Abstract

The rendezvous search problem asks how two blind searchers in a known search region, having maximum speed one, can minimize the expected time needed to meet. Suppose that two players are placed an arc-distance $x \in [0, 1/2]$ apart on a circle of circumference 1, and faced in random directions. If x has a continuous density function h which is either decreasing and satisfies $h(1/2) \geq h(0)/2$,or increasing, we determine an optimal rendezvous strategy. Furthermore if h is strictly monotone, this strategy (which depends in a simple manner on h) is uniquely optimal. This work extends that of J. V. Howard, who showed for the uniform density $h(x) = 2$ that 'search and wait' is optimal, with expected search time $1/2$. We also show that the uniform density is the only counterexample on the circle to S. Gal's conjecture (which he proved for the line) on the nonoptimality of 'search and wait'.

Asymmetric Rendezvous Search on the Circle

Steve Alpern
Department of Mathematics
The London School of Economics
London, WC2A 2AE, U.K.
email: S.Alpern@lse.ac.uk

Author Supported by NATO grant CRG 972991

October 7, 1999

1 Introduction and Main Results

The theory of rendezvous search, as proposed by the author in [1], asks how two agents who are randomly placed in a known region X can optimally find each other. It is assumed that the agents have unit speed, cannot see each other, and that when the search region is a network they 'find each other' when they first have a common position. We will be concerned here with the *asymmetric* version of the problem, where the players are allowed to meet before the game begins to decide which role each one will take; for example one might stay still while the other searches the whole search region, the so called 'go-stay' strategy. The optimality notion we will adopt here is the most common one, namely minimization of the expected meeting time. We call this optimal time the asymmetric rendezvous value R^a. As shown in the above cited introductory paper, the value R^a depends to a great extent on the amount of the symmetry of X which the players are able to 'see through'. The example given in that paper is the one we deal with here, namely the case where X is a circle of circumference 1, or equivalently, the unit interval mod 1. For asymmetric rendezvous, there are three cases to consider. In the case where the region is a 'clockface', the players have a common labeling of all the points, and they could agree for example to both head for 'three o'clock'. In the case where they have no common labeling of points of X, but have a common notion of direction, they could for example decide for one player to go clockwise and the other to go counter-clockwise, until they meet. (In fact for the uniform initial placement of the players this was shown to be optimal by Howard [11].) The final case, which we consider here, is where they have no common labeling of points and no common notion of direction. (These symmetry notions are discussed more formally in the cited paper [1], in terms of subgroups of the isomorphism group of X.) For this final case, it was conjectured in [1] that for a uniform initial placement of the players, the optimal solution was for one player to go around the circle at maximal speed (in either direction), while the other one waits (go-stay). If true, this would imply that $R^a = 1/2$. J. V. Howard subsequently verified this conjecture in [11]. Howard's result is in sharp contrast with a result of S. Gal [10], who proved for the case where the search region is a line that it is *never* optimal for one player to stay still until found. In this context *never* means *for no distribution of initial distance.*

In order to state our main results, we must introduce a little notation. We begin the game by picking an arc of length x, $0 \leq x \leq 1/2$, placing the two players randomly at either end, and facing them in random directions. Suppose that the distribution of the random variable x is given by a continuous positive density function h on $[0, 1/2]$, with $\int_0^{1/2} h(x)\,dx = 1$. The uniform initial placement is given by the function $\bar{h}(x) = 2$. The directed clockwise distance y, $0 \le y \le 1$, from player I to player II has density $h(y)/2$ and a cumulative distribution function we denote by $H(y)$, $0 \le y \le 1$, given by

$$H(y) = \frac{1}{2}\int_0^y h(x)\,dx, \text{ taking } h(x) \equiv h(1-x) \text{ for } 1/2 \le x \le 1.$$

Observe that whether or not the distribution H of y comes from a density function $h(y)/2$, it satisfies the equation

$$H(y) = 1 - H(1-y). \tag{1}$$

Note that h is the density of the initial undirected distance between the players, and that $h/2$ is the density of the directed distance from I to II. So $h/2$ is the density corresponding to the distribution function H.

Our first result, Theorem 1, determines an optimal rendezvous strategy for a large class of monotone densities h which includes the uniform density $\bar{h}(x) = 2$. The strategy we determine is in fact uniquely optimal for all the densities we consider which are strictly monotone. The optimal strategies will have one player going all around the circle at maximum speed one, while the other player will have to determine a cutoff time c where his motion changes. The time c depends on whether h is increasing or decreasing, as follows. If h is increasing, define c to be the smallest number in $[1/2, 1]$ which maximizes the quotient $H(y)/y$. If h is decreasing, define c to be the largest number in $[0, 1/2]$ which minimizes the quotient $(1 - H(y))/(1-y)$. This is the main result of the paper.

Theorem 1 (monotone densities) *If the initial distance between the players is given by a monotone density h, $0 \le x \le 1/2$, the following strategies are optimal, and ensure a meeting by time 1. In both cases one player (I) goes all around the circle at maximum speed one.*

- *If h is increasing, then player II moves at maximum speed in a random direction for time $c/2$, then reverses direction at maximum speed for time $c/2$, and finally stays still until time 1. The rendezvous value is given by*

$$R^a = -\frac{1}{4}\int_0^{1-c} xh(x)\,dx + \frac{2+2c}{8} + \frac{2-c}{4}H(1-c).$$

- *If h is decreasing and $h(1/2) > h(0)/2$, player II stays still for time c, then moves at speed one in a random direction for time $(1-c)/2$, then reverses direction until time 1. In this case*

$$R^a = \frac{1}{4}\int_0^{c} xh(x)\,dx + \frac{2c+4}{8} - \frac{c+1}{4}H(c).$$

- *Furthermore, if h is* ***strictly*** *increasing or decreasing, the relevant above strategy is* ***uniquely*** *optimal.*

- *If h is increasing and decreasing, it is the uniform density $\bar{h}(x) = 2$, both the above strategies are optimal with $c = 1/2$, and both formulae for R^a give $1/2$.*

The next result applies to continuous densities h on $[0, 1/2]$ which are *slowly varying* in the sense that *their maximum divided by their minimum is less than two.* Such densities are in a sense close to the uniform density, where this ratio has the minimum possible value of 1. (The two conditions in the previous theorem, that h is decreasing and $h(1/2) > h(0)/2$, together imply that h is slowly varying.) The following extends a similar result of Howard [11] for the uniform density.

Theorem 2 (slowly varying densities) *If the initial distance between the players is given by a slowly varying density, then any optimal strategy has one player going all around the circle at speed one, and the other player back at his starting point at time 1, by which time they have met.*

Our last result applies to arbitrary distributions of the initial distance (not necessarily given by a density) and shows that Howard's counterexample (uniform distribution on the circle) to Gal's conjecture on the non-optimality of the go-stay strategy is the only counterexample for the circle.

Theorem 3 (arbitrary distributions) *The only initial distribution of initial distance between the players for which the go-stay strategy is optimal is the uniform distribution.*

In order to analyze the circular rendezvous problem for the density h, we will use an idea of the author which converts some asymmetric rendezvous problems on regions with group structure into an *alternating search* problem of two searchers on disjoint regions trying to locate an object hidden in one of the regions. This idea has already been used in [2] to prove a conjecture of Baston and Gal [7] concerning rendezvous on the line. Once the problem has been reduced to one of alternating search, the techniques recently developed by the author and Howard [3] can be applied to good effect.

Other approaches to continuous rendezvous problems are due to Hulme and Thomas [14], Anderson and Essegaier [4], Anderson and Fekete [5], Baston [6] and Lim *et. al.* [12]. More generally, the wider field of two-sided search is covered in Gal's monograph [9], and geometric games are covered in Ruckle's book [13].

2 Formalization of the problem

The analysis given in this section will not assume that H is given by a density function.

A strategy $r = (r_1, r_2)$ in the rendezvous problem is a pair of paths $r_i : [0, \infty) \to R$, with $r_i(0) = 0$, and $|r_i(t) - r_i(s)| \leq |t - s|$, $i = 1, 2$, and $t, s \geq 0$. The number $r_i(t)$ is the net distance that player i will have travelled by time t, in the direction he was initially facing. Often we will only have to define these paths up to a time t by which the players will have certainly met. Define new variables z_j by

$$\begin{aligned} z_1 &= r_1 + r_2 \\ z_2 &= r_1 - r_2, \text{ and consider the extreme values,} \end{aligned} \tag{2}$$

$$\begin{aligned} M_1(s) &= \max_{t \leq s} z_1(s), \\ M_2(s) &= \max_{t \leq s} -z_1(s), \\ M_3(s) &= \max_{t \leq s} z_2(s), \\ M_4(s) &= \max_{t \leq s} -z_2(s). \end{aligned} \tag{3}$$

If the two players start with I facing clockwise and II facing counter-clockwise, then they will have met by time s if either $M_1(s) \geq y$ or $M_2(s) \geq 1 - y$. In particular, they will definitely have met if $M_1(s) + M_2(s) \geq 1$, so we may restrict strategies r_1, r_2 so that $M_1(s) + M_2(s)$ never exceeds 1. In general, they will have met in this case with probability

$$p_1 = H(M_1(s)) + [1 - H(1 - M_2(s))].$$

There are four possible ways of initially facing the players, and the meeting probabilities in the other three are as follows:

$$\begin{aligned} p_2 &= H(M_2(s)) + [1 - H(1 - M_1(s))], \\ p_3 &= H(M_3(s)) + [1 - H(1 - M_4(s))], \text{ and} \\ p_4 &= H(M_4(s)) + [1 - H(1 - M_3(s))]. \end{aligned}$$

Since the four ways of initially facing the players are equally probable, it follows that the probability that players using rendezvous strategy $r = (r_1, r_2)$ will have met by time s is given by $P_r(s)$, where

$$\begin{aligned} P_r(s) &= (p_1 + p_2 + p_3 + p_4)/4, \text{ which by (1),} \\ &= \frac{H(M_1(s)) + H(M_2(s))}{2} + \frac{H(M_3(s)) + H(M_4(s))}{2}. \end{aligned} \tag{4}$$

The expected time required for players using the strategy $r = (r_1, r_2)$ to meet is then given by

$$E_r = \int_0^\infty P_r(s)\ ds.$$

3 Alternating search on two circles

In [3] the general problem of alternating search is introduced. Two coordinated searchers, starting at given points in disjoint regions, alternate searching (at speed 2) for a single object placed according to a known distribution in the union of the two regions. More precisely, they move so that the sum of their speeds is bounded above by 2. For our current purposes, the two search regions will be circles of circumference 1, and the distribution on each circle is $H/2$. A joint search strategy is a pair of paths $z = (z_1, z_2)$, where $z_i(t)$ denotes the net clockwise motion of searcher i, by time $t \geq 0$, and the paths are restricted by the constraint

$$|z_1'(t)| + |z_2'(t)| \leq 2,$$

for all t except for a discrete set of *switching points* at which a derivative may fail to exist because a searcher is reversing. A more general formulation of the problem is given in the cited paper.

An equivalent linear version of this circular problem is the two searchers start at the origins of two lines, and that Nature places two objects on one of the lines, one at distance y and the other in the opposite direction at distance $1-y$, $0 \leq y \leq 1$. Note that after choosing the number y, there are in general two ways of placing the two objects, and we assume they are equally likely. The two searchers wish to minimize the expected time taken to find *one* of the objects.

As in the previous section, we observe that as soon as searcher 1 (on line 1) has searched an interval around the origin of length one, that is, when $M_1(s) + M_2(s) = 1$, he will certainly have found an object if the two were placed on his line (or if the object was placed on his circle). Hence we can assume that at this point he would rest and let searcher 2 do the rest of the searching. Similarly, if $M_1(s) + M_2(s) \leq 1$ and the distribution H of the distance y is nonatomic, then the probability of searcher 1 having found *both* of the objects is zero. Observe that if the objects are placed on line 1, and searcher 1 has searched the interval $[-M_2, M_1]$, then the probability that he has found an object is given by $H(M_1) + H(M_2)$. Hence the probability that an object has been found by time s, using joint search strategies z_1 and z_2, is given by

$$\frac{H\left(M_1\left(s\right)\right)+H\left(M_2\left(s\right)\right)}{2}+\frac{H\left(M_3\left(s\right)\right)+H\left(M_4\left(s\right)\right)}{2},$$

which of course is the same as the second part of the formula (4) given in the previous section for $P_r\left(s\right)$. Thus we have shown that asymmetric rendezvous on the circle is equivalent to the alternating search problem on two circles, or its linearized version as the search for one of two objects placed on two lines. The equivalence of the two problems is given by the simple change of variables (2). The importance of the new scenario is that we may use the solution concepts for alternating search established in (Alpern and Howard, 1998).

To illustrate the connection between the two problems, consider the asymmetric rendezvous problem on the circle with y chosen uniformly on $[0,1]$. In the two circle problem (this is easier to visualize than its linearized version) it seems intuitively clear (proofs will wait until later sections) that each searcher should pick a single direction, and that the alternation between the two searchers' motion is irrelevant. In any alternation scheme the object will be found by time t with probability t, and hence in expected time $1/2$. One such scheme is for both searchers to move at speed one for times t, $0\le t\le 1$. That is, $z_1\left(t\right)=z_2\left(t\right)=t$. Solving for r_1 and r_2 from equation (2), we obtain $r_1\left(t\right)=t$ and $r_2\left(t\right)=0$, or simply that player II waits for player I to find him. Another scheme in the alternating search problem is for searcher 1 to search his circle first, and then for searcher 2 to searcher his circle. That is, $z_1\left(t\right)=2t$ for $t\le 1/2$ and $z_2\left(t\right)=2\left(t-1/2\right)$ for $t\ge 1/2$. This translates in the rendezvous problem to both players moving forward at speed one ($r_1\left(t\right)=r_2\left(t\right)=t$) for $t\le 1/2$ and then player II reverses while player I continues in the same direction ($r_1'=-1, r_2'=1$) for $t\ge 1/2$. Actually neither of these solutions represents the general solution that we shall find later, these are unstable solutions.

4 Searching a single circle

In this section we consider the problem faced by a single searcher on a circle who wishes to find an object hidden at clockwise distance y, with symmetric distribution H (that is, $H\left(y\right)=1-H\left(1-y\right)$) in least expected time. We assume the searcher has a maximum speed of one. Every search strategy w determines a distribution $F_w\left(t\right)$ giving the probability that the object will be found by time t using search path w. We say that a strategy $\breve{w}$ dominates strategy w if $F_{\breve{w}}\left(t\right)\ge F_w\left(t\right)$ for all $t\ge 0$, with strict inequality for some t. We devote this section to determining a sufficient condition on H such that the strategy $\tilde{w}(t) = t$, $0 \le t \le 1$, (or its reverse) dominates every other strategy. (The strategy $\tilde{w}$ is the one which goes all around the circle in a clockwise direction at maximum speed one, and satisfies $F_{\tilde{w}}(t) = H(t), 0 \le t \le 1$.)

Theorem 4 *Suppose that the distribution H satisfies the condition*

$$H(a) + H(s - 2a) < H(s), \text{for all } a, s \text{ with } 0 < a < s/3 \le 1/3. \quad (5)$$

Then the search strategy $\tilde{w}$ which goes all around the circle clockwise at speed one dominates every other strategy. The associated distribution of detection is $F_{\tilde{w}}(s) = H(s)$.

P roof. w be any other search strategy. For any time s when $w(s) = \tilde{w}(s) = s$, we clearly have $F_w(s) = F_{\tilde{w}}(s) = s$. Consider any time $s < 1$, when we have $|w(s)| < s$. By changing the sign of w, if necessary, we can assume that $0 \le w(s)$. Define $a = \max_{t \le s} -w(t)$ and let $b = \max_{t \le s} w(t)$, where $b < s$.

$$F_w(s) = H(a) + H(b).$$

Furthermore the total variation of w, up to time s, is at least $2a + b$. Since w has maximum speed one, we have

$$2a + b \le s.$$

By the hypothesis (5) we have the required domination inequality

$$F_w(s) \le H(a) + H(b) \le H(a) + H(s - 2a) < H(s) = F_{\tilde{w}}(s).$$

■

Corollary 5 *Suppose that the symmetric distribution H is given by a continuous density function $f(y)$ which is either increasing on $[0, 1/2]$ or is slowly varying ($\max_{y \in [0,1/2]} f(y) / \min_{y \in [0,1/2]} f(y) < 2$). Then the search strategy $\tilde{w}$ dominates every other strategy.*

P roof. erve that

$$\begin{aligned} & H(s) - [H(a) + H(s - 2a)] && (6) \\ &= \int_0^s f(y)\, dy - \int_0^a f(y)\, dy - \int_0^{s-2a} f(y)\, dy && (7) \\ &= \int_{s-2a}^s f(y)\, dy - \int_0^a f(y)\, dy && (8) \end{aligned}$$

so to establish the condition (5) of the previous theorem we need to show that (8) is positive for all a, s with $0 < a < s/3 \leq 1/3$. But since the ratio

$$\frac{\text{length}\left([s-2a, s]\right)}{\text{length}\left([0, a]\right)} = 2,$$

the slowly varying hypothesis ensures that the expression (8) is positive. Now assume that f is increasing on $[0, 1/2]$. If $s \leq 1/2$ the same observation (which would work even with ratio 1) shows that (8) is positive. If $s > 1/2$, the fact that $f(y) = f(1-y)$ ensures that any interval (arc) on the circle $[0, 1]$ (mod1), with length at least a and not containing the point 0, has an f integral exceeding that of $[0, a]$. Since this is true of the interval $[s, s-2a]$, we are done. ∎

It is worth noting that if $f(y)$ is decreasing on $0 \leq y \leq 1/2$, $f(1-y) = f(y)$, and $f(1/2) > f(0)/2$, then f is slowly varying (and also $h = 2f$).

5 Optimal interleaving of two distributions

In this section we consider the alternating search problem on two circles, where the distribution H of the hidden object is assumed to satisfy condition (5). We apply the general results obtained in [3].

Under the hypothesis (5), we may assume that each searcher moves around his circle in a monotone fashion, which we may take to be clockwise, so that

$$z_1'(t) \geq 0, z_2'(t) \geq 0, |z_1'(t)| + |z_2'(t)| = 2, \text{ for all } t \in [0, 1],$$

except possibly for a discrete set of times t when the derivative may fail to exist. The proof that we need consider only undominated ways of searching each circle is given in [3]. The possible ways of combining the individual searches of the two searchers may be described by an increasing function α, called an *alternation rule*, with $\alpha(0) = 0, \alpha(1) = 1$, and $|\alpha(t_2) - \alpha(t_1)| \leq 2|t_2 - t_1|$. Given such an alternation rule α, we define

$$z_1(t) = \alpha(t), z_2(t) = 2t - \alpha(t). \tag{9}$$

Periods when $\alpha'(t) = 2$ correspond to searcher 1 moving clockwise around circle 1 at speed 2, while searcher 2 waits; when $\alpha'(t) = 0$ the roles are reversed. The rule $\alpha'(t) = 1$ corresponds to both searchers moving on their respective circles at speed one.

If the alternation rule α is being used, then the object will be found by time t if it lies in the interval $[0, \alpha(t)]$ on circle 1 (written as $[0, 1] \bmod 1$) or the interval $[0, 2t - \alpha(t)]$ on circle 2. Hence the probability that is will be found by time t is given by

$$H_\alpha(t) = \frac{1}{2}\left[H(\alpha(t)) + H(2t - \alpha(t))\right].$$

An alternation rule is called *optimal* if it minimizes the expected time $E(\alpha) = \int_0^1 t \, dH_\alpha(t)$ to find the object.

The theory developed in [3] determines optimal alternation rules for a slightly more general situation with possibly distinct distributions H_1 and H_2, with $H_\alpha(t) = \frac{1}{2}\left[H_1(\alpha(t)) + H_2(2 - \alpha(t))\right]$. We apply only the symmetric case $H_1 = H_2 = H$ here. (That paper had a maximum derivative 1 for α, whereas we have a maximum derivative 2 here.) We summarize the results from that paper in the following theorem. Note that the concavification of a function is the smallest concave function greater than or equal to the given function.

Theorem 6 *Let H be a cumulative probability distribution function with $H(1) = 1$, and let $\bar{H}$ denote its concavification.*

1. *If $\bar{H}(t) > H(t)$ for $a < t < b$, $\bar{H}(a) = H(a)$, and $\bar{H}(b) = H(b)$, then any optimal alternation rule α satisfies either $\alpha' = 2$ on $\left[a, \frac{a+b}{2}\right]$ and $\alpha' = 0$ on $\left[\frac{a+b}{2}, b\right]$, or the reverse. That is, the intervals $[a, b]$ on the two circles are searched one at a time at speed 2.*

2. *If H is concave then the strategy $\alpha'(t) = 1$ is optimal. If H is strictly concave then $\alpha'(t) = 1$ is uniquely optimal. (This is the rule which searches both circles in parallel, $z_1(t) = z_2(t) = t$, $0 \leq t \leq 1$.)*

6 Proofs of main results

We are now in a position to give proofs of the Theorems (1,2,3) which we stated in the Introduction. These theorems are all stated in terms of optimal solutions to rendezvous on the circle, but in each case we will consider the equivalent alternating search problem on two circles, find the optimal solution, and then find the corresponding solution in the rendezvous context.

The reader may find it helpful, in understanding the proof of Theorem 1, to consider the examples and figures in the next section. However we will not refer to them in the proofs themselves.

Proof of Theorem 1 for increasing densities h. Consider the alternating search problem on two circles with object distribution H with density

$f(x) = h(x)/2$ for $x \leq 1/2$ and $f(y) = f(1-y)$ for $y \geq 1/2$. The assumption that h is increasing on $[0, 1/2]$ assures (by Corollary 5) that condition (5) holds, and that the only undominated way of searching for a single object on one circle with distribution H is a simple search in a single direction at speed 1. So the analysis of section 5, on the determining the optimal alternation rule α, may be applied. The assumption that h is increasing, together with the definition of the cutoff time $c \geq 1/2$ for this case given in the Introduction, implies that the concavification $\bar{H}$ of H has the following form.

$$\bar{H}(y) = \begin{cases} yH(c)/c, & \text{if } 0 \leq y \leq c, \text{and} \\ H(y), & \text{if } c \leq y \leq 1. \end{cases}$$

By Theorem 6 (part 1, with $0 = a, c = b$) we see that the intervals $[0, c]$ on the two circles must be searched alternately at speed 2. That is,

$$z_1' = 2, z_2' = 0 \text{ on } [0, c/2] \text{ and } z_1' = 0, z_2' = 2 \text{ on } [c/2, c].$$

The interleaving problem that remains at time c (normalized to time zero) has a concave function H (which is strictly concave if h is strictly increasing). Hence the second part of Theorem 6 show that an optimal strategy (uniquely if h is strictly increasing) for the remaining time is for both searchers to move at speed one, so

$$z_1' = z_2' = 1 \text{ on } [c, 1].$$

If we interpret this solution in terms of the original rendezvous problem, by solving (2) for r_1, r_2 in terms of z_1, z_2, we get the claimed optimal strategy

$$r_1' = 1 \text{ on } [0,1], \; r_2' = 1 \text{ on } [0, c/2], \; r_2' = -1 \text{ on } [c/2, c], \; r_2' = 0 \text{ on } [c, 1].$$

The expected time resulting from these optimal rendezvous strategies is as claimed. ■

Proof of Theorem 1, for increasing density h. We begin as in the above proof, applying Corollary 5 to the assumption that h is (strictly) decreasing and satisfies $h(1/2) \geq h(0)/2$ to show that condition (5) holds. In this case we have $c \leq 1/2$ and

$$\bar{H}(y) = \begin{cases} H(y), & \text{if } 0 \leq y \leq c, \text{and} \\ H(c) + \frac{(y-c)(1-H(c))}{1-c} & \text{if } c \leq y \leq 1. \end{cases}$$

H is (strictly) concave on $[0, c]$ so Theorem 6 (part 1) gives the (uniquely) optimal strategy $z_1' = z_2' = 1$ there. H lies below $\bar{H}$ on the interior of $[c, 1]$

so Theorem 6 (part 2) gives the (uniquely) optimal strategy $z_1' = 2, z_2' = 0$ on $\left[c, \frac{c+1}{2}\right]$ and $z_1' = 0, z_2' = 2$ on $\left[\frac{c+1}{2}, c\right]$. Solving (2) for the corresponding rendezvous strategy, we get

$$r_1' = 1 \text{ on } [0,1], \; r_2' = 0 \text{ on } [0,c], r_2' = 1 \text{ on } \left[c, \frac{c+1}{2}\right], r_2' = -1 \text{ on } \left[\frac{c+1}{2}, 1\right].$$

It is not difficult to calculate that the expected meeting time resulting from these optimal rendezvous strategies is as claimed. ■

Proof of Theorem 2. If h is slowly varying, then Corollary 5 says that condition (5) holds, and hence the analysis of section 5 is applicable. For any alternation rule α we have equation (9), and the alternating search problem has solution $z_1(t) = \alpha(t), z_2(t) = 2t - \alpha(t)$. If we reinterpret this solution in terms of the asymmetric rendezvous problem using (2), we get $r_1(t) = 2t/2 = t$ and $r_2(1) = (\alpha(1) - (2t - \alpha(1)))/2 = (2\alpha(1) - 2)/2 = 0$, since $\alpha(1) = 1$. ■

Proof of Theorem 3. Suppose that $r_1' = 1, r_2' = 0$ ('go-stay') is the optimal strategy for some distribution H. Then in the alternating search problem on the circle, the solution is $z_1' = z_2' = 1$, which corresponds to $\alpha' = 1$ on all of $[0,1]$. By Theorem 6 (part 1) this implies that H never lies below its concavification $\bar{H}$, because in that case we would have $\alpha' = 0$ on some nontrivial interval. Since by definition H is never above its concavification, it follows that H must be concave. However, since $H(1-y) = 1 - H(y)$, the concavity of H on $[1/2, 1]$ implies the convexity of H on $[0, 1/2]$. So we must have that H is concave and convex on $[0, 1/2]$, which together with the fact that $H(1/2) = 1/2$ implies that $H(y) = y$ on $[0, 1/2]$. For $y \geq 1/2$ we then have $H(y) = 1 - H(1-y) = 1 - (1-y) = y$, since $1 - y \leq 1/2$. We have shown that if 'go-stay' is optimal, then the initial distribution must be the uniform distribution given by $H(y) = y$, $0 \leq y \leq 1$. ■

7 Examples

In this section we give two examples of monotone densities h on the initial undirected distance between the players, where we can determine unique and explicit optimal rendezvous strategies. In each case (one example each for increasing and decreasing densities) we determine the cutoff value c and the concavification $\bar{H}$. In both cases the time interval $[0,1]$ consists of two subintervals where we can apply the parts of Theorem 6. On one of these $\bar{H}$ lies strictly above H, so the players alternate moving at full speed on this time interval. On the other subinterval H is concave and equals $\bar{H}$, so that the players move in parallel at speed one on this interval.

Consider first the density $h(x) = 2 + 2\sin(2\pi[x - 1/4])$ which is increasing on the interval $[0, 1/2]$. The corresponding symmetric density on $[0, 1]$ is given by $H(y) = y - \cos(2\pi[y - 1/4])/2\pi$. We determine the cut-off time $c \approx .715$ by maximizing $H(y)/y$ on $[.5, 1]$. For $y \leq c$ we have $\bar{H}(y) = H(c)\,y/c \approx 1.217\,2y$ which is above $H(y)$, and for $y \geq c$ we have $\bar{H}(y) = H(y)$. This is shown in Figure 1 below.

Figure 1: $\bar{H}(y), H(y), y$, for $h(x) = 2 + 2\sin(2\pi[x - 1/4])$

Suppose the initial undirected distance density function $h(x) = 2 + 2\cos(2\pi x)/n$, $0 \leq x \leq 1/2$, which is strictly decreasing. Then the symmetric distribution H on $[0, 1]$ which has density $f(y) = h(y)/2$ is given by $H(y) = y + \frac{\sin 2\pi y}{2n\pi}$. To make h slowly varying we need $n > 2$, but the picture is easier to see for $n = 1$. To determine c (for $n = 1$) we minimize the quotient $(1 - H(y))/(1 - y) = \left(1 - y - \frac{\sin 2\pi y}{2n\pi}\right)/1 - y$, giving $c \approx .285$. Hence for $y \geq c$,

$$\bar{H}(y) = H(c) + \frac{(y - c)(1 - H(c))}{1 - c} \approx .217\,23 + .782\,77y,$$

which lies above the curve $H(y)$. For $y \leq c$, $\bar{H}(y) = H(y)$. These curves, together with the identity function y, are shown in Figure 2 below.

Figure 2: $H(y), y, \bar{H}(y)$ for $h(x) = 2 + 2\cos(2\pi x)$

8 Conclusions

In this paper we have considered the asymmetric rendezvous problem on the circle for distributions of initial distance where the optimal solution has one player going all around the circle. If the initial distance is given by a monotone density function h (increasing or decreasing with $h(1/2) > h(0)/2$), we have determined a simple optimal solution which is also uniquely optimal when the monotonicity is strict. We have also shown that Howard's counterexample (uniform distribution) to Gal's conjecture on the sub-optimality of the go-stay strategy is the only counterexample for the circle, establishing the Gal conjecture is generically true even on the circle. It remains to be seen whether this interesting conjecture is also generically true on other networks.

We have not studied cases where the initial distance between the players is small with respect to the circumference of the circle, where the players could probably ignore the circle topology and pretend they are playing on a line. These 'pre-Columbian' strategies should resemble those for the Linear Search Problem (see [8]) when the initial distance is bounded above by a small number. Intermediate cases, between those studied in this paper and those for which the Linear Search Problem is a good model, will probably pose difficulties that cannot be handled by our methods.

We conclude by noting that the symmetric version of the rendezvous search problem is still unsolved (as far as the author is aware), even for the uniform distribution or the case where the players know their initial distance is half the circumference.

References

[1] Alpern,S. (1995). The rendezvous search problem, *SIAM Journal of Control and Optimization,* **33** No. 3, 673-683.

[2] Alpern, S., A. Beck, (1997). Rendezvous search on the line with bounded resources: Expected time minimization. *European Journal of Operational Research*,Vol. 101, No. 3.

[3] Alpern, S., J.V. Howard (1998). Alternating search at two locations, preprint.

[4] Anderson, E. J., S. Essegaier (1995). Rendezvous search on the line with indistinguishable players, *SIAM Journal of Control and Optimization*, **33**, 1637-1642.

[5] Anderson, E. J., S. Fekete (1998). Two-dimensional rendezvous search. *Proceedings, 14'th ACM Symposium on Computational Geometry*, 365-373. *Operations Research*, to appear.

[6] Baston, V. J. (1997). Two rendezvous search problems on the line, *Naval Research Logistics* **46**, 335-340.

[7] Baston, V. J., S. Gal (1998). Rendezvous on the line when the players' initial distance is given by an unknown probability distribution. *SIAM Journal of Control and Optimization* 36, no. 6, 1880-1889.

[8] Beck, A (1964). On the linear search problem. *Israel Journal of Mathematics* **2**, 221-228.

[9] Gal, S. (1980). *Search Games*. Academic Press, New York.

[10] Gal, S. (1998). Rendezvous search on the line. *Operations Research,* to appear.

[11] Howard, J. V. (1998). Rendezvous search on the interval and circle. *Operations Research,* **47**, No. 4, 550-558.

[12] Lim,W. S., S. Alpern, A. Beck (1997). Rendezvous search on the line with more than two players. *Operations Research* 45 , No. 3, 357-364.

[13] Ruckle, W. H., (1983). *Geometric Games and Their Applications*. Pitman, Boston.

[14] Thomas, L. C., and P. B. Hulme (1997). Searching for targets who want to be found. *Journal of the O. R. Society* 48, Issue 1, Pages 44 - 50