Abstract

Let μ be a locally positive Borel measure on a σ-compact $n-$manifold X, $n \geq 2$. We show that there is always a $\mu-$preserving homeomorphism of X which is *maximally chaotic* in that it satisfies Devaney's definition of chaos, with the sensitivity constant chosen maximally. Futhermore, maximally chaotic homeomorphisms are compact-open topology dense in the space of all μ-preserving homeomorphisms of X if and only if (X, μ) has at most one end of infinite measure. (For example, for Lebesgue measure λ on $X = R^2$, but *not* for λ on the strip $X = R \times [0,1]$.) This work extends that of Aarts and Daalderop, Daalderop and Fokkink, Kato *et. al.*, and Alpern, regarding chaotic phenomena on compact manifolds, and that of Besicovitch and Prasad for other dynamical properties on noncompact manifolds.

keywords: *chaos, manifold, noncompact, homeomorphism*

Maximally Chaotic Homeomorphisms of Sigma-Compact Manifolds

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August 12, 1998

1 Introduction

Let $\mathcal{M}[X,\mu]$ denote the space of all homeomorphisms of the σ-compact manifold X which preserve a given locally positive nonatomic Borel measure μ, which is zero for the manifold boundary. We show that the space $\mathcal{M}[X,\mu]$ always contains a homeomorphism which possesses a property we call *maximally chaotic,* and give necessary and sufficient conditions for such homeomorphisms to be dense in $\mathcal{M}[X,\mu]$, with respect to the compact-open topology. A maximally chaotic map is one which, roughly speaking, satisfies Devaney's definition of chaos [13], with the constant in the definition of sensitive dependence on initial conditions (s.d.i.c.) chosen maximally.

More formally, we define a map $h: X \to X$ to satisfy the *prime cycle condition* if given nonempty open sets U, V, and W, there exist h-periodic points p and q in U, with relatively prime periods, whose orbits respectively enter V and W. It follows from the relatively prime hypothesis that there is an arithmetic sequence of iterates of h which simultaneously place p in V and q in W. If δ is any number less than $d(X)$ (the diameter of the metric manifold (X,d), which may be infinite), then we may choose V and W with $\inf_{v\in V, w\in W} d(v,w) > \delta$, and consequently $d(h^i(p), h^i(q)) > \delta$ for infinitely many i. Consequently any homeomorphism h satisfying the prime cycle condition has the following three properties, which we collectively call *maximally chaotic*:

1. The periodic points of h are dense in X.

2. The map h is transitive

3. For any nonempty open set U, $\overline{\lim}_i\, d(h^i(U)) = d(X)$.

Condition 3 and the triangle inequality imply that given any $\delta < d(X)/2$, $x \in X$, and $\varepsilon > 0$, there is some point y with $d(x,y) < \varepsilon$ such that $d(h^i x, h^i y) > \delta$ (we may take the above p or q for y). This is equivalent to saying that h has sensitive dependence on initial conditions, with a constant of $d(X)/2$. Since a chaotic map might have a fixed point at a center of the space X, we cannot in general hope to obtain a sensitivity constant larger than half the diameter of the X. We also note that if h satisfies the prime cycle condition then it is not only transitive, but also topologically weak mixing.

The aim of this paper is to construct maximally chaotic homeomorphisms on any sigma-compact connected metric manifold (X,d) of dimension $n \geq 2$. This problem is made more accessible by adding the additional requirement that the homeomorphism must preserve a given Borel measure μ on X which is locally positive (positive on all nonempty open sets). It is easily seen that the manifold X always supports such a measure (even one with $\mu(X) < \infty$), since it can be covered by countably many homeomorphic images of open balls, on which suitable multiples of Lebesgue measure can be defined. Most of our constructions are carried out using successive approximations in the space $\mathcal{M} = \mathcal{M}[X, \mu]$, which we endow with either the uniform topology or the compact-open topology.

Our first result (Theorem 1) is that there is always a maximally chaotic homeomorphism on X. Moreover any homeomorphism in $\mathcal{M}$ which has compact support can be uniformly approximated such homeomorphisms. This result is proved by an elementary combinatorial algorithm (Lemma 3) which has been used often in an ergodic theoretic context. We modify the traditional method by controlling (in an 'alternating' fashion) the period of the approximating orbit.

We then consider the problem of whether maximally chaotic homeomorphisms are dense in the space $\mathcal{M}[X, \mu]$, with respect to the compact open topology.. Using results from our previous work, we show that this is true if and only if (X, μ) has at most one end of infinite measure. When there are at least two ends of infinite measure, even the recurrent homeomorphisms are not dense. The notions of ends (roughly, a way of going to infinity) and measures on ends needed to understand the last claim are defined in section 5, where the needed work of the authors on these subjects is stated. To illustrate the application of these results, and to see an intuitive notion of ends, consider the following examples. Any volume-preserving homeomorphism of $R^n, n \geq 2$, can be compact-open approximated by a chaotic one, since R^n has only one end. However observe that any homeomorphism in a small enough compact open neighborhood of the translation $(x, y) \longmapsto (x+1, y)$ of the strip $R \times [0,1]$ cannot have any periodic points, and hence cannot be chaotic (in the sense of Devaney).

The results of this paper can be considered extensions of earlier work in two different directions. In the domain of dynamics on R^n or noncompact manifolds, they are extensions of previously demonstrated dynamical properties (transitivity, ergodicity) to chaos. In the domain of the existence or denseness of chaos, they can be considered as an extension of earlier results on compact manifolds to the noncompact case and to maximal sensitivity. An account of this earlier work is given in section 2.

The remainder of the paper is organized as follows. Section 2 gives some more detailed historical background, and section 3 states some known results. Section 4 presents a self contained constructive proof of the existence of maximally chaotic homeomorphisms on arbitrary $\sigma-$compact manifolds. Section 5 gives necessary and sufficient conditions for such homeomorphisms to be dense among all measure preserving homeomorphisms, relying on previous results of the authors.

2 Historical Background

Various authors have considered the question of the existence of transitive homeomorphisms, or other dynamics, on different manifolds. As far back as 1936, Kerekjarto mentioned the existence of a transitive homeomorphism for a closed circular region as an unsolved problem in [15]. Problem 115 from the *Scottish Book*, posed by S. Ulam in 1935, asks for the existence of a transitive homeomorphism of Euclidean n-space R^n [18, p. 201]. In 1937, Besicovitch [9] gave an explicit example of a transitive homeomorphism for the plane (later modified by Xu [22] for the square). Later that same year, Oxtoby showed [19] that the transitive homeomorphisms of the Euclidean n-cube, $n \geq 2$, are dense G_δ in the group of measure preserving homeomorphisms (with the uniform topology). The latter result was extended in the 1941 paper of Oxtoby and Ulam [20], where they showed that for any compact connected manifold with a nonatomic, locally positive measure which is zero on its boundary, the ergodic homeomorphisms are dense G_δ in the space of measure preserving homeomorphisms. Note that a transformation is ergodic if its only invariant sets are those which have zero measure or their complement has zero measure. Ergodic homeomorphisms are transitive, as almost every point has a dense orbit. This celebrated theorem of Oxtoby and Ulam was extended to both the noncompact setting, as well as to measure theoretic properties other than ergodicity, in a series of papers (among these are [5] and [6]) by the authors– these various extensions and applications of the Oxtoby-Ulam theory form the focus of our forthcoming book "Typical Properties of Volume Preserving Homeomorphisms" [8].

The interest in manifold homeomorphisms with chaotic properties is more recent. Aarts and Daalderop [1] have shown that Devaney-chaotic homeomorphisms are dense in $\mathcal{M}[X,\mu]$ for compact manifolds X, and Kato *et.al.* [14] have done this for compact Menger manifolds. Very recently, Daalderop and Fokking [12] have added the property G_δ property for compact manifolds. (The last result can be extended to the noncompact case under the assumptions of section 4, but we have decided not to pursue these matters here.) In [7], we proved the existence of Devaney-chaotic volume preserving homeomorphisms of R^n, $n \geq 2$, by showing that in the space of volume preserving homeomorphisms of the torus T^n which have mean rotation zero, those homeomorphisms with chaotic lifts are dense (with the uniform topology). Although chaos was not a subject of study at that time, the construction given in Theorem 5 of [20], and extended by the first author in his Ph.D. thesis (see [3] for a discussion) has a chaotic homeomorphism as a degenerate case of one that permutes a countable family of subcubes of I^n.

3 Dynamics of Measure Preserving Homeomorphisms

In this section we review three known results regarding measure preserving homeomorphisms which we shall need later. For the present purpose they will be called lemmas. To make this paper more self contained we shall present the proof of the last of these. We use the notation $||f||_S = \sup_{x\in S} d\left(f\left(x\right),x\right)$, for any subset S of X, with the subscript S taken as X when it is omitted.

It will be useful to represent the manifold X as the union of compact manifolds, and to represent each of the latter as unit cubes with boundary identifications. We summarize all the parts of this representation that we will need in the following result, which combines the topological result of Brown [10] with the so called 'homeomorphic measures theorem' of Oxtoby and Ulam [20] as applied in [5, Lemma 0]. A particular example, which provides a nice special case for the reader to consider in the proof of Theorem 1, is Lebesgue measure on the infinite strip $R \times [0,1]$, where the compact submanifolds are simply $K_i = [-i, i] \times [0,1]$.

Lemma 1 *We may represent the σ-compact connected $n-$manifold X as a countable increasing union of compact connected manifolds K_i, $i = 1, \ldots, \infty$, for which $\mu(\partial K_i) = 0$, where ∂ denotes the manifold boundary. Each of these compact manifolds K_i can be represented as the unit $n-$cube I^n with boundary identifications. More precisely, there exist continuous maps $\phi_i : I^n \to K_i$ such that*

1. *ϕ_i maps onto K_i.*

2. *ϕ_i restricted to the interior of I^n is a homeomorphism onto its image.*

3. *$\mu(\phi_i(A)) = \mu(K_i) \cdot \lambda(A)$, for any Borel set A in the interior of I^n.*

A well known method of decomposing the cube I^n into small sets of equal measure is the so called dyadic decomposition, into dyadic cubes (of rank α) defined a n-fold products of intervals of the type $[j/2^\alpha, (j+1)/2^\alpha]$. (If we replace the 2 by a 3 we call this a triadic decomposition.) The map ϕ_i described in the above lemma gives us a way of obtaining measurable partitions of K_i into compact sets of small diameter and equal measure. A quasi-dyadic (or triadic) decomposition of K_i is simply the ϕ_i image of a dyadic (or triadic) decomposition of I^n. These are measurable partitions in the sense that the elements may overlap, but only in sets of measure zero.

In order to create periodic orbits with specified properties, we shall sometimes need to extend a small map defined on a finite set to a small measure preserving homeomorphism. The precise statement of this result of Oxtoby and Ulam [20, Lemma 5] is given below (for a simple proof, see [4]).

Lemma 2 *Let $K \subset X$ be a compact connected $n-$manifold with $\mu(\partial K) = 0$, where ∂K denotes the manifold boundary, and let F be a finite subset of $K - \partial K$. Then any injection $\psi : F \to K - \partial K$ with $\max_{x \in K} d(\psi(x), x) < \delta$ can be extended to a $\mu-$preserving homeomorphism f of K,which equals the identity on ∂K and satisfies $||f|| < \delta$. We can further assume that f fixes any given finite set which is disjoint from F and $\psi(F)$.*

The next result (due to Lax [17] and Alpern [2]) says that cyclic dyadic permutations form good uniform approximations to measure preserving homeomorphisms. A fast algorithm to construct these approximations (without cyclicity) has recently been given by Kloeden and Mustard [16].

Lemma 3 *Let $K \subset X$ be a compact connected $n-$manifold with $\mu(\partial K) = 0$, and let F be a finite subset of K. Let $h \in \mathcal{M}[X,\mu]$ be a homeomorphism with support in K, and let ε be any positive number. Then if $\mathcal{P} = \{c_j\}_{j=1}^m$ is a measurable partition of K into compact sets of equal measure $\mu(c_j) = \mu(K)/m$, and sufficiently small diameter, then there is a homeomorphism f in $\mathcal{M}[X,\mu]$ which fixes the set F, has support in K, satisfies $||f||_K < \varepsilon$, and such that under fh a set of points $\{x_j\}_{j=1}^m$ with $x_j \in c_j$ is cyclically permuted.*

Proof. Although this result appears several times in the literature, the proof is short enough to be outlined again. We first apply the so called 'Marriage Theorem' by saying that c_j *knows* c_k if $h(c_j) \cap c_k \neq \varnothing$. Since any set of r of the c's has combined measure $r\mu(K)/m$, so does their image, which therefore cannot be contained within fewer than r of the c's. Thus any set of r c's together know at least r of the c's, and hence by the Marriage Theorem there is a permutation $\sigma : \{1,\ldots,m\} \to \{1,\ldots,m\}$ such that c_j knows $c_{\sigma(j)}$ for every j, or equivalently, $h(c_j) \cap c_{\sigma(j)} \neq \varnothing$. A simple result of Alpern [2] says that any permutation σ can be approximated by a cyclic permutation τ (of the same set) such that $|\tau(j) - \sigma(j)| \leq 2$ for all j. Consequently if the c's are numbered so that those with consecutive indices are adjacent then the diameter of the set $h(c_j) \cup c_{\tau(j)}$ is small for every j. If we pick $x_j \in c_j$ avoiding the finite set F, then this implies that $d\left(h(x_j), x_{\tau(j)}\right)$ is small for each j, and hence by the previous lemma that there is a small $f \in \mathcal{M}[X,\mu]$ with $f(h(x_j)) = x_{\tau(j)}$. In particular, we have $||f|| < \gamma(d) + 3d$, where d is the maximum diameter of the c's and γ is the uniform modulus of continuity of h on the compact set K. ■

4 Construction of Chaos

In this section we give a construction of uniform approximation by maximally chaotic homeomorphisms. We use the technique of successive approximations, under which a sequence $\lim_i f_i f_{i-1} \circ \ldots \circ f_2 f_1$ with $f_j \in$M$[X,\mu]$ will converge to a homeomorphism in M$[X,\mu]$ if successive f_i are chosen with $||f_i||$ sufficiently small in terms of the previous choices of f_j. This can be justified by observing that the uniform topology is given by the complete metric ρ_X, where ρ_S is defined by $\rho_S(f,g) = \max\left(\max_{x\in S}|f(x) - g(x)|, \max_{x\in S}|f^{-1}(x) - g^{-1}(x)|\right)$. See also [11]. We can now prove the following.

Theorem 1 *Let μ be a locally positive Borel measure on a $\sigma-$compact $n-$manifold X, $n \geq 2$. Given any compactly supported μ-preserving homeomorphism $h \in \mathcal{M}[X,\mu]$ and $\delta > 0$, there is a homeomorphism $f \in$M$[X,\mu]$, with $||f|| < \delta$, such that fh is satisfies the prime cycle condition.*

Proof. Let $K_1 \subset K_2 \subset \ldots$ be the increasing sequence of compact $n-$manifolds given in Lemma 1 whose union is the given manifold X, renumbered such that

K_1 contains the support of h. We begin the approximation process by choosing a quasi-dyadic partition $\mathcal{P}_1$ of K_1 with diameters ε_1 sufficiently small so we may apply Lemma 3 (with $F = F_1 = \emptyset$) to obtain an $f_1 \in \mathcal{M}[X, \mu]$ with $||f_1|| < \delta/2$ and support in K_1 such that $f_1 h$ has a periodic orbit $B = B_1$ entering each the m_1 elements of P_1 exactly once. Next we choose $\varepsilon_2 < \delta/2^2$ and sufficiently small so that $||f_2|| < \varepsilon_2$ implies that $\rho_X (f_2 f_1, f_1) < \delta/2^2$. Let $\mathcal{P}_2$ be a quasi-triadic partition of K_2 with sufficiently small diameters so that we may apply Lemma 3 to $f_1 h$ with $F = F_2 = F_1 \cup B_1$ to obtain an $f_2 \in \mathrm{M}[X, \mu]$ with support in $K_2, ||f_2|| < \varepsilon_2$, and such that $f_2 f_1 h$ has a periodic orbit B_2 which enters each of the m_2 elements of $\mathcal{P}_2$ exactly once. We continue recursively, finding for each k a homeomorphism f_k with support in K_k, satisfying $\rho_X (f_k f_{k-1} \ldots f_1, f_{k-1} \ldots f_1) < \delta/2^k$ and $||f|| < \delta/2^k$, which fixes the finite set $F_k = F_{k-1} \cup B_{k-1}$ and such that $f_k f_{k-1} \ldots f_1 h$ has a periodic orbit B_k which enters every one of the m_k elements of the partition $\mathcal{P}_k$ exactly once. For odd k, we choose $\mathcal{P}_k$ as a quasi dyadic partition, and for even k we choose it as quasi-triadic. The completeness of $\mathrm{M}[X, \mu]$ ensures that $f = \lim_k f_k f_{k-1} \ldots f_1$ exists, with$||f|| < \delta$.

To see that fh satisfies the prime cycle condition, let U, V, and W be any nonempty open subsets of X. For any sufficiently large odd k, both U and V contain elements of $\mathcal{P}_k$, and consequently $f_k f_{k-1} \ldots f_1 h$ has a periodic orbit B_k of length $2^{n\alpha}$ for some α, entering U and V. Similarly, for sufficiently large even j, the homeomorphism $f_j f_{j-1} \ldots f_1 h$ has a periodic orbit B_j of length $3^{n\beta}$ for some β, which enters U and W. But for $i > \max(k, j)$, f_i fixes B_k and B_j so that these sets are also periodic orbits for fh. Since these orbits have relatively prime lengths $2^{n\alpha}$ and $3^{n\beta}$, the prime cycle condition is satisfied. ■

Corollary 1 *If μ is a locally positive measure on a σ-compact n-manifold X, $n \geq 2$, then there is a maximally chaotic μ-preserving homeomorphism on X.*

Proof. Apply the previous theorem to the case where h is the identity homeomorphism, and recall from the Introduction that the prime cycle condition implies maximal chaos. ■

5 Dense Chaos

In this section we determine conditions on (X, μ) under which chaos is a dense phenomenon. From the work of Aarts and Daalderop [1] we know that compactness is sufficient (with uniform topology). We will determine a necessary and sufficient condition for maximal chaos to be dense, and show furthermore that when this condition fails even Devaney chaos is not dense. We will need to use the compact-open topology (uniform convergence on compact sets). A basis for this topology is given by sets of the form

$$B(h, K, \varepsilon) = \{f \in \mathcal{M}[X, \mu] : d(f(x), h(x)) < \varepsilon, \text{ for all } x \in K\},$$

where K is a compact set and $\varepsilon > 0$. If K_i, $i = 1, \ldots, \infty$, is a sequence of compact sets with union X, then the compact-open topology is induced by the complete metric ρ given by

$$\rho(f,g) = \sum_{i=1}^{\infty} \frac{\rho_{K_i}(f,g)}{1+\rho_{K_i}(f,g)}, \text{ where}$$

$\rho_S(f,g) = \max\left(\max_{x\in S}|f(x)-g(x)|, \max_{x\in S}|f^{-1}(x)-g^{-1}(x)|\right)$. The results on successive approximation by homeomorphisms which we used for Theorem 1 hold with respect to this topology, for the same reason (completeness).

Our condition for dense chaos involves the notion of an *end* of a noncompact space. An end is, roughly speaking, a way of going to infinity. More precisely, an end e of the space X is a function which assigns to each compact subset K of X a nonempty unbounded connected component, denoted $e(K)$, of the complementary set $X - K$, in such a way that

$$K_1 \subset K_2 \text{ implies } e(K_2) \subset e(K_1).$$

As an example, the strip $R \times [0,1]$ has two ends, which we will call e_+ and e_-. If K is the compact set $[5,6] \times [0,1] \cup [8,9] \times [0,1]$, its complement has the three components $(-\infty, 5) \times [0,1]$, $(6,8) \times [0,1]$, and $(9,\infty) \times [0,1]$. The first and last of these are, respectively, $e_-(K)$ and $e_+(K)$.

Let $E = E(X)$ denote the set of all the ends of the manifold X. We say that an end $e \in E(X)$ has *infinite measure* if $\mu(e(K)) = \infty$ for all compact sets K. Thus, with respect to Lebesgue measure, both ends of the strip $R \times [0,1]$ have infinite measure.

A homeomorphism $h \in \mathcal{M}[X,\mu]$ is said to be *weakly mixing* if its Cartesian square $h \times h : X^2 \to X^2$ is ergodic, with respect to the product measure $\mu \times \mu$. This implies, in particular, that given any positive measured sets U, V, and W, there is a positive integer i with $\mu(h^i(U) \cap V)$ and $\mu(h^i(U) \cap W)$ both positive. The authors have proved the following result [5] for a very wide class of measure theoretic properties, which we specialize the to case of weak mixing.

Theorem 2 *If (X,μ) has at most one end of infinite measure, then the weakly mixing homeomorphisms form a compact-open topology dense G_δ subset of $\mathcal{M}[X,\mu]$. If (X,μ) has two or more ends of infinite measure, then there is an open subset consisting of homeomorphisms which are not recurrent.*

We can apply this result on dense weakly mixing for noncompact manifolds in much the same way that dense transitivity (Oxtoby's result cited above) is applied in [1], to obtain dense chaos.

Theorem 3 *Maximally chaotic homeomorphisms are dense in $\mathcal{M}[X,\mu]$, with respect to the compact-open topology, if and only if (X,μ) has at most one end of infinite measure.*

Proof. If (X, μ) has two or more ends of infinite measure then any homeomorphism in the open set cited in the previous theorem cannot have any periodic points, and cannot be chaotic even in the sense of Devaney. So suppose that (X, μ) has at most one end of infinite measure. By the previous result, the set $\mathcal{W}$ of weakly mixing homeomorphisms is dense in $\mathcal{M}[X, \mu]$. Let (U_k, V_k, W_k), $k = 1, \ldots, \infty$, be an enumeration of all triples of a countable basis of open subsets of X. For simplicity, we may assume that these open sets have compact closures. Let $h \in \mathcal{M}[X, \mu]$ be given, together with a compact set $K \subset X$ and a positive number δ. We will construct a homeomorphism $f \in \mathcal{M}[X, \mu]$ with $||f||_K < \delta$, such that for every k there are $fh-$periodic points p_k and q_k in U_k and a positive integer m_k with $(fh)^{m_k}(p_k) \in V_k$ and $(fh)^{m_k}(q_k) \in W_k$. This condition clearly implies that the period points of fh are dense, that fh is transitive (it is 'topologically weak mixing'), and following the same reasoning as for the prime cycle condition, that fh has maximal dependence on initial condition (condition 3 of maximal chaos).

By the previous theorem, there is a $g_1 \in \mathcal{M}[X, \mu]$ with $\rho(g_1, \mathit{identity})$ arbitrarily small, such that $g_1 h$ is weakly mixing. It follows from the property of weak mixing mentioned above that there exist points p_1 and q_1 in U_k (which we can assume for simplicity are distinct) such that $(g_1 h)^{m_1}(p_1) \in V_1$ and $(g_1 h)^{m_1}(q_1) \in V_1$ for some m_1, and $(g_1 h)^{u_1}(p_1)$ and $(g_1 h)^{v_1}(q_1)$ are arbitrarily close to p_1 and q_1, respectively. Hence by Lemma 2 we can find an arbitrarily small (close to identity) homeomorphism f_1 which maps $(g_1 h)^{u_1}(p_1)$ to p_1 and $(g_1 h)^{v_1}(q_1)$ to q_1, and hence for which p_1 and q_1 are periodic with respect to $f_1 g_1 h$. Furthermore note that we can perform this construction so that any given finite set is fixed by $f_1 g_1$ by requiring that f_1 take $g_1(a)$ into a for any point a in this given finite set. We continue recursively as in the proof of Theorem 1, at each step fixing the periodic orbits of the points $p_1, q_1, p_2, q_2, \ldots, p_{i-1}, q_{i-1}$. ■

Corollary 2 *Maximally chaotic homeomorphisms are dense, with respect to the compact-open topology, in the space of all volume preserving homeomorphism of Euclidean space* R^n, $n \geq 2$.

This result could also be proved using a variant of the construction given in Theorem 1 by using the result of the second author [21] that the homeomorphisms with an arbitrarily large invariant cube are dense in the above space. Such a proof would be similar to that of the proof in [21] that ergodicity is generic for volume preserving homeomorphisms. It is also worth observing that if μ is finite, then of course X has no ends of infinite measure, so the condition of Theorem 3 are satisfied trivially.

In the case that (X, μ) has more than one end of infinite measure, much more can be said about the denseness of maximal chaos within certain components of $\mathcal{M}[X, \mu]$. Of course, Theorem 1 gives denseness within the compact-open closure of the homeomorphisms of compact support. For more results on denseness of weak mixing within certain components of $\mathcal{M}[X,\mu]$, see our forthcoming book [8].

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