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The chromatic roots of generalised dodecahedra

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Abstract

The aim of this paper is to study the chromatic polynomials of a family of cubic graphs $\{D_n\}$ with $4n$ vertices. In particular D_5 is the graph of the regular dodecahedron.
We use the compatibility matrix method to obtain a third-order recursion for the leading terms of the chromatic polynomial of D_n, as well as information about the remaining terms. Then we apply a theorem of Beraha, Kahane and Weiss to describe the curves on which the limit points of the chromatic roots lie.
In Section 6 we analyse a method of Salas and Sokal, in order to provide a good description of the limit curves.

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1 Introduction

The aim of this paper is to study the chromatic polynomials of a family of cubic graphs $\{D_n\}$ with $4n$ vertices. In particular D_5 is the graph of the regular dodecahedron. The motivation for studying the family $\{D_n\}$ is the question of whether chromatic roots can have negative real parts. Many years ago it was thought that η_5, one of the chromatic roots of D_5, had this property. But there was a small error in the computation, and in fact the real part of η_5 is approximately 0.04. Subsequently, empirical work by Read and Royle [4] showed that cubic graphs can have chromatic zeros with negative real parts, and more recently, Salas and Sokal [5] have found families of regular graphs of degree four that have this property. But the situation is not well understood, and in the cubic case examples are scarce. All the Read-Royle examples have girth at least five, which is the girth of D_n when $n \geq 5$. This suggests that it is worth studying the behaviour, as $n \to \infty$, of the chromatic root η_n of D_n that has the smallest real part. (Of course, we exclude 0, which is always a chromatic root.)

We use the compatibility matrix method [2] to obtain a third-order recursion for the leading terms of the chromatic polynomial of D_n, as well as information about the remaining terms. Then we apply a theorem of Beraha, Kahane and Weiss [1] to describe the curves on which the limit points of the chromatic roots lie. In fact, the leftmost point η on these curves is approximately $0.13 + 1.13i$. This suggests that the real part of η_n increases as $n \to \infty$, and no D_n has a root with negative real part.

In Section 6 we analyse a method of Salas and Sokal [5], in order to provide a good description of the limit curves.

2 The compatibility matrix

Let B be a path with vertex-set $V = \{1, 2, 3, 4\}$ (1 and 4 being the end-vertices), and let $L = \{(1, 1), (3, 2), (4, 4)\}$. Given an integer $n \geq 3$, take n disjoint copies of B and link them by extra edges as follows. For every $i = 1, 2, \ldots, n$ and each pair $(v, w) \in L$ join vertex v in the ith copy of B, to the vertex w in the $(i + 1)$th copy of B, with the convention that $n + 1 = 1$. We obtain a 'ring' of n copies of B linked by edges in the manner prescribed. This graph will be denoted by D_n; it is a cubic graph with $4n$ vertices, and in particular D_5 is the graph of the regular dodecahedron.

Throughout this section k will be a natural number. We consider the functions $\alpha : V \to \{1, 2, \ldots, k\}$ that are (proper) k-colourings of the path B. We denote by P_k the set of all such k-colourings, and by $\mathcal{V}_k$ the vector space of real-valued functions defined on P_k. The *standard basis* for $\mathcal{V}_k$ contains for every $\alpha \in P_k$, the function $[\alpha]$ defined as

$$[\alpha](\beta) = \begin{cases} 1 & \text{if} \quad \alpha = \beta \ ; \\ 0 & \text{otherwise.} \end{cases}$$

For the given B and L we say that a pair of (α, β) of members of P_k is *compatible with* L if the following three conditions hold:

$$\alpha(1) \neq \beta(1), \quad \alpha(2) \neq \beta(3), \quad \alpha(4) \neq \beta(4).$$

This means that if one copy of B is coloured according to α, a second copy of B according to β, and they are linked according to L, the resulting graph is properly k-coloured by α and β.

We define a *compatibility matrix* $T = T(k)$ as follows. The rows and columns of T correspond to the elements of P_k, and the entries are given by:

$$(T)_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \text{ is compatible with } L \ ; \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to use the same symbol T for the linear operator represented by the matrix T, with respect to the standard basis of $\mathcal{V}_k$. The connection between $T(k)$ and the chromatic polynomial of D_n arises from the formula [2]:

$$P(D_n; k) = \mathrm{trace}(T(k)^n).$$

If the eigenvalues of $T(k)$ are denoted by $\lambda_1(k), \lambda_2(k), \ldots, \lambda_s(k)$ and their respective multiplicities by $m_1(k), m_2(k), \ldots, m_s(k)$, we get:

$$P(D_n; k) = \sum_{r=1}^{s} m_r(k) \lambda_r(k)^n. \tag{1}$$

Hence the calculation of $P(D_n; k)$ is reduced to the calculation of the spectrum of $T(k)$. Note that k is any positive integer, and the left-hand side in the formula above is known to be a polynomial function. The individual eigenvalues $\lambda_r(k)$ are not necessarily polynomials, but by collecting them in sets of algebraic conjugates we get a sum of polynomials. It follows that we can replace k by a complex variable z.

We describe the action of T on $\mathcal{V}_k$ by using the sieve formula established in Theorem 1 of [3]. Some preparatory definitions are needed. We shall write

$$[1|h] \ = \ \sum_{\gamma(1)=h} [\gamma].$$

Thus $[1|h]$ is the function which takes the value 1 on the colourings that assign the colour h to vertex 1, and the value 0 otherwise. Similarly, we write $[1,2|h,i]$ for the sum of all $[\gamma]$ with $\gamma(1) = h$ and $\gamma(2) = i$; and so on. If $\alpha \in P_k$ is defined by $\alpha(i) = \alpha_i$ $(1 \le i \le 4)$, then, in this notation, $[\alpha] = [1,2,3,4|\alpha_1,\alpha_2,\alpha_3,\alpha_4]$. (Note that $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are not necessarily all distinct.) Finally, $u = [-|-]$ is the sum of all $[\gamma]$ without restriction, that is, the function which takes the value 1 on all colourings.

Using Theorem 1 in [3] we get for any given $\alpha \in P_k$:

$$\begin{aligned} T[\alpha] = u &- [1|\alpha_1] - [2|\alpha_3] - [4|\alpha_4] \\ &+ [1,2|\alpha_1,\alpha_3] + [1,4|\alpha_1,\alpha_4] + [2,4|\alpha_3,\alpha_4] \\ &- [1,2,4|\alpha_1,\alpha_3,\alpha_4]. \end{aligned} \tag{2}$$

3 The invariant subspace at level 0

Following the approach outlined in Section 4 in [3] we shall begin by studying the action of T on the invariant subspace generated by u, that is

$$\langle u \rangle_T = \langle u, Tu, T^2u, \dots \rangle.$$

Since in $u = [-|-]$ no colour is specified we refer to this as the subspace at 'level 0'.

We have

$$Tu = T\left(\sum_{\alpha \in P_k} [\alpha]\right) = \sum_{\alpha \in P_k} T[\alpha],$$

so it is necessary to evaluate the sum over all $\alpha \in P_k$ for each of the 8 terms in the formula (2) for $T[\alpha]$. Clearly

$$\sum_{\alpha \in P_k} u = k(k-1)^3 u,$$

and

$$\sum_{\alpha \in P_k} [1|\alpha_1] = \sum_{\alpha \in P_k} [2|\alpha_3] = \sum_{\alpha \in P_k} [4|\alpha_4] = (k-1)^3 u.$$

Next, consider the sum over $\alpha \in P_k$ of the terms $[1,2|\alpha_1,\alpha_3]$. In any colouring the colours i and j assigned to vertices 1 and 2 must be different, and hence the sum of $[1,2|i,j]$ over all pairs with $i \neq j$ is just u. For each such pair the number of $\alpha \in P_k$ such that $\alpha_1 = i$ and $\alpha_3 = j$ is $(k-1)(k-2)$. Hence

$$\sum_{\alpha \in P_k} [1,2|\alpha_1,\alpha_3] = (k-1)(k-2)u.$$

For the remaining terms, it is helpful to define

$$E_{14} = \{\gamma \in P_k \mid \gamma(1) = \gamma(4)\}, \quad E_{24} = \{\gamma \in P_k \mid \gamma(2) = \gamma(4)\}.$$

(Note that the analogous set E_{12} is empty.) Let e_{14} and e_{24} be the characteristic functions of the sets E_{14} and E_{24}. Now consider the sum over $\alpha \in P_k$ of the terms $[1,4|\alpha_1,\alpha_4]$. We split P_k into two parts, E_{14} and $P_k \setminus E_{14}$. For the first part, the sum of $[1,4|i,i]$, taken over all colours i, is e_{14}, and the number of α such that $\alpha_1 = \alpha_4 = i$ is $(k-1)(k-2)$. For the second part, the sum of $[1,4|i,j]$, taken over all pairs $i \neq j$, is $u - e_{14}$, and the number of α such that $\alpha_1 = i, \alpha_4 = j$ is $k^2 - 3k + 3$. Hence

$$\sum_{\alpha \in P_k} [1,4|\alpha_1,\alpha_4] = (k-1)(k-2)e_{14} + (k^2 - 3k + 3)(u - e_{14}).$$

Similarly we find

$$\sum_{\alpha \in P_k} [2,4|\alpha_1,\alpha_4] = (k-1)^2(u - e_{24}).$$

$$\sum_{\alpha \in P_k} [1,2,4|\alpha_1,\alpha_3,\alpha_4] = (k-2)(u - e_{24}).$$

Collecting up the terms, and writing $c = k - 2$, we get

$$T(u) = (c^4 + 2c^3 + 3c^2 + c + 1)u - e_{14} - (c^2 + c + 1)e_{24}.$$

The following expressions for $T(e_{14})$ and $T(e_{24})$ can be obtained in a similar way:

$$T(e_{14}) = (c^2 + c)u + c^2 e_{14} - ce_{24},$$

$$T(e_{24}) = (c^3 + c^2 + 2c)u - ce_{14} - ce_{24}.$$

Since the vertices 1 and 2 are adjacent it follows immediately that e_{14} and e_{24} are linearly independent. Let α be any colouring in P_k where every vertex gets a different colour. Clearly $u(\alpha) = 1$ and $e_{14}(\alpha) = e_{24}(\alpha) = 0$. It follows that $\langle u \rangle_T$ is the 3-dimensional space $\langle u, e_{14}, e_{24} \rangle$. The action of T on this subspace is represented by the matrix

$$\hat{T}_0 = \begin{pmatrix} c^4 + 2c^3 + 3c^2 + c + 1 & c^3 + c & c^3 + c^2 + 2c \\ -1 & c^2 & -c \\ -(c^2 + c + 1) & -c & -c \end{pmatrix}.$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are the the roots of the characteristic equation:

$$\det(\lambda I - \hat{T}_0) = \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0,$$

where

$$\begin{aligned} a_1 &= c^4 + 2c^3 + 4c^2 + 1 \\ a_2 &= c(c+1)(c^4 + c^3 + 2c^2 + 2) \\ a_3 &= c^2(c+1)^2. \end{aligned}$$

The characteristic polynomial is irreducible, so its roots are algebraic conjugates and have the same multiplicity m. Thus $m(\lambda_1^n + \lambda_2^n + \lambda_3^n)$ contributes to the chromatic polynomial. Newton's formulae can be used to evaluate $A_n = \lambda_1^n + \lambda_2^n + \lambda_3^n$ recursively, thus:

$$A_1 = a_1, \quad A_2 = a_1 A_1 - 2a_2, \quad A_3 = a_1 A_2 - a_2 A_1 + 3a_3,$$

$$A_n = a_1 A_{n-1} - a_2 A_{n-2} + a_3 A_{n-3} \quad (n > 3).$$

It follows, in particular, that A_n is a monic polynomial of degree $4n$ in c. Since the chromatic polynomial of D_n is also monic of degree $4n$, it follows that $m = 1$. Thus the leading terms of the chromatic polynomial are given by A_n.

4 The invariant subspaces at levels 1, 2 and 3

In this section we investigate the action of T on elements such as $[1|h]$, $[1,2|h,i]$ and $[1,2,3|h,i,j]$ for any $h,i,j \in \{1,2,\ldots,k\}$. Since elements of this type specify one, two or three colours, we refer to the subspaces

$$\langle [1|h] \rangle_T, \quad \langle [1,2|h,i] \rangle_T \text{ and } \langle [1,2,3|h,i,j] \rangle_T$$

as the invariant subspaces at 'levels 1, 2, 3' respectively.
The invariant subspace at level 1 is

$$\langle [1|h] \rangle_T = \langle [1|h],\, T[1|h],\, T^2[1|h], \ldots \rangle.$$

As we shall see, this contains the subspace at level 0. Starting from the relation

$$T[1|h] = T\left(\sum_{\substack{\alpha \in P_k \\ \alpha_1 = h}} \right) = \sum_{\substack{\alpha \in P_k \\ \alpha_1 = h}} T[\alpha],$$

we can calculate as follows, using the formula (2).

For the first term we have

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} u = (k-1)^3\ u,$$

and for the second term

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [1|\alpha_1] = (k-1)^3\ [1|h].$$

For the third term, we note that, since vertices 1 and 3 are not adjacent, there are two possibilities: $h=\alpha_1=\alpha_3$ or $h=\alpha_1\neq\alpha_3$. Hence

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [2|\alpha_3] = (k-1)^2\ [2|h] + (k-1)(k-2)\ (u-[2|h]),$$

and similarly

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [4|\alpha_4] = (k-1)(k-2)\ [4|h] + ((k-1)+(k-2)^2)\ (u-[4|h]).$$

Let $[1|h]_{E_{14}}$ denote the sum of $[\alpha]$'s in E_{14} subject to the constraint that $\alpha_1=h$; in other words the sum over all $[\alpha]$'s with $\alpha_1=\alpha_4=h$. Similarly for $[2|h]_{E_{24}}$, $[2|h]_{E_{14}}$ and $[1|h]_{E_{24}}$. (As noted in the previous section, $E_{12}=\emptyset$, so the corresponding expressions are not needed.) With this notation we obtain the following results:

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [1,2|\alpha_1,\alpha_3] = (k-1)(k-2)\ [1|h],$$

$$\begin{aligned}\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [1,4|\alpha_1,\alpha_4] &= (k-1)(k-2)\ [1|h]_{E_{14}}\\ &\quad + ((k-1)+(k-2)^2)\ ([1|h]-[1|h]_{E_{14}}),\end{aligned}$$

$$\begin{aligned}\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}} [2,4|\alpha_3,\alpha_4] &= (k-1)\ ([2|h]-[2|h]_{E_{24}})\\ &\quad + (k-1)\ ([4|h]-[2|h]_{E_{24}})\\ &\quad + (k-1)\ (u-[2|h]-[4|h]+[2|h]_{E_{24}}-e_{24}),\end{aligned}$$

and

$$\sum_{\substack{\alpha\in P_k\\ \alpha_1=h}}[1,2,4|\alpha_1,\alpha_3,\alpha_4] = (k-2)\ [1|h]_{E_{14}} \\ + (k-2)\ ([1|h]-[1|h]_{E_{14}}-[1|h]_{E_{24}}).$$

Putting $c=k-2$, and collecting terms, and inserting signs according to (2), we obtain:

$$\begin{aligned} T[1|h] = (c^3+c^2+2c+1)u& \\ -c(c^2+c+2)(k-2)\ [1|h] - c\ [2|h] + [4|h]& \qquad (3)\\ -[1|h]_{E_{14}} - [2|h]_{E_{24}} + c\ [1|h]_{E_{24}}.& \end{aligned}$$

We observe that elements containing the vertex 3, such as $[3|h]$ do not appear in (2) or (3). It remains to calculate the action of T on the other elements that occur in (3). We find that only three further elements e_{14}, e_{24} and $[2|h]_{E_{14}}$ arise. The elements u, e_{14}, e_{24} are in the invariant subspace at level 0, and the action of T on these elements has already been determined. Hence $\langle[1|h]\rangle_T$ can be decomposed as

$$\langle[1|h]\rangle_T = \langle u\rangle_T \oplus \mathcal{W}_1(h),$$

where $\mathcal{W}_1(h)$ is the space spanned by

$$[1|h], [2|h], [4|h], [1|h]_{E_{14}}, [2|h]_{E_{24}}, [1|h]_{E_{24}}, [2|h]_{E_{14}}.$$

It is easy to show that these elements form a basis for $\mathcal{W}_1(h)$. For example, let α be any colouring in P_k such that every vertex gets a different colour and $\alpha_1 = h$. Then $[1|h](\alpha) = 1$ and every other element of the spanning set evaluates to zero at α. Hence $[1|h]$ is linearly independent of all the other elements.

The action of T on the 10-dimensional space $\langle[1|h]\rangle_T$ can therefore be represented by a matrix of the form

$$M = \begin{pmatrix} \hat{T}_0 & * \\ 0 & \hat{T}_1 \end{pmatrix}.$$

Thus we obtain 7 new eigenvalues of T, specifically the eigenvalues of the

7×7 matrix $\hat{T}_1$, which is: :

$$\begin{pmatrix} -c(c^2+c+2) & -c & 1 & -1 & -1 & c & 0 \\ c^2+1 & c^2 & -c & -1 & c & 0 & 1 \\ 1 & c^2+c+1 & -c^3-c^2-c+1 & -2c-1 & -c^2-c-1 & 0 & 0 \\ -c^2 & c & -c^2 & c(c+2) & -c & 0 & 0 \\ c & c & -c^2 & -c & -c & 0 & 0 \\ -c^2-1 & -1 & c & -c & 1 & 1 & 0 \\ c-1 & c-2 & c-1 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of $\hat{T}_1$ takes the form

$$\mu^7 - b_1\mu^6 + b_2\mu^5 - b_3\mu^4 + b_4\mu^3 - b_5\mu^2 + b_6\mu - b_7,$$

where the coefficients are polynomials in c. For example, $b_1 = -2c^3 - 2c + 3$ and $b_2 = c^6 - 2c^5 - 2c^4 - 10c^3 - 6c^2 - c + 4$.

The sum of the nth powers of the roots (the trace of $(\hat{T}_1)^n$) is a polynomial B_n in c, which can be calculated by using Newton's formulae. That is

$$B_1 = b_1, \quad B_2 = b_1B_1 - 2b_2, \quad B_3 = b_1B_2 - b_2B_1 + 3b_3, \ \ldots$$

and

$$B_n = b_1B_{n-1} - b_2B_{n-2} + b_3B_{n-3} - b_4B_{n-4} + b_5B_{n-5} - b_6B_{n-6} + b_7B_{n-7}$$

for $n > 7$. Since deg $b_1 = 3$, deg $b_2 = 6$ and deg $b_i < 3i$ for $i = 3, 4, 5, 6, 7$, it follows that B_n is a polynomial of degree $3n$ for all $n \geq 1$.

In order to determine the multiplicities of the roots, as eigenvalues of T, we argue as follows. Let μ be an eigenvalue of $\hat{T}_1$. Let h be any element of $\{1, 2, \ldots, k\}$, so that there exists an eigenvector $b_h \in \mathcal{W}_1(h)$, $\hat{T}_1 b_h = \mu b_h$. Since $\hat{T}_1$ does not depend on the choice of h it follows b_h is a linear combination of the form

$$\alpha_1[1|h] + \alpha_2[2|h] + \ldots + \alpha_7[2|h]_{E_{14}},$$

where coefficients α do not depend on h. Then

$$b = \sum_{h=1}^{k} b_h = \alpha_1 \sum_{h=1}^{k}[1|h] + \alpha_2 \sum_{h=1}^{k}[2|h] + \ldots + \alpha_7 \sum_{h=1}^{k}[2|h]_{E_{14}}$$

and since, for example

$$\sum_{h=1}^{k}[1|h] = u \qquad \text{and} \qquad \sum_{h=1}^{k}[1|h]_{E_{14}} = e_{14},$$

it is easy to see that $b \in \langle u \rangle_T$. On the other hand $b \in \bigoplus_{h=1}^{k} \mathcal{W}_1(h)$, thus $b = 0$. It follows that the dimension of the eigenspace corresponding to μ is bounded above by $k-1$, i.e. the multiplicity of μ as eigenvalue of T is at most $k-1$. In fact the multiplicity is $k-1$. Hence the degree of $(k-1)B_n$ is $3n+1$.

For the invariant subspaces at level 2 we proceed as follows. Analogously to level 1, let $[1,2|h,i]_{E_{24}}$ denote the sum of $[\alpha]$'s in E_{24} subject to the constraint that $\alpha_1 = h$ and $\alpha_2 = i$. Similarly for $[1,2|h,i]_{E_{1,4}}$ and $[1,2|i,h]_{E_{1,4}}$. Let

$$\begin{aligned} \mathcal{W}_2(h,i) = &\big\langle [1,2|h,i], [1,2|i,h], [1,4|h,i], [2,4|h,i], \\ &[2,4|i,h], [1,2|h,i]_{E_{2,4}}, [1,2|h,i]_{E_{1,4}}, [1,2|i,h]_{E_{1,4}} \big\rangle. \end{aligned}$$

We can write

$$\langle [1,2|h,i] \rangle_T = \langle u \rangle_T \oplus \mathcal{W}_1(h) \oplus \mathcal{W}_1(i) \oplus \mathcal{W}_2(h,i),$$

The action of T on $\langle [1,2|h,i] \rangle_T$ can be represented by a matrix

$$\begin{pmatrix} \hat{T}_0 & * & * & * \\ 0 & \hat{T}_1 & 0 & * \\ 0 & 0 & \hat{T}_1 & * \\ 0 & 0 & 0 & \hat{T}_2 \end{pmatrix}$$

where

$$\hat{T}_2 = \begin{pmatrix} -c & 0 & -c & -2 & -2 & -1 & 1 & 0 \\ 0 & -c & -c & -2 & -2 & 1 & -1 & 0 \\ -c & 0 & c^2+1 & 1 & 0 & c & 0 & 0 \\ 0 & 1 & -c+1 & -c & 0 & -1 & 0 & -1 \\ 1 & 0 & -c+1 & 0 & -c & -1 & -1 & 0 \\ -1 & 0 & c & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using methods similar to ones already used at level 1 and 2, we find

$$\begin{aligned} \langle [1,2,4|h,i,j] \rangle_T = &\langle u \rangle_T \oplus \mathcal{W}_1(h) \oplus \mathcal{W}_1(i) \oplus \mathcal{W}_1(j) \\ &\oplus \mathcal{W}_2(h,i) \oplus \mathcal{W}_2(h,j) \oplus \mathcal{W}_2(i,j) \oplus \langle [1,2,4|h,i,j] \rangle \end{aligned}$$

and $\hat{T}_3 = 1$, the constant 1×1 matrix. Note that (2) implies $\langle [1,2,3,4|h,i,j,l] \rangle_T$ is a direct sum of level 3 spaces such as $\langle [1,2,4|h,i,l] \rangle_T$, and a matrix $\hat{T}_4$ would be of rank 0.

The contribution of the eigenvalues at level 2 to the chromatic polynomial is a polynomial of degree $2n+2$, and at level 3 (just) 3.

5 The location of the chromatic roots

It follows from the results in the previous sections that the polynomial $A_n = A_n(c)$ determines the leading terms of the chromatic polynomial of D_n. Specifically, replacing c by a complex variable z, we have shown that

$$P(D_n; z+2) = A_n(z) + R_n(z),$$

where $R_n(z)$ is a polynomial of degree $3n+1$. In this section we attack the problem of locating the chromatic roots of D_n, using only this fact and the explicit form of $A_n(z)$ obtained in Section 3.

The analysis is based on a theorem of Beraha, Kahane, and Weiss [1]. Suppose we are given a sequence of complex polynomials, expressed in the form

$$P_n(z) = \sum_{i=1}^{r} m_i(z)\lambda_i(z)^n.$$

Then the theorem says that ζ is limit points of the zeros of $P_n(z)$ if and only if (i) two of the terms $\lambda_i(\zeta)$ and $\lambda_j(\zeta)$ have equal modulus and dominate the others, or (ii) $m_i(\zeta) = 0$ and $\lambda_i(\zeta)$ dominates the other terms. In practice, case (ii) provides only a small number of isolated points, which are easily found, and so we concentrate on case (i).

In Section 2 we observed that the chromatic polynomial of D_n can be expressed as a sum of nth powers, with suitable coefficients. In Section 3 we showed that the polynomial $A_n(z)$, which determines the leading terms of the chromatic polynomial, can be expressed as the sum of three nth powers:

$$A_n(z) = \lambda_1(z)^n + \lambda_2(z)^n + \lambda_3(z)^n.$$

The terms $\lambda_1(z)$, $\lambda_2(z)$, $\lambda_3(z)$, are the roots of a cubic equation whose coefficients $a_1(z)$, $a_2(z)$, $a_3(z)$ are polynomial functions. But it must be stressed that the roots are not holomorphic functions, that is, they cannot be defined consistently in the entire complex plane. Informally we can say that, although there are three eigenvalues at each point z, we cannot give them names that remain valid as we move around the plane. The situation could be described in terms of a Riemann surface, but a more elementary treatment is sufficient for this particular example.

In the light of the Beraha-Kahane-Weiss theorem, define the *equimodular set* E to be the set of complex numbers z for which the equation $\lambda^3 - a_1(z)\lambda^2 + a_2(z)\lambda - a_3(z) = 0$ has two roots of equal modulus. It is permissible to think of E as a set of curves, although the precise justification for this will appear only in the next section.

A simple practical method of determining the equimodular curves involves the the *discriminant* of the equation $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$. This is defined to be the function D of the coefficients given by

$$3D = 4(a_1^2 - 3a_2)(a_2^2 - 3a_1a_3) - (a_1a_2 - 9a_3)^2,$$

that is,

$$D = -27a_3^2 + 18a_1a_2a_3 - 4a_2^3 - 4a_1^3a_3 + a_1^2a_2^2.$$

If $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation, then it can be shown that

$$D = (\lambda_1 - \lambda_2)^2\,(\lambda_2 - \lambda_3)^2\,(\lambda_3 - \lambda_1)^2.$$

So $D = 0$ is a necessary and sufficient condition for two of the roots to be equal.

In our case the coefficients a_1, a_2, a_3 are known polynomial functions of a complex variable z, and we can thus consider D as a function $D(z)$. If $D(z) = 0$ two of the roots are equal at z, and they certainly have equal modulus, so z belongs to the equimodular set.

Inserting the relevant expressions for a_1, a_2, a_3 gives

$$\begin{aligned} -3D(z) = {} & z^3(z+1)^3(z^3+z-1)^2 \\ & (z^8 + 5z^7 + 14z^6 + 24z^5 + 31z^4 + 19z^3 + 3z^2 - 27z + 4). \end{aligned}$$

The roots of $D(z) = 0$ are 0 and -1 (three times each), and the points we shall denote by $\alpha, \beta, \bar{\beta}, \gamma, \delta, \epsilon, \bar{\epsilon}, \zeta, \bar{\zeta}, \eta, \bar{\eta}$, whose approximate values and multiplicities are as follows:

α	$\beta, \bar{\beta}$	γ	δ	$\epsilon, \bar{\epsilon}$	$\zeta, \bar{\zeta}$	$\eta, \bar{\eta}$
0.68	$-0.34 \pm 1.16i$	0.15	0.61	$0.14 \pm 1.64i$	$-1.02 \pm 1.51i$	$-1.87 \pm 1.13i$
2	2, 2	1	1	1, 1	1, 1	1, 1

It follows from the preceding discussion that these points all belong to the equimodular set E.

Given these points, and assuming that E is a union of piecewise-smooth arcs, the equimodular curves can be constructed by a simple local search method. We choose one of the points, call it σ, and look at the roots of the cubic equation on a circle $z = \sigma + r\exp(i\phi)$, $0 \le \phi \le 2\pi$. In general, if r is small enough we shall find at most 2 points on this circle where the cubic has two roots of equal modulus. If there are no such points we conclude that σ is an isolated point of E. (In fact, this does not happen in the present example.) If there is at least one such point, say σ', we repeat the procedure with σ replaced by σ'. Assuming that σ' is an internal point of an arc with continuous derivative, we can restrict the search to a small circular arc $z = \sigma' + r\exp(i\phi)$, $\phi_1 \le \phi \le \phi_2$, where the bounds of the range, ϕ_1 and ϕ_2, are determined by the direction of the line $\sigma\sigma'$.

It turns out that the set E is the union of five continuous curves, as listed below and illustrated in Figure 1. In Section 6 we shall give a more thorough analysis of these curves.

•1 the interval $[-1, 0]$ of the real axis. On this arc two of the roots are complex conjugates.

•2 a curve joining ζ to $\bar{\zeta}$, which intersects $[-1, 0]$ at the point $-0.543\ldots$.

•3 the interval $[\gamma, \delta]$ of the real axis. On this arc two of the roots are complex conjugates.

•4 a curve joining η to $\bar{\eta}$, passing through the points β and $\bar{\beta}$, and intersecting $[\gamma, \delta]$ at the point $0.337\ldots$.

•5 a curve joining ϵ to $\bar{\epsilon}$, which crosses the real axis at the point α.

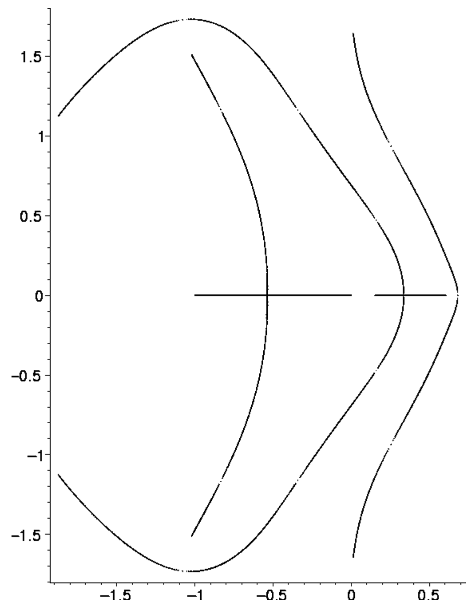

Figure 1: The five equimodular curves

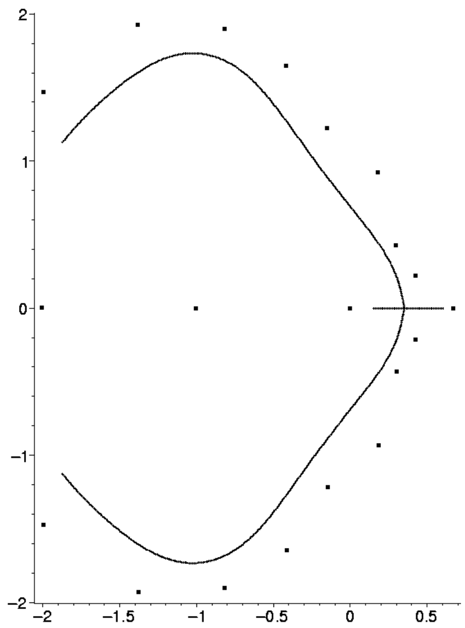

Figure 2: The chromatic roots of D_5 and the limit curves.

It remains to invoke the dominance condition, which can be done by direct calculation. It turns out that the curves on which the two roots with equal modulus dominate the third root are •3 and •4. Thus we should expect that chromatic roots of the graphs D_n lie close to these two curves. The results for D_5 (Figure 2), confirm the conclusion. (It must be remembered that that the variable z in the preceding calculation corresponds to $k-2$, where k is the number of colours, and that a non-trivial graph always has a chromatic root at $k=0$.)
The diagram indicates that, if η_n denotes the non-zero chromatic root of D_n with smallest real part, then $\eta_n \to \eta$ as $n \to \infty$. Since η_5 has positive real part, this suggests all roots η_n have positive real part.

6 Further discussion of the equimodular curves

The empirical description of the equimodular curves given in Section 5 does not fully explain their structure. In order to make further progress we shall develop a technique introduced by Salas and Sokal [5], who give numerous examples and discuss the case of quadratic eigenvalues in detail. For the generalised dodecahedra the eigenvalues are the roots of a cubic equation, and we shall analyse this case in detail.

The idea stems from the observation that if $|\lambda_A| = |\lambda_B|$, then there is a complex number s with $|s| = 1$ such that $\lambda_A = s\lambda_B$. So if λ_A and λ_B are roots of the polynomial equation $p(\lambda) = 0$, where

$$p(\lambda) = \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3,$$

it follows that λ_B is a common root of $p(\lambda) = 0$ and $p_s(\lambda) = 0$ where

$$p_s(\lambda) = s^3\lambda^3 - s^2 a_1\lambda^2 + s a_2\lambda - a_3.$$

It is a classical result that a necessary and sufficient condition for a common root is that the resultant $R(p, p_s)$ vanishes, where

$$R(p,p_s) \;=\; \det \begin{pmatrix} 1 & -a_1 & a_2 & -a_3 & 0 & 0 \\ 0 & 1 & -a_1 & a_2 & -a_3 & 0 \\ 0 & 0 & 1 & -a_1 & a_2 & -a_3 \\ s^3 & -s^2a_1 & sa_2 & -a_3 & 0 & 0 \\ 0 & s^3 & -s^2a_1 & sa_2 & -a_3 & 0 \\ 0 & 0 & s^3 & -s^2a_1 & sa_2 & -a_3 \end{pmatrix}.$$

Expanding, we obtain a polynomial of degree 9 in s:

$$R(s, a_1, a_2, a_3) = (s-1)^3 a_3\, R_0(s, a_1, a_2, a_3).$$

Evidently, R_0 is a reciprocal polynomial of degree 6 in s, explicitly

$$\begin{aligned} s^{-3}R_0(s,a_1,a_2,a_3) = {} & a_3^2(s^3+s^{-3}) \\ & + (3a_3^2 - a_1a_2a_3)(s^2+s^{-2}) \\ & + (6a_3^2 - 5a_1a_2a_3 + a_2^3 + a_1^3a_3)(s+s^{-1}) \\ & + (7a_3^2 - 6a_1a_2a_3 + 2a_2^3 + 2a_1^3a_3 - a_1^2a_2^2). \end{aligned}$$

Changing the variable to $t = s + s^{-1} + 2$, and collecting the coefficients of the monomials in a_1, a_2, a_3, gives the expression

$$(t-1)^3a_3^2 - (t-1)(t+2)a_1a_2a_3 + ta_2^3 + ta_1^3a_3 - a_1^2a_2^2,$$

which we shall denote by $r_0 = r_0(t, a_1, a_2, a_3)$. Since s has unit modulus we can write $s = \exp(i\theta)$, so that $t = 2\cos\theta + 2 = 4\cos^2(\theta/2)$. Thus the relevant range of t is $0 \le t \le 4$. Note that when $t = 4$, r_0 is equal to $-D$, where D is the discriminant defined in Section 5.

In Section 3 we obtained explicit formulae for the coefficients a_1, a_2, a_3 in the case of the generalised dodecahedra:

$$\begin{aligned} a_1(z) &= z^4 + 2z^3 + 4z^2 + 1 \\ a_2(z) &= z(z+1)(z^4 + z^3 + 2z^2 + 2) \\ a_3(z) &= z^2(z+1)^2. \end{aligned}$$

Observing that every term in r_0 has a factor $z^2(z+1)^2$, we write $a_2 = z(z+1)q_2$, where $q_2 = q_2(z)$ is the quartic polynomial given above. Then

$$r_0(t, a_1(z), a_2(z), a_3(z)) = z^2(z+1)^2 v(t, z),$$

where

$$v(t,z) = (t-1)^3 z^2(z+1)^2 - (t-1)(t+2)z(z+1)a_1 q_2 + tz(z+1)q_2^2 + ta_1^3 - a_1^2 q_2^2.$$

We have shown that the set $E \subseteq \mathbf{C}$ of points z where the equation $\lambda^3 - a_1(z)\lambda^2 + a_2(z)\lambda - a_3(z) = 0$ has two roots of equal modulus comprises the points 0 and -1, together with

$$\{z \in \mathbf{C} \mid v(t,z) = 0 \text{ for some real } t,\ 0 \leq t \leq 4\}.$$

For a given t_0, $v(t_0, z) = 0$ is a polynomial equation of degree 16 in z. Let z_0 be any one of its 16 roots. By the implicit function theorem, if the Jacobian of the mapping $z \mapsto v(t_0, z)$ is not zero at z_0, then there is a neighbourhood N of t_0 and a continuously differentiable function $z(t)$ defined on N, such that $z(t_0) = z_0$ and $v(t, z(t))$ is identically zero for $t \in N$. This means that E is a union of 16 homeomorphic images of the interval $[0, 4]$, and these arcs are differentiable, except possibly at the end-points and points where the Jacobian is zero.

In Section 5 we listed the roots of the discriminant. Noting that the removal of the factor $z^2(z+1)^2$ means that 0 and -1 become simple roots, we have the 16 roots of $v(4, z) = 0$. For the roots of $v(0, z) = 0$, we remark that

$$v(0, z) = -(a_1 q_2 - z(z+1))^2,$$

so there are 8 distinct roots, each with multiplicity 2. These roots are (approximately)

$$\phi, \bar{\phi} = -1.08 \pm 1.73i, \quad \chi, \bar{\chi} = -0.84 \pm 1.17i,$$
$$\psi, \bar{\psi} = 0.16 \pm 0.47i, \quad \omega, \bar{\omega} = 0.25 \pm 0.93i.$$

All these points are interior points of the curves $\bullet 2$, $\bullet 4$ and $\bullet 5$. The fact that they occur with multiplicity 2 means that two images of $[0, 4]$ join up at the endpoints where $t = 0$, and the join is smooth. For example the curve $\bullet 5$ is the union of four images of $[0, 4]$, comprising arcs joining ϵ to ω, ω to α, α to $\bar{\omega}$, and $\bar{\omega}$ to $\bar{\epsilon}$.

In order to decompose the other curves in a similar fashion, it is necessary examine the two points where the curves appear to intersect. What happens at these points can be explained by looking at the Jacobian.

For each t the Jacobian of the mapping $z \mapsto v(t,z)$ is $|v'(t,z)|^2$, where v' denotes the derivative with respect to z. So the Jacobian vanishes if and only if $v'(t,z) = 0$. Thus the condition that there is a point which lies on one of the equimodular curves, and where the Jacobian vanishes, is that there exist t^* and z^* such that $v(t^*, z^*) = 0$ and $v'(t^*, z^*) = 0$. In other words, there is a value t^* such that the equations $v(t^*, z) = 0$ and $v'(t^*, z) = 0$ have a common root.

We write

$$v(t,z) = c_0(t)z^{16} + c_1(t)z^{15} + \ \dots \ + c_{16}(t),$$

where the coefficients $c_i(t)$ $(1 \le i \le 16)$ are known polynomials in t with degree at most 3. Then

$$v'(t,z) = 16c_0(t)z^{15} + 15c_1(t)z^{14} + \ \dots \ + c_{15}(t),$$

and the condition for these two polynomials in z to have a common root is that their resultant is zero.

The resultant is a polynomial in t, the determinant of a 31×31 matrix whose non-zero entries are the known polynomials $c_i(t)$. Using MAPLE 6 this determinant was found to be a polynomial of degree 60 in t, and its real roots were evaluated to a high level of precision. The real roots in the open interval $(0,4)$ are

$$t_c = 2.3586986002\dots\ , \text{and } t_d = 3.0850160178\dots\ .$$

The common roots corresponding to t_c, t_d are the real numbers $0.337\dots$ and $-0.534\dots$. These are the points where, in Figure 1, the equimodular curves appear to cross but, in fact, the behaviour at these point is more complicated.

Consider what happens when we start at the end-points γ and move along the equimodular curve •3 in the positive direction. Initially $t = 4$, and t then decreases until it reaches the value $t_c = 2.358\dots$ at the point $0.337\dots$. Now, if we continue along •3, t increases until it reaches the value 4 again at the other end-point δ. Thus we cannot consider •3 as the homeomorphic image of $[0,4]$. The answer is that we must turn sharply to the right at $0.337\dots$, onto the curve •4. The value of t then continues to decrease until it reaches 0 at the point $\hat{\psi}$. Thus we have a homeomorphic image of $[0,4]$, comprising parts of •3 and •4; it is not smooth at the point $0.337\dots$, precisely because the Jacobian vanishes there. Similarly, if we start from δ and move along •3 in the negative direction, turning sharp right at $0.337\dots$ onto •4, then we obtain another homeomorphic image of $[0,4]$.

A similar technique deals with the 'intersection' of •1 and •2 at $-0.534\ldots$.

The outcome of this analysis is that we have decomposed the five equimodular curves shown in Figure 1 into 16 proper arcs, 12 of which are smooth. The other four each have a point of non-differentiability, and they intersect in pairs at these points The arcs intersect smoothly at their end-points when these correspond to double roots of $v(0, z) = 0$ or $v(4, z) = 0$. Figure 3 is a schematic representation of this decomposition.

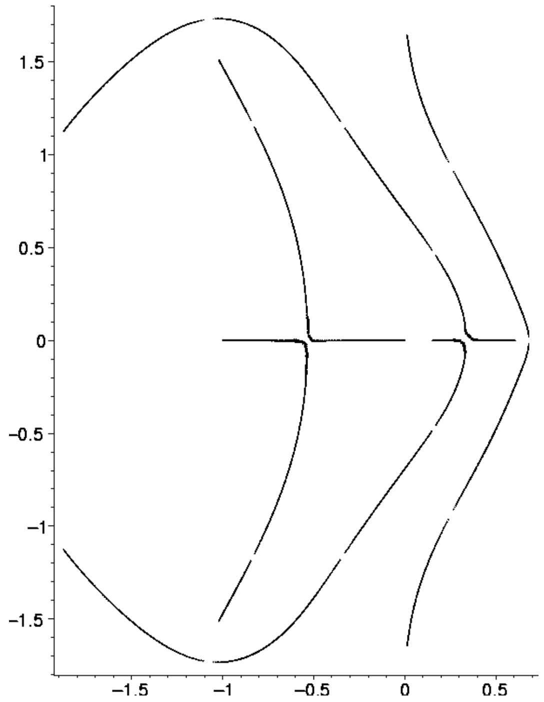

Figure 3: Decomposition of the equimodular curves into 16 arcs.

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