominator, increasing the order of the pole at $m=1$. Since the Searcher meets Cooperators at times z_1, z_2, z_5, z_6 and Evaders at times z_1, z_3, z_4, z_7, the expected meeting time for the one-turn strategy is given by

$$
\begin{aligned}
v_1(m,c) &= c(z_1+z_2+z_5+z_6)/4+(1-c)(z_1+z_3+z_4+z_7)/4 \qquad (18)\\
&= \frac{-4cm^2-4cm^3-4cm^4+4cm^5+2+5m+3m^2-m^3-m^4}{(1+m)^4(-1+m)^2}
\end{aligned}
$$

The single-turn strategy derived here for the Searcher will be denoted by s_1, and the corresponding Cooperator strategy will be denoted by r_1. The resulting paths are shown below in Figure 7. It can be readily seen that the single-turn strategy s_1 is efficient in meeting the Evader paths early, so we can expect the pair (s_1, r_1) to be optimal for small enough c.

Figure 7: Single turn strategies

4.4 Combined Solution

In the two previous subsections we found the optimal search paths s and r when the Searcher was restricted to paths with one or two turns. (Paths with more than two turns are too inefficient and are never globally optimal.) We now combine the two solutions, for the one and two-turn strategies, to find the global optimum. We recall the value of v_2 for small c in equation (17) as

$$v_2 = \frac{2cm^4 + m^3 + 3cm^3 - 10cm^2 + 5m - 5cm + 6 - 2c}{2\left(m+1\right)^2\left(m-1\right)^2}.$$

Consequently the one and two-turn solutions are equally good when $v_2 = v_1$ (for this form of v_2), on the curve given by

$$c = \rho\left(m\right) = \frac{-\left(m^2 + m + 2\right)\left(1+m\right)^3}{-14m^2 - 14m^3 + 6m^4 - m^5 + 2m^6 - 9m - 2}.$$

Figure 8: Plots of $c = \phi(m)$, $c = \tau(m)$.

Figure 8 above plots the curves $c = \phi(m)$ (the upper curve) and $c = \rho(m)$ (the lower curve). These curves partition the (m, c) square into three regions where the solutions to the rendezvous-evasion game on the line are of the following three types. We can use this figure to summarize the solution to the rendezvous-evasion problem on the line, as follows.

Theorem 5 *(i) Below the lower curve the single-turn strategy (s_1, r_1) is optimal. Here the Searcher goes in one direction until he is sure the Agent (either type) is not in that direction.*

(ii) Between the two curves, the two-turn strategy $(s_{2,w}, r_{2,w})$ with a Cooperator who waits for time 1 is optimal. Here the Searcher goes in one direction until he knows the Cooperator is not in that direction (though the Evader might be).

(iii) Above the upper curve, the two-turn strategy (s_2, r_2) with the Cooperator always moving at maximum speed m is optimal.

In all cases, the evader moves at maximum speed without turning, a strategy we have been calling h_0.

Observe that for fixed values of c between the minimum values of the two curves (e.g., $c = .9$) the single-turn strategy is optimal for extreme values of m. If m is close to zero it is clear that the Searcher should go the little bit beyond $+1$ required to catch the forward evader, before turning. Also, if m is close to 1 then the capture times for the Cooperators are all negligeable compared with those for the two escaping evaders, so that the Searcher should minimize the expected time to catch these two, which involves a single-turn as in the solution to the pure search game.

An important distinction between this solution and that given earlier for an immobile hider on a tree is that here, for large enough evader speed m, there is an intermediate qualitative solution for intermediate values of the cooperation probability c, namely the solution described in part (ii) of the Theorem.

5 Conclusions

This paper initiates the study of optimal search for an Agent of unknown aims. We have restricted the possible aims to two: minimization or maximization of the time until found. However more general aims (or the lack of any - that is, stochastically known motions) might be considered in the future. Even more general uncertainties are possible, as with two Agents, where *either one* may or may not wish to be found. The particular model chosen in this paper combines the theory of Search Games (zero-sum games with capture time T as payoff) with the theory of Rendezvous Search (two Agents who want to find each other) to find intermediate type games. We considered two search contexts, an immobile hider on a tree and a mobile hider on an infinite line. In both cases we were able to obtain full solutions to the general problem. In the former, we found a cutoff value of the cooperation probability c which divides two qualitatively distinct solutions. In the latter problem we found, on the contrary, a qualitatively different solution that occurs only for intermediate value of the cooperation probability.

We believe that the main contribution of this paper is not in the particular search problems that we solved, but rather the introduction into search theory of models with uncertain target motives. Though undoubtably realistic, such models present many difficulties that this paper only begins to deal with.

6 Problems for further study

Rendezvous-evasion in two cells: The Searcher and Agent are initially placed at time $t = 0$ into distinct cells, which we (and the players) respectively call 1 and 2. At each time $t = 1, 2, 3, \ldots$, Searcher and Agent locate in cell x_t and y_t, and meet at the first time when they are in the same cell. We may assume (by relabeling in each period) that the Cooperator strategy is $(2, 2, \ldots)$. For c close to 1, the optimal Searcher strategy is $(2, b, b, \ldots)$ and the optimal Evader strategy is $(1, b, b, \ldots)$, where b denote the Bernoulli strategy of picking either location equiprobably and independently of previous choices. For c near 0 the optimal strategy for both Searcher and Evader is $(b, b, b, \ldots)$. These strategy pairs give respective expected meeting times of $c\,(1) + (1-c)\,(1+2) = 3 - 2c$ and 2, which give the same time 2 when $c = 1/2$. However in this intermediate case ($c = 1/2$) both Searcher strategies are dominated by the mixed strategy $.9\ \ (2, b, b, \ldots) + .1\ \ (1, 2, b, b, \ldots)$. Since a best response of the Evader is $(1, b, b, \ldots)$, this mixed Searcher strategy ensures that the expected meeting time is no more than $(.9)\,(3-2c) + (.1)\,(2c + (1-c)) = 2.8 - 1.7c = 1.95 < 2$, for $c = 1/2$. This analysis shows that for intermediate values of c the optimal Searcher strategy is not optimal for either extreme case ($c = 0$ or 1). The determination of optimal strategies for all c, and the corresponding optimal expected meeting time $v\,(c)$, seems an interesting problem.

Rendezvous-evasion in n cells: The problem is the same as for two cells, except that for $n > 2$ the two players can never achieve a common labeling of the cells. For $c = 1$ the Cooperator stays still while the Searcher picks a random permutation of the remaining cells. For $c = 0$ both Searcher and Evader should move randomly. What is the solution for general c? See [7] for a discussion of a related problem.

Mobile rendezvous-evasion on the circle: The Searcher and Agent are placed randomly on a circle of unit circumference and can both move with unit speed. For the search game $c = 0$ it has been shown [1][23] that the optimal search strategy is 'cohatu'. This is short for 'coin half tour': At times $i/2$, $i = 0, 1, 2, \ldots$, go half way around the circle equiprobably in either direction. All evader responses give the same expected meeting time of $v(0) = 3/4$. For the rendezvous problem $c = 1$ [2] the optimal strategy is for the Searcher to move clockwise while the Cooperator moves counterclockwise (for a proof see [15]), with expected meeting time $v(1) = 1/4$. For c near to 1 the Searcher should use 'clockwise-cohatu': this means clockwise for $t \leq 1/2$ (by which time he will have met the Cooperator) and then use cohatu. Any evader strategy which goes clockwise for $t \leq 1/2$ is an optimal response. Thus this strategy has an expected meeting time of

$$c\left(\frac{1}{4}\right) + (1-c)\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{5-4c}{4}.$$

The two strategies, cohatu and clockwise-cohatu, give the same expected time of $3/4$ when $c = 1/2$. However neither is optimal in this case ($c = 1/2$), since both are dominated by a mixture of .9 (clockwise-cohatu) and .1 (counterclockwise-clockwise-cohatu). The latter strategy goes in the indicated directions for times $(0, 1/2)$ and $(1/2, 1)$ and then in random directions. Against this mixture, an optimal Evader must go clockwise for the first half time unit. Thus if clockwise-cohatu is used the expected meeting time is the same as for either of the two Searcher strategies already calculated, that is, $3/4$. However in the case that counterclockwise-clockwise-cohatu is used, the Searcher does better: The Cooperator will be found in the same expected time $\frac{1}{2} + v(1) = \frac{3}{4}$ (no improvement), but the Evader will be found in the smaller expected time $\frac{1}{4}$. This analysis is similar to that for two cells (there are two directions on the circle), and as in that problem the solution for intermediate values of c would be interesting. This analysis assumes that the Searcher and Agent have a common notion of direction around the circle. The problem is also well defined (but distinct) without this assumption. (In that case the Cooperator stays still - see [15].)

Immobile Rendezvous-evasion on networks: To what extent does the dichotomy of search strategies found for trees apply to other networks?

Two-sided ambiguity of aims: In the analysis given in the article, only the aims of the Agent are uncertain. How should the problem be analyzed if the Agent has a probability c_A of wanting to minimize T (otherwise he wants to maximize) and the Searcher has a similar probability c_S?

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