Abstract

Following Schelling (1960), coordination problems have mainly been considered in a context where agents can achieve a common goal (*e.g.* rendezvous) only by taking *common* actions. . Dynamic versions of this problem have been studied by Crawford and Haller (1990), Ponssard (1994), and Kramarz (1996). This paper considers an alternative dynamic formulation in which the common goal (dispersion) can only be achieved by agents taking *distinct* actions. The goal of spatial dispersion has been studied in static models of habitat selection, location or congestion games, and network analysis. Our results show how this goal can be achieved gradually, by indistinuishable non-communicating agents, in a dynamic setting.

Spatial Dispersion as a Dynamic Coordination Problem

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1 Introduction

Thomas Schelling's classic book [16] gave a cogent if not formal presentation of the problem faced by two non-communicating players who wish to meet at a common location. He emphasized the importance of a common culture in producing 'focal points' which enable the players to distinguish among several possible meeting points, that is, among equilibria of the associated common interest game. Subsequently two dynamic models of Schelling's static problem were developed in which the *a priori* focal points, or notions of a common culture, were replaced by coordination principles which developed over time. The first of these, informally proposed in [1], analyzed for discrete settings in [3], and explicitly modelled in [2], is called the 'rendezvous search problem' (see also [8]). It places Schelling's problem in the context of geometric search theory and more specifically that of search games [7]. As appropriate in a search theory model, the players in this formulation obtain no information regarding the whereabouts of the other until they meet. The possibility of culturally biased focal points is removed in this model through formal symmetry assumptions regarding the locations. The second family of models, called 'coordination games', was introduced by Crawford and Haller [4] and extended to the multiple player setting by Kramarz [14]. These models are put into a unified context by Ponssard in [15]. In these games, information about players' choices is revealed after each period. In all these problems (or games), the aim of the coordination is to produce a *common* choice or location for all the players.

This article introduces the opposite coordination aim, that of attaining *distinct* locations (spatial dispersion), while maintaining similar dynamics and information to the problems discussed above. This aim has often been considered in a static context, as in the following situations.

1. Location Games: Retailers(or candidates for election) simultaneously choose their positions within a common space so as to maximize the area for which they are the closest. For an excellent survey, see [6].

2. Habitat Selection: Males of a species choose territories where there are no other males; animals choose feeding patches with low population density with respect to food supply. Fretwell and Lucas [5] introduced the equilibrium notion of the *ideal free distribution.* (originally in terms of birds choice of nesting sites). A recent application of this concept is given in [11].

3. Congestion Games: Individuals seek faciities or locations of low population density, for example unemployed workers willing to migrate to find jobs. A recent article in this area is [**?**].

4. Network Problems: Travelers choose routes with low congestion levels. (See [11].)

If we seek a spatial dispersion setup which mirrors Schelling's choice of a common location in New York for two friends to meet, the following seems a good story. A group of friends living in New York awaits the arrival of an old friend who settled elsewhere but is visiting. The New Yorkers would like to disperse among the potential locations that the visitor might first approach. Similarly, the aims of Schelling's paratroopers , who in [16] wish to meet after landing, might be modified so that they wish instead to find a wounded colleague - for this purpose they would want to disperse rather than to meet.

We will establish that, just as in coordination games, the players may use spatial configurations that arise in the course of the game through chance and choice to coordinate their actions. Such configurations have been called 'dynamic focal points' by Kramarz [14], and we shall define them for our purposes. For example in the two person setup of [4], if the players find themselves at distinct locations out of a total of $n = 3$ locations, a state we denote by $[1, 1, 0]$, they can coordinate in the next period by going to the currently empty position and thus ensuring a meeting. Less obvious is that in the $n = 5$ location case (with memory) they can go independently from $[1, 1, 0, 0, 0]$ to the three empty locations. If they are unlucky, and choose distinct locations, then they can surely meet in the next period by choosing the unique location which has been unoccupied in both periods. (It turns out that this leads to a shorter expected meeting time than if they coordinate immediately on the two occupied locations, meeting with conditional probability $1/2$ in each period.)

The spatial dispersion problem $\Gamma(m, n)$ introduced here begins with an initial random placement of m indistinguishable agents onto n indistinguishable locations. For example if $m = 8$ and $n = 4$,then the locations might have the following populations (written in decreasing order): $[4, 2, 2, 0]$. This distribution (and all subsequent one) becomes common knowledge for the next period. The agents move according to a common rule (depending only on the current distribution) until they reach the equidistribution $[2, 2, 2, 2]$. We consider which common strategy (mixed) takes the least expected time to reach the equidistribution. In fact, to simplify the analysis, we always take $m = n$ so that in the resulting problem $\Gamma(n)$ the aim of the players is to attain the distribution $[1, 1, \ldots, 1]$ with a single agent at each location. In other words, we seek common rules for non-communicating memoryless agents to achieve distinct locations in least expected time. An important example of a dynamic focal point for this dispersion problem occurs when $n = 5$ and the initial (or later) distribution of populations is given by $[3, 2, 0, 0, 0]$. In this situation, agents at the population 3 location should go to the empty locations (they cannot distinguish between these) while those at the population 2 location should go equiprobably to the locations with populations 3 and 2. (Obviously for such a strategy to be feasible, agents must be able to distinguish the population at their current location and at the other locations, and we assume this.) This local strategy for the configuration $[3, 2, 0, 0, 0]$ does not ensure that they achieve distinct locations in the next period, but it does help to reduce the expected time to achieve distinct locations, and will satisfy our formal definition (given in Section 3) of a dynamic focal point. In fact, this is the simplest such dynamic focal point, as no other ones exist for $n \leq 5$.

2 The Optimal Dispersion Problem $\Gamma(n)$

In this section we formally define the dynamic coordination problem $\Gamma(n)$, in which n agents are randomly (equiprobably and independently) placed at time zero onto n unordered locations, and seek to minimize the time T (called the *dispersal time*) taken to achieve distinct locations. At the end of each period each agent chooses which location to move to for the next period based solely on the population at his current location and the populations at the other locations. The agents have no common *a priori* labelling of the locations, though they can distinguish locations by their current populations. A *societal state* (or simply *state)* of the game is simply a list of the numbers of population at different locations, given in decreasing order. For example, when $n = 4$, a state $[3, 1, 0, 0]$ describes the situation where one of the locations has three agents, another (called a *singleton location*) has one agent (called a *singleton agent*), and two of the locations are empty. The current societal state is always common knowledge. In general, the set $\mathcal{S}_n$ of all societal states for the problem $\Gamma(n)$ can be described as follows, where s_i is the number of agents at the i'th most populated location.

$$\mathcal{S}_n = \left\{ s = [s_1, s_2, \ldots, s_n] : s_1 \geq s_2 \geq \cdots \geq s_n \geq 0, \sum_{i=1}^{n} s_i = n \right\}.$$

The objective described above as 'achieving distinct locations' can now be rephrased as reaching the state $\bar{s} = [1, 1, \ldots, 1]$ consisting of entirely of singletons. An important subset of $\mathcal{S}_n$ for our purposes will be determined by those states which have exactly r non-singleton locations, which we denote by $\mathcal{S}_n^r$. Formally we have,

$$\mathcal{S}_n^r = \{ s = [s_1, s_2, \ldots, s_n] \in \mathcal{S}_n : \# \{ i : s_i \neq 1 \} = r \}. \tag{1}$$

As this is a spatial model, we assume that each agent can distinguish his current location. For example, the strategy of 'staying still' should always be allowed. So we need to define a *personalized state* as a societal state together with an indication of where a particular agent is located. To do this, we will modify the notation for a societal state by adding a * after the number indicating the population at an agent's location. So for example the personalized state $[3, 2^*, 2, 1, 0, 0, 0, 0]$ describes the situation of an agent who is at a location of population 2 and can see that there is a location with population 3, another location which also has population 2, one singleton location, and the rest empty.

We assume that agents base their decision on where to move only on their position within the current population distribution, that is, on their personalized state. Consequently we restrict our attention to Markov strategies. A (Markov) strategy Q tells each agent how to move probabilistically from each of his personalized states. It is symmetric with respect to agents, in that all agents must be given the same strategy. In this sense, we are considering the problem with indistinguishable agents. (See section 8.2 for a discussion of a similar problem with distinguishable agents who need not all adopt the same strategy.) For example, in the personal state $[3,3,2^*,2,1,0,0,0,0,0,0]$, a strategy will specify the probability q_3 of going (equiprobably) to one of the locations where there are three agents, a probability q_* of remaining at the current location, a probability q_2 of going to the *other* location with two agents, q_1 of going to a singleton location, and q_0 of going to an empty location. The agents cannot distinguish between locations (other than their own) with the same population.

It is worth observing that the random strategy of picking at each stage equiprobably among the n locations will produce a permutation (one agent at each location) with probability $n!/n^n$, and consequently the expected time required is $(n^n/n!)-1$ (since we start at time 0). We list these expected times in Table 1 so the reader will see the level of improvement given later by the basic simple strategy (see Table 2).

Table 1– Expected Dispersal Times for Random Strategy

n	2	3	4	5	6	7	8	9	10
$(n^n/n!)-1$	1	3.5	9.6	25.0	63.8	162.4	415.1	1066.6	2754.7

The random strategy is clearly inefficient since even when agents attain a singleton status, it may be lost in the next period. Since the aim of the problem is for each agent to become a singleton, we will henceforth assume that, for any individual, becoming a singleton is an absorbing state. We require that singleton agents should stay still and non-singleton agents should not move to singleton locations. In our formal notation this means that if a personalized state has a 1^* then $q_*=1$ and that no player should go to a singleton location, that is, $q_1=0$. Such strategies will be called *progressive*. We denote the set of all progressive Markov strategies for the problem $\Gamma(n)$ as $\mathcal{Q}_n$. A strategy $Q\in\mathcal{Q}_n$ determines a Markov chain on the set of states $\mathcal{S}_n$, with the *distinct location state* $\bar{s}=[1,\ldots,1]$ as the unique absorbing state. Since all agents use Q with independent randomization, the transition probabilities of the chain depend in a polynomial fashion on the probabilities q_i specified in Q as functions of the personalized states. Our restriction to progressive strategies ensures that in the resulting Markov chain on the states $\mathcal{S}_n$, the number of singleton locations is nondecreasing in time. Let $T(Q,s)$ denote the expected time for the Markov chain based on the strategy Q to reach the absorbing state $\bar{s}$. Similarly, we define $T(Q)$ to be the expected time for the Markov chain based on Q to reach $\bar{s}$ starting from the distribution on $\mathcal{S}_n$ corresponding to the initial random placement of the agents on the n locations. The notation $T(Q)$ is really shorthand for $T(Q,\ldots,Q)$, since our assumption of indistinguishable agents required that all agents follow the same strategy. Later, when we briefly look at game theoretic versions of the problem, we shall use the latter notation with arguments that are not necessarily all the same.

For any societal state $s\in\mathcal{S}_n$, let $v(s)=\min_{Q\in\mathcal{Q}_n}T(Q,s)$ denote the least expected time required to reach $\bar{s}$ from s. In particular, we have $v(\bar{s})=0$. Observe that since we are using progressive strategies (which preserve singleton locations), we can evaluate $v(s)$ by deleting singletons and thereby reducing the dimension of the problem. For example $v([2,1,0]) = v([2,0])$. More generally, if $u > 1$ is the smallest such population in a state, we have the 'singleton deletion property',

$$v([s_1, s_2, \ldots, u, 1, 1, \ldots, 1, 0, \ldots, 0]) = v([s_1, s_2, \ldots, u, 0, \ldots, 0]). \tag{2}$$

Let

$$v_n = \min_{Q \in \mathcal{Q}_n} T(Q) \tag{3}$$

denote the least time to achieve the distinct location state $\bar{s}$, starting from a random initial placement of the agents, where each state s occurs with probability $p(s)$ based on an independent and equiprobable placement of the players. For example when $n = 3$ the initial distribution p over $\mathcal{S}_3$ is given by

$$p([3,0,0]) = \frac{1}{9},\ p([2,1,0]) = \frac{2}{3},\ \text{and } p([1,1,1]) = \frac{2}{9}. \tag{4}$$

A strategy $Q \in \mathcal{Q}_n$ which achieves this minimum time v_n in (3) will be called *optimal*. So for any fixed n, we have

$$v_n \geq \sum_{s \in \mathcal{S}_n} p(s)\, v(s), \tag{5}$$

with equality if and only if there is a common strategy Q with $T(Q,s) = v(s)$ for all s.

To illustrate some of these ideas we first examine the trivial case $n = 2$. There are two societal states, the absorbing state $\bar{s} = [1,1]$ and $[2,0]$, and they have unique personalized states $[1^*, 1]$ and $[2^*, 0]$. In the initial random distribution each occurs with probability $p = 1/2$. Of the two personalized states, only the latter has any strategic latitude. Define a strategy Q_x by setting $q_* = x$ and $q_0 = 1 - x$ (that is, x is the probability of staying still and $1 - x$ is the probability of moving to the other location). If we let $T_x = T(Q_x, [2,0])$ denote the expected time to reach $\bar{s}$ from $[2,0]$ when using the strategy Q_x described above, we obtain

$$T_x = [2x(1-x)](1) + [1 - 2x(1-x)](1 + T_x), \tag{6}$$

as the Markov chain determined by Q_x on $\mathcal{S}_2$ has transition probability $[2x(1-x)]$ from state $[2,0]$ to state $[1,1]$ (corresponding to exactly one of the two agents moving) and complementary probability from state $[2,0]$ to itself (corresponding to both agents moving or both staying). Solving the recursive equation (6) for T_x gives $T_x = \frac{1}{2x(1-x)}$,which has a minimum of

$$v([2,0]) = 2 \tag{7}$$

at $x = 1/2$. Consequently $v_2 = [1/2]\,0 + [1/2]\,2 = 1$.

It is worth noting that for $n = 2$ the random strategy of moving or staying still with equal probability is also optimal for agents starting at distinct nodes who *want to be together* (the opposite aim), both in the rendezvous search context [3] (without revealed actions) and the coordination game [4] (with revealed actions). Even from this simple example, it is clear that the determination of v_n will involve the optimal control of a Markov chain and is related to Markov decision problems. The fact that the transition probabilities can not be chosen directly but only as polynomials in the decision variables (x in the above example) means that those advanced theories cannot be directly applied. In any case we will retain in this paper a direct probabilistic approach that avoids those theories and is consequently more transparent. Future work in this area will however have to be more technical.

3 Simple Strategies and Equilibrium Properties

The use of general Markovian strategies ($\mathcal{Q}_n$) requires the agents to accomplish accurate calculation of arbitrary probabilities. In some cases we may wish to consider simpler agents or alternatively want to see how much optimality is lost by using simpler strategies. To this end we define a *simple strategy* as one which assigns to each personalized state a subset of the populations (other than 1 of course) occurring in that state. The understanding is that the agent moves equiprobably to a location having one of those populations. For example, if we assign the population set $\{2,0\}$ to the personalized state $[3,2^*,2,1,0,0,0,0]$ then an agent at a location with population 2 will go equiprobably to one of the six locations with population 2 or 0. We will write this assignment as $[3,2^*,2,1,0,0,0,0] \to \{2,0\}$. Note that in a simple strategy an agent cannot distinguish (in his actions) between his current location and other locations with the same population. In keeping with our restriction to progressive strategies, we will only define simple strategies on non-singleton personalized states (no 1^*) and we will not allow a 1 to be included in the set of acceptable populations.

We are particularly interested in the efficacy of the *basic simple strategy,* the simple strategy in which non-singleton agents always choose equiprobably over the non-singleton locations. That is, the set of acceptable populations assigned to a personalized state includes all the populations other than 1 occurring in that state. We shall denote the basic simple strategy by $\tilde{Q}$. We have seen in the previous section that the basic simple strategy $\tilde{Q}$ is optimal when $n=2$ (which is the only n where it agrees with the random strategy). We shall discuss the *optimality* properties of the basic simple strategy for various n in subsequent sections. For the time being we give some easy results concerning the weaker *equilibrium* properties that it possesses.

We first observe that in cases like $n=2$, where $\tilde{Q}$ is optimal, we do not have to worry about individual agents trying to improve the group outcome (i.e., to lower the expected dispersal time) by unilaterally deviating from $\tilde{Q}$. This is an immediate consequence of the following easy general result relating optimal strategies (universally adopted) to symmetric Nash equilibria.

Lemma 1 *If $Q \in \mathcal{Q}_n$ is optimal for $\Gamma(n)$, then the symmetric strategy profile*

$(Q, Q, \ldots, Q)$ *is a Nash equilibrium for the game* $G(n)$ *with the same dynamics where each player's utility is the negative of the dispersal time* T.

Proof. This is a well known argument. Suppose on the contrary that $T(Q', Q, \ldots, Q) < T(Q, Q, \ldots, Q)$ for some Q'. Consider a new strategy $\hat{Q}$ for $\Gamma(n)$ which is a mixture of Q' and Q with respective probabilities ε and $1 - \varepsilon$, with ε close to zero. If all agents adopt this strategy then the expected dispersal time is given by

$$T\left(\hat{Q}, \ldots, \hat{Q}\right) = (1-\varepsilon)^n T(Q, Q, \ldots, Q) + n(1-\varepsilon)^{n-1} \varepsilon T(Q', Q, \ldots, Q) + \ldots,$$

where we have left out terms with higher powers of ε. For $\varepsilon = 0$ clearly $T\left(\hat{Q}, \ldots, \hat{Q}\right)$ equals $T(Q, Q, \ldots, Q)$, but for sufficiently small positive ε it is strictly less than $T(Q, Q, \ldots, Q)$. This contradicts our assumption that Q was optimal for the problem $\Gamma(n)$. ■

In the cases where $\tilde{Q}$ is optimal for $\Gamma(n)$, the above lemma establishes that there is no incentive for any agent to deviate from this instruction. However for n where $\tilde{Q}$ is not optimal (see Corollary 8) the following argument is needed.

Lemma 2 *For all* n *the basic simple strategy* $\tilde{Q}$ *(when adopted by all agents) forms a symmetric Nash equilibrium for the game* $G(n)$. *(We assume, as in the team problem, that all agents must use progressive strategies.)*

Proof. Observe that the dispersal time T is the first passage time to $\bar{s}$ starting from a random distribution in the Markov chain on $\mathcal{S}_n$ determined by the chosen strategy. Consequently if two strategies determine the same Markov chain on $\mathcal{S}_n$ then they have the same expected dispersal time. We claim that if a single agent deviates from the basic simple strategy, then the resulting Markov chain is the same as if no agent deviates. To see this, consider any state, and let r denote the number of its non-singleton locations. If the deviating player moves to say the first of these locations (in some ordering common to this agent and an observer), and the remaining $r - 1$ non-singleton agents move randomly to these r locations, the resulting distribution of populations over these unordered locations is the same as if the deviating agent also chose randomly. (The expected population of the 'first' location is not the same, but the distribution over unordered locations will be the same.) ■

We can also examine the equilibrium properties of the basic simple strategy $\tilde{Q}$ in the game $\bar{G}(n)$ in which the players are more selfish and pay a cost (negative utility) in each period equal to the number of their neighbors (fellow occupants of the same location). In this game we have no *a priori* restriction to progressive strategies - players are allowed to move to a singleton location.

Proposition 3 *The basic simple strategy profile (when* $\tilde{Q}$ *is universally adopted) forms a subgame perfect symmetric Nash equilibrium of the game* $\bar{G}(n)$.

Proof. Suppose there are n players, one of whom is considering deviating. Clearly if this player ever becomes a singleton, his future costs are zero if he never deviates, which is a minimum. So assume that at some time this player is a non-singleton in a state with r non-singletons (including himself). Such a player will have on average $(r-1)/r$ neighbors next period if he moves to any non-singleton state, and exactly 1 neighbor if he moves to a singleton state. So a deviator can not improve his immediate cost. So we have to check whether a deviation which takes him to a current singleton state (it will have population 2 if he moves there) can lower his future costs by decreasing r, and hence decreasing $(r-1)/r$. So suppose that when he doesn't deviate (follows the basic simple strategy) he has x neighbors in the following period. If instead of this he moves to a singleton, we ask how the number of singletons will change. If $x > 1$ the deviation will result in one fewer singleton, if $x = 1$ the number of singletons will be unchanged, and if $x = 0$ the deviation will produce two fewer singletons. So in no case will the future situation for this player be improved by having more singletons and decreasing the number r of non-singletons. ■

4 Dynamic Focal Point States

The basic simple strategy $\tilde{Q}$ is a natural response to a situation where agents cannot distinguish between locations, and indeed for some small n it is even optimal (see Proposition 5). However as a result both of random outcomes in the dynamics on $\mathcal{S}_n$ and of the strategies that produce these dynamics, the agents may be able to make use of the population distribution to coordinate their actions. This possibility is entirely analogous to the coordination suggested in [4] whereby two rendezvousers placed at two out of three possible locations can use this distribution to coordinate on the unique unoccupied location as a meeting point. An analogous situation arises in the dispersion problem on $n = 5$ locations when the state $[3, 2, 0, 0, 0]$ is arrived at. The players at the location with population 3 can go independently to the empty locations and those at the population 2 location can go independently to the two locations with nonzero population. This corresponds to the simple strategy which is described at this state by

$$\begin{aligned} [3^*, 2, 0, 0, 0] &\rightarrow \{0\}, \\ [3, 2^*, 0, 0, 0] &\rightarrow \{3, 2\}. \end{aligned}$$

Unlike the situation in the example of Crawford and Haller [4], this doesn't necessarily produce the desired result (dispersion to $[1, 1, 1, 1, 1]$) in a single move, but as we shall see later it reduces the expected dispersal time relative to the basic simple strategy. With this example in mind, we make the following definition.

Definition 4 *A societal state $s \in \mathcal{S}_n$ is called a **dynamic focal point state** if there is a simple strategy which deviates from the basic simple strategy only on* *personalized states corresponding to s, and which has a lower expected dispersal time than the basic simple strategy.*

To determine whether such dynamic focal point states s can arise in our problem, we will obviously need to evaluate the expected dispersal time corresponding to the basic simple strategy $\tilde{Q}$. To this end, define $w(s) = T\left(s, \tilde{Q}\right)$ to be the expected (dispersal) time to reach the distinct location state $\bar{s}$ starting from s and using the basic simple strategy, and define w_n to be this expected time when starting from the initial random distribution over $\mathcal{S}_n$. If we review the analysis for $n = 2$ given earlier in terms of this new notation, we see that $w_2 = v_2 = 1$ (and we have trivially that $w_1 = v_1 = 0$). To evaluate w_n for larger n we need to consider the probability $p_{n,r}$ that if n agents are placed randomly on n locations, exactly r of these locations will be singleton locations. This probability can be computed by the following formula (kindly shown to us by Graham Brightwell):

$$p_{n,r} = \sum_{s=r}^{n} \left(\frac{n!}{(n-s)!r!}\right)^2 \frac{s!}{(n-r)!} \frac{(n-s)^{n-s}}{n^n} (-1)^{s-r}. \tag{8}$$

For example when $n = 2$, we have $p_{2,1} = 0$ (it is impossible to have exactly one singleton) and $p_{2,0} = p_{2,2} = 1/2$.

Observe that when the basic simple strategy is employed starting from a state $s \in \mathcal{S}_n$ with exactly $r > 0$ nonsingleton locations (which implies $r \geq 2$), the distribution at the next period over the r locations which are not presently singletons will be the same as the initial distribution on $\mathcal{S}_r$, and consequently the expected time to reach $\bar{s} = [1, \ldots, 1]$ from the state s using the basic simple strategy is $1 + w_r$. Since this observation will be used often we state it in formal notation, using the definition (1) as follows:

$$\text{For } s \in \mathcal{S}_n^r,\ T\left(\tilde{Q}, s\right) = 1 + w_r. \tag{9}$$

Consequently we have the following formula for w_n.

$$\begin{aligned} w_n &= p_{n,0}(1 + w_n) + p_{n,1}(1 + w_{n-1}) + \cdots + p_{n,n-2}(1 + w_2), \text{or} \\ w_n &= \left(\frac{1}{1 - p_{n,0}}\right)\left(p_{n,0} + \sum_{r=1}^{n-2} p_{n,r} w_{n-r}\right), \end{aligned} \tag{10}$$

which gives w_n as a function of $w_2, \ldots, w_{n-1}$. The first ten values of w_n are given below in Table 2.

Table 2: Expected Dispersal Times for the Basic Simple Strategy $\tilde{Q}$

n	1	2	3	4	5	6	7	8	9	10
w_n	0	1	$\frac{13}{8} = 1.625$	2.1	2. 5	2.8	3.1	3.4	3.6	3.8

It is worth observing that leaving singletons alone (the only difference with the random strategy whose times are given in Table 1) certainly makes a large reduction in the expected dispersal time.

5 Analysis for $n = 3$

In this section we determine the strategy for $n = 3$ agents to achieve the distinct location state $\bar{s} = [1, 1, 1]$ in least expected time v_3. We find that it is the basic simple strategy $\tilde{Q}$ defined in the previous section.

We begin the analysis by noting that we already know the least expected time for two of the states: $v([1, 1, 1]) = v(\bar{s}) = 0$, by definition; and $v([2, 1, 0]) = v([2, 0]) = 2$, by the singleton deletion property (2) and (7). In the latter case (analyzed in section 2) the agents stay or move to the unoccupied location equiprobably. In the remaining state $[3, 0, 0]$, with unique personalized state $[3^*, 0, 0]$, there is a single strategic variable $a = q_*$, with complementary probability $b = 1 - a = q_0$. That is, each agent stays still with probability a, goes equiprobably to the unoccupied locations with probability $1 - a$. Let T_a denote the expected time to get to $\bar{s}$ from $(3, 0, 0)$ when using the strategy with parameter a. Then we have that

$$\begin{aligned}
T_a &= \left[a^3 + b^3/4\right](1 + T_a) + \\
&\quad \left[3ab^2/2 + 3b^3/4 + 3a^2b\right](1 + v[2, 1, 0]) + \left[3ab^2/2\right](1 + v[1, 1, 1]) \\
&= \left[a^3 + b^3/4\right](1 + T_a) + \left[3ab^2/2 + 3b^3/4 + 3a^2b\right](3) + \left[3ab^2/2\right](1),
\end{aligned}$$

with solution

$$T_a = \left(\frac{2}{3}\right)\left(\frac{-5 + 3a - 9a^2 + 9a^3}{a^3 - 1 - a + a^2}\right),$$

which has a minimum of $T_a = 21/8 = v([3, 0, 0])$ at $a = 1/3$. At the single previously undetermined state $[3^*, 0, 0]$ the optimal strategy of staying still with probability $1/3$ corresponds exactly to the basic simple strategy of going to non-singleton locations equiprobably. Thus the basic simple strategy $\tilde{Q}$ is optimal from any state in $\mathcal{S}_3$. Consequently from (5) and (4) we have

$$\begin{aligned}
v_3 &= p([3, 0, 0])\left(\frac{21}{8}\right) + p([2, 1, 0])(2) + p([1, 1, 1])(0) \qquad (11) \\
&= \frac{1}{9}\left(\frac{21}{8}\right) + \frac{2}{3}(2) + \frac{2}{9}(0) = \frac{13}{8},
\end{aligned}$$

which of course agrees with w_3 (see Table 2) since the basic simple strategy has been shown to be optimal. This implies that there cannot be any dynamic focal point states in $\mathcal{S}_3$. We summarize this in the following.

Proposition 5 *The basic simple strategy $\tilde{Q}$ is optimal for the spatial dispersion problem $\Gamma(3)$. There are no dynamic focal points for the problem $\Gamma(3)$.*

6 Analysis for $n = 4$

In this section we will establish that for spatial dispersion problem $\Gamma(4)$ the basic simple strategy $\tilde{Q}$ is still optimal among all simple strategies but is no longer optimal (within the full set $\mathcal{Q}_n$ of all progressive Markov strategies). We begin by considering only simple strategies. Observe that once a singleton location is attained, the singleton deletion property (2) reduces the problem to the case $n = 3$, where we showed in the previous section that the basic simple strategy is optimal. So we need consider only the two states in $\mathcal{S}_4$ which have no singleton locations, namely $[4, 0, 0, 0]$ and $[2, 2, 0, 0]$. Each of these corresponds to a unique personalized state, namely $[4^*, 0, 0, 0]$ and $[2^*, 2, 0, 0]$. We need to check whether there is any way to define a simple strategy over these two personalized states which improves on the basic simple strategy of $[4^*, 0, 0, 0] \to \{4, 0\}$ and $[2^*, 2, 0, 0] \to \{2, 0\}$, which corresponds to a completely random selection of locations.

Observe first that any simple strategy with $[4^*, 0, 0, 0] \to \{4\}$ has $[4, 0, 0, 0]$ as an absorbing state, and consequently infinite expected dispersion time! Next, observe that $[2^*, 2, 0, 0] \to \{2\}$ is equivalent to $[2^*, 2, 0, 0] \to \{0\}$, since either can be described by saying that all four agents move randomly to a common set of two locations. So we need not consider simple strategies involving the latter of these two possibilities. We are left with only three modifications of the basic simple strategy on the two new personalized states to consider as possible improvements, and it is easily calculated that all three do strictly worse than the basic simple strategy:

$$
\begin{array}{lccc}
\text{strategy 1:} & [4^*, 0, 0, 0] \to \{0\} & \text{and} & [2^*, 2, 0, 0] \to \{2\}\,. \\
\text{strategy 2:} & [4^*, 0, 0, 0] \to \{0\} & \text{and} & [2^*, 2, 0, 0] \to \{2, 0\}\,. \\
\text{strategy 3:} & [4^*, 0, 0, 0] \to \{4, 0\} & \text{and} & [2^*, 2, 0, 0] \to \{2\}\,.
\end{array}
$$

We now show that while for $n = 4$ the basic simple strategy is optimal among simple strategies, it is not optimal within the full strategy set $\mathcal{S}_n$. For $n = 4$ the method of improving on the basic simple strategy uses a localized strategy from the societal state $[2, 2, 0, 0]$, that is, one which distinguishes an agent's *own* location of population 2 from the *other* location of population 2. Suppose that agents use the basic simple strategy, except from state $[2, 2, 0, 0]$. From here, they use the localized strategy of staying still with probability $1/2$ and moving to a 0-location with probability $1/2$ (equiprobably to each). They never go to the other location of population 2. Then the transitions from $[2, 2, 0, 0]$ are summarized in the following table, according to the independent coin tossing (move-stay) of the four agents.

prob	type	result	cond prob	total prob
$\frac{1}{16}$	none move	$[2,2,0,0]$	1	$\frac{8}{128}\left(=\frac{1}{16}\right)$
$\frac{4}{16}$	one moves	$[2,1,1,0]$	1	$\frac{32}{128}$
$\frac{2}{16}$	two move, from	$[2,2,0,0]$	$\frac{1}{2}$	$\frac{8}{128}$
	same location	$[2,1,1,0]$	$\frac{1}{2}$	$\frac{8}{128}$
$\frac{4}{16}$	two move, from	$[2,1,1,0]$	$\frac{1}{2}$	$\frac{16}{128}$
	distinct locations	$[1,1,1,1]$	$\frac{1}{2}$	$\frac{16}{128}$
$\frac{4}{16}$	three move	$[3,1,0,0]$	$\frac{1}{4}$	$\frac{8}{128}$
		$[2,1,1,0]$	$\frac{3}{4}$	$\frac{24}{128}$
		$[4,0,0,0]$	$\frac{1}{8}$	$\frac{1}{128}$
$\frac{1}{16}$	all four move	$[3,1,0,0]$	$\frac{3}{8}$	$\frac{3}{128}$
		$[2,2,0,0]$	$\frac{4}{8}$	$\frac{4}{128}$

Combining the different ways of getting to the five possible states in $\mathcal{S}_4$ using this strategy, we get the following transition probabilities from $[2,2,0,0]$.

$[4,0,0,0]$	$\frac{1}{128}$
$[2,2,0,0]$	$\frac{20}{128}$
$[3,1,0,0]$	$\frac{11}{128}$
$[2,1,1,0]$	$\frac{80}{128}$
$[1,1,1,1]$	$\frac{16}{128}$

If this strategy is used for only one period and then the basic simple strategy is used, the expected time to reach $\bar{s}=[1,1,1,1]$ from $[2,2,0,0]$ can be computed using singleton deletion (2) as

$$\begin{aligned}
& \frac{21}{128}\left(1+T\left(\tilde{Q},[4,0,0,0]\right)\right)+\frac{11}{128}\left(1+T\left(\tilde{Q},[3,0,0]\right)\right)+ \\
& \frac{80}{128}\left(1+T\left(\tilde{Q},[2,0]\right)\right)+\frac{16}{128}\left(1+T\left(\tilde{Q},[1]\right)\right) \\
= \;& \frac{21}{128}\left(2+w_4\right)+\frac{11}{128}\left(2+w_3\right)+\frac{80}{128}\left(2+w_2\right)+\frac{16}{128}\left(1+w_1\right), \text{ by } (9) \\
= \;& \frac{21}{128}\left(2+\frac{227}{108}\right)+\frac{11}{128}\left(2+\frac{13}{8}\right)+\frac{80}{128}(3)+\frac{16}{128}(1) \doteq 2.984\,5
\end{aligned}$$

which is less than the time of $1 + w_4 = \frac{335}{108} = 3.1019$ when using the basic simple strategy. Consequently using the above strategy in all periods will yield an expected time less that of the basic simple strategy. So we have shown that the basic simple strategy is not optimal for $n = 4$. Summarizing the results of this section for $n = 4$, we have the following.

Theorem 6 *For the spatial dispersion problem $\Gamma(4)$ the basic simple strategy is optimal among simple strategies and consequently there are no dynamic focal point states. However the basic simple strategy is not optimal (among all strategies in $\mathcal{S}_4$).*

7 Analysis for $n \geq 5$

For the spatial dispersion problem $\Gamma(5)$ we will now show that the basic simple strategy is not even optimal among the simple strategies because the state $\hat{s} = [3, 2, 0, 0, 0]$ is a dynamic focal point. We do this by finding a simple strategy which differs from the basic simple strategy only on the state $\hat{s} = [3, 2, 0, 0, 0]$ and has a lower dispersal time. Since there are no singleton locations in $[3, 2, 0, 0, 0]$, it follows from (9) that $T\left(\tilde{Q}, [3, 2, 0, 0, 0]\right) = 1 + w_5 \doteq 3.5$. We will show that the agents can improve by employing the simple strategy $\check{Q}$ which is defined as the basic simple strategy in all other states, but at state $\hat{s} = [3, 2, 0, 0, 0]$ is given by the rules mentioned earlier of

$$\begin{aligned} [3^*, 2, 0, 0, 0] &\rightarrow \{0\}, \text{ and} \\ [3, 2^*, 0, 0, 0] &\rightarrow \{3, 2\}. \end{aligned}$$

In other words, agents at the location with population 3 go to empty locations equiprobably, and agents at the location with population 2 equiprobably stay still or go to the location with population 3. The group of three will go to one of the $n = 3$ substates $[1, 1, 1]$, $[2, 1, 0]$, or $[3, 0, 0]$. The group of two will go either to $[1, 1]$ or $[2, 0]$. The possible states are given below in a matrix which shows how they arise from the random actions of the two groups of agents.

	$[1,1,1]$	$[2,1,0]$	$[3,0,0]$
$[1,1]$	$[1,1,1,1,1]$	$[2,1,1,1,0]$	$[3,1,1,0,0]$
$[2,0]$	$[2,1,1,1,0]$	$[2,2,1,0,0]$	$[3,2,0,0,0]$

Note that under this strategy the system cannot return to $\hat{s} = [3, 2, 0, 0, 0]$ once it leaves $\hat{s}$, because all the distinct successors of $\hat{s}$ have at least one singleton (which can never be lost in the future). Consequently after leaving $\hat{s}$ the strategy $\check{Q}$ is identical to the basic simple strategy $\tilde{Q}$ and hence the expected time to $\bar{s} = [1, 1, 1, 1, 1]$ using $\check{Q}$ from the other five states in the table is given by the function w whose values were computed in Table 2. Since the distribution over the columns is $\left(\frac{2}{9}, \frac{6}{9}, \frac{1}{9}\right)$ (see (4)), the distribution over the rows is $\left(\frac{1}{2}, \frac{1}{2}\right)$, and the motion of the two groups of agents is independent, we obtain the following formula for the expected time $T\left(\check{Q}, \hat{s}\right)$ from $\hat{s}$ to $\bar{s}$ when using strategy $\check{Q}$. The six terms are listed going left to right in the top row and then the bottom row. The subscript of w gives the number of non-singleton locations.

$$18 \quad T\left(\check{Q},\hat{s}\right) = 2\left(1+w_0\right)+6\left(1+w_2\right)+1\left(1+w_3\right) \\ +2\left(1+w_2\right)+6\left(1+w_4\right)+1\left(1+T\left(\check{Q},\hat{s}\right)\right).$$

Solving for $T\left(\check{Q},\hat{s}\right)$and then substituting the known values $w_0 = 0$, $w_2 = 1$, $w_3 = 13/8$, $w_4 = \frac{227}{108}$, we obtain,

$$\begin{aligned} T\left(\check{Q},\hat{s}\right) &= \frac{18}{17}+\frac{2}{17}w_0+\frac{8}{17}w_2+\frac{1}{17}w_3+\frac{6}{17}w_4, \text{ or} \\ T\left(\check{Q},\hat{s}\right) &= \frac{2897}{1224} = 2.3668, \end{aligned}$$

which is considerably less than $T\left(\tilde{Q},\hat{s}\right) = 1+w_5 \doteq 3.5$. This analysis shows that the state $[3,2,0,0,0]$ is a dynamic focal point state. We showed earlier that there are no dynamic focal point states for $n \leq 4$. Furthermore, since states of the type $[3,2,1,1,1,\ldots 1,0,0]$ occur for all values of n greater than or equal to 5, and all of these configuration are dynamic focal points, we have

Theorem 7 *The spatial dispersion problem $\Gamma(n)$ has a dynamic focal point state if and only if $n \geq 5$.*

Since we have previously shown that the basic simple strategy is optimal for $n = 2$ and 3 but not for $n = 4$, and the existence of a dynamic focal point state precludes its optimality, we have the following.

Corollary 8 *The basic simple strategy $\tilde{Q}$ is optimal for the spatial dispersion problem $\Gamma(n)$ if and only if $n \leq 3$, and optimal within the class of simple strategies if and only if $n \leq 4$.*

8 Non-Progressive Strategies

We have been assuming thus far that the agents are restricted to using progressive strategies which maintain singleton states once they are achieved. In this section we first discuss this assumption within our previous model (where we conjecture that it is harmless) and then present two variations on our previous model where restricting to progressive strategies definitely is suboptimal. The first of these models individuals have 'limited habitats' in that they may go only to certain locations. The second model has 'distinguished agents' who may use distinct strategies.

First observe that even in our basic model, there may be non-progressive strategies which are optimal. Consider for example the basic simple strategy $\tilde{Q}$ (which is optimal for $n = 3$) as it relates to the personalized states corresponding to the state $[2,1,0]$, but considered within the wider space of *generalized strategies* (progressive and non-progressive strategies). We would write it as follows (if working within progressive strategies the second line would be omitted):

$$\begin{aligned} [2^*,1,0] &\to \{2,0\}, \text{ and} \\ [2,1^*,0] &\to \{1\}. \end{aligned}$$

Suppose we modify it on this state to

$$\begin{aligned} [2^*,1,0] &\to \{1,0\}, \text{ and} \\ [2,1^*,0] &\to \{2\}. \end{aligned}$$

This is a non-progressive strategy which produces the same Markov chain on $\mathcal{S}_3$ as $\tilde{Q}$, since the single agent will remain a singleton agent (albeit at a new location), while the two neighbors move randomly to a common set of two locations (which no one else will enter). So it is certainly possible for a non-progressive strategy to be optimal among generalized strategies. However we conjecture the following:

Conjecture 9 *For any n, there is always a progressive strategy for the spatial dispersion problem $\Gamma(n)$ which is optimal among all generalized strategies.*

However there are two variations on our problem $\Gamma(n)$ for which we can show that progressive strategies are always suboptimal: the version with limited habitats and the version with distinguishable agents.

8.1 Limited Habitats

Up to this point, we have been assuming that a satisfactory outcome arises from any configuration with one agent per location, regardless of which agents are where. We now consider situations in which each agent has only certain locations, called acceptable habitats, where he may be placed. For example, the agents may be people and the locations houses, and perhaps not all houses are acceptable to all people. Another version of this restricted matching problem is called the Marriage Problem, where say the agents are a set of boys and the locations are girls. Each boy knows a certain subset of the girls, and the problem is to find a matching (each boy assigned to a different girl) with the property that every boy is assigned to a girl that he knows. In some cases, such a matching is impossible. Necessary and sufficient conditions for the existence of such a matching were established by P. Hall (see [10]) in the so called 'Marriage Theorem': An acceptable matching exists if and only if it is true that for every set of k boys, $k = 1, \ldots, n$, together they know at least k girls. There are efficient algorithms for finding such matchings (see [9]) but they cannot be implemented by the type of agents we are modelling in this paper.

Here we consider a dynamic version of the Marriage Problem (sometimes called the Assignment Problem, with a story involving workers and jobs). Initially every boy is assigned at random to one of the girls that he knows. He knows the distribution (of boys) over his set of acceptable girls (those whom he knows) and over the unacceptable girls. In each period he may move to another acceptable girl, but based only on the population distributions (or more formally, the two population distributions). We seek the strategy which, if adopted by all the boys, takes the least expected time to reach an acceptable matching. It should be observed that a finite minimum exists, since the random strategy of going to a random acceptable location (independent of the configuration) has finite expected time, less than that given by Table 1.

We will not consider this problem in complete generality, only in a family of situations which we call the circular dispersion problem $C(n)$. In this problem there are n boys, boy i knows girl j if and only if $j = i$ or $j = i + 1$ (with arithmetic modulo n). The problem facing any two agents are equivalent via a circular permutation of the boys and girls. The aim is to minimize the time required to achieve a matching with each boy assigned to a girl that he knows. There are two such matchings, one with each boy i assigned to girl i and another with each boy i assigned to girl $i+1 \pmod n$. However in keeping with our previous language, we will refer to agents and locations. The information available to each boy is the population distribution over the two acceptable locations (one of these will have a star indicating his own location) and the population distribution over the remaining $n-2$ locations.

We first consider the case $n = 3$. The labeling of the boys and girls (or agents and locations) is from the point of view of an observer. Each boy can simply recognize the two girls that he knows, and he can't even distinguish between them. A state of the system (called a configuration) can be represented by a $3-$tuple of sets A_1, A_2, A_3, where A_i denotes the set of boys at location i. From the observer's point of view, there are two solutions, namely $\{1\}, \{2\}, \{3\}$ and $\{3\}, \{1\}, \{2\}$. Since no locations is acceptable to all three agents, it is not possible to have a state with a population of 3 at any location, that is, $\#(A_i) \leq 2$. Consequently there are four personalized states in this problem, namely $([2^*, 0], [1])$, $([2^*, 1], [0])$, $([1^*, 0], [2])$, and $([1^*, 1], [1])$. For example, the first of these says that I have one neighbor at my current location, no one is at the other location acceptable to me, and one agent is at the location which is not acceptable to me. A little analysis shows that the last of these, namely $([1^*, 1], [1])$, actually corresponds to an acceptable matching for all agents, and consequently an agent should stay still from this personalized state. An optimal strategy which results in an acceptable matching in one step is the following:

(i) If alone (singleton), stay still.

(ii) If not alone and your other acceptable location is occupied, stay still.

(iii) If not alone and your other acceptable location is not occupied, move to it.

Note that this is a progressive strategy. Our claim regarding this strategy is easily checked. Since each agent has two acceptable locations, there are eight possible initial configurations, which we list as three sets: agents at location i,

$i = 1, 2, 3$. For each, we give the acceptable matching that results in a single step from the strategy. Of course the first two are acceptable initial configurations.

initial configuration	next configuration
$\{1\},\{2\},\{3\}$	$\{1\},\{2\},\{3\}$
$\{3\},\{1\},\{2\}$	$\{3\},\{1\},\{2\}$
$\{1,3\},\{2\},\{\}$	$\{1\},\{2\},\{3\}$
$\{1,3\},\{\},\{2\}$	$\{3\},\{1\},\{2\}$
$\{1\},\{\},\{2,3\}$	$\{1\},\{2\},\{3\}$
$\{\},\{1\},\{2,3\}$	$\{3\},\{1\},\{2\}$
$\{\},\{1,2\},\{3\}$	$\{1\},\{2\},\{3\}$
$\{3\},\{1,2\},\{\}$	$\{3\},\{1\},\{2\}$

To explain this table, consider for example the third initial configuration, $\{1,3\}$, $\{2\}$, $\{\}$. Since agent 2 is alone, he stays still by rule (i). Since agent 1 is not alone and his other acceptable location (location 2) is occupied (by agent 2), he stays still by rule (ii). Since agent 3 is not alone, and his other acceptable location (location 3) is empty, he goes there.

Next we consider the case $n = 4$. The acceptable matchings (absorbing configurations) in this case are $\{1\},\{2\},\{3\},\{4\}$ and $\{4\},\{1\},\{2\},\{3\}$. There are six personalized states for this problem, as given below.

Table 3: Personalized states for $C(4)$

label	personalized state
a	$[1^*,1],[1,1]$
b	$[1^*,1],[2,0]$
c	$[1^*,0],[2,0]$
d	$[2^*,0],[1,1]$
e	$[2^*,0],[2,0]$
f	$[2^*,1],[1,0]$

A strategy for this problem tells an agent whether to stay still or to move to his other acceptable location, depending only on his personalized state (it need not be defined for the personalized state 'a', since this corresponds only to an absorbing state). A pure strategy will give a definite answer, a mixed (or Markov) one a probability.

The societal states may be visualized as in Figure 1 by listing agents 1 to 4 down a column on the left and locations 1 to 4 in another column to the right. In every state an agent is either linked by a horizontal line to the location on his right (with the same number) or by a slanted line to the location below (with the next higher number). Using the circular symmetry of the problem we may chose the list starting with any agent on top, so as to maximize the number of horizontal lines starting from the top. This gives a reduced set of six configurations with respectively 4,3,2,1,1,0 horizontal lines at the top of the picture. The first and last of these are the absorbing final states. For each agent in each state, we give his personalized state according to the labelling of Table

3.

Figure 1: Six states for $C(4)$

In this figure the bottom (fifth) location in each right-hand column is the same as the top (first) location. The labeling of the personalized states shows that the absorbing state 6 is strategically equivalent to state 1 and state 5 is strategically equivalent to state 2.

Consider state 3, and the two singleton agents (both with personalized state c). If a progressive strategy is adopted, their assignments will never change. However these assignments conform neither to the absorbing state 1 or to the absorbing state 6, so the dispersal time from state 3 will be infinite. Consequently no progressive strategy for the problem $C(4)$ can be optimal (or even have finite dispersal time). Another observation is that no pure strategy has a finite expected dispersal time. To see this, suppose state 4 is reached (this will occur with positive probability at time zero regardless of the strategy used). Regardless of whether the pure strategy says to stay still or move from personalized state e, state 4 will be an absorbing state of the Markov chain. (If the instruction is to move, then the resulting state will not initially be listed in Figure 1, but after moving one of the horizontal lines to the top of the figure it will again correspond to state 4.)

8.2 Distinguishable Agents

Up to now we have been assuming that the same strategy is given to each agent at the beginning of the problem $\Gamma(n)$ (or we have looked for symmetric equilibria in an associated game). If we are allowed to give different strategies to the agents, we call this problem $\Gamma'(n)$. In keeping with the similar assumption that has sometimes been made in rendezvous problems we will call this the problem 'with distinguishable agents'. For example a book on this topic for $n = 3$, held by all agents, could say what the tallest agent should do, the middle agent, and the shortest agent. Obviously the expected dispersal time cannot be larger for this problem, since it is still possible to tell all the agents to follow the same strategy. We shall not give a complete analysis of $\Gamma'(n)$, but we shall show that even for $n = 3$ an optimum cannot be attained by progressive strategies.

Theorem 10 *For the spatial dispersion problem* $\Gamma'(3)$ *with distinguishable agents,* *the optimal expected dispersal time is* $v'_3 = \frac{5}{6}$. *No strategy profile consisting of progressive strategies can attain this time.*

Proof. We first establish that v'_3 is $\frac{5}{6}$ and then we show that progressive strategy profiles have higher expected meeting times. First consider the state $[2,1,0]$. Since this is not the absorbing state $[1,1,1]$, we clearly have $v'[2,1,0] \geq 1$. The (partial) strategy profile in which agent $i = 1,2,3$ moves from wherever he happens to be in the state $[2,1,0]$ to the unique location with population $i-1$ ensures an immediate transition from $[2,1,0]$ to $[1,1,1]$ and therefore establishes that $v'[2,1,0] = 1$. From state $[3,0,0]$ we tell agent 1 to stay still and agents 2 and 3 to move to a zero population location. Half the time this leads to the absorbing state $[1,1,1]$ and half the time to $[2,1,0]$, and so this strategy profile goes from $[3,0,0]$ to $[1,1,1]$ in expected time

$$\frac{1}{2}(1 + v'[1,1,1]) + \frac{1}{2}(1 + v'[2,1,0]) = \frac{1}{2}(1) + \frac{1}{2}(1+1) = \frac{3}{2}.$$

To show that this time cannot be improved, consider the most general strategy profile from $[3,0,0]$, in which agents 1,2,3 move to a zero population location with respective probabilities p, q, r. This leads immediately to the absorbing state $[1,1,1]$ if one agent stays still and the other two move to distinct locations, an event that occurs with probability

$$P(p,q,r) = \frac{1}{2}[p(1-q)(1-r) + (1-p)(1-q)r + (1-p)q(1-r)].$$

Since from any other state it takes at least one period to get to $[1,1,1]$ it follows that

$$\begin{aligned}
v'[3,0,0] &\geq 1(P(p,q,r)) + 2(1 - P(p,q,r)) \\
&= \frac{1}{2}[4 - 2P(p,q,r)] \\
&= \frac{1}{2}[4 - p(1-q)(1-r) + (1-p)(1-q)r + (1-p)q(1-r)] \\
&= \frac{1}{2}[4 - p(1-qr) + r - 2qr + q] \\
&\geq \frac{1}{2}[4 - (1-qr) + r - 2qr + q] \quad \text{(taking } p = 1\text{)} \\
&= \frac{1}{2}[3 - qr + r + q] \\
&\geq \frac{1}{2}[3] = \frac{3}{2} \text{ (taking } q = r = 0\text{)}.
\end{aligned}$$

So we have $v'[3,0,0] = \frac{3}{2}$ and, since the same strategy profile is used to obtain $v'[3,0,0]$ and $v'[2,1,0]$, we have by (5) that

$$\begin{aligned}
v'_3 &= p[3,0,0]\ v'[3,0,0] + p[2,1,0]\ v'[2,1,0] + p[1,1,1]\ v'[1,1,1] \\
&= \frac{1}{9}\left(\frac{3}{2}\right) + \frac{2}{3}(1) = \frac{5}{6}.
\end{aligned}$$

(This time should be compared with the much larger optimal dispersal time of $v_3 = \frac{13}{8}$ for the case of indistinguishable agents obtained in equation (11).)

We now show that no progressive strategy profile can give such a low dispersal time. Given the above analysis, it is enough to demonstrate that no strategy profile which uses all progressive strategies can ensure getting from $[2,1,0]$ to $[1,1,1]$ in a single step. In order to achieve this for certain, one of the agents at the population 2 location must move to the population 0 location, while the other must stay still. So we must assign rules (move or stay) to the three agents so that any two of them (the two that arrive at the personalized state $[2^*,1,0]$) will be assigned distinct rules. However the 'pigeon-hole principle' ensures that if three agents are placed among (given) two rules (move or stay from $[2^*,1,0]$) two of them will be placed at the same rule. In other words, two of the agents will be given the same rule. If these two agents share a location (with the other agent elsewhere) then the state $[1,1,1]$ will not be attained at the next move. Consequently no progressive strategy profile can attain the expected dispersal time of $\frac{5}{6}$. ■

It is worth noting that the analysis of the last paragraph of the above proof does not imply that a progressive profile cannot be optimal for $\Gamma'(3)$ if the problem is started from the state $[3,0,0]$. The following progressive strategy profile is indeed optimal from $[3,0,0]$.

1. Agent 1 stays still in $[3^*,0,0]$ and stays still in $[2,1^*,0]$.

2. Agent 2 moves in $[3^*,0,0]$,stays still in $[2,1^*,0]$, and moves empty location in $[2^*,1,0]$.

3. Agent 3 moves in $[3^*,0,0]$,stays still in $[2,1^*,0]$, and stays still in $[2^*,1,0]$.

Suppose this strategy profile is adopted. From $[3,0,0]$ we move to $[1,1,1]$ half the time and to $[2,1,0]$ half the time. In the latter case the singleton agent will be agent number 1. Of the remaining two agents who are together, one (agent 2) will move to the unoccupied location, and one (agent 3) will stay still, so in the next period we arrive at $[1,1,1]$. So the expected time to reach $[1,1,1]$ from $[3,0,0]$ using this progressive strategy profile is $\frac{1}{2}(1) + \frac{1}{2}(2) = \frac{3}{2} = v'[3,0,0]$. The reason this approach cannot be used in the usual formulation of $\Gamma'(3)$ is that if $[2,1,0]$ is attained as an initial position (rather than as a follower of $[3,0,0]$) we cannot be sure of the agent locations.

8.3 Distinguished Locations

We have been assuming throughout that agents can distinguish between locations only through the differing populations at these locations. Suppose however that the agents have a common *a priori* numbering of the locations. This would obviously help for *any* n in the rendezvous problem, as they could all immediately go to say location 1. But does this initial common labeling help in the spatial dispersion problem? The answer depends on the number n (of agents and of locations), as we shall see in this section.

First observe that having the locations distinguished would not help in the case $n = 2$, because in the only nonabsorbing state $[2, 0]$ the locations are *already* distinguished by their differing populations.

for $n = 3$, the same observation applies to the state $[2, 1, 0]$. However the state $[3, 0, 0]$ has to be analyzed further, as in the new formulation the agents can distinguish between the two empty locations. Consider the strategy which equals the basic simple strategy except from $[3, 0, 0]$, where it tells the agents to go to the commonly labeled locations 1,2,3 with respective probabilities x, y, z. Since the expected time to reach $\bar{s} = [1, 1, 1]$ from $[2, 1, 0]$ is 2 and from $[1, 1, 1]$ is 0, the expected time T to reach $\bar{s} = [1, 1, 1]$ from $[3, 0, 0]$ using this strategy (playing the optimal random strategy at states with at most two non-singletons) is given by the equation

$$\begin{aligned} T &= \left(x^3 + y^3 + z^3\right)(1 + T) \\ &\quad + \left(3x^2(1 - x) + 3y^2(1 - y) + 3z^2(1 - z)\right)(1 + 2) + (6xyz)(1 + 0), \text{ or} \\ T &= \left(\frac{1}{3}\right)\left(\frac{1 + 6x + 6y - 6x^2 - 6y^2 - 24xy + 18x^2y + 18xy^2}{x + y - x^2 - 2xy - y^2 + x^2y + xy^2}\right). \end{aligned}$$

But this expression has a minimum at $x = y = 1/3$ (and hence also $z = 1 - x - y = \frac{1}{3}$) of $21/8$, which is the same optimum and the same (basic simple) strategy as in the case with indistinguishable locations. So we have shown the following.

Proposition 11 *For $n \leq 3$ the basic simple strategy remains optimal even when strategies that distinguish between locations are allowed.*

This result cannot, however, be extended beyond $n = 3$. For example suppose we are in the (unordered, as usual) state $[2, 2, 0, 0]$, with locations 1 and 2 having population 2 and locations 3 and 4 empty. The strategy in which an agent at location 1 equiprobably stays or moves to location 3, and an agent at location 2 equiprobably stays or moves to location 4, has expected dispersal time $8/3$, assuming the basic simple strategy is used whenever there are no more than three non-singleton states. No strategy which does not distinguish between locations can achieve such a low dispersal time. A full analysis of the problem with distinguished locations is beyond the scope of this article. However the problem appears as interesting as the one with indistinguishable locations that we have examined.

Of course if we have both distinguishable agents and distinguishable locations, the problem disappears, as we simply have agent i go to location i. This is what happens in a company parking lot where each space is marked with the name of its approved occupant.

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