The Telephone Coordination Game

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July 27, 1998

1. Introduction

In this paper we extend the work of several authors on the 'Telephone Coordination Game' originally proposed as follows by the first author: In each of two rooms, there are n telephones randomly strewn about. The phones are connected pairwise in some unknown fashion. There is a player in each room. In each period $0, 1, 2, \ldots$, each player picks up a phone and says 'hello', until the first period when they hear each other (called a meeting). The common aim of the two players is to minimize the (expected) number of periods T required to meet. The players do not have any common labeling of the telephones on which they can coordinate. Of course a good simple solution to this problem would be for one player to pick up the same phone each period, and for the other to pick different ones each period (a random permutation). However this solution requires the players to determine in advance which role each will take, and we rule out such solutions by requiring a symmetric solution in which both players employ the same mixed strategy. For example, this could be the strategy written down in a common guidebook, copies of which are placed in both rooms. What should this guidebook recommend? This problem is a paradigm of coordination without communication. The problem has led to a number of papers [4][1][5][2][3](described below), some based on modifications of the original version, but (for $n > 2$) the original Telephone Problem remains open.

The first real progress on this problem was made by Anderson and Weber [4], who simply consider that there are n 'locations' and the initial one is chosen randomly. We may consider that the players are moving on the nodes of a complete graph K_n until they arrive at a common node. They demonstrated the efficacy of strategies (called here AW strategies) in which the players wait where they are for $n-1$ periods with some common probability (separately realized by independent randomization), and with the complementary probability they choose a random ordering of the remaining nodes. If, after $n-1$ of these periods they have not found each other, they repeat the process by again choosing (with the same probability) between waiting or searching during the next group of $n-1$ periods, and so on. They calculated the waiting probability θ_n which minimizes the expected meeting time, with exact calculations for small n and numerical calculations for large n. Then they calculated the expected meeting time for the strategy (which we call the optimal AW strategy, and denote by w_n) which waits for the optimal time θ_n. They showed that the expected meeting time for players using their strategy w_n is asymptotically $(0.8288497)\,n$, and no one has found strategies giving a lower time than this.

Only the cases $n = 2, 3, 4$ have been subjected to rigorous individual analysis (4 for the first time here), so we review these cases briefly here. For the case $n = 2$ the optimal AW strategy w_2 reduces to the random walk strategy, as $\theta_2 = 1/2$, and Anderson and Weber demonstrated [4, Theorem 1] that this is optimal among all strategies. For $n = 3$ the optimal waiting time is given by $\theta_3 = 1/3$. They claimed that w_3 is also optimal among all strategies (*i.e.*, not just AW strategies), and while the demonstration they gave is faulty, subsequent work (including that given here) tends to support this claim. For $n = 4$ (which is the largest we shall consider) the optimal waiting probability in an AW strategy is given by $\theta_4 \simeq 0.32198$. The main results of this paper show that the optimal AW strategies for $n = 3$ and 4 have additional optimality properties (stated precisely in the next section) with respect to a wider class of Markov strategies. We do not, however, prove that w_3 or w_4 are optimal among all strategies. So even for these small values of n the Telephone problem remains open.

2. Classes of Strategies

In this section we formalize the original Telephone Coordination Game, which we denote by Γ_n, and consider other versions where the players' strategies are more restricted. We will consider that the games all begin with an initial random placement of the players at distinct nodes (locations). This is equivalent to choosing initial locations randomly (and independently), and beginning again if the two locations are the same. This is the version of the game considered in [3], whereas in [4] the possibility exists that the players are initially placed at the same location and the game ends immediately. The two problems are of course equivalent.

In this paper (following [3]) we shall consider $m-$Markov strategies on n locations, collectively denoted by $S_{n,m}$. These are, roughly speaking, strategies which repeat after each group of m periods, independently of previous choices. We first define a *primary strategy* in $S_{n,m}$ which is like a pure strategy except that, to reduce the dimensions involved, it uses a bit of randomization. To describe a primary strategy in say $S_{3,4}$ we proceed as follows. At the beginning of each grouping of $m = 4$ periods (at the beginning of periods $4k$, $k = 1, 2, \ldots$) a player denotes his present location as A. All he can do in the next period is to stay where he is (A), or move to another location. Assuming he does not use information about what happened in previous groupings of four consecutive periods, all other locations are equivalent, call such a move B. If he moves at the first step (B) he can now go back to his original location (BA), stay where he is (BB), or go to a new location (BC). Thus a primary strategy in $S_{3,4}$ is any sequence of length $m = 4$, consisting of the first $n = 3$ letters (A,B,C, though not all need occur), in which the first occurrence of C follows the first occurrence of B. In general, the first occurrence of a later (alphabetically) letter must follow the first occurrence of an earlier one - except for A which may be thought to be un-named in position zero. We give below a lexicographic listing of primary strategies in $S_{3,2}$ and $S_{3,3}$.

AA,AB,BA,BB,BC

AAA,AAB,ABA,ABB,ABC,BAA,BAB,BAC,BBA,BBB,BBC,BCA,BCB,BCC

A general strategy in $S_{n,m}$ is a probability distribution over the primary strategies in $S_{n,m}$. For example, the optimal AW strategy for three locations can be written as an element in $S_{3,2}$ as $w_3 = (1/3, 0, 0, 0, 2/3)$.

We define the game $\Gamma_{n,m}$ to be the original game Γ_n in which the players are constrained to use the same strategy in $S_{n,m}$. The value $v_{n,m}$ is defined by

$$v_{n,m} = \min_{s \in S_{n,m}} T(s, s),$$

where $T(s, s)$ denotes the expected meeting time when both players use the strategy s. Any strategy s for which the minimum is attained will be called an *optimal strategy* for the game (problem) $\Gamma_{n,m}$. Formulae for T will be given as required in the later sections. In this notation, the result obtained in [3, Theorem 4] can be stated as

Theorem 1. *The Anderson-Weber (AW) strategy* $w_3 = (1/3, 0, 0, 0, 2/3)$ *is optimal for* $\Gamma_{3,2}$.

In this paper we will not obtain any new results on optimality of strategies in the games $\Gamma_{n,m}$. However we will obtain results with respect to the weaker notion of Nash Equilibrium. The relation between these concepts, which essentially says that the optimal strategy, as a pair, is the best symmetric Nash Equilibrium, is given in the following easy lemma.

Lemma 2. *If s is an optimal strategy in $S_{n,m}$, then for any $t \in S_{n,m}$ we have*

$$T(s, s) \leq T(s, t).$$

That is, (s, s) is a symmetric Nash Equilibrium in the normal form game $T : S_{n,m} \times S_{n,m}$.

Proof. Suppose that $T(t,t) < \infty$ and $T(s,t) < T(s,s)$. For $x \in [0,1]$, define

$$\begin{aligned} f(x) &= T((1-x)s + xt, (1-x)s + xt) \\ &= (1-x)^2 T(s,s) + 2x(1-x)T(s,t) + x^2 T(t,t), \text{ so that} \\ f'(0+) &= 2(T(s,t) - T(s,s)) < 0. \end{aligned}$$

Since $f(0) = T(s,s)$, it follows that for sufficiently small positive values of x, the strategy $(1-x)s + xt$ contradicts the assumption that s is optimal.

Next suppose that for some $t \in S_{n,m}$ we have $T(t,t) = \infty$ and $T(s,t) < T(s,s)$. For example t might be the waiting strategy (AAA...) which never results in a meeting when both players use it. However for sufficiently small positive values of y, we have that $T(s,(1-y)t + yr) < T(s,s)$ by continuity of T and the fact that $T(s,r) < \infty$ for any s, where r is the random strategy (choose a random location each move). Furthermore $T((1-y)t + yr, (1-y)t + yr) < \infty$ since in each group of m periods the players will meet with probability at least $y^2\left(1 - \left(\frac{n-1}{n}\right)^m\right)$. Hence the result in the first paragraph applies to the strategy $(1-y)t + yr$. ■

This lemma leads to a partial test for optimality within $S_{n,m}$: we see whether any of the primary strategies in $S_{n,m}$ do better against a proposed solution s than s does against itself. If there is a better response than s itself, then s cannot be a Nash Equilibrium and hence cannot be optimal. On the other hand, if *none of the primary strategies do better against s than s does against itself*, the directional derivative of $T(s, xs + (1-x)t)$ at $x = 0$ is nonnegative for any $t \in S_{n,m}$, since any such t is a convex average of primary strategies. We shall say that a strategy $s \in S_{n,m}$ satisfying the above italicized condition is *locally Nash*. In the following two sections we shall show respectively that w_4 is locally Nash in $\Gamma_{4,3}$ and that w_3 is locally Nash in $\Gamma_{3,4}$.

3. The Game $\Gamma_{4,3}$.

In this section we consider the Telephone Game when $n = 4$. Since the Anderson-Weber strategy w_4 is a 3-step Markov strategy, we consider its optimality properties with respect to the game $\Gamma_{4,3}$, that is, with respect to the strategy space $S_{4,3}$. Actually Anderson and Weber do not make any optimality claims at all for w_4 - in fact they conjecture [4, Conjecture 4, p. 851] that it is not optimal. Hoping to find a better strategy for $\Gamma_{4,3}$ by making a small change to w_4, we applied the test outlined in the last paragraph of the previous section. We found to our (mild) surprise that it passed this test, and so we establish that:

Theorem 3. *The Anderson-Weber strategy w_4 is locally Nash for the game $\Gamma_{4,3}$.*

We now outline the theory, programming, and calculations that we did to justify the above assertion. First the basic theory needed to calculate the expected time function T. Let p, q, and r, denote the probabilities that the two players (following simultaneously some strategy in $S_{4,3}$) will meet respectively for the first time at the end of periods 1, 2, or 3. Since they are using some 3-Markov strategy, the expected meeting time T is given by the following geometric series, where $a = 1 - p - q - r$.

$$\begin{aligned} T &= (1p + 2q + 3r) + (4ap + 5aq + 6ar) + \left(7a^2p + 8a^2q + 9a^2r\right) \dots \\ &= (p + 2q + 3r)\left(1 + a + a^2 + \dots\right) + (p + q + r)(3a)\left(1 + 2a + 3a^2 + \dots\right) \\ &= \frac{p + 2q + 3r}{1 - a} + \frac{(p + q + r)(3a)}{(1 - a)^2} = \frac{-2p - q + 3}{p + q + r}. \end{aligned}$$

In order to evaluate the probabilities p, q, r, we must evaluate them for all pairs of primary strategies in $S_{4,3}$. There are 15 of these, which we number in the following (lexicographic) order:

AAA, AAB, ABA, ABB, ABC, BAA, BAB, BAC, BBA, BBB, BBC, BCA, BCB, BCC, BCD

A program was written by the second author which computes, for any pair of primary strategies in $S_{n,m}$, the probability that their first meeting is at the end of period i, $i = 1, \dots, m$. To illustrate the nature of the calculation, consider that player I employs ABC (strategy number 5) and II employs BAA (strategy 6). They will meet at the end of period 1 if and only if II's random move to a new location is to I's starting location. Since there are three possible new locations, this probability is $p = 1/3$. With probability $2/3$ this doesn't occur, and at the beginning of period 2, player II is randomly placed at one of the three locations player I did not start at. In the next two moves, player I (following BC) will go to two of these locations. It follows that $q = r = (2/3)(1/3) = 2/9$. The numbers $1/3$, $2/9$, $2/9$ are indicated in position 5,6 of the15×15 matrices P, Q, R given below.

$$9P = \begin{pmatrix} 0&0&0&0&0&3&3&3&3&3&3&3&3&3&3 \\ 0&0&0&0&0&3&3&3&3&3&3&3&3&3&3 \\ 0&0&0&0&0&3&3&3&3&3&3&3&3&3&3 \\ 0&0&0&0&0&3&3&3&3&3&3&3&3&3&3 \\ 0&0&0&0&0&3&3&3&3&3&3&3&3&3&3 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&2&2&2&2&2&2&2&2&2&2 \end{pmatrix}.$$

$$18Q = \begin{pmatrix}
0 & 0 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\
0 & 0 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\
6 & 6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\
6 & 6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\
6 & 6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\
0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 6 & 6 & 6 & 4 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 6 & 6 & 6 & 4 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 6 & 6 & 6 & 4 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 6 & 6 & 6 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 6 & 6 & 6 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 6 & 6 & 6 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\
6 & 6 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \\
6 & 6 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \\
6 & 6 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \\
6 & 6 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3
\end{pmatrix}.$$

$$18R = \begin{pmatrix}
0 & 6 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & 0 & 6 \\
6 & 4 & 4 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 2 & 2 & 2 & 0 \\
0 & 4 & 0 & 6 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 6 & 4 \\
0 & 4 & 6 & 0 & 4 & 0 & 4 & 2 & 4 & 0 & 2 & 4 & 4 & 0 & 2 \\
6 & 2 & 4 & 4 & 3 & 4 & 2 & 1 & 2 & 2 & 1 & 3 & 3 & 2 & 2 \\
0 & 4 & 0 & 0 & 4 & 0 & 6 & 4 & 0 & 0 & 4 & 0 & 6 & 0 & 4 \\
0 & 4 & 0 & 4 & 2 & 6 & 0 & 4 & 4 & 0 & 2 & 4 & 0 & 4 & 2 \\
6 & 2 & 4 & 2 & 1 & 4 & 4 & 3 & 2 & 4 & 1 & 3 & 2 & 3 & 2 \\
0 & 4 & 0 & 4 & 2 & 0 & 4 & 2 & 0 & 6 & 4 & 0 & 4 & 4 & 2 \\
0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 6 & 0 & 4 & 6 & 0 & 0 & 4 \\
6 & 2 & 4 & 2 & 1 & 4 & 2 & 1 & 4 & 4 & 3 & 2 & 3 & 3 & 2 \\
0 & 2 & 0 & 4 & 3 & 0 & 4 & 3 & 0 & 6 & 2 & 0 & 4 & 4 & 3 \\
0 & 2 & 0 & 4 & 3 & 6 & 0 & 2 & 4 & 0 & 3 & 4 & 0 & 4 & 3 \\
0 & 2 & 6 & 0 & 2 & 0 & 4 & 3 & 4 & 0 & 3 & 4 & 4 & 0 & 3 \\
6 & 0 & 4 & 2 & 2 & 4 & 2 & 2 & 2 & 4 & 2 & 3 & 3 & 3 & 2
\end{pmatrix}.$$

In fact we shall only need the first and last rows of these matrices (corresponding to Player I following the strategy w_4) but these may be useful for investigators with more powerful optimization programs who wish to calculate optimal strategies in $S_{4,3}$. We now calculate how each of the 15 primary strategies does against $w_4 = \theta_4 [AAA] + (1 - \theta_4) [BCD]$ with respect to p, q, r, and T. We also calculate how well w_4 does against itself.

	w_4	AAA	AAB	ABA	ABB	ABC	BAA	BAB
p	.2477	.2260	.2260	.2260	.2260	.2260	.2580	.2580
q	.2222	.2260	.2260	.1827	.1827	.1827	.1507	.1507
r	.1967	.2260	.1073	.1507	.0753	.1827	.1507	.0753
T	3.4247	3.4247	4.151	4.229	4.887	4.000	4.1716	4.821

	BAC	BBA	BBB	BBC	BCA	BCB	BCC	BCD
p	.2580	.2580	.2580	.2580	.2580	.2580	.2580	.2580
q	.1507	.1507	.1507	.1507	.2203	.2203	.2203	.2203
r	.1827	.0753	.1507	.1827	.1130	.1130	.1130	.1827
T	3.946	4.821	4.1716	3.946	3.828	3.828	3.828	3.4247

We see that all except the two primary strategies on which w_4 is supported have a larger expected time against w_4 than w_4 has against itself. This completes the demonstration that w_4 is locally Nash.

Another elementary observation obtained from this table is that w_4 does not maximize p, all strategies beginning with B do better in this respect against w_4. This is in contrast to w_3, which simultaneously maximizes the probability of meeting within one step (p) and two steps $(p+q)$ - see [3].

Further numerical work idicates that not only is $s = w_4$ a local minimum of $T(s,s)$ for $s \in S_{4,3}$ (with the least value, 3.4247, that we have found numerically), but also the strategy which plays ABA with probability .35942, BAC with probability .32757, and BCC with probability .31301, although the latter gives a higher value (expected time).

4. FURTHER OPTIMALITY OF w_3

The strategy given by Anderson and Weber for three locations, w_3, says to stay put for two steps with probability $1/3$, and to search the remaining two locations with probability $2/3$; then repeat as long as necessary. Following a false demonstration of optimality in Γ_3 given in [4], optimality within $\Gamma_{3,2}$ was demonstrated in [3]. However the possibility remains that if longer groups of periods are allowed (larger m) then a better strategy might be possible. One path of investigation might be to evaluate $v_{3,3}$; if it is lower than $v_{3,2} = T(w_3, w_3)$ then of course we would know that w_3 is not optimal in Γ_3.

In this section we pursue a different line of inquiry. Observe that there is a natural embedding of $S_{n,m}$ in $S_{n,km}$: for any integer k - simply repeat a strategy $s \in S_{n,m}$ for

k times. With this in mind, we see that $w_3 \in S_{3,4}$, which implies that $v_{3,4} \leq v_{3,2}$. So a natural question to ask is whether w_3 is optimal for the game $\Gamma_{3,4}$, that is, within $S_{3,4}$. At the moment, this is beyond our reach, but we can establish the following.

Theorem 4. *The Anderson-Weber strategy w_3 for three locations is locally Nash in the game $\Gamma_{3,4}$.*

To prove this result we must analyze the 41 primary strategies in $S_{3,4}$ which we list in such a way that w_3 gives positive weight to the first five:

AAAA, AABC, BCCC, BCAB, BCBA, AAAB, AABB, ABAA, ABAB, ABAC, ABBB, ABBC, ABCA, ABCB, ABCC, BAAA, BAAB, BAAC, BABA, BABB, BABC, BACA, BACB, BACC, BBAA, BBAB, BBAC, BBBA, BBBB, BBBC, BBCA, BBCB, BBCC, BCAA, BCAC, BCBB, BCBC, BCCA, BCCB.

Now we consider how a repeated w_3 can be represented in terms of these 4-Markov strategies. To distinguish the two-step from the four-step versions, represent the former by lower case letters. Hence aa+aa has probability $1/9$, aa+bc and bc+aa have probability $2/9$, and bc+bc has probability $4/9$. The plus sign separates the first choice of primary strategy for moves 1 and 2 from the second choice, for moves 3 and 4.

Clearly aa+aa stays put for 4 moves, and corresponds to AAAA. Similarly aa+bc corresponds to staying put and then searching, or AABC. The pair bc+aa corresponding to a random search on moves 1 and 2 followed by staying at the location picked for move 2. In capital letters, this is BCCC. Finally, we consider bc+bc. Clearly this starts with BC.. . After the first two moves, all three nodes are labelled, and the current node has label C. A random search of the remaining two continues either with AB or with BA, that is BCAB or BCBA. Hence the strategy w_3 can be represented in $S_{3,4}$ as $(1/9, 2/9, 2/9, 2/9, 2/9, 0, 0, \ldots)$.

In a similar manner to the previous section, we can compute the formula for the expected meeting time T in terms of the probabilities p, q, r, s that the players meet in time 1,2,3, and 4, as follows:

$$T = \frac{-3p - 2q - r + 4}{p + q + r + s}.$$

We now compare w_3 and each of the 41 primary strategies in $S_{3,4}$ in terms of how they do against w_3.

	w_3	AAAA	AABC	BCCC	BCAB	BCBA	AAAB	AABA
$18p$	6	6	6	6	6	6	6	6
$18\,(p+q)$	12	12	12	12	12	12	12	12
$18\,(p+q+r)$	14	14	14	14	14	14	14	14
$18\,(p+q+r+s)$	16	16	16	16	16	16	15	15
T	2+1/2	2+1/2	2+1/2	2+1/2	2+1/2	2+1/2	2+2/3	2+2/3
T (numeric)	2.5	2.5	2.5	2.5	2.5	2.5	2.67	2.67

	AABB	ABAA	ABAB	ABAC	ABBA	ABBB	ABBC
$18p$	6	6	6	6	6	6	6
$18\,(p+q)$	12	9	9	9	9	9	9
$18\,(p+q+r)$	14	12	12	12	12	12	12
$18\,(p+q+r+s)$	15	13	14	15	14	15	13
T	2+2/3	3+6/13	3+3/14	3	3+3/14	3	3+6/13
T (numeric)	2.67	3.46	3.21	3	3.21	3	3.46

	ABCA	ABCB	ABCC	BAAA	BAAB	BAAC	BABA
$18p$	6	6	6	6	6	6	6
$18\,(p+q)$	9	9	9	9	9	9	9
$18\,(p+q+r)$	12	12	12	12	12	12	12
$18\,(p+q+r+s)$	15	13	14	15	13	14	13
T	3	3+6/13	3+3/14	3	3+6/13	3+3/14	3+6/13
T (numeric)	3	3.46	3.21	3	3.46	3.21	3.46

	BABB	BABC	BACA	BACB	BACC	BBAA	BBAB
$18p$	6	6	6	6	6	6	6
$18\,(p+q)$	9	9	9	9	9	9	9
$18\,(p+q+r)$	12	12	12	12	12	12	12
$18\,(p+q+r+s)$	14	15	14	15	13	14	13
T	3+3/14	3	3+3/14	3	3+6/13	3+3/14	3+6/13
T (numeric)	3.21	3	3.21	3	3.46	3.21	3.46

	BBAC	BBBA	BBBB	BBBC	BBCA	BBCB	BBCC
$18p$	6	6	6	6	6	6	6
$18\,(p+q)$	9	9	9	9	9	9	9
$18\,(p+q+r)$	12	12	12	12	12	12	12
$18\,(p+q+r+s)$	15	13	15	14	15	14	13
T	3	3+6/13	3	3+3/14	3	3+3/14	3+6/13
T (numeric)	3	3.46	3	3.21	3	3.21	3.46

	BCAA	BCAC	BCBB	BCBC	BCCA	BCCB
$18p$	6	6	6	6	6	6
$18\,(p+q)$	12	12	12	12	12	12
$18\,(p+q+r)$	14	14	14	14	14	14
$18\,(p+q+r+s)$	15	15	15	15	15	15
T	2+2/3	2+2/3	2+2/3	2+2/3	2+2/3	2+2/3
T (numeric)	2.67	2.67	2.67	2.67	2.67	2.67

A quick glance at the final rows shows that only the first five primary strategies, those on which w_3 is supported, give the same value of 2.5 as the first (w_3) column. This verifies the claim that w_3 is locally Nash in $\Gamma_{3,4}$. Furthermore, the column vector $(6, 12, 14, 16)'$ dominates all the others, so that even in the game $\Gamma_{3,4}\,(j)$ in which the players win if and only if they meet by time $j = 1, 2, 3, 4$, the strategy w_3 is Nash. This is related to conjecture (2) of [4], that w_3 maximizes the probability of meeting by time j for all j. It is claimed there that the conjecture is known to be true for $j \leq 4$.

5. Overlook Probabilities

The very last conjecture in [4] is the following:

> In the case that players who search the same location only meet with a probability α, we have shown that the optimal strategy for $n = 2$ is unchanged, regardless of the value of α. We conjecture that even for larger n the optimal strategy is the same for all α.

It is difficult to attack this question directly, as we don't know what the optimal strategy *is* for $n \geq 2$. So we consider the easier question of whether the optimal AW strategy $w_n\,(\alpha)$ depends on α. It turns out that the answer depends on n.

We first consider the case $n = 3$. If the players are both using w_3 then they cannot meet both at time 1 and time 2. This is clearly impossible (for *any* n) if they both stay put (there will be no meetings) or one stays put and one searches (there will be one meeting when $n = 3$, at most one when $n > 3$). Suppose $n = 3$ and they both search (that is, use BC) for the first two periods. Assume player I starts at location 1 and II starts at location 2 (numbered by an observer). The meeting times are as follows:

$$\begin{array}{ccc} & 1,2 & 2,1 \\ 2,3 & \{\} & \{1\} \\ 3,2 & \{2\} & \{\} \end{array}$$

In no case is there more than one meeting time. It follows that the probabilities that the first meeting occurs at time 1,2 are given by $\alpha p\,(\theta)$, and $\alpha q\,(\theta)$, where θ is the waiting probability in an AW strategy and p and q are the probabilities when $\alpha = 1$. So the optimal value of θ (=1/3) is unaffected by α.

We now consider the case $n = 4$. The probabilities of meeting for the first time at the end of periods 1,2,3 can be calculated as functions of α and the probability θ that they stay put.

$$\begin{aligned} p &= \frac{2\alpha + 2\alpha\theta - 4\alpha\theta^2}{9} \\ q &= \frac{4\alpha - \alpha^2 + (4\alpha + 2\alpha^2)\,\theta + (-8\alpha - \alpha^2)\,\theta^2}{18} \\ r &= \frac{(2\alpha - \alpha^2) + (2\alpha + 2\alpha^2)\,\theta + (-4\alpha - \alpha^2)\,\theta^2}{9}. \end{aligned}$$

Since the expected meeting time is given by

$$\begin{aligned} T &= \frac{-2p - q + 3}{p + q + r}, \text{ we have} \\ T &= -\frac{1}{3}\frac{-12\alpha - 12\alpha\theta + 24\theta^2\alpha + \alpha^2 - 2\theta\alpha^2 + \theta^2\alpha^2 + 54}{\alpha\left(-4 - 4\theta + 8\theta^2 + \alpha - 2\alpha\theta + \theta^2\alpha\right)}, \text{with min at} \\ \theta_4(\alpha) &= \frac{-72 - 9\alpha + 4\alpha^2 + 3\sqrt{(576 + 144\alpha - 39\alpha^2)}}{4\alpha^2}. \end{aligned}$$

The formula for the optimal waiting (stay put) probability $\theta_4(\alpha)$ gives a strictly increasing function in the range $0 \le \alpha \le 1$. In any case, the curve is certainly not horizontal, and the optimal waiting probability does depend on α.

6. Conclusions

The results obtained in this paper are very modest, in that only weak notions of optimality of strategies are considered. However they do tend to support the usefulness of the Anderson-Weber strategies in the Telephone Coordination Game. Much further work will be needed to completely solve this surprisingly difficult coordination problem.

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