Alternating Search at Two Locations

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Abstract. In this paper we introduce a new class of search problem which we call 'alternating search'. Two searchers starting at given points in different search regions, who can move alternately with speed one, or (as a limiting case) simultaneously with combined speed one, seek to find (reach) an object in least expected time. The hidden object is stationary and its location is given by a known distribution over the union of the two search regions. An important special case is the 'Double Linear Search Problem' in which both search regions are infinite lines, and which has been shown to be equivalent to the 'Asymmetric Rendezvous Search Problem on the Line' (ARSPL). The general results proved here are applied in a concurrent paper of Alpern and Beck to prove that the strategy conjectured by Baston and Gal to be optimal for the convex ARSPL is indeed optimal. Our general results are concerned with determining the method of interleaving two given distributions so as to minimize the first moment of the resulting distribution.

1. Introduction

In this paper we introduce a new class of search problem that we call *alternating search.* In this problem, a single stationary object is placed according to a known probability distribution in one of two disjoint, separated regions (usually taken as networks). Two searchers, starting at given points in each region, can alternate moving at unit speed in their region, until the first time T that one of them reaches the location at which the object is hidden. The problem is to find the object in the minimum expected time. An equivalent formulation is that there is a continuous valuable material (e.g. coal) whose mass is distributed according to the given distribution (no longer interpreted probabilistically) along some network, which is to be mined as quickly as possible. A slow moving borer is placed at each location, but given constraints either on electric power or on crew, only one can be operated at a time. If there is a substantial travel cost or travel time for the crew, then this must be modeled, and the problem becomes more complicated. However even the basic problem presents significant difficulties. These difficulties appear, for example, in a simple problem which we analyze where searchers start at the origins of two lines, and the object is hidden equiprobably on each line and in each direction, at distances 1 and 2 from the origin with respective probabilities .7 and .3.

An interesting question in alternating search is how the joint motions of the two searchers compare to how each would individually search his own region if he knew the object was there. We show that in general there is no simple relation between the individual and joint (global) problems by giving examples of symmetric problems (the regions and local distributions are identical) where the joint solution has the searchers employing distinct local searches, one of which is not optimal for the local problem. The problem mentioned in the previous paragraph is one such example. Another significant problem arises as a subproblem in alternating search after the individual motion of each searcher within his region has been selected: How should the pattern of alternating movement between the searchers be determined? This subproblem, for which we will derive a fairly simple solution, can be interpreted as a nonspatial search problem as follows: An object is lost in one of two regions. The probability that it will be found after searching region i for time t is a given function $F_i(t)$. If by time t region 1 has been searched for a cumulative time $\alpha(t)$ (and hence region 2 for time $t-\alpha(t)$) the probability that the object has been found is $F^\alpha(t) = F_1(\alpha(t)) + F_2(t-\alpha(t))$. How should we choose the alternation rule α to minimize the expected time $\int_0^\infty t \, dF^\alpha(t)$ required to find the object?

An important motivation for this problem is its connection with the asymmetric rendezvous search problem on the line ([Alpern, 1995],[Alpern and Gal, 1995]), where two searchers who cannot see each other try to meet in minimal time. We show in this paper, and in a companion one [Alpern and Beck, 1998], that such rendezvous problems have an equivalent formulation in terms of alternating search. Indeed, as an application of results on alternating search obtained in this paper, a positive solution to a conjecture in [Baston and Gal, 1998] on asymmetric rendezvous search on the line has been obtained in [Alpern and Beck, 1998]. The reduction of rendezvous search to alternating search depends on a certain group structure in the rendezvous region. This structure is also present in the rendezvous problem on the circle, as proposed in [Alpern, 1995] and solved for the uniform initial distribution in [Howard, 1999]. The results of this paper can also be used to extend Howard's proof to more general distributions, though that application will appear in a subsequent paper. To avoid a natural confusion it should be said that the two searchers in the alternating search problem *do not* correspond to the two searchers in the rendezvous problem.

The paper is organized as follows. Section 2 gives a formalization of the general problem of alternating search at two (or more) regions. It establishes certain relations between local and global search strategies. Section 3 introduces the double linear search problem, where both search regions are (infinite) lines. A family of symmetric (with respect to the lines and their origins) distributions are considered, for which the global solutions involve distinct local searches, which may be locally suboptimal. This distribution family was constructed to counter a conjecture of S. Gal concerning symmetric solutions of symmetric problems. Section 4 explains the connection between double linear search and asymmetric rendezvous on the line. Section 5 is the most important. It deals with the case where each region is an infinite ray $[0, \infty)$ and each searcher starts at 0. In fact, *every* alternating search problem reduces to this one after the regional search trajectories have been chosen if the distance along ray i is identified with the time spent searching region i according to the chosen regional strategy. So this corresponds to the pure alternation subproblem mentioned above. Section 6 discusses the application of the results of section 5 to the double linear search problem, completing the analysis begun in section 3. Section 7 briefly summarizes the article and suggests some possible extensions.

We conclude this Introduction with a brief word concerning the connections between this paper and its sister paper [Alpern and Beck, 1998]. While working on the asymmetric rendezvous problem on the line for the article [Alpern and Beck, 1998], it was observed that it could be reduced to a problem of alternating search on two lines, for which there was no theory. Such a theory was constructed by the present authors (this paper) and then used to complete the analysis in [Alpern and Beck, 1998]. The results of this paper are more general, and are applied in [Alpern and Beck, 1998], but the results of that paper are not used here.

2. FORMALIZATION OF THE PROBLEM

In this section we formalize the alternating search problem and explain the connections between the global solution and the combination of regional solutions with an alternation pattern.

For each $i = 1, 2$, there is a *search region* X_i which we assume to be arcwise connected with respect to a metric d_i. In each region, there is a given *starting point* 0_i. In practice, we will always take X_i to be a connected network (union of arcs), with arc length as distance. We assume there is no path connecting X_1 with X_2. There is no need to restrict the number of regions to 2, this is just for simplicity of presentation. There is a known Borel probability measure μ on the disjoint union $X = X_1 \cup X_2$ which describes the location of the 'object', and is written as $\mu_1 + \mu_2$, where μ_i has support in X_i. If there is an isomorphism from $(X_1, 0_1, \mu_1)$ onto $(X_2, 0_2, \mu_2)$ (preserving distances and measures) we will call the problem *symmetric*. We shall assume that $\mu_i (X_i) > 0$, $i = 1, 2$.

A *global strategy* is a path $g = (g_1, g_2) : [0, \infty) \to X_1 \times X_2$ with $g_i (0) = 0_i$, which describes the positions of the two searchers. We further assume that the sum of the speeds of the two searchers g_i is equal to 1. When X_i are lines (or circles obtained by identifications) we require the g_i to be Lipschitz continuous with

$$|g_1'| + |g_2'| = 1 \ ,$$

(when the derivatives exist). More generally we define speed as the derivative of the total variation, assuming g_i to have bounded variation. This is a departure from the description of the problem in the Introduction, where we said that only one searcher moved at a time. If that is the case, *i.e.* if either $|g_1' (t)| = 0$ or $|g_2' (t)| = 0$ except for a discrete set of *switching times* t, then we say that g is a *simple* strategy, or $g \in \mathcal{S}$. We need the more general strategies because the simple strategies are not closed, indeed the closure of $\mathcal{S}$ in the topology of uniform convergence on compact sets is the set $\mathcal{G}$ of global strategies, which are compact in that topology. Given any location $x \in X$ and $g \in \mathcal{G}$, we define the capture time T by $T (g, x) = \min \{t : d_i (x, g_i (t)) \leq \varepsilon\}$, when $x \in X_i$, where $\varepsilon \geq 0$ is a given capture radius. In practice, when we deal with one dimensional spaces, we will take $\varepsilon = 0$ (called point capture) so that the condition becomes $x = g_i (t)$. More generally we define $T (g)$ as the expected capture time,

$$T (g) = \int_X T (g, x) \ d\mu (x) .$$

Given the data $(X_i, 0_i, \mu_i)$, $i = 1, 2$, the *value* v, giving the least expected search time, is defined by

$$v = \min_{g \in \mathcal{G}} T (g) = \inf_{g \in S} T (g) ,$$

and any global strategy g with $T (g) = v$ is called optimal, or globally optimal. The use of the minimum is justified by the upper semicontinuity of T and the compactness of $\mathcal{G}$ with respect to the topology we are using.

Given the condition $|g_1'| + |g_2'| = 1$ a.e., we may assign to every global strategy g an *alternation rule* $a(t)$ by $a_i(t) = \int_0^t |g_i'(s)|\, ds$. When g is a simple strategy, $a_i(t)$ represents the total amount of time that searcher i has been moving, up to time t. More generally, for an arbitrary global strategy g, $a_i(t)$ represents the part of the total length t (searched by the two players) that player i has searched. In either case, we have $a_1(t) + a_2(t) = t$ for all t. A path which searches the same set at speed one in time $a_i(t)$ is given by $r_i(s) = g_i\left(a_i^{-1}(s)\right)$. Note that this is well defined even though $a_i^{-1}(s)$ need not be unique. We call the path r_i a *regional strategy*, and more specifically the regional strategy induced on X_i by the global strategy g. In general, a regional strategy on X_i is a speed one path in X_i which starts at 0_i. Summarizing this paragraph, we see that a global strategy g uniquely determines an alternation rule a and two regional strategies r_1 and r_2.

In a similar manner, if we are given two regional strategies r_1 and r_2 and an alternation rule a, we get a global strategy $g = g[r,a]$ by the formula

$$g_i(t) = g_i[r,a](t) = r_i(a_i(t)), \ i = 1, 2.$$

Consequently, we may reformulate the minimization problem in two parts: first determine the regional strategies to be followed by the two searchers, and then determine the optimal way in which to have the two searchers alternate moving. That is,

$$v = \min_{r_1, r_2} \left(\min_a T(g[r,a]) \right).$$

In section 5 we give a solution to the subproblem of finding the alternation rule a which minimizes $T(g[r,a])$ for a given pair r_1, r_2 of regional search strategies. To solve this problem, we do not have to know r_i itself, but only the rate at which it finds the object, that is, the distribution function

$$F_i(t) = \mu_i \{x \in X_i : x = r_i(s), \text{ for some } s \le t\}.$$

For any alternation rule a, we can calculate $T(g[r,a])$ as a function only of the distributions F_i corresponding to the regional strategies r_i, by the formula

$$T(g[r,a]) = \int_0^\infty t \ d(F^a(t)), \text{where } F^a(t) = F_1(a_1(t)) + F_2(a_2(t)).$$

In section 5 we will show how to find the optimal alternation rule $\bar{a} = \bar{a}(F_1, F_2)$ and the value function $v = v(F_1, F_2)$ corresponding to a given pair of distribution functions F_1, F_2, defined by

$$v(F_1, F_2) = \min_a \int_0^\infty t \ d(F^a(t)). \tag{1}$$

We think of $\bar{a}(F_1, F_2)$ of representing the optimal way of interleaving two given distribution functions, in the sense of producing a combined distribution function $F^{\bar{a}}$ with minimal first moment. In section 5 we will use the notation $(a_1, a_2) = (\alpha, \beta)$ and identify a strategy with its first coordinate α.

We conclude this section with a brief discussion of domination of search strategies, or their resulting distributions, which will reduce the set of potential optimal strategies. We say that one distribution dominates another if it is at least as large for all t and larger for some t. Consequently if one distribution dominates another it has a smaller first moment. Similarly, we say that one search strategy dominates another if it has a dominating distribution. Regional search strategies which are dominated cannot be part of a global optimum, as stated precisely below.

Lemma 1. *If $\tilde{r}_1$ dominates $\bar{r}_1$ then no globally optimal strategy g induces the regional strategy $\bar{r}_1$ on X_1.*

Proof. Given any regional strategy r_2 on X_2, and any alternation rule a, the global strategy $\tilde{g}$ generated by $\tilde{r}_1, r_2$ and a dominates the strategy $\bar{g}$ generated by $\bar{r}_1, r_2$ and a, since its distribution $\tilde{F}^a$ dominates $\bar{F}^a$.

3. Double Linear Search

In this section we consider an important special case of alternating search which we introduce concurrently in [Alpern and Beck, 1998] as the 'Double Linear Search Problem' (DLSP). In this problem, two searchers starting at the origins of distinct lines must move at combined speed one to find a single object placed according to a known distribution on the union of the lines. If there is only one line (or the object has zero probability of lying on line 2), then this is the well known Linear Search Problem introduced in [Bellman, 1963] and [Beck, 1964].

We consider here a particular one parameter family of distributions μ^p which is a kind of universal counterexample to numerous possible conjectures regarding globally optimal solutions to alternating search problems. In was constructed in particular to produce a counterexample to a conjecture of S. Gal that a symmetric DLSP problem must have a symmetric solution.

For μ^p, the object is hidden equiprobably on the two lines, and equiprobably on each side. Its distance to the origin is either 1 or 2, with respective probabilities p and $1-p$. In the notation of the previous section μ_i is an atomic measure with atoms of probability $p/4$ at ± 1 and $(1-p)/4$ at ± 2. All the interesting behavior is for $p > 1/2$, so the reader should think of a big atom at ± 1 and a small one at ± 2. The treatment given in this section will not be exhaustive, in that we will not consider all possible optimal interleavings. This gap will be remedied in section 6, where we will have access to the theory of optimal interleavings developed in section 5.

According to Lemma 1, we need only consider strategies which are undominated at the regional level, that is, for searching on a single line in the Linear Search Problem. It is easy to see that there are only two of these (up to symmetry): the *efficient* strategy A which goes directly to +2 and then back to -2, and the *greedy* strategy B which gets the big atoms first by going to +1, then to -1, continuing to -2 and then +2. At the regional level, with probabilities normalized, we see that the expected capture times for the efficient strategy A and the greedy strategy B are given by

$$\begin{aligned} T_A &= \frac{p}{2}1 + \frac{1-p}{2}2 + \frac{p}{2}5 + \frac{1-p}{2}6 = -p + 4, \text{ and} \\ T_B &= \frac{p}{2}1 + \frac{p}{2}3 + \frac{1-p}{2}4 + \frac{1-p}{2}8 = -4p + 6. \end{aligned}$$

Consequently the efficient strategy A is locally optimal for $p \leq 2/3$ and the greedy strategy B is optimal for $p \geq 2/3$. According to the general analysis given in the previous section, we need only consider global strategies based on the regional strategies A and B (both A, both B, or one A and one B).

If both players use the efficient strategy A as their regional strategy, there are two global strategies which are optimal for some p, which we will call $[AA]_1$, $[AA]_2$. These are among the strategies plotted parametrically in Figure 2 below. In the global strategy $[AA]_1$, the searchers go in turn to 1, then in turn to 2, then in turn to -2. In the global strategy $[AA]_2$, they alternate going to 2, then they alternate going to -2. The global strategy $[AA]_1$ is illustrated in the following table, which gives the positions of players I and II at times 0,1,...,12. The superscript * indicates a possible finding of the object at ± 2 and $^{\#}$ indicates a possible finding at ± 1. (Note that the table has four entries with a $^{\#}$ and four with a *.)

time	0	1	2	3	4	5	6	7	8	9	10	11	12
I's location, using $[AA]_1$	0	$1^{\#}$	1	2*	2	1	0	$-1^{\#}$	-2*	-2	-2	-2	-2
II's location, using $[AA]_1$	0	0	$1^{\#}$	1	2*	2	2	2	2	1	0	$-1^{\#}$	-2*

The expected capture time for strategy $[AA]_1$ is calculated by multiplying the times with * by $(1-p)/4$ and the times with # by $p/4$, giving the eight terms as follows:

$$1/4\,(1p + 2p + 3\,(1-p) + 4\,(1-p) + 7p + 8\,(1-p) + 11p + 12\,(1-p)) = -\frac{3}{2}p + \frac{27}{4}.$$

A similar table is shown below for the strategy $[AA]_2$.

time	0	1	2	3	4	5	6	7	8	9	10	11	12
I's location, using $[AA]_2$	0	$1^{\#}$	2*	2	2	1	0	$-1^{\#}$	-2*	-2	-2	-2	-2
II's location, using $[AA]_2$	0	0	0	$1^{\#}$	2*	2	2	2	2	1	0	$-1^{\#}$	-2*

Among the global strategies which use B at both locations, there are three which give a minimal expected time for some value of p : $[BB]_1$ in which they alternate going to 1, then they alternate going to -2, then to 2; $[BB]_2$ in which they alternate going to 1 and -2, then they alternate going to 2; $[BB]_3$ in which they alternate going to 1, then alternate going to -1, then to -2, and then to 2.

If player I uses A (efficient) and II uses B (greedy) the two interleavings which are best in this class for some p are as follows. $[AB]_1$, where they alternate going to 1, then I goes to 2, then II goes to -2, then I goes to -2, then II goes to 2; and $[AB]_2$, they alternate going to 1, then II goes to -1, then I goes to 2 , then II goes to -2,then I goes to -2, then II goes to 2.

strategy	$[AA]_1$	$[AA]_2$	$[BB]_1$	$[BB]_2$	$[BB]_3$	$[AB]_1$	$[AB]_2$
time T	$-\frac{3}{2}p+\frac{27}{4}$	$-p+\frac{13}{2}$	$-\frac{27}{4}p+\frac{41}{4}$	$-6p+10$	$-\frac{15}{2}p+\frac{43}{4}$	$-4p+\frac{33}{4}$	$-\frac{19}{4}p+\frac{35}{4}$

In Figure 1 below, we plot these expected capture times as functions of p.

After solving for some relevant intersection points, we determine the optimal strategies for different p ranges, as follows.

p range	local optimum	global optimum
$[0, 1/2]$	A (efficient)	$[AA]_2$
$[1/2, 3/5]$	A (efficient)	$[AA]_1$
$[3/5, 2/3]$	A (efficient)	$[AB]_1$
$[2/3, 8/11]$	B (greedy)	$[AB]_2$
$[8/11, 1]$	B (greedy)	$[BB]_3$

We see from this table that for $p \leq 3/5$ both searchers should follow the efficient strategy (A) and that for $p \geq 8/11$ both should follow the greedy strategy (B). However in the intermediate range $3/5 \leq p \leq 8/11$, one should follow the greedy strategy and the other the efficient strategy. This contradicts Gal's conjecture regarding symmetric solutions and also shows that regional strategies that are locally suboptimal (that is, B for p in $[3/5, 2/3]$, or A in $[2/3, 8/11]$) may form part of a globally optimal strategy.

We conclude this section with a drawing (Figure 2) which parametrically plots the five global strategies which are optimal for some value of p. In these drawings, line 1 is the horizontal axis and line 2 is the vertical axis. Note that each strategy starts at the center of the of the square $[-2, 2] \times [-2, 2]$ (drawn with a thin line) and must meet all four of its sides to ensure finding the object.

4. Asymmetric Rendezvous on the Line

As indicated in the Introduction, the alternating search problem is motivated by its connection with rendezvous search. We give here a brief discussion of this connection when the search region is the line. Suppose that two players are placed a distance d apart on the line and faced in random directions (equiprobably for the four possible ways). Let us suppose each player has maximum speed $1/2$. If their positions at time t are respectively $h_i(t)$ in the direction they were initially facing, $i = 1, 2$, then if they began by facing each other they will have met *by* time t if and only if

$$\max_{s \leq t} \left(h_1(s) + h_2(s)\right) \geq d. \tag{2}$$

Define $g_1(t) = h_1(t) + h_2(t)$ and $g_2(t) = h_1(t) - h_2(t)$. Note that the global strategy $g = (g_1, g_2)$ in the double linear search problem of the previous section will have found an object placed on line 1 at a distance d to the right of the origin, if and only if condition(2) holds. The four possible ways of facing the two rendezvous searchers correspond to the four possible ways of placing the object at distance d in the symmetric alternating search problem on two lines. In particular, the expected time for two rendezvous searchers on the line to find each other when using strategies (h_1, h_2) if their initial distance d is chosen from a cumulative distribution function F is the same as the expected search time $T(g)$ for a symmetrically placed object at distance d in the DLSP. The details of this correspondence are explained more fully in [Alpern and Beck, 1998].

Consider the special case where d is known to be 2 (a point distribution). This corresponds to the case $p = 0$ in the previous section, where the double linear search problem (alternating search on two lines) has the solution we called $[AA]_2$. If we solve for h_1 and h_2 in terms of g_1 and g_2, we obtain $h_1 = (g_1 + g_2)/2$ and $h_2 = (g_1 - g_2)/2$. If we take for g the global strategy $[AA]_2$, the associated rendezvous strategies h_1 and h_2 are as graphed below:

Figure 3: Optimal rendezvous strategies, $d = 2$

Rendezvous player 1 follows the thin line $h_1(t) = 2 - |t-4|/2$ which means he goes forward for 4 hours and the reverses for 8 hours, always at maximum speed 1/2. Rendezvous player 2 follows the thick line $h_2(t) = (-6 + t + |t-8| - |t-2|)/2$ which means he goes forward for 2 hours, then backwards for 6 hours, and finally forward for 4 hours. If, as assumed, they start 2 units apart, then they meet equiprobably at the four times 2,4,8, and 12, when $\pm h_1 \pm h_2 = 2$,which are the same times that the searchers in the double linear search problem might find the object if it is placed at ± 2 (see the starred numbers in the table for the global strategy $[AA]_2$ in the previous section). The intersection times can be seen from the figure below, in which the origin is placed at player 2's starting point, player 2 is facing up, and player 1 paths corresponding to the four possible initial conditions are given.

Figure 4: Player 2 and four paths of player 1

This is the solution to the asymmetric rendezvous problem on the line with a known initial distance $d = 2$, given in [Alpern and Gal, 1995]. It has been conjectured in [Baston and Gal, 1998] that the above rendezvous strategy remains optimal if the initial distance d between the players has a distribution which is convex on $[0, 2]$. This conjecture is true, and it can be proved using the theory of optimal interleaving of distributions that we develop here in the next section, in particular the result on optimal interleaving of a convex distribution (the last assertion in Corollary 6). The full proof of the correctness of the Baston-Gal conjecture is contained in the companion paper to this one, [Alpern and Beck, 1998].

5. Theory of Optimal Interleaving of Distributions

The results of this, the main section of the paper, can be thought of as the special case where the spaces X_i are both the ray $[0, \infty)$, with both searchers starting at the origin. In this case, there is only one undominated regional strategy, namely the simple speed one path $r_i(t) = t$. So the only problem is the selection of the alternation rule a, which will depend on the distributions F_1 and F_2 corresponding to the measures μ_1 and μ_2 on the two rays. Of course the significance of this analysis is that once regional strategies r_i are selected for *any* search spaces, this is exactly the subproblem that the searchers face.

For this reason, we prefer to interpret the results of this section in the following manner. Suppose that two distributions F_1 and F_2 (or equivalently, the two measures μ_1 and μ_2) are given. What is the alternation rule a which minimizes the first moment M of the amalgamated distribution $F^a(t) = F_1(a_1(t)) + F_2(a_2(t))$, that is

$$M(a) = \int_0^\infty t \; dF^a(t).$$

When the sum $\mu = \mu_1 + \mu_2$ is a probability measure, the moment M defined above coincides with the expected time operator T introduced in section 2. To simplify our notation in this section, we will identify the alternation rule a with its first coordinate a_1, which we will denote by α, and use the notation $\beta(t) = a_2(t) = t - \alpha(t)$ for the search position on line 2 at time t. We will call the associated moment simply $M(\alpha)$. We will call α a *strategy*, since in the context of this section it is the only strategic variable.

More precisely, a strategy is a monotonic increasing function $\alpha : R^+ \to R^+$ such that

$$\alpha(0) = 0, \text{ and } \alpha(t) - \alpha(s) \leq t - s, \qquad \forall s \leq t.$$

The collection of all strategies is denoted by $\mathcal{A}$, a set which we endow with the topology of uniform convergence on compact sets. The Arzela-Ascoli Theorem ensures that $\mathcal{A}$ is compact in this topology. A strategy α has a derivative almost everywhere, which must lie within the closed unit interval. When $\alpha'(x) = 1$, the searcher is looking in region 1, and when $\alpha'(x) = 0$, he is looking in region 2. A noted earlier, we call a strategy *simple* if it is doing one of these two things except at a discrete set of 'switching times'.

Of course if one of the distributions F_i has an infinite first moment, then $M(\alpha)$ cannot be finite for any strategy α. On the other hand, if both F_i have finite moments then the symmetric strategy $\alpha(t) = t/2$ will have a finite moment (twice the sum of the moments of the F_i). So we assume that both F_i have finite moments. The fact that M is lower semicontinuous in α and $\mathcal{A}$ is compact combine to guarantee the existence of

$$v = v(F_1, F_2) = \min_{\alpha \in \mathcal{A}} M(\alpha).$$

5.1. Simple Strategies. We can get as close to v as we like using simple strategies, using the following approximation result.

Lemma 2. *Suppose we are given a strategy $\alpha \in \mathcal{A}$, a positive number ε, and a discrete set of nonnegative 'times' $\mathcal{T}$. Then there is a simple strategy $\hat{\alpha} \in S$ which agrees with α on $\mathcal{T}$ and satisfies $\hat{\alpha}(t + \varepsilon) \geq \alpha(t)$ and $\hat{\beta}(t + \varepsilon) \geq \beta(t)$ for all $t \geq 0$. Since $\hat{\alpha}$ reaches any point on line 1 or line 2 at in time at most ε greater than α does, $M(\hat{\alpha}) \leq M(\alpha) + \varepsilon$.*

Proof. By adding additional points to $\mathcal{T}$, if necessary, we may assume that every time $t \notin \mathcal{T}$ lies in a unique interval $[t_i, t_{i+1}]$ whose endpoints are in $\mathcal{T}$ and satisfy $t_{i+1} - t_i \leq \varepsilon$. The required simple approximation $\hat{\alpha}(t)$ may be defined as $\alpha(t_i) + (t - t_i)$ for $t \in [t_i, t_i + \alpha(t_{i+1}) - \alpha(t_i)]$ (searching line 1) and as $\alpha(t_{i+1})$ on $[t_i + \alpha(t_{i+1}) - \alpha(t_i), t_{i+1}]$ (searching line 2). ■

The above proof leads to an interesting question regarding the best order of searching the two lines. Suppose strategy α first searches the interval $[0, L_1]$ on line 1 and then the interval $[0, L_2]$ on line 2, and strategy γ searches them in the opposite order.

After time $L_1 + L_2$ the two strategies agree. Which method is better? To answer this we calculate the difference

$$\begin{aligned} M(\alpha) - M(\gamma) &= \left[\int_0^{L_1} t\, F_1(t) + \int_0^{L_2} (t + L_1)\ dF_2(t)\right] \\ &\quad - \left[\int_0^{L_2} t\ dF_2(t) + \int_0^{L_1} (t + L_2)\ dF_1(t)\right] \\ &= \int_0^{L_2} L_1\ dF_2(t) - \int_0^{L_1} L_2\, dF_1(t) \\ &= L_1 \cdot F_2(L_2) - L_2\ \cdot F_1(L_1). \end{aligned} \tag{3}$$

Note that this will be positive if $F_2(L_2)/L_2 > F_1(L_1)/L_1$, that is, when $[0, L_2]$ on line 2 has a higher density than $[0, L_1]$ on line 1. *So to minimize M one should search the interval with higher density first.* Another way to calculate (3) is to use the fact that a mass contributes to a moment as if it were concentrated at its center of mass. Calling this center c_i, we can do the above calculation without integrals as

$$\begin{aligned} M(\alpha) - M(\gamma) &= F_1(L_1) \cdot c_1 + F_2(L_2)(L_1 + c_2) - [F_2(L_2) \cdot c_2 + F_1(L_1) \cdot (L_2 + c_1)] \\ &= L_1 \cdot F_2(L_2) - L_2\ \cdot F_1(L_1). \end{aligned}$$

Note that the center of masses c_i cancel out. Although we shall not need to use the above observation in our subsequent analysis, it is very intuitive and worth recording as the following.

Proposition 3. *If a strategy begins by searching an initial interval $[0, L_1]$ on line 1, and then searches a higher density initial interval $[0, L_2]$ on line 2, its moment can be reduced by the positive amount $L_1 \cdot F_2(L_2) - L_2\ \cdot F_1(L_1)$ by reversing the order of these searches.*

A far reaching generalization of the above idea of searching intervals in order of their density is the following.

Lemma 4. *Let α be a simple strategy with switching times which include $t_0 \leq t_1 < t_2 < t_3 \leq t_4$. Denote $x_i = \alpha(t_i)$ and $y_i = \beta(t_i)$, for $i = 0 \ldots, 4$. Let $0 < w < r$. Assume that the ratio $(F_1(x) - F_1(x_0)) / (x - x_0)$ is bounded above by r for all x in $(x_0, x_4]$, and by w on $[x_1, x_3]$. Similarly, assume that the ratio $(F_2(y_4) - F_2(y)) / (y_4 - y)$ is bounded below by r for all y in $[y_0, y_4)$. Let $\hat{\alpha}$ denote the simple strategy which agrees with α outside of $[t_0, t_4]$ and first searches region 2 ($\hat{\alpha}' = 0$) from y_0 to y_4 for t in $[t_0, t_0 + y_4 - y_0]$ and then region 1($\hat{\alpha}' = 1$) from x_0 to*

x_4 *for* t *in* $[t_0 + y_4 - y_0, t_4]$. *Then the moment reduction satisfies*

$$\begin{aligned} M(\alpha) - M(\hat{\alpha}) \geq & \left(1 - \frac{w}{r}\right)(F_2(y_3) - F_2(y_1))(x_1 - x_0) \\ & + \left(1 - \frac{w}{r}\right)(F_2(y_3) - F_2(y_2))(x_2 - x_1). \end{aligned}$$

Proof. We restrict our attention entirely to the time interval $[t_0, t_4]$, as the two strategies are identical outside this interval. In this period the strategy α will alternate between searching the two regions in a finite number of left-open right-closed intervals which we label $I_1, J_1, I_2, J_2, \ldots, I_n, J_n$, searching region 1 during the I intervals and region 2 during the J intervals. Since the t_i are assumed to be switching points of α, we may if necessary insert empty J intervals to be searched just prior to times $t_1, \ldots, t_4$, and empty I intervals to be searched starting at times $t_0, \ldots, t_3$ so that J_d, J_k, J_e, and J_n are the last intervals searched by times t_1, t_2, t_3, and t_4. In any case, we have for line 1 that $(x_0, x_1] = \bigcup_{i=1}^{d} I_i$, $(x_1, x_2] = \bigcup_{i=d+1}^{k} I_i$, and so on.

We now calculate the portion of the total moment of α that is contributed between times t_0 and t_4. To do this, we must know the length, mass, and center of gravity of the intervals I_i and J_i with respect to the distributions F_1 (for the I's) and F_2 (for the J's). Denote the length and mass (measure) of I_i by a_i and p_i, and denote the sum of t_0 $(= x_0 + y_0)$ plus the distance of the center of mass of I_i from its left end point by a_i^*. For J_i denote these by b_i, q_i, and b_i^*. In calculating a moment, the mass of an interval behaves as if it were all at the center of mass. In the strategy α, the center of mass of the interval I_i is reached at time $a_1 + b_1 + \ldots + a_{i-1} + b_{i-1} + a_i^*$. Similarly for the J_i. Hence the moment contributed by α between times t_0 and t_4 is calculated as

$$M_{t_0}^{t_4}(\alpha) = p_1 a_1^* + q_1(a_1 + b_1^*) + p_2(a_1 + b_1 + a_2^*) + \ldots + q_n(a_1 + b_1 + \ldots + a_n + b_n^*).$$

In the strategy $\hat{\alpha}$, the intervals are searched in the order $J_1, \ldots, J_n, I_1, \ldots, I_n$ so the corresponding moment is given by

$$\begin{aligned} M_{t_0}^{t_4}(\hat{\alpha}) = & \; q_1(b_1^*) + \ldots + q_n(b_1 + \ldots + b_{n-1} + b_n^*) + \\ & p_1(b_1 + \ldots + b_n + a_1^*) + \ldots + p_n(b_1 + \ldots + b_n + a_1 + \ldots + a_{n-1} + a_n^*). \end{aligned}$$

The reduction R in moment is therefore

$$\begin{aligned} R &= M_{t_0}^{t_4}(\alpha) - M_{t_0}^{t_4}(\hat{\alpha}) && (4) \\ &= q_1 a_1 + q_2(a_1 + a_2) + \ldots + q_n(a_1 + \ldots + a_n) \\ &\quad - [p_1(b_1 + \ldots + b_n) + p_2(b_2 + \ldots + b_n) + \ldots + p_n b_n] \\ &= \sum_{i=1}^{n} q_i \sum_{j=1}^{i} a_j - \sum_{i=1}^{n} p_i \sum_{j=i}^{n} b_j && (5) \end{aligned}$$

Note that the center of mass positions cancel out. Translating the hypotheses regarding the ratios $(F_1(x) - F_1(x_0))/(x - x_0)$ into our interval notion, we have that

$$\frac{p_1 + \ldots + p_i}{a_1 + \ldots + a_i} \le \lambda_i, \text{ where } \lambda_i = \begin{cases} r, & \text{for } i = 1, \ldots, d, \text{and } e+1, \ldots, n. \\ w, & \text{for } i = d+1, \ldots, e. \end{cases} \tag{6}$$

Similarly, for the line 2 ratios we have

$$\frac{q_i + q_{i+1} + \ldots + q_n}{b_i + \ldots + b_n} \ge r, \text{ for } i = 1, \ldots, n. \tag{7}$$

Substituting the second ratio inequality (7) into the formula (4) for the moment reduction R, we obtain the inequality

$$\begin{aligned} R &\ge \sum_{i=1}^{n} q_i \sum_{j=1}^{i} a_j - \sum_{i=1}^{n} \frac{p_i}{r} \sum_{j=i}^{n} q_j \\ &= \sum_{i=1}^{n} q_i \left[\sum_{j=1}^{i} a_j - \frac{1}{r} \sum_{j=1}^{i} p_j \right] \end{aligned}$$

Now substituting the first ratio inequality (6) into the above expression for R, we get

$$\begin{aligned} R &\ge \sum_{i=1}^{n} q_i \left[\sum_{j=1}^{i} a_j - \frac{\lambda_i}{r} \sum_{j=1}^{i} a_j \right] \\ &= \sum_{i=1}^{n} q_i \left(1 - \frac{\lambda_i}{r}\right) \sum_{j=1}^{i} a_j \\ &= \left(1 - \frac{w}{r}\right) \sum_{i=d+1}^{e} q_i \sum_{j=1}^{i} a_j, \text{ since } r = \lambda_i \text{ if } i \notin \{d+1, \ldots, e\} \\ &= \left(1 - \frac{w}{r}\right) \sum_{i=d+1}^{e} q_i \sum_{j=1}^{d} a_j + \left(1 - \frac{w}{r}\right) \sum_{i=d+1}^{e} q_i \sum_{j=d+1}^{i} a_j \\ &\ge \left(1 - \frac{w}{r}\right) (F_2(y_3) - F_2(y_1))(x_1 - x_0) + \left(1 - \frac{w}{r}\right) \sum_{i=d+1}^{e} q_i \sum_{j=d+1}^{i} a_j \\ &\ge \left(1 - \frac{w}{r}\right) (F_2(y_3) - F_2(y_1))(x_1 - x_0) + \left(1 - \frac{w}{r}\right) \sum_{i=k+1}^{e} q_i \sum_{j=d+1}^{k} a_j \\ &\ge \left(1 - \frac{w}{r}\right) (F_2(y_3) - F_2(y_1))(x_1 - x_0) + \left(1 - \frac{w}{r}\right) (F_2(y_3) - F_2(y_2))(x_2 - x_1). \end{aligned}$$

■

5.2. Optimal Strategies. We now begin to determine some conditions that an optimal strategy α must satisfy. To do this, let $\bar{F}_i(x)$ be the concavification of $F_i(x)$, that is, the smallest concave function with $\bar{F}_i(x) \geq F_i(x)$ for all x. Such a concavification always exists and whenever $\bar{F}_i(x) > F_i(x)$ there are always $x_* < x < x^*$ with $\bar{F}_i(x_*) = F_i(x_*)$ and $\bar{F}_i(x^*) = F_i(x^*)$. Since each $\bar{F}_i$ is concave, its left and right hand derivatives (denoted D^- and D^+) exist everywhere and are equal except on a countable set. Both $D^-\left(\bar{F}_i(x)\right)$ and $D^+\left(\bar{F}_i(x)\right)$ are nonincreasing, with $D^-\left(\bar{F}_i(x)\right) \geq D^+\left(\bar{F}_i(x)\right)$. See, for example (Royden, 1968) for demonstrations of these properties of concave functions.

The next theorem shows that an optimal strategy searches the regions using the derivative intervals D_i defined by $D_i(x) = \left[D^+\left(\bar{F}_i(x)\right), D^-\left(\bar{F}_i(x)\right)\right]$ as priority indices. Note that $D_i(x)$ is a single point except for countably many x.

Theorem 5. *If the strategy α (with $\beta(t) = t - \alpha(t)$) is optimal for the distributions F_1, F_2, then for all t with $\alpha(t)$ and $\beta(t)$ both positive, we have*

$$D_1(\alpha(t)) \cap D_2(\beta(t)) \neq \emptyset.$$

Furthermore, if $\alpha(t) = t > 0$ (and hence $\beta(t) = 0$), we have $D^-\left(\bar{F}_1(\alpha(t))\right) \geq D^+\left(\bar{F}_2(0)\right)$.

Proof. Suppose either statement is false. Then for some time $t_2 > 0$ and positive numbers r and w, we have $\alpha(t_2) > 0$ and

$$D^-\left(\bar{F}_1(\alpha(t_2))\right) < w < r < D^+\left(\bar{F}_2(\beta(t_2))\right).$$

(or a similar statement holds interchanging 1, 2 and α, β). Let $\alpha(t_2) = x_2 > 0$ and $\beta(t_2) = y_2$. The supporting line to $\bar{F}_1(x)$ with slope w touches $F_1(x)$ at some point $x_1 = \alpha(t_1) < x_2$. Note that if $w > D^+\left(\bar{F}_1(0)\right)$, then $x_1 = 0$. Similarly the supporting line to $\bar{F}_2(y)$ with slope r touches $F_2(y)$ at some point $y_3 = \beta(t_3)$ for some $y_3 > y_2$. Define $y_1 = \beta(t_1)$ and $x_3 = \alpha(t_3)$. To apply Lemma 4 formally we will need to set up artificial subscripts 0,4, equal respectively to $1, 3$, so set $t_0 = t_1$, $x_0 = x_1$, $y_0 = y_1$, $t_4 = t_3$, $x_4 = x_3$, and $y_4 = y_3$.

Since each concave function $\bar{F}_i$ lies on or below its supporting lines, and $F_i(x)$ lies on or below $\bar{F}_i$, we have that

$$\begin{aligned} F_1(x) - F_1(x_1) &\leq w(x - x_1) \text{ for } x > x_1 \text{ and} \\ F_2(y_3) - F_2(y) &\geq r(y_3 - y) \text{ for } y < y_3. \end{aligned}$$

Set $\varepsilon = \left(1 - \frac{w}{r}\right)(F_2(y_3) - F_2(y_2))(x_2 - x_1)$, which is positive, and use Lemma 2 to approximate α by a simple strategy $\tilde{\alpha}$ with $M(\tilde{\alpha}) \leq M(\alpha) + \varepsilon/2 = v + \varepsilon/2$, since

α was assumed to be optimal. Apply Lemma 4 to the simple strategy $\tilde{\alpha}$ to obtain a strategy $\hat{\alpha}$ with $M(\tilde{\alpha}) - M(\hat{\alpha}) \geq \varepsilon$, which implies the impossibility $M(\hat{\alpha}) < v$. Hence for optimal α, the conditions stated in the Theorem must hold. ■

The above theorem gives a static characterization of an optimal strategy in terms of the global nature of the distributions F_i. The following result is useful in determining the an optimal strategy dynamically. It deals with the problem of how to proceed for some initial interval of time, starting at time zero. Of course the problem can always be renormalized to a problem starting at time zero if one has been proceeding optimally up to some time t.

Corollary 6. *Suppose that $F_1(x)/x$ has a maximum of r_1 at x_1. Let $r_2 = D^+\left(\bar{F}_2(0)\right)$.*

$$\begin{aligned} (i)\ \text{If } r_1 &> r_2 \text{ then any optimal strategy satisfies } \alpha(x_1) = x_1. \\ (ii)\ \text{If } r_1 &\geq r_2 \text{ then some optimal strategy satisfies } \alpha(x_1) = x_1. \end{aligned}$$

In particular, if F_1 is convex on some interval $[x_2, x_3]$, then some optimal strategy α satisfies $\alpha(t_2) = x_2$ and $\alpha(t_2 + x_3 - x_2) = x_3$, for some time t_2.

Proof. (i) Suppose on the contrary that α is an optimal strategy with $\alpha(x_1) = s_1 < x_1$, and hence $\beta(x_1) = t_1 > 0$. Then since $\bar{F}_2$ is the concavification of F_2,we have

$$D^-\left(\bar{F}_2(t_1)\right) \leq D^+\left(\bar{F}_2(0)\right) = r_2.$$

But the assumption regarding x_1 implies that $\bar{F}_1(x) = r_1 x$ for $0 \leq x \leq x_1$, and consequently

$$D^+\left(\bar{F}_1(s_1)\right) = r_1 > r_2 \geq D^-\left(\bar{F}_2(t_1)\right).$$

But since $\alpha(x_1) = s$, and $\beta(x_1) = t_1$ this violates the optimality condition for α given in the previous theorem, so our assumption that $\alpha(x_1)$ could be less than $x_1$was false.

(ii) Suppose on the contrary that the best moment that can be attained by an α with $\alpha(x_1) = x_1$ is $v + \delta$, where $\delta > 0$ and v is the unrestricted minimum moment. Choose $\varepsilon > 0$ sufficiently small so that the value v^*, of the *-problem with an added atom of mass ε placed at distance x_1 on line (region) 1, satisfies $v^* < v + \delta$. (To do this note that any optimal strategy for the original problem reaches x_1 on line 1 at some first time τ and hence achieves the moment $v + \tau\varepsilon$ for the *-problem, so simply take $\varepsilon < \delta/\tau$.) Observe that for the *-problem the distribution F_2 is unchanged so that $r_2^* = r_2$ while

$$r_1^* = F_1^*(x_1)/x_1 = (F_1(x_1) + \varepsilon)/x_1 > r_1.$$

Consequently $r_1^* > r_2^*$ and so by part (i) there is a strategy α satisfying the condition $\alpha(x_1) = x_1$ which is optimal for the *-problem. Consequently it achieves a moment of

v^* in the *-problem and hence less than v^* in the original problem. Since $v^* < v + \delta$, this contradicts our assumption that $v+\delta$ was the minimum achievable in the original problem by any strategy with $\alpha(x_1) = x_1$.

To establish the final assertion, consider the problem remaining (after beginning with any optimal strategy) at the last time t_2 for which $\alpha(t_2) = x_2$. The convexity assumption implies that smallest x_1 which maximizes $(F_1(x_1) - F_1(x_2)) / (x_1 - x_2)$ (makes it equal to r_1) satisfies $x_1 \geq x_3$. At time t_2 we must have that $r_1 \geq r_2$, so that condition (ii) ensures that some optimal method of continuing the strategy searches the interval $[x_2, x_3]$ without interruption. ■

Theorem 5 gave a complete determination of the unique optimal strategy in the case that the concavifications $\bar{F}_1$ and $\bar{F}_2$ are not linear with the same slope on nondegenerate closed intervals. In the excluded case the optimal strategy is not unique. For example if $F_1(x) = F_2(x) = x/2$ on common support intervals $[0, 1/2]$, then clearly *any* strategy (which doesn't go past 1/2) is optimal. Corollary 6 showed furthermore that if the concavifications do have such intervals of common slope then it is optimal for either to be searched first in a block. The next result shows that if one of the distributions lies below its concavification on an interval, then that interval *must* be searched as a block by any optimal strategy.

Theorem 7. *Suppose the strategy α is optimal for distributions F_1, F_2. Suppose that $F_1(x) < \bar{F}_1(x)$ over the interval (x_0, x_4) with $F_1(x_0) = \bar{F}_1(x_0)$ and $F_1(x_4) = \bar{F}_1(x_4)$. Then the interval $[x_0, x_4]$ on line 1 is searched without interruption by α. That is, if t_0 is the largest t with $x_0 = \alpha(t)$, we must have $x_4 = \alpha(t_0 + x_4 - x_0)$.*

Proof. Let t_0 be the last t when $\alpha(t) = x_0$ and let t_4 be the first t when $\alpha(t) = x_4$. Assume the assertion of the theorem is false, so that α is optimal but $y_4 \equiv \beta(t_4) > y_0 \equiv \beta(t_0)$. Let r denote the slope of the straight line $\bar{F}_1(x)$ for x between x_0 and x_4, and note that Theorem 5 implies that $\bar{F}_2(y)$ is also a straight line with slope r between y_0 and y_4. If $F_2(y_0)$ were less than $\bar{F}_2(y_0)$ we could apply the previous theorem, normalized with $t_0 = 0$, to show that any optimal strategy must begin by searching line 2 for a positive time, contradicting our assumption that t_0 was the last time with $\alpha(t_0) = x_0$. So we may assume that $F_2(y_0) = \bar{F}_2(y_0)$ and similarly $F_2(y_4) = \bar{F}_2(y_4)$. This already gives us part of what we shall need to apply Lemma 4, namely

$$\begin{aligned} (F_1(x) - F_1(x_0)) / (x - x_0) &\leq r, \text{ for } x \in (x_0, x_4], \text{ and} \\ (F_2(y_4) - F_2(y)) / (y_4 - y) &\geq r, \text{ for } y \in [y_0, y_4). \end{aligned}$$

Since neither α nor β are constant on the interval $[t_0, t_4]$, they will not be constant on any subinterval $[t_1, t_3]$ with $t_1 - t_0$ and $t_4 - t_3$ sufficiently small. Choose any such times t_1 and t_3 with $t_1 - t_0$ positive and $F_2(y_3) - F_2(y_1) > 0$. For the second choice we may always take $y_3 < y_4$ unless the interval $[y_0, y_4]$ on line 2 has all its measure at y_4 (that is, $F_2(y) = F_2(y_0)$ for $y \in [y_0, y_4)$), a case which we will exclude for the moment. Define $w = \sup_{x_1 \le x \le x_3} (F_1(x) - F_1(x_0)) / (x - x_0)$. Since the distribution function F_1 is continuous from the right and nondecreasing, it is upper semicontinuous. Consequently the quotient in the definition of w is also upper semicontinuous, and hence assumes its maximum on the compact interval $[x_1, x_3]$. Since this F_1 lies strictly below $\bar{F}_1$ on $[x_1, x_3]$, this maximum must be less than r, so we have that $w < r$.

Let $\varepsilon = \left(1 - \frac{w}{r}\right)(F_2(y_3) - F_2(y_1))(x_1 - x_0)$, which is positive by our above construction, and apply Lemma 2 to obtain a simple strategy $\tilde{\alpha}$ with switching points t_0, t_1, t_3, t_4 satisfying $M(\tilde{\alpha}) - M(\alpha) < \varepsilon/2$. Then apply Lemma 4 to obtain a strategy $\hat{\alpha}$ with

$$M(\tilde{\alpha}) - M(\hat{\alpha}) \ge \left(1 - \frac{w}{r}\right)(F_2(y_3) - F_2(y_1))(x_1 - x_0) = \varepsilon.$$

Consequently $M(\alpha) - M(\hat{\alpha}) \ge \varepsilon/2 > 0$, which contradicts the optimality of α. Hence the assumption that α did not satisfy the asserted property must have been false.

Now consider the excluded case where $F_2(y) = F_2(y_0)$ for $y \in [y_0, y_4)$. If we have $F_1(x_3) > F_1(x_0)$ for some $x_3 \in (x_0, x_4)$, then the above proof will work if we reverse the roles of F_1 and F_2. So we need only consider the case of locating an object which is either at distance a $(= x_4)$ on line 1 or at a distance b $(= y_4)$ on line 2. Since the second location will be reached by any optimal strategy at time $a + b$, the unique optimum is to reach the first of these in minimum time, that is, without any interruption of search of that line. ■

In the double linear search problem, a typical search strategy for one of the lines involves reversing directions and hence retracing an interval that has already been searched. The probability distribution function for such an interval remains constant between the local times in which it is searched. In the analysis of such problems (Alpern and Beck, 1998) it is essential to know that the search of this interval on is not interrupted by motion of the other searcher. This is established here, as follows.

Corollary 8. *If F_1 is constant on some interval $[x_1, x_2]$, then any optimal strategy α searches that interval without interruption. That is, there exist times t_1, t_2, with $\alpha(t_i) = x_i$ and $t_2 - t_1 = x_2 - x_1$.*

Proof. On the interval (x_1, x_2) we must have $F_1 < \bar{F}_1$, so the conditions of the previous theorem will be satisfied for some $x_0 \le x_1$ and some $x_4 \ge x_2$. ■

Another result that is needed in analyzing rendezvous on the circle is the special case of two identical strictly concave distributions. The solution is an easy consequence of Theorem 5.

Corollary 9. *Let F_1 and F_2 be identical concave distribution functions. Then the strategy $\alpha(t) = \beta(t) = t/2$ is an optimal strategy. If the distribution functions F_i are moreover strictly concave, then this is the unique optimal solution.*

Proof. Since the distributions F_i are already concave, we have $F_i = \bar{F}_i$. If F_i is strictly concave, then for $x_1 < x_2$ we have $D_i(x_1) \cap D_i(x_2) = \emptyset$, so that Theorem 5 requires that $\alpha(t) = \beta(t)$ for all t, or $\alpha(t) = t/2$. If the F_i are flat on a maximal closed interval $[x_1, x_2]$, then the optimal strategy is not unique for $t \in [2x_1, 2x_2]$, and as observed earlier *any* strategy α which is otherwise optimal and satisfies $\alpha(2x_1) = x_1$, $\alpha(2x_2) = x_2$ is optimal. ■

The results of this section can be applied in a variety of search problems. We believe that similar methods can be used to obtain results for finitely many search locations, as the improvement techniques for suboptimal strategies need only compare the search at any two locations.

5.3. Searching in boxes. The problem of optimal interleaving of distributions was motivated by the continuous time search problems discussed in our earlier sections. However for discrete distributions it can be interpreted as "searching in boxes" problem. The following description of the boxes problem is taken from the abstract of [Wegener, 1980].

> "Suppose one object is hidden in the k'th of n boxes with probability $p(k)$. The boxes are to be searched sequentially. Associated with the j'th search of box k is a cost $c(j,k)$ and a conditional probability $q(j,k)$ that the first $j-1$ searches of box k are unsuccessful while the j'th search is successful given that the object is hidden in box k. The problem is to minimize the expected cost of a strategy which at last surely finds the object. ... We give a procedure for the computation of an optimal strategy."

Wegener's problem can be seen as a special case of the problem posed in this section (for $n = 2$) by defining F_k to be the distribution associated with the atomic measure μ_k which for each j has an atom at $x_{k,j} = \sum_{i \le j} c(i,k)$ with mass $\mu_k(x_{k,j}) = p_k q(j,k)$. Our analysis for the continuous problem is independent of the box search problem, and indeed we only became aware of the latter problem after S. Gal suggested a possible connection after hearing a talk given by one of the authors. A good survey of the box-search literature is given (in the context of a larger survey) in section 4.2 ('Stationary Target - Discrete Time and Space') of [Benkoski et al, 1991].

6. Double linear search revisited

We now apply the results of the previous section to the example of double linear search studied in section 3. This will enable us to justify the restriction to the interleavings evaluated in section 3, as there are other possible interleavings of AA, BB and AB which we did not consider. We restrict our analysis to the range $2/3 < p < 3/4$, where we found a counterexample to Gal's conjecture, and normalize so that each line has total mass 1. The distribution associated with the efficient strategy A has atoms of mass $p/2$ at $x = 1, 5$ and atoms of mass $(1-p)/2$ at $x = 2, 6$. The concavification $\bar{F}_A$ consists of three straight lines as follows: slope $p/2$ for $x \in (0,1)$, slope $(1-p)/2$ for $x \in (1,2)$, and slope $1/8$ for $x \in (2,6)$. The distribution associated with the greedy strategy B has atoms of mass $p/2$ at $y = 1, 3$ and atoms of mass $(1-p)/2$ at $y = 4, 8$. The concavification $\bar{F}_B$ consists of four lines: the slope is $p/2$ on $(0,1)$, $p/4$ on the interval $(1,3)$, $(1-p)/2$ on $(3,4)$, and finally $(1-p)/8$ on $(4,8)$. These distributions and their concavifications are shown in Figure 5.

Theorem 5 (actually the simple Proposition 3 would suffice here) says that for an optimal interleaving of the distributions $F_1 = F_A$ and $F_2 = F_B$ we must search these seven intervals in order of decreasing slope. For the interval $2/3 < p < 3/4$, the decreasing order of these seven slopes (including intervals of the same slope on the two lines) is

$$p/2 \geq p/2 \geq p/4 \geq (1-p)/2 \geq (1-p)/2 \geq 1/8 \geq (1-p)/8.$$

An optimal method of searching is given by the following table, with the decreasing slopes in the top row and the associated local time intervals (the subscript indicates the relevant line) in the bottom row.

$p/2$	$p/2$	$p/4$	$(1-p)/2$	$(1-p)/2$	$1/8$	$(1-p)/8$
$[0,1]_2$	$[0,1]_1$	$[1,3]_2$	$[1,2]_1$	$[3,4]_2$	$[2,6]_1$	$[4,8]_2$

In the double linear search problem, this corresponds to the strategy we called $[AB]_2$: they alternate going to 1, then II goes to -1, then I goes to 2 , then II goes to -2,then I goes to -2, then II goes to 2.

If we optimally interleave two copies of F_A, the associated double linear search strategy is for the searches to alternate going to 1, then alternate going to 2, and finally alternate going to -2. This is the strategy we called $[AA]_1$. Similarly, if we optimally interleave to copies of F_B, the corresponding double linear search strategy is for the searchers to alternate going to 1, then alternate going to -1, then alternate going to -2, and finally alternate going to +2. This is the strategy we called $[BB]_3$.

In summary, we have applied the theory established in the previous section to show that for $2/3 < p < 3/4$, the optimal interleaving of the distributions corresponding to A and B, A and A, B and B, correspond respectively to the strategies defined in section 3 as $[AB]_2$, $[AA]_1$, and $[BB]_3$. It is now easy to calculate that $[AB]_2$ is the best of these three alternatives for the range $2/3 < p < 3/4$, and similar calculations complete the table below figure 1.

7. Conclusions

In this paper we have introduced the alternating search problem, explained its connection with rendezvous search, and given a complete solution to the simple case of a symmetric double linear search with two possible hider distances. The subproblem of determining an optimal interleaving of two given regional search strategies has been solved completely. The general results obtained in this paper on interleaving have been applied in a sister paper [Alpern and Beck, 1998] to prove a conjecture given in [Baston and Gal, 1998] regarding the optimal strategy pair in asymmetric rendezvous problem on the line with a convex distribution of the initial distance between the rendezvousers.

Two extensions of the problem studied here seem worth pursuing. The first and most obvious is an $n-$location version. While most of the analysis given here is immediately extendable, some of the results on interleaving must be treated more subtly to treat more than two locations. The second extension is to consider that the two (or more) searchers are placed in a common region, that is, that there is a path between their starting points. This extension has more uncertainties, as it is not clear a priori which searcher should be the first to reach a given location. Of course once a global strategy is chosen, the regions searched by the two searchers can be 'separated' (even if they overlap), and if the global strategy is optimal for the new problem, it must also be optimal for the separated version, which is a problem of the type considered here.

The first author (alphabetically) wishes to take this opportunity to announce that he has applied the interleaving theory developed here to solve a version of the asymmetric rendezvous problem on the circle without the assumption of a uniform initial distribution of the players which was made in [Howard, 1998]. This will appear in [Alpern, 2000].

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