Abstract

The rendezvous search problem asks how two unit speed players can find each other in least expected time when randomly placed in a known region. This article surveys recent work on rendezvous search and its offshoots, from the perspective of one of its founders.

Rendezvous Search: A Personal Perspective

Steve Alpern
Department of Mathematics
The London School of Economics
Houghton Street
London WC2A 2AE
United Kingdom
email: alpern@lse.ac.uk

February 1, 2000

Contents

----

1 Introduction

The theory of optimal search is a long established field (see [52], [40]) which overlaps the areas of operations research, applied mathematics, and statistics. It asks the question of how one can optimally find an immobile or mobile object or agent. The word 'optimally' can have various meanings, but a typical one is 'in least expected time'. In the earlier stages of search theory, the hider was assumed to have no aims of its own - all of the optimization was on the part of the searcher. However, beginning with some problems proposed by Isaacs in his book on Differential Games [44], games have been studied in which the searcher (minimizer) and the hider (maximizer) play a zero sum game with the (capture) time T (required for the searcher to find the hider) as payoff. These games, called 'search games' have received much recent attention, and are the subject of the definitive text by Gal [43].

It was at the speculative final portion of a talk on search games given by the author (at the Institute for Advanced Studies, Vienna, 1976) that rendezvous search was first introduced as an optimization problem. After a survey of work on search games, the question was raised as to what would happen if *both* players sought to minimize the capture time T. This is the problem that arises, for example, when two people shopping together in a department store look around and see that they are no longer together. How should they proceed to find each other? Should one stay still and hope that the other is searching the whole store? What should they agree in advance to help them if this situation arises.

At the Vienna talk, two specific problems were suggested by the author, as paradigms of rendezvous search. We restate them here.

Problem 1 (Astronaut Problem) *Two astronauts land on a spherical body that is larger than the detection radius (within which they can see each other), but small enough so that the entire surface can be searched several times over. The body does not have a fixed orientation in space, nor does it have an axis of rotation, so that no common notion of position or direction is available to the astronauts for coordination. Given unit walking speeds for both astronauts, how should they move about so as to minimize the expected meeting time T (before they come within the detection radius)?*

Since this problem appeared difficult at the time, an apparently easier discrete problem with a similar flavor was also proposed.

Problem 2 (Telephone Problem) *In each of two rooms, there are n telephones randomly strewn about. They are connected in a pairwise fashion by n wires. At discrete times $t = 0, 1, 2, \ldots$ players in each room pick up a phone and say 'hello'. They wish to minimize the time t when they first pick up paired phones and can communicate. (This problem is equivalent to spatial rendezvous on a complete graph.)*

For both problems, it was supposed that the two players could not distinguish between themselves, and consequently would have to use the same mixed strategy. So the problem was to find that mixed strategy which *if employed simultaneously by both players* would minimize the expected time required for them to meet. This version of the problem was later defined precisely and called the *symmetric rendezvous problem*. The version where they could use *distinct strategies* (hence no need for mixing) was only introduced later in **??** and called the *asymmetric rendezvous problem*. It is also worth noting that in both problems the formulation is carefully worded so as to indicate that the players do not have a common labeling for positions or directions on the sphere, or for the telephones - otherwise the solution would be trivial (meet at the North Pole for example, or both pick phone number 1). This notion of symmetry was later formalized in [1] by giving a particular subgroup of the symmetry group of the search region, and considering this subgroup as a parameter of the problem. All these difficulties in formulation will be discussed in Section 2.

The reader may be wondering what progress, if any, has been made on these two problems which initiated the field of rendezvous search. It must be admitted that no significant progress has been made on the Astronaut Problem, although Anderson and Fekete [18] have obtained other results on related two-dimensional problems, and the author, Baston and Essegaier [3] have results on graph rendezvous which might have some use if the sphere were approximated by a graph. Even for the supposedly easier Telephone Problem only bounds [19], rather than exact solutions, have been obtained. However the strategy class for this problem proposed in the pioneering paper of Anderson and Weber [19] has recently been shown to be optimal in successively wider domains [3],[15]. Despite this lack of progress on the motivating problems, much has been achieved in this field, as will be outlined in this survey. The phenomenon of our motivating problems resisting solution themselves while at the same time spawning subsidiary problems with solutions and new theory is not unique to our field but is actually common in many other areas of mathematics.

Due to its obvious connection with problems in everyday life (how to find your child when lost in a supermarket) rendezvous search theory has attracted some attention outside the usual academic circles, including radio and the popular press [31]. However in this survey we will confine our attention to technical developments.

This paper will survey the positive results that have been obtained in the field of rendezvous search. We shall give the full proofs of results only when they are very short and particularly instructive. The reader interested in a deeper understanding of this field can find the full proofs in the original cited papers or in the forthcoming book [11]. The paper contains a few new results, and many new observations or connections between disparate parts of the theory.

It should be noted that while the formal mathematical analysis of rendezvous search is relatively new, and covered fairly comprehensively in this survey, the problems at the heart of rendezvous search have been dealt with by man and animal down the ages. Migrating animals which get separated from the herd may have to find it again if they are to survive, and as the herd size gets small the problem becomes more symmetric (between herd and individual) and approaches our formulation (which is exact when there is simply a migrating pair). Similarly, ships at sea have been travelling in convoy for a long time, and have had to attempt to regroup after being separated in storms (e.g. Columbus' ships). When travelling along coastlines they had agreed upon rendezvous points on the shore (avoiding the problems dealt with here), but when far at sea this was not practicable. Similarly,the military has always had protocols for rendezvous in unfamiliar territory (see [47, p. 96] for the SAS version), and the same goes for explorers. A famous example is that of Schelling [51], who considered the problem of two parachutists landing in a field with several landmarks. He discussed how they might try to figure out where to meet (assuming no prior agreement). However Schelling's analysis differs radically from ours in that he views the problem as a one-shot game, where the players either succeed (both go to same place) or fail. Our version of the problem, on the contrary, is a dynamic model within the theory of optimal search in which the players keep trying to meet even if they are not successful at first. Schelling's analysis depends heavily on shared 'focal points' on which players with certain common backgrounds can use to coordinate, whereas our analysis specifically rules out such spatial coordination through the assumption of symmetry of the region as formalized in a given group of isometries. The important economic field of 'coordination games' (see [41], [45], [48]) also gives a dynamic extension of Schelling's problems. However, since its assumption that the moves of the players are revealed periodically is hard to justify within our context of spatial search theory, we will cover this field only briefly. In fact the researchers in rendezvous problems and coordination games have up to now been oblivious to the other's work – a situation that we hope to correct here.

This survey article is not arranged chronologically but logically by topic. In Section 2 we will present the formal model for the rendezvous search problem that was first introduced in [1]. This section considers the basic problem on a compact space, and defines precisely the symmetric and asymmetric rendezvous problems and the significance of symmetry groups.

Section 3 deals with the area of rendezvous search that has attracted the most attention - rendezvous on the line. It discusses the work on the symmetric version of the problem by the author [1], Anderson and Essegaier [17] and Baston [20], as well as the work on the asymmetric version by the author with Gal [9] and with Beck [5], and by DiCairano [22]. In all of these models, the two players are placed a distance d apart on the line (where the distribution of d or its exact value are known to both players), and faced in random directions. They seek to meet in least expected time.

Section 4 discusses rendezvous on other networks. First we show how the general formulation given in Section 2 for compact metric spaces can be specialized to finite networks (one dimensional metric spaces), as in [1]. Then we discuss Howard's solution [26] to the rendezvous problem on the closed interval, where the two players are initially placed randomly and independently on the interval, at which time they know their own position but not their partner's. Finally we discuss the case, mentioned as an example in Section 2, where rendezvous takes place on the circle. We cover both Howard's analysis [26] for the case that the initial placement of the players is uniform and the author's analysis [2] for monotone distributions.

Section 5 discusses the papers of Anderson and Fekete [18] and Thomas and Hulme [34] concerning rendezvous on the plane. This marks the beginning of a theory that may in time be sufficiently developed to deal with questions like Problem 1 (astronaut problem) mentioned above.

Section 6 discusses the combinatorial problems of rendezvous on a graph, where at each step both players simultaneously move to a node adjacent to their current node, until their positions coincide. We show how the pioneering work of Anderson and Weber [19] for the complete graph (this is equivalent to the Telephone Problem discussed above) has been continued by the author and Pikounis, and extended to more general classes of graphs by the author, Baston, and Essegaier [3]. A connection between the rendezvous theory of this article and the work on coordination games by Crawford and Haller [41], Kramarz [45] and Ponssard [48] is established by considering the latter as rendezvous search on a complete graph with a periodic searchlight. Finally, rendezvous on a cycle graph is discussed.

All the above work is defined by the search space (line, circle, plane, graph). The remaining sections are distinguished not by the space they work on but by other variations on the basic theme of rendezvous. Usually the search space for these models is the line or the complete graph. Section 7 considers the rendezvous problem for *more than two players*, where the problem is for a certain given number of the players to meet at the same location. We consider both the model of expected time minimization and maximum time minimization, respectively studied by the author with Beck and Lim [30] and with Lim [29]. Part of the latter paper has been improved upon by Baston [20], who identified an optimal strategy for 3-player rendezvous which bettered a claimed bound in [29] and has subsequently [14] been shown to be uniquely optimal. We also give Gal's algorithm for multiple minimax rendezvous on the line [24]. For the circle, we present a new (unpublished) algorithm of the author for multi-player rendezvous. For the complete graph we describe both the searchlight version of multi-player rendezvous given in a coordination game context by Kramarz [45] and Ponssard [48] and the simulation results of Pikounis and Thomas [**?**] comparing strategies in which players who meet stick together versus separating and meeting again later.

Section 8 considers developments arising from the problem of 'adversary rendezvous' posed by the author in [1], in which the initial placement of the players is performed by an antagonistic Nature. Baston and Gal [21] were the first to produce rendezvous strategies for the line which are robust in the sense that they are effective against all initial distributions of initial distance. Hence these strategies are useful when this initial distribution is not known to the players. The strategy for one of the versions they consider has since been optimized by the author and Beck [8].

Section 9 considers the work of the author and Beck ([4], [6], [8]) on rendezvous when the players have given bounds on the total distance each can travel, modelling players travelling in cars on a road with limited fuel. When these bounds (fuel levels) are sufficiently large the players can ensure a meeting, and in this case the expected time minimization is considered [4]. However for smaller bounds, the problem considered ([6] [8]) is that of maximizing the probability that they will eventually meet.

Section 10 considers the so called rendezvous-evasion game, where two rendezvousers attempt to meet (on a complete graph) before either of them meets an enemy agent. This problem, originally posed by the author, produces two distinct versions of a zero sum game (between the rendezvous team and the enemy searcher), depending on whether the rendezvous aspect is considered asymmetrically or symmetrically. The former has been analyzed by Lim [27] and the latter by the author and Lim [13].

2 Formalizing the Problem

In this section we present the formal model for rendezvous search on a compact metric space (X, d) that was introduced in [1]. We will define the symmetric and asymmetric rendezvous values, R^s and R^a, which depend on X, the detection radius δ, and a given group G of symmetries of X. In the symmetric version, the players cannot distinguish between themselves and are therefore constrained to employ the same strategy, which will usually have to be a mixed one. In the asymmetric version they can employ distinct strategies (for example the 'wait for mummy' strategy where one searches while the other waits). We will always be assuming that each player can see the whole space X but cannot see the other player. The players are assumed to have a maximum speed of one.

To illustrate these ideas informally, we consider the case where the search region is the circle, and one of three symmetry groups is given: G_1 consists of just the identity, G_2 consists of all rotations, and G_3 is the group of all symmetries of X - rotations and reflections. The group G_1 corresponds to the case where the circle is a clockface with a common labeling of all points. So the players could agree to meet at the 3 o'clock position for example. We call such a common strategy **GO TO Z**. The group G_2 corresponds to the case where points on the circle are not marked, but the players have a common notion of 'up', or equivalently, clockwise. In this case the strategy of going to a common agreed point is not available to the players, but in the asymmetric problem they could agree that one would go clockwise while the other goes anti-clockwise, a strategy pair we will call **OP DIR**. The group G_3 corresponds for example to rendezvous on a space station built around a circle, where the players have no common notions of 'up' or clockwise. A symmetric strategy available in this case is for each player to do the following: Flip a fair coin that says 'clockwise' and 'anti-clockwise' and go half way around the circle in this direction - repeat indefinitely. We call this strategy (which was shown by the author to be optimal for both players in the search game on the circle) **COHATO**, for 'coin-half-tour'. One last strategy that is always available for the asymmetric problem consists of one player staying still while the other searches the entire space. We usually call this 'wait for mummy' and will shorten it to **GO-STAY** in the table below. According to a recent National Geographic Video, the mother kangaroo teaches its baby to find the nearest bush and hide (wait) when the two become separated. The following table, taken from [1] (where the circumference is 2 and all numbers are twice as large as here) lists the six versions of rendezvous on a circle of circumference 1 arising from the three possible symmetry groups and the symmetric and asymmetric player situation.

Table 1: Rendezvous Bounds for the Circle

	G_1	G_2	G_3
R^s	$\frac{1}{3}$ **(GO TO Z)**	$\frac{3}{4}$**(COHATO)**	$\frac{3}{4}$ **(COHATO)**
R^a	$\frac{1}{4}$ **(OP DIR)**	$\frac{1}{4}$ **(OP DIR)**	$\frac{21}{1}$ **(GO-STAY)**

In each of the six scenarios, we name what we believe to be the best available strategy, and give the corresponding expected meeting times. In the top row, the strategy is a single one that is supposed to be used by both players (but with independent coin tosses in the latter two, which are mixed). The strategies in the bottom row are actually pairs of strategies. For example, OP DIR means player I clockwise and player II anti-clockwise. The expected meeting times are calculated based on an initial placement which is uniform and independent. For example, the $1/3$ in the upper left entry is the expected value of the maximum of two independent random variables which are uniformly distributed on the interval $[0, 1/2]$. These numbers are upper bounds for the corresponding rendezvous values, which are defined more precisely below. Note that the given values are nondecreasing to the right and up, which must be true for the optimal values.

It is worth observing that both G_2 and G_3 act transitively on the circle. This means that given any two points on the circle there is a group element taking one into the other. In such cases the player acquires no new information after his initial placement on the circle, and consequently his strategy cannot be a function of this placement. Such situations are easier to model, and we will confine ourselves to transitive groups G here. The non transitive case is analyzed in [1].

So assuming a transitive symmetry group G, which means that essentially all points in X look alike, the players may pick paths in X starting at some particular point x_0, even if that is not their initial placement. With this in mind, we define the path or strategy space $\mathcal{S}$ to consist of all Lipshitz continuous paths starting at x_0,

$$\mathcal{S} = \{p : [0, \infty) \to X, p(0) = x_0, d(p(t), p(t')) \leq |t - t'|\}. \tag{1}$$

The actual path followed by a player will depend both on p and his initial placement in terms of position and orientation in X. The latter will be determined by the element g chosen from the group G, so the path is $g(p(t))$, $t \geq 0$.

For any two paths p and q in X (starting anywhere), we define their *meeting time* T by

$$T(p, q) = \min\{t : d(p(t), q(t)) \leq \delta\},$$

where $\delta \geq 0$ is the given detection radius, usually taken to be zero in one dimensional problems. If the players choose strategies (paths) $s_1, s_2 \in \mathcal{S}$ then the expected time for them to meet is given by

$$\begin{aligned} \hat{T}(s_1, s_2) &= \int_{g_1 \in G} \int_{g_2 \in G} T(g_1(s_1), g_2(s_2))\, d\nu(g_1)\, d\nu(g_2) \\ &= \int_G T(g(s_1), s_2)\, d\nu(g), \end{aligned}$$

where ν denotes Haar measure on the group G – which is simply the uniform distribution for the examples in this paper. The reason that we need only let G act on one of the paths is that for $g, h \in G$, $T(gp, hq) = T(h^{-1}gp, q)$.

We can now define the symmetric and asymmetric rendezvous problems (and values). The symmetric problem models situations where a single plan (strategy) must be broadcast to two lost individuals whose names are not known. So one can't say 'Jim stay still and Jack search'. For this reason a pure strategy may not be very advisable. For example, if both players are on a line, then in the case where they start facing the same direction (which has probability $1/2$) they will always remain at their initial distance apart if they follow the same pure strategy. So mixed strategies are in general needed. With this in mind, let $\mathcal{S}^*$ denote the set of all mixed search strategies, that is, the set of all Borel probability measures on $\mathcal{S}$. We then define the symmetric rendezvous value R^s by

$$\begin{aligned} R^s &= R^s(X,G,\delta) = \min_{s^*\in\mathcal{S}^*} \int_{\mathcal{S}}\int_{\mathcal{S}} \hat{T}(s_1,s_2)\, ds^*(s_1)\, ds^*(s_2)\,, \text{ or simply} && (2)\\ &= \min_{s^*\in\mathcal{S}^*} \hat{T}(s^*,s^*)\,, \text{if we allow } \hat{T} \text{ to act on mixed strategies.} && (3)\end{aligned}$$

The use of min rather than inf (the existence of optimal symmetric rendezvous strategies) is justified by the observation that T is lower semicontinuous in each variable on the strategy space $\mathcal{S}$ when it is given the topology of uniform convergence on compact intervals.

Observe that if s^* is the optimal symmetric strategy, then the pair (s^*,s^*) is a Nash equilibrium (but not conversely). Otherwise if $\hat{T}(s^*,s') < \hat{T}(s^*,s^*)$ then $\hat{T}((1-\varepsilon)s^*+\varepsilon s',(1-\varepsilon)s^*+\varepsilon s') < \hat{T}(s^*,s^*)$ for sufficiently small positive ε, contradicting the optimality of s^*. This argument requires that $\hat{T}(s^*,s') < \infty$, so that for noncompact spaces an additional argument may be necessary.

For the asymmetric problem, there is no need to mix, and the asymmetric rendezvous value R^a is given simply by

$$R^a = R^a(X,G,\delta) = \min_{s_1,s_2\in\mathcal{S}} \hat{T}(s_1,s_2)\,. \tag{4}$$

Of course since both players could use the same mixed strategy if they wanted to, we have the inequality $R^a \leq R^s$. For an example where equality holds, see Section 4.2.

Although the formal ideas given in this section are derived only for compact spaces, most of the ideas carry through to all the topics discussed in the paper. Here are a few general ideas the reader should keep in mind in thinking about the rest of the paper. To gain upper bounds for the rendezvous value, it is enough to give a rendezvous strategy: better strategies yield lower upper bounds. To obtain a lower bound some other idea is needed. In some contexts, we can determine the rendezvous value exactly, in others progress has been made by sharpening the bounds. For the asymmetric problem, the 'wait for mummy' (called G0-STAY earlier) strategy is always available. How far from optimal is this strategy? Can it actually be optimal? In the symmetric problem, the players always have a version of 'wait for mummy' that they can use: First determine the time $t_{\max}$ required to search the whole region. Then flip a coin (possibly a weighted one) and if heads wait, if tails search. Repeat after time $t_{\max}$. How far from optimal is the best weighting of this class of 'symmetrized wait for mummy' strategies?

3 Rendezvous on the Line

The main problem considered in this section is the following: Two players are placed a distance d apart on the real line, and faced randomly in either direction (which they then regard as positive). Hence they have no *common* notion of a positive direction on the line. They both know the cumulative distribution function F for the initial distance d. They have unit speed and wish to meet in least expected time. What should they do? We denote the symmetric and asymmetric rendezvous values for this problem as R_F^s and R_F^a. The symmetric version of this problem was introduced in [1], and is still open even in the apparently simple case that the initial distance is known to be 1. The asymmetric version, first considered in [9] has proved somewhat more tractable, with optimal solutions determined for certain classes of distributions F, and reasonable bounds on the rendezvous value obtained for others. For rendezvous on the line, we may take $x_0 = 0$ in the general definition (1) of the pure strategy (path) space S, which we write more simply as

$$\mathcal{S} = \{s : [0, \infty) \to X, s(0) = 0, |s(t) - s(t')| \leq |t - t'|\}. \tag{5}$$

In some cases of bounded distributions, it will not be necessary to define strategies on the full time domain $[0, \infty)$, as there may be a time by which the players are certain to have met. The number $s(t)$ should be interpreted as the position of the player at time t relative to his starting position, in the direction he was initially pointed.

We define $\hat{T}(s_1, s_2; F)$ to be the expected meeting time for rendezvousers using the (pure or mixed) strategies s_1 and s_2 when the distribution of their initial distance is given by F.

3.1 Asymmetric Version

The first question one might ask in the asymmetric context is whether the players can achieve a finite expected meeting time for a given distribution F? That is, when is R_F^a finite? This question is best answered by a comparison with the following spatially symmetric version of the Linear Search Problem (LSP) introduced by Bellman [39] and Beck [38]: An object is hidden on the real line, equiprobably at $\pm d$, where d is drawn from F. The least expected time required for a unit-speed searcher, starting at 0, to find the object is denoted by L_F. It was shown by Beck that a simple geometric strategy, of say doubling the distances of successive searches to the left and right, will find the object in finite expected time if F has finite mean. Furthermore, L_F is finite if and only if F has finite mean μ_F.

To compare the problems of asymmetric rendezvous on the line and the LSP, we consider giving the players more information, namely, a common notion of a positive direction on the line. For example, they might be rendezvousing on a flowing river. (This corresponds to the symmetric group consisting of translations but not reflections.) Call the rendezvous value for this problem $\hat{R}_F^a$. In this case it is easy to see that periods of time when the two players are both moving in say the positive direction are wasted. So they should always move at speed one in opposite directions, so that the difference in their velocities is ± 2. They will find each other when the vector difference in their positions has either increased by d or decreased by d (depending on which one started to the right). Hence the problem they face is identical to that of a speed 2 searcher in the LSP with the same F. Consequently we have that (see [9, Lemma 5.1] for a formal proof)

Lemma 3 *For any distribution F, an optimal strategy in the common-direction rendezvous problem has both players always going in opposite directions at speed 1. Consequently $\hat{R}_F^a = L_F/2$, where L_F is the optimal value of the corresponding Linear Search Problem.*

Of course $\hat{R}^a_F \leq R^a_F$, because the common-direction version of the problem has more feasible strategies. For the same reason we have that $R^a_F \leq L_F$ (by restricting to strategies where one player is stationary). Consequently we have the easy but important result

Theorem 4 *$L_F/2 \leq R^a_F \leq L_F$, for any distribution F.*

One consequence of this result is that R^a_F is finite if and only if the mean μ_F of F is finite, since this fact is known for L_F. Another way of looking at this result is to say that the 'wait for mummy' strategy is never too bad on the line. Since we may restrict ourselves to consideration of distributions with finite mean, we define the set $\mathcal{F}$ consisting of all such distributions.

3.1.1 Bounded distributions

We now assume that there is a maximum initial distance $D = D_F$ between the players, and consequently a finite mean initial distance $\mu = \mu_F$. To motivate the subsequent discussion, consider for the time being that in fact the actual distance is known to be $d = D$. In this case the 'wait for mummy' solution has the searcher (player I) going a distance D forward, and then going back $2D$ in the other direction, a strategy we will denote by $\tilde{g} = \tilde{g}_D$,

$$\tilde{g}(t) = \begin{cases} t, & \text{if } 0 \leq t \leq D \\ 2D - t, & \text{if } D \leq t \leq 3D. \end{cases} \tag{6}$$

If Player II simply waits, the expected meeting time (assuming for our argument that $d = D$) will be $\frac{1}{2}D + \frac{1}{2}3D = 2D$. But of course the waiting strategy for II is dominated by one that goes forward for time $D/2$ to possibly (probability $1/4$) meet Player I, while having just enough time ($D/2$) to get back to his starting point for I to meet him from the other direction (also probability $1/4$). If II has not met I by time D, he knows that I initially headed away from him and is now a distance $2D$ from him, and coming back to him. So II can do the same thing as before, with twice the distances. That is, he goes in some direction (either one) a distance D (where he meets with probability $1/4$) and then returns again to his starting point by time $3D$ (where he meets I with probability $1/4$). This strategy for II is the (nonuniquely) optimal response to $\tilde{g}$. We call it $\tilde{h} = \tilde{h}_D$, and for completeness give it explicitly as

$$\tilde{h}(t) = \begin{cases} t, & \text{if } 0 \leq t \leq D/2, \\ D - t, & \text{if } D/2 \leq t \leq 2D, \\ t - 3D, & \text{if } 2D \leq t \leq 3D. \end{cases} \tag{7}$$

We call the strategy pair $\left(\tilde{g}, \tilde{h}\right)$ a 'modified wait for mummy' strategy, because $\tilde{g}$ is optimal against a waiting partner, while $\tilde{h}$ is optimal against the pure searcher $\tilde{g}$. (It is called the 'greedy' strategy in [21]). Observe (from the above discussion or directly from the definitions) that if the initial distance is $d = D$, the modified wait for mummy strategy achieves a meeting equiprobably at times $D/2$, D, $2D$, and $3D$. The expected meeting time $T\left(\tilde{g}, \tilde{h}\right)$ is therefor $13D/8$. Similar calculations easily establish the four possible meeting times when the initial distance is $d < D$, giving the first part (i) of the following result. The second part is proved as [9, Theorem 3.2]. The third part (iii) was conjectured by Baston and Gal in [21, Conjecture 6.1]. It was proved by the author and Beck [5, Theorem 12, Corollary 13] using the equivalence of asymmetric rendezvous on the line to a problem on 'double linear search' and using results on the latter problem due to the author and Howard [12]. For parts (ii) and (iii) the difficult part is establishing a lower bound for the rendezvous value.

Theorem 5 *Let F be a bounded distribution with maximum D and mean μ. Then*

(i) $R_F^a \leq (4\mu + 9D)/8$.

(ii) *If F is the atomic distribution with $d = D$ certainly, then the modified wait for mummy strategy $\left(\tilde{g}, \tilde{h}\right)$ is optimal and $R^a = 13D/8$.*

(iii) *If F is convex on $[0, D]$ then the modified wait for mummy strategy $\left(\tilde{g}, \tilde{h}\right)$ is optimal and $R_F^a = (4\mu + 9D)/8$. In particular, the rendezvous value for the uniform distribution on $[0, 1]$ is 11/8.*

We observe in the above theorem that the modified wait for mummy strategy is sometimes (for some F) optimal. We may ask if the original 'wait for mummy' strategy (where one player always waits) is ever (for any F) optimal. However Gal [24, Theorem 1] has answered this question in the negative.

Theorem 6 *The 'wait for mummy' strategy pair is never optimal (for no F) for the asymmetric rendezvous search problem on the line.*

We note that Howard [26] has shown that this result does *not* hold for asymmetric rendezvous on the circle, though the author [2] has shown that it almost holds there. See section 4.3 for these results. A slightly stronger positive version of Gal's negative result was proved by the author and Beck [5, Theorem 11].

Theorem 7 *If (f, g) is an optimal strategy pair for some distribution F, then there exists a non-trivial time interval on which $|f'| = |g'| = 1$ (when both players are moving at maximum speed).*

We conclude this section with a brief summary of a simple case of a more complicated result [10] regarding the point distribution (known initial distance 1), where the players have *unequal speeds,* 1 for player I and $m \leq 1$ for player II.

Theorem 8 *If m is less than the golden mean $\frac{\sqrt{5}-1}{2} \doteq .618\,03$ then II should stay still for time 1, and then move at maximum speed in a certain way, and the rendezvous value is $\frac{m^2+4m+2}{(m+1)^2}$. If m exceeds the golden mean, then a modification of the above strategy $\left(\tilde{g}, \tilde{h}\right)$ is optimal, and the rendezvous value is $\frac{4m^2+7m+2}{(m+1)^3}$.*

In solving the above problem, there is a subgame Γ' that occurs after the searching player I learns that he has gone in the wrong direction (not found II in his forward direction). At this point it is common knowledge that player I knows the direction of player II, and the distance d between them is also known. However, II still does not know the direction of I. In this subgame Γ' the least expected meeting time is $\frac{(2m+1)d}{(1+m)^2}$, or more generally $\frac{(v_1+2v_2)d}{(v_1+v_2)^2}$ for any speeds of the players.

3.1.2 Discrete distributions

Suppose that the initial distance d between the players can take on only a countable number of possible values $0 \leq d_1 \leq d_2 \ldots$. Without loss of generality we may choose a coordinate system (as an observer) with player I starting at the origin and pointing right, and player II starting at $\pm d_i$ and facing left or right. Thus if the strategies chosen are g and h, then $g(t)$ becomes the actual path of player I, while II's path will be of the form $\pm d_i \pm h(t)$. We call these distinct possible paths of player II, the agents of player II. We label the agents of player II in the order that they meet the path $g(t)$. In this setting the following optimality condition has been established by the author and Gal [9, Theorem 4.1], which greatly simplifies the search for optimal strategies.

Theorem 9 *Suppose that F is a discrete distribution and (g, h) is an optimal strategy against F. Then player I and agent $m+1$ (of II) approach each other at combined speed 2 as soon as agent m has been met.*

In the case of a discrete distribution F algorithms for determining optimal rendezvous strategies have been developed and programmed by DiCairano [22]. Gal has found a general polynomial algorithm (computationally effective for any distribution function) for rendezvous on the line. It will be explicitely described [11].

3.2 Symmetric Version

The symmetric version of rendezvous on the line, although more difficult than the asymmetric, was actually proposed earlier, in the original continuous rendezvous paper [1]. The basic setup is as presented at the beginning of this section. This is an example where pure strategies are useless in symmetric search: if the two players are initially pointed in the same direction and use the same pure strategy, then they will never meet.

We will restrict our attention to distributions F (of the initial distance d between the players) which have a finite maximum value D. The first strategy for symmetric rendezvous on the line was suggested by the author and called 1F2B, for 1 step forward and two steps backwards. We define a *step* to be a distance $D/2$. The strategy proceeds as follows: Pick a random direction, go one step in this direction, then go two steps in the opposite direction. Repeat if necessary.

If the two players start out in opposite directions they will meet by time $3D/2$, otherwise their distance at time $3D/2$ will be the same as their initial distance. From this simple observation it is an easy matter to calculate the expected meeting time for the common use of this strategy, yielding the following result (from [1, Proposition 3]).

Theorem 10 *If F is a bounded distribution with maximum D and mean μ, the symmetric rendezvous value for the line satisfies $R_F^s \leq 2D + \mu/2$.*

It was suggested in [1] that indeed this strategy might be optimal, at least for the atomic distribution. However, this was a short lived conjecture, as Anderson and Essegaier [17, Proposition 1] made a more sustained attack on the problem which yielded the following improved bound.

Theorem 11 *For any bounded distribution F, the symmetric rendezvous value on the line satisfies $R_F^s \leq 1.78388D + \mu/2$, where $D = \inf\{x : F(x) = 1\}$ and μ is the mean initial distance.*

Anderson and Essegaier obtain this estimate with a strategy which uses a probability distribution over the following motions, with step length $D/2$,

$$\begin{aligned}\gamma_1 &= 2F4B \\ \gamma_2 &= 1F3B2F \\ \gamma_3 &= 1F3B2F \\ \gamma_4 &= 1F1B1F3B\end{aligned}$$

which each take a total time equal to $3D$. Anderson and Essegaier have this procedure repeated exactly after each period of time $3D$, without making use of any information that might have been acquired during these periods. For example, a player who used γ_1 in the first period and did not meet his partner knows that his partner has either also used γ_1 and chosen the same direction as forward, or has used one of γ_3, γ_4 and started in the direction chosen as forward. By making these observations regarding the Anderson-Essegaier strategy, Baston [20] came up with the following further improved strategy based on four motions of period time $7D/2$:

$$\begin{aligned}\alpha_1 &= 1F2B2F2B, \\ \alpha_2 &= 1F3B3F, \\ \alpha_3 &= 1F1B1F3B1F, \\ \alpha_4 &= 2F5B.\end{aligned}$$

By optimizing the probability of these four motions, and by making use of observations similar to those given above regarding their interaction, Baston [20, Theorem 2] obtained the current best estimate for the symmetric rendezvous value on the line.

Theorem 12 *For any bounded distribution F with maximum D and mean μ, the symmetric rendezvous value on the line satisfies $R_F^s \leq 1.7091D + \mu/2$.*

4 Rendezvous on Networks

In this section we consider the rendezvous search problem on compact networks X. By a network, we mean a finite union of closed intervals (arcs) which may meet at their endpoints (nodes). The arcs have lengths, so that the network as a whole is a compact metric space. As such, these problems fit the general formulation of the rendezvous search problem on a compact metric space given in Section 2, with the detection radius assumed to be zero. This means that the players meet when their positions coincide. Aside from some elementary results for general networks, we will mainly be concerned with the cases of a single closed interval and of the circle.

4.1 General Networks

The problem of rendezvous on a general network is one that we hope will be analyzed by combinatorists. In the mean time, we offer some elementary results that are at least much better than random walks. These results are taken from [1, Proposition 1]. We assume the game begins with an initial uniform distribution of the players, and that the players have unit speed.

Theorem 13 *Let X be a connected network consisting of m unit-length arcs, and let d denote the arc-length distance between points. Let G be the group of all symmetries (isomorphisms) of X. Let D denote the metric diameter of X. Then*

1. *There is a search strategy for I which finds II in expected time $\leq 2mD \leq 2m^2$ regardless of the motion of II.*
2. *$R^s(X, G) \leq 4m$.*
3. *$R^a(X, G) \leq m$.*

These are all easy to prove. For (1), which is not a rendezvous result but a useful comparison from search games, simply go to the middle of a random arc and then to one of its endpoints; repeat until II is found. For (2), each player should randomize equiprobably between searching the entire network or waiting for a time period of $2m$, with independent repetitions. For (3), use the 'wait for mummy' asymmetric strategy pair.

4.2 The Closed Interval

The problem of rendezvous on the closed interval $[0, 1]$ was proposed by the author, who worked unsuccessfully on it with Essegaier. The successful work presented in this subsection is due to Howard [26]. We assume that the two players have a common labeling of $[0, 1]$ so that after their initial placement, they know exactly where they are. This corresponds to the symmetry group $\bar{G}_1$ containing only the identity. (The only nontrivial symmetry is the reflection $x \to 1 - x$, and if this is added the group is called $\bar{G}_2$.) As Howard puts it, two individuals arrive at a (finite) street where all the houses are visibly numbered.

As neither $\bar{G}_1$ nor $\bar{G}_2$ acts transitively, the treatment given in Section 2 does not apply (a general formulation of the problem for arbitrary symmetry groups is given in [1], if the reader is interested). The only extension to the theory needed to cover this situation is the notion that a strategy depends on the initial placement of the player (up to symmetry). Howard begins his analysis by noting that a player who is initially placed at an endpoint should obviously move at unit speed to the other endpoint. These two paths, $\alpha(t) = t$ and $\beta(t) = 1 - t$, are respectively referred to as the *left sweeper* and the *right sweeper*.

The most natural distribution with which to initially place the players is the uniform distribution. Howard's result [26, Theorem 3] is more general however, as can be seen from the following theorem which was proved by an elegant discrete and geometric argument.

Theorem 14 *Suppose the players are initially placed independently on the interval* $[0,1]$ *according to a distribution* F *whose density is nondecreasing. Suppose the players know their exact position, or equivalently, that the given symmetry group is* $\bar{G}_1$ *consisting only of the identity. Then an optimal strategy pair in the asymmetric rendezvous problem is for each player to move at unit speed to the right until he meets the right sweeper, and then to return with him to the center. Consequently* $R^a\left([0,1],\bar{G}_1\right) = \frac{1}{2}\left(1 - \int_0^1 (1-F(x))^2\,dx\right)$.

It is interesting to note that the optimal strategy pair for this asymmetric problem consists of two identical strategies. Consequently these strategies must also be optimal for the symmetric problem, which in this case has a solution in pure strategies. This is an example where the symmetric rendezvous value is not more than (equals) the asymmetric rendezvous value. If F is not the uniform distribution, it cannot be symmetric with respect to the reflection $x \to -x$, and consequently it is not reasonable to consider the symmetry group $\bar{G}_2$, since the distribution should be invariant with respect to the symmetry group.

Now let us specialize to the case of the uniform distribution $F(x) = x$ on $[0,1]$, where we have

$$R^a\left([0,1],\bar{G}_1\right) = \frac{1}{2}\left(1 - \int_0^1 (1-x)^2\,dx\right) = \frac{1}{3}$$

Another strategy which is optimal in this case (in the symmetric or asymmetric version) is for each player to go directly to the middle point $1/2$ and wait there. The optimality is simply a matter of checking that the expected time is $1/3$, which is just the expected value of the maximum of two independent random variables which are uniformly distributed on $[0,1/2]$. (Note that this is the same calculation as was needed to put the $1/3$ time next to the GO TO Z in Table 1. Observe that the strategy of going to $1/2$ does not require a knowledge of which side of the center of the interval you are on. Consequently it is a feasible strategy even for the symmetry group $\bar{G}_2$. So as Howard observes [26, p. 555], we have the following.

Theorem 15 *Suppose the players are initially placed uniformly and independently on the interval* $[0,1]$. *Then*

$$R^a\left([0,1],\bar{G}_1\right) = R^s\left([0,1],\bar{G}_1\right) = R^a\left([0,1],\bar{G}_2\right) = R^s\left([0,1],\bar{G}_2\right) = 1/3.$$

We may rephrase this as follows: Relative to the most difficult problem (that of $R^s\left([0,1],\bar{G}_2\right)$) where the players must use the same strategy, which must be the same either side of $1/2$, the problem does not get easier if the players are allowed to use distinct strategies, or if these strategies may be different either side of $1/2$.

There are other optimal strategies given by Howard for the uniform distribution. For these, and for a discussion on how these strategies may and may not be combined, see the original paper.

4.3 The Circle

We have already given a brief discussion of rendezvous on the circle in Section 2, where it was used to illustrate the formulation of the problem for an arbitrary compact metric space. We suggest that the reader reviews that material, particularly Table 1, before returning here. The reader is reminded that for the circle the symmetry groups are as follows: G_1 consists of only the identity isomorphisms and corresponds to the scenario where there is a common labeling of all points on the circle, G_2 consists of all rotations and corresponds to the scenario where points are not labelled but the players can agree on what 'clockwise' means, G_3 includes both rotations and reflections and in this scenario the players have no common notion of direction of the circle.

4.3.1 Uniform Initial Placement

We first consider the case where the game begins with the players being placed on the circle independently with a uniform distribution. This is the situation described in Table 1. The six strategies entered in Table 1 represented the best strategies the author could think of at the time of writing [1]. In [26], Howard established that three of the six are in fact optimal in their context. The following is [26, Theorem 4].

Theorem 16 *Suppose the players are placed uniformly and independently on the circle C of unit circumference. Consider the scenario, formalized by the symmetry group G_1, where the players have a common labeling of points. Then the asymmetric rendezvous value is given by*

$$R^a(C, G_1) = \frac{1}{4},$$

and the unique optimal strategy is (OP DIR) for one to go clockwise while the other goes anti-clockwise. Since this strategy (pair) does not require a common labeling of points on the circle, it is also admissable in the scenario with no common labeling of points, but only a common notion of direction (formalized by G_2). Hence it is also optimal in that context, and

$$R^a(C, G_2) = \frac{1}{4}.$$

Howard established this result by the simple observation if the initial position of II is more distant than t from I's position at time t, then the players cannot have met by time t. For the uniform distribution, the set of such initial positions has probability $1 - 2t$. Consequently, the expected meeting time, which is the integral of the probability of not having met by a particular time, it at least $\int_0^{1/2} (1 - 2t)\, dt = \frac{1}{4}$. It should be observed that this proof implicitly uses the assumption that I cannot see II (but not the reverse assumption) in that I's position at time t cannot depend on II's initial position. If we consider the problem where I can actually see II but II cannot see I, the solution is for II to remain still while I goes directly to meet him. The expected time is therefore the same as the expected initial distance between the players, which is $1/4$ for the uniform distribution. Since this gives the players additional information, it must be a lower bound for the original game with less information.

Another approach is to use the idea of Lemma 3. When players have a common notion of direction, they should always move in opposite directions at maximum speed. What results is an equivalent problem in which a searcher with speed 2 seeks to find a stationary hidden object whose distribution is the same as the distribution of the difference of the two rendezvousers. In this case, the problem reduces to the trivial one of how a Searcher on the circle should seek an object hidden uniformly on the circle. The obvious solution is simply to move at maximum speed (2) around the circle. For the original problem, this translates into the two rendezvousers moving at speed 1 in fixed and opposite directions. The advantage of this approach is that it derives the optimal rendezvous strategies, rather than simply giving a lower bound on the rendezvous value.

A much more involved argument was necessary for Howard to establish the third of the six entries in Table 1, to get [26, Theorem 5] the following.

Theorem 17 *$R^a(C, G_3) = \frac{1}{2}$, and the 'wait for mummy' strategy pair (where one player waits while the other goes around the circle at unit speed - called GO-STAY in Table 1) is optimal. More generally, a strategy is optimal if and only if it has one players going around the circle at unit speed while the other player adopts any strategy that returns him to his starting position at time 1.*

4.3.2 Other Initial Placements

In this section we consider asymmetric rendezvous on the circle in the most difficult scenario for the players: they have no common labeling of points on the circle and no common notion of direction around the circle. This corresponds to the given symmetry group being G_3. The results discussed here were obtained by the author [2] using the notion of 'alternating search at two locations' which was developed in a separate article [12] jointly with Howard.

We begin the game by picking an arc of length $x, 0 \leq x \leq 1/2$, placing the two players randomly at either end, and facing them in random directions. We suppose that the random variable x is given by a continuous positive density function h on $[0, 1/2]$, with $\int_0^{1/2} h(x)\, dx = 1$.

The first result determines an optimal rendezvous strategy for a large class of monotone densities h which includes the uniform density $\bar{h}(x) = 2$. The strategy is in fact uniquely optimal for all the densities which are strictly monotone. The optimal strategies will have one player going all around the circle at maximum speed one, while the other player will have to determine a cutoff time c where his motion changes. The time c depends on whether h is increasing or decreasing, as follows. If h is increasing, define c to be the smallest number in $[1/2, 1]$ which maximizes the quotient $H(y)/y$. If h is decreasing, define c to be the largest number in $[0, 1/2]$ which minimizes the quotient $(1 - H(y))/(1 - y)$. With this definition of the cutoff value c, the solutions given by the author in [2] are as follows.

Theorem 18 (monotone densities) *If the initial distance between the players is given by a monotone density h, $0 \leq x \leq 1/2$, the following strategies are optimal, and ensure a meeting by time 1. In both cases one player (I) goes all around the circle at maximum speed one.*

- *If h is increasing, then player II moves at maximum speed in a random direction for time $c/2$, then reverses direction at maximum speed for time $c/2$, and finally stays still until time 1.*
- *If h is decreasing and $h(1/2) > h(0)/2$, player II stays still for time c, then moves at speed one in a random direction for time $(1 - c)/2$, then reverses direction until time 1.*

- *Furthermore, if h is* ***strictly*** *increasing or decreasing, the relevant above strategy is* ***uniquely*** *optimal.*
- *If h is increasing and decreasing, it is the uniform density $\bar{h}(x) = 2$, in which case both the above strategies are optimal with $c = 1/2$.*

The next result applies to continuous densities h on $[0, 1/2]$ which are *slowly varying* in the sense that *their maximum divided by their minimum is less than two*. Such densities are in a sense close to the uniform density, where this ratio has the minimum possible value of 1. (The two conditions in the previous theorem, that h is decreasing and $h(1/2) > h(0)/2$, together imply that h is slowly varying.) The following extends the similar result of Howard for the uniform density given in the previous section.

Theorem 19 (slowly varying densities) *If the initial distance between the players is given by a slowly varying density, then any optimal strategy has one player going all around the circle at speed one, and the other player back at his starting point at time 1, by which time they have met.*

The last result of this section applies to arbitrary distributions of the initial distance (not necessarily given by a density) and shows that Howard's counterexample (uniform distribution on the circle) to Gal's conjecture on the non-optimality of the 'wait for mummy' strategy is the *only* counterexample for the circle.

Theorem 20 (arbitrary distributions) *The only distribution of initial distance between the players for which the 'wait for mummy' strategy is the optimal asymmetric rendezvous strategy is the uniform distribution.*

5 Asymmetric rendezvous in the plane

It is interesting to note that although the first formal rendezvous problem (see Problem 1 – the Astronaut problem) was posed for a two dimension search region (the sphere), the initial attacks on the rendezvous problem were all in a zero (graph) or one dimensional context. In this section we describe the work of Anderson and Fekete [18] and Thomas and Hulme [34] on rendezvous search in the plane. We begin by discussing the analytical presentation of [18], saving the simulation work of [34] to the final subsection.

Just as for the work on the line, the initial placement of the players is described by a distribution of their initial distance, or of the vector difference between their initial positions. Everything about the problem is invariant under translations and sometimes also rotations of the plane, so these are usually the relevant isometry groups G in the sense of Section 2.

5.1 Known distance with common directions

Consider the case where Players I and II have respective maximum speeds $v_1 \leq v_2$, and know that the distance between them is d and the direction of their difference is uniformly distributed. Equivalently, II is placed uniformly on a circle of radius 1 centered at I. The players are assumed to share a common notion of compass directions in the plane. (This assumption is similar to that of Lemma 3.)

First observe that the 'wait for mummy' strategy, with the searcher (I) going along an arbitrary radius of the circle (containing player II) and then going around the circle at maximum speed in either direction, yields a maximum meeting time of $\frac{d}{v_1}(1+2\pi)$ and an expected meeting time of $\frac{d}{v_1}(1+\pi)$. However, since the players have a common notion of direction, they can make use of the observation of Lemma 3, and ensure that they always move at maximum speed in opposite directions (this is observed in [18, Theorem 2]). In the resulting problem, a searcher with speed v_1+v_2 seeks a stationary hidden object placed uniformly along a circle of radius d, centered at the searcher's initial position. In this scenario, 'wait for mummy' is the only strategy allowed, and hence the same reasoning as above shows that the minimax meeting time is $\frac{d}{v_1+v_2}(1+2\pi)$ and that the least expected meeting time (rendezvous value) is $\frac{d}{v_1+v_2}(1+1\pi)$. If the 'wait for mummy' strategy in the problem with the stationary object is translated back to the original rendezvous problem, the solution pair has an elegant geometrical interpretation in terms of 'kissing circles'. The details are given in [18, Lemma 2, Theorem 1], and the main results may be summarized as follows.

Theorem 21 *The 'kissing circles' strategy pair is optimal for the known distance, common direction planar rendezvous problem, in both the minimax meeting time and expected meeting time contexts. These optimal times are respectively* $\frac{d}{v_1+v_2}(1+2\pi)$ *and* $\frac{d}{v_1+v_2}(1+1\pi)$.

In the case that the players have bounded resources in the sense of Section 9, a modification of the kissing circles strategy maximizes the probability of a meeting, has smallest worst case time and optimal expected time, assuming the players meet [18, Theorem 3]. Similar techniques to those described here can also be used to analyze problems with common directions (orientations) and a finite set of initial vector differences between the players.

5.2 No common directions

When the players have a common sense of orientation, the problem is not truly one of rendezvous – it is really (can be reduced to) a search problem for a stationary object. However the following problem (from [18, Section 4.2] introduced and solved by Anderson and Fekete is a true rendezvous problem and the first one which is two-dimensional.

The search region is the grid $\{(x,y) : x \text{ or } y \text{ is an integer}\}$. Suppose the initial vector difference between the two players is taken equiprobably from the set $Y = \{(1,1),(1,-1),(-1,-1),(-1,1)\}$. In other words, the graph distance between their starting nodes is 2. Assume the players are independently faced in random directions, which they each call 'North'. Assume that the players each have maximum speed one and have no common notion of orientation. Then the solution is as follows [18, Theorem 4].

Theorem 22 *For the above asymmetric rendezvous problem on the grid, the optimal solution pair is (NWSSEENN,NSNSNSNS), where each letter denotes a move of distance 1 in time 1 in that compass direction.*

In terms of the general theory of rendezvous, it is useful to observe that this strategy (like the optimal strategy $\left(\tilde{g}, \tilde{h}\right)$ in Theorem 5 (ii)) is a modified wait for mummy strategy. Player I searches optimally for a stationary object hidden equiprobably at the four nodes at distance 2 (which he reaches equiprobably at times 2,4,6 or 8), while Player II oscillates from his initial position (at even time) to an adjacent one (at odd times).

5.3 Known initial position

Consider the following asymmetric information problem. Player II is placed uniformly on a circle of known radius d centered at Player I. Then I is told the position of II. All this is common knowledge. Assuming the players have maximum speeds v_1 and v_2, what is their least expected meeting time? This formulation corresponds to a problem in which the last known location of player II (using GPS to broadcast a rescue request after an accident) is common knowledge, as is the location of the rescue base. However II's compass does work after the accident.

Recall from the discussion at the end of Section 3.1.1 that if this problem took place on the line, with I at 0 and II at $\pm d$, the least expected meeting time is $\frac{(v_1+2v_2)d}{(v_1+v_2)^2}$. Clearly this time is a lower bound for the planar rendezvous problem, as it would be minimal if II were told the line connecting him to I (but not the direction to I). On the other hand, a strategy available in the planar problem is the simple one of wait for mummy which has meeting time d/v_1. So very simple reasoning puts the rendezvous value in the interval $\left[\frac{(v_1+2v_2)d}{(v_1+v_2)^2}, \frac{d}{v_1}\right]$.

Anderson and Fekete show that an optimal solution must have certain properties [18, Lemma 7]. Their strategy uses the assumption of a common notion of clockwise (or *up*) in the plane.

Theorem 23 *Suppose players I and II are placed randomly in the plane at a known distance d, and that I is told the direction to II. Under optimal play both players should have the same distance to II's starting point from time $\frac{d}{v_1+v_2}$ onwards, when they should be moving at maximum speed in opposite rotational directions.*

By formulating the minimum expected meeting time problem as a differential equation, and numerically solving it, they obtain an optimum spiral solution to the following case.

Theorem 24 *In the situation described in the previous theorem with $d = 2$ and $v_1 = v_2 = 1$, the least expected meeting time (rendezvous value) is approximately* 1.97.

Anderson and Fekete pose the interesting question of whether the 'wait for mummy' strategy becomes optimal when v_1 is sufficiently large compared with v_2.

5.4 Helicopter-walker simulation

A problem with a very similar flavor as that given by Anderson and Fekete in the previous subsection was considered earlier by Thomas and Hulme [34] in the realistic context of a fast helicopter searching for a lost slow walker. As in the previous problem, it is assumed that the region of search is the 'further on circle' (meaning disk) that the walker can have moved to from his last known (to the helicopter) position. However this known initial position "is sufficiently out–of–date that he [helicopter] makes the assumption the other player [walker] is equally likely to be anywhere within the 'further on circle'. It is assumed the search region stays constant during the search." The detection distances are asymmetric; the walker can see (or maybe hear) the helicopter from a long way off, while the helicopter must be very close to spot the walker. There is also a probability that even if they are close they may overlook each other. The problem is modeled as a search on a hexagonal grid. Obviously this is too complicated to be solved analytically, so a simulation is used instead.

For given pairs of strategies, the probabilities that a meeting has occurred up to various times are estimated. For the walker, the following strategies are considered:

1. Stationary.
2. Random Walk (go with probability 1/6 to each of the neighboring hexagons).
3. Head When Audible (stationary except when helicopter is heard, then head in that direction).

For the helicopter the following are considered.

1. Scanning Search (moves in horizontal paths, going from bottom to top of region).
2. Expanding Square Search (moves in a spiral out from center of region).
3. Local versions of the above two.
4. Drag strategy (decreasing spiral).

An interesting observation is the following (p. 48):

> "Examining the walker's paths in the simulation shows that he keeps tacking back and forth as he is confused by the helicopter becoming audible on different bearings. This suggests there may be an advantage in role-reversal. The helicopter should try to drag the walker towards it. Thus it should search less and make its presence audible more and eventually stay within a small area waiting for the walker to rendezvous with it."

6 Rendezvous on a Graph

In this section we consider the rendezvous problem on a connected combinatorial graph $X = (\mathcal{V}, \mathcal{E})$ consisting of vertices $\mathcal{V}$ and edges $\mathcal{E}$ together with a given group G of symmetries (isometries). That is, $(x, y) \in \mathcal{E}$, $g \in G \Rightarrow (gx, gy) \in \mathcal{E}$. Let $\mathcal{P} = \mathcal{P}(X)$ denote the set of all paths in X, functions from the natural numbers into V which preserve adjacency. We interpret this to include paths which stay at the same vertex, by considering vertices to be self-adjacent. The players meet at the first (discrete) time that they occupy the same vertex. All this is really just a discrete version of the set-up given for arbitrary connected compact spaces in Section 2, with integrals replaced by finite averages. For pairs (p,q) of paths we define

$$\begin{aligned}
T(p,q) &= \min\{i : p(i) = q(i)\}, \\
\hat{T}(p,q) &= \frac{1}{(\#(G))^2} \sum_{g_1,g_2\in G} T(g_1 p, g_2 q) = \frac{1}{\#(G)} \sum_{g\in G} T(gp,q), \\
R^a(X,G) &= \min_{p,q\in\mathcal{P}(X)} \hat{T}(p,q), \\
R^s(X,G) &= \min_{\nu} \int_{\mathcal{P}(X)} \int_{\mathcal{P}(X)} \hat{T}(p,q)\, d\nu(p)\, d\nu(q),
\end{aligned}$$

where ν varies over probability measures on the path space $\mathcal{P}(X)$. There is one important difference between this formulation and that given above in the case of networks. In the network scenario, if the players are respectively at adjacent vertices v_1, v_2 at time t (and have not met by then), and have transposed their respective positions to v_2, v_1 at time t', then we know (Intermediate Value Theorem) that $T < t'$. However in the analogous situation where players at adjacent vertices of a graph transpose their positions, they are *not* assumed to have met in the intervening time - the game continues. This distinction was observed by Anderson and Weber [19], who first proposed the graph formulation.

We should note that Problem 2 (the telephone problem) mentioned in the introduction can be viewed as the complete graph K_n, with the group of all permutations of the vertices. We note that in Anderson and Weber's paper [19] on this subject, they began the game at time $i = 1$ and assumed that the initial positions of the players are chosen randomly but distinctly. We follow a notation more consistent with the continuous problem, where the game begins at time $i = 0$ and the initial positions are chosen randomly with the possibility that both players are placed at the same node and $T = 0$. Consequently their formulae differ by 1 from the ones given here, which are as in the author's paper with Baston and Essegaier [3].

6.1 Asymmetric Rendezvous

Anderson and Weber considered the asymmetric problem for the complete graph and showed the following [19, Lemma].

Theorem 25 *In the asymmetric rendezvous problem on the complete graph K_n with the group G^* consisting of all permutations of the vertices, the 'wait for mummy' strategy is optimal, and consequently*

$$R^a(K_n, G^*) = (n-1)/2.$$

Using the same idea as Anderson and Weber (see proof below), their rendezvous time was shown [3, Lemma 1] to be a lower bound for a certain class of graphs.

Lemma 26 *Let $X = (\mathcal{V}, \mathcal{E})$ be a graph with G transitive (and hence X vertex transitive) and $\#V = n$. Then*

$$R^a(X,G) \geq (n-1)/2.$$

Proof. Let A_j denote the event that the two players occupy the same vertex at time j. If player 1 chooses the strategy (path) $p = (p_0, p_1, \dots)$ and player 2 chooses $q = (q_0, q_1, \dots)$ then

$$A_j = \{g \in G : g(p_j) = q_j\},$$

which has probability (normalized counting measure on G) $1/n$ regardless of p_j and q_j. Hence $F(k)$, defined as the probability that they meet within the first k steps $0, 1, \dots, k-1$ (or that $T \leq k$) is less than or equal to k/n. Consequently, the expected value $\hat{T}$ of T, for any strategy pair (p, q) is given by

$$\hat{T} = \sum_{k=0}^{\infty} (1 - F(k)) \geq \sum_{k=1}^{n-1} \frac{n-k}{n} = \frac{n-1}{2}. \quad \blacksquare$$

For Hamiltonian graphs (containing a Hamiltonian *circuit*) the reverse inequality can be established [3].

Lemma 27 *If X is Hamiltonian, then for any (not necessarily transitive) isometry group G, $R^a(X, G) \leq (n-1)/2$. In the case that G does not act transitively on X, this result holds for either way of starting the game.*

Proof. The proof is simple, but varies a bit depending on the case. In all cases one of the players stays still on his initial position, however it is chosen. First suppose that G acts transitively. In this case the moving player simply uses a pure strategy of traversing a Hamiltonian circuit in its given order. Since the stationary player's position is random, he will be met with probability $1/n$ at each of the steps $0, 1, \dots, n-1$. If G does not act transitively but the game starts with an independent random placement of the agents, the same argument works. If the players choose their initial locations, the stationary player should choose randomly. $\blacksquare$

Observe that the above lemma fails if there is only a Hamiltonian *path* but still holds if there is a Hamiltonian path beginning at any node. Combining our two lemmas, we have

Theorem 28 *If X is a Hamiltonian graph and G is a transitive group of isometries of X, then the asymmetric rendezvous value is given by*

$$R^a(X, G) = (n-1)/2.$$

At this point it might seem appropriate to give an example of a non-Hamiltonian graph X, with a transitive group G of isometries, where $R^a(X, G) > (n-1)/2$. However if G is transitive then the graph X must be vertex transitive (this means the group of *all* isometries of X is transitive). And as far as we know, Lovasz's conjecture [46] that all vertex transitive graphs have Hamiltonian paths (starting at any node) is still open. If this conjecture is true, then the above theorem holds without the assumption that X is Hamiltonian. The conjecture has in fact been established for many classes of graphs, for example if n is prime. See [36] for more in this direction.

6.2 Symmetric Rendezvous

In the symmetric rendezvous context, the 'wait for mummy' strategy is not available. A natural strategy to consider is a simultaneous random walk, in which the players pick an adjacent vertex randomly. Tetali and Winkler [54] have shown that this always gives an expected meeting time no more than $16n^3/27$, and give an example where the meeting time is about $2n^3/27$. A more complicated strategy was introduced in [35] which finds another player following *any path* in time less than $576e^2 (n-1)$. Of course if the agents cooperated, they can do better. To see this observe that every connected graph with n nodes can be searched from any starting point within $2n$ steps (double every arc of a minimal spanning tree and follow an Eulerian circuit). Suppose that in each time interval of length $2n$ the agents search exhaustively with probability $1/2$ and wait with probability $1/2$. Then in each time interval of length $2n$, they will meet with probability at least $1/2$, and hence in expected time no more than $(2n)(1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 1/8 \ldots) = 4n$. This discussion is taken from [3], where more details are given.

In the remainder of this section we will concentrate on two classes of graphs: the complete graphs K_n (the telephone problem) and the cycle graphs C_n.

6.2.1 Symmetric search on K_n

After many years without progress on the 'telephone problem', the author luckily mentioned it to Eddie Anderson, who then with his Cambridge colleague Richard Weber produced the first results in rendezvous search theory [19]. We present their results (and subsequent modifications) as the symmetric rendezvous problem on a complete graph K_n, specializing to the case K_3 of three nodes. In all cases we assume the group of all permutations of the n vertices is the given symmetry group. Anderson and Weber [19] proposed the following strategy: if the search has not been successful by time $(n-1)k$, the next $n-1$ steps consist either of remaining still , or going randomly to the $n-1$ remaining locations, with these two probabilities optimized to minimize the expected meeting time in the case that both searchers use it. We call this the Anderson-Weber strategy and denote it π_n. In the case $n=2$ it is stays or searches with probabilities $(1/2, 1/2)$ and in the case $n=3$ these probabilities are $(1/3, 2/3)$. Anderson and Weber (see also Crawford and Haller's equivalent result [41] for $n=2$ in the next section) showed [19, Theorem 1] the following.

Theorem 29 $R^s(K_2) = 1$, *and the random strategy is optimal.*

For general n the family π_n gives the following upper bound [19] for $R^s(K_n)$.

Theorem 30 $R^s(K_n) \le \hat{T}(\pi_n, \pi_n)$ *which is asymptotic to* $0.82888497n$.

In addition to demonstrating that π_2 is optimal, Anderson and Weber [19] claimed that π_3 is optimal for K_3. Their own work fails to establish this, but subsequent work has thus far tended to support their claim. In particular, it has been shown that π_3 is optimal for K_3 if the strategy space available to the players is limited in certain natural ways.

Their proposed strategy is what we call a 2-Markov strategy, in which the same probability vector $v = (v_1, \ldots, v_5)$, over five possible 2-step strategies σ_i

cited below, is used for each of the time intervals 1,2 and 3,4, and more generally $2k+1, 2k+2$.

$$\sigma_1 = [a,a],\ \sigma_2 = [b,c],\ \sigma_3 = [a,b],\ \sigma_4 = [b,a],\ \sigma_5 = [b,b]\,.$$

This notation, due to Anderson and Weber, is explained as follows. The location a is the location at time $2k$. The location b is a randomly chosen location from the other two possibilities. The location c is the unique remaining location after b has been determined in the first step. Thus the Anderson-Weber strategy is given by $\pi_3 \equiv (1/3, 2/3, 0, 0, 0)$. Let $\mathcal{S}_{n,m}$ denote the set of all strategies for K_n which are $m-$step Markovian in the above sense. The following result [3, Theorem 4] established its optimality within that class.

Theorem 31 *The Anderson-Weber strategy π_3 is optimal with respect to the class $\mathcal{S}_{3,2}$.*

Recall that we have discussed in Section 2 the fact that a mixed strategy solution (ν,ν) to the symmetric rendezvous problem must be a symmetric Nash equilibrium in the game where each player has utility given by the expected meeting time function $\hat{T}$. So to check that a strategy is a solution to the symmetric problem we can first ask if it is Nash. If it fails this easier test, it is not the symmetric solution. Pikounis and the author have applied an easier version of this test to the Anderson-Weber strategy (π_3, π_3) - which they passed. For the weaker test, we see if any of the primary strategies σ_k do better against a proposed mixed strategy s than does s itself. If so, s cannot be Nash and hence cannot be optimal. If none of the primary strategies does better against s than s itself, we will say s is *locally Nash.* The justification of this term is given in [15], where the following result is proven as Theorem 3.

Theorem 32 *The Anderson-Weber strategy π_4 is locally Nash with respect to the class $\mathcal{S}_{4,3}$.*

We note that Anderson and Weber do not make any optimality claims at all for π_4- in fact they conjecture [19, Conjecture 4] that it is not optimal. We return now to a consideration of the strategy π_3. It was shown earlier that π_3 was optimal among 2 step Markov strategies. Since a two-step Markov strategy is also a four step Markov strategy, it is natural to ask whether it is still optimal in the larger class of four step Markov strategies. This question has not been answered, but the weaker form of optimality within this class has been established in [15, Theorem 4].

Theorem 33 *The Anderson-Weber strategy π_3 is locally Nash with respect to the class $\mathcal{S}_{3,4}$.*

6.2.2 Searchlight rendezvous on K_n

One of the basic assumptions of the rendezvous search problem is that the players cannot see each other and receive no information during the search process (except of course the knowledge that they have not yet found their partner). However the notion of a *searchlight* which periodically illuminates the search region is well known if not common in search theory, *e.g.* [37]. If the search region is the line, then the subsequent play after a searchlight is turned on is trivial - this is not so for case of graphs however. In the case of the complete graph K_n this might be a sort of 'ringing telephone' version of Problem 1. Suppose that if the players choose non-corresponding phones to pick up in a period they hear (or *see* a flashing visible ringer) the ring of the phone in their room corresponding to the one chosen by their partner. This information might be useful in the future.

The work which we describe below, taken from Ponssard [48] and covering results of Crawford and Haller [41] and Kramarz [45], belongs to the significant economic literature on coordination games which is a different offshoot of Schelling's version of rendezvous. It concerns a repeated game model of $n \times n$ coordination, where players wish to use the same strategy but have no common labelling of strategies (rows, columns). After each period, the strategies that were used are revealed. While not so appropriate for the spatial search models considered in rendezvous search, this revelation (searchlight, in our terms) accurately represents information available to players when the strategic choice made in each period are publicly visible things such as prices set by firms or the choice of products to supply.

If $n = 2$, the random strategy is optimal, as in Anderson and Weber's result of $R^s(K_2) = 1$ in the previous subsection (Our Theorem 29). (For $n = 2$ the searchlight version is identical to the original, as players who have not met can obviously deduce the location of their partner.) For $n = 3$ the optimal symmetric move from the position $(1, 0, 1)$ (from the point of view of an observer who puts a 1 in occupied positions) is clearly to the (middle) unoccupied position, which results in a definite meeting in the next time step. For $n = 5$ consider the position $(1, 1, 0, 0, 0)$ (there is really only one position, up to symmetry). Two strategies immediately present themselves. The first is to coordinate on the two occupied positions, giving an expected additional time T to meet of

$$T = 1\,(1/2) + (1 + T)\,(1/2)\,, \text{with } T = 2.$$

The second is to coordinate on the three unoccupied positions, giving the lower (better) time

$$T = 1\,(1/3) + 2\,(2/3) = 5/3.$$

Crawford and Haller [41] show that the cases up to $n = 5$ are sufficient for the general case. Thus the searchlight version of symmetric search on K_n (ringing telephone problem) does not present all the difficulties of the unsolved original telephone problem. However the multi-agent version of this problem is only partially solved – see the last part of Section 7.4.2.

6.2.3 Symmetric search on C_n

The cycle graph C_n is given by the vertex set $\{0, 1, 2, \ldots, n-1\}$ with i adjacent to j if $|i - j| = 1 \pmod{n}$, and is best pictured by arranging these numbers (vertices) around a circle. Ruckle ([50],[49]) has considered the search game on C_n where the players are initially randomly placed and are constrained to use symmetric Markovian strategies (1 step) of the following type. They pick probability $p \leq 1/2$, and in each step move clockwise with probability p, anticlockwise with probability p, and remain still with probability 1-2p. The author recast the same dynamics in a rendezvous context and found that for $n = 4$ each of the three types of moves should have the same probability $1/3$, giving a symmetric rendezvous value for the problem of $27/8 = 3.375$. For $n = 5$ the symmetric rendezvous is approximately 4.88. These numbers are given for comparison with the results for more general strategies given below.

If the above strategies are modified so that with probability p the searcher moves not just one step but k steps, the expected meeting times can be reduced. The following least expected times for such strategies on C_n were calculated by Essegaier in his MSc dissertation [23], supervised by the author.

Table 2: Expected Meeting Times For Symmetric Rendezvous

Step Length	C_4	C_5	C_6	C_7	C_8
1	3.37500	4.87655	6.64000	8.57987	10.7400
2	4.24141	4.12982	7.68495	7.61800	11.6400
3	3.37955	4.03499	6.26342	6.85714	8.8175
4	4.02473	3.60000	8.80062	6.55032	9.8037
5		4.00000	5.84508	6.13841	9.1100
6			6.53709	5.57143	9.5330
7				6.00000	8.3319
8					9.0472

A few observations can be made from this small table. Aside from $n = 4$, the smallest number comes from a step length of $n - 1$. For odd n, the expected meeting time decreases in step length up to $n - 1$. For even n, odd step lengths perform better than even lengths.

Using the above analysis to concentrate on step lengths of $n - 1$, and fairly complicated calculations, the following bounds for the symmetric rendezvous value of C_n were obtained in [3, Corollary 6].

Theorem 34 $R^s(C_n) \leq \begin{cases} (n-1)(2n-1)/(2n), & \textit{if } n \textit{ is odd,} \\ 1.254 + O(1), & \textit{if } n \textit{ is even.} \end{cases}$

A sharper bound for even n of $1.15n + O(1)$ is also claimed in [3].

7 Multi-Agent Rendezvous

This simplest form of rendezvous consists of two players who want to meet. However there are many examples where a population of n players needs to have m of them at a single location in order to accomplish some task, and such problems were formally introduced in two articles of the author with Lim [29] and with Lim and Beck [30]. We will denote this problem by $\Gamma_{n,m}$ and denote the corresponding least expected times in the symmetric and asymmetric cases by $R^s_{n,m}$ and $R^a_{n,m}$. In the 'traditional' case where $n = m = 2$, we will suppress these subscripts. We will also consider, in the asymmetric context only, the problem of minimizing the maximum time required for all n players to meet at a single location, denoting this minimax time as M_n. Both problems will be considered mainly in the scenario where the n players are initially randomly placed at n consecutive integers on the line, and (as usual) faced in random directions. Note that in both problems the case $m > 2$ may involve transfer of information between pairs of players that meet prior to an m-way meeting. When $m = 2$ no information transfer is relevant.

In the final subsection, we will briefly consider multi-agent rendezvous on the complete graph K_n, covering the simulation work of Pikounis and Thomas [33] and the 'searchlight' version (from the field of coordination games) of Kramarz [45].

7.1 Asymmetric multi-rendezvous on the line

In this and the following subsection we consider the problem of minimizing the expected time for m players to rendezvous. In the asymmetric setting only the $m = 2$ version of the game , $\Gamma^a_{n,2}$, has been studied. In this problem n players are randomly placed at distinct integers $1, 2, \ldots, n$, and are faced equiprobably in either directions. For $m = 2$ there is no information transfer, so a strategy s_i of player i is chosen from the same strategy space S as in (5) for the two-player problem. (In the case $m > 2$ discussed later we will call this part of the strategy the Stage 1 strategy.) Nature thus starts the game by placing player i at integer c_i, where $(c_1, \ldots, c_n)$ is a random permutation of the integers 1 to n, and pointing each player i independently in direction $c_{n+i} \in \{-1, +1\}$ equiprobably. We may assume that player 1 is pointed in the positive direction ($c_{n+1} = 1$) so that the set $C = C_n$ of all initial configurations has cardinality $n!2^{n-1}$. Note that the first n coordinates describe positions while the last n coordinates describe directions.

The problem described here was introduced in [30, Section 1]. In was shown there that an optimal strategy profile consists of paths which are piecewise linear with slopes either -1 or +1, which can change directions only at times which are integer multiples of 1/2. So in some ways these problems can be considered as rendezvous on a line graph rather than the real line itself. A useful way of representing such individual strategies is by a sequence of -1 and $+1$'s. Thus $s = [+1, -1, +1, -1, \ldots]$ represents a path that oscillates between a player's starting position and a point one half unit distance in the forward direction. Using a branch and bound technique, the game $\Gamma_{3,2}$ was solved in [30, Theorem 5], as follows.

Theorem 35 *The asymmetric rendezvous value for three person search on the line, where two are required to meet, is given by $R^a_{3,2} = 47/48$. Up to permutations of the labels of the players, there are six optimal strategy profiles, which we give in the [] notation below:*

- $\{[0, 1, 0, 0], [1, 0, 0, 1], [1, 0, 1, 0]\}$,
- $\{[0, 1, 0, 0], [1, 0, 0, 0], [1, 0, 1, 0]\}$,
- $\{[0, 1, 0, 0], [1, 0, 0, 0], [1, 1, 0, 1]\}$,
- $\{[1, 1, 0, 1], [1, 0, 0, 0], [1, 1, 0, 1]\}$,
- $\{[0, 1, 0, 0], [1, 0, 0, 0], [1, 1, 1, 1]\}$,
- $\{[0, 1, 0, 0], [1, 0, 0, 1], [1, 1, 1, 1]\}$.

7.2 Symmetric multi-rendezvous on the line

In [30, Section 2], the problem $\Gamma^s_{n,n}$ is discussed, in which n players placed on the integers $1, 2, \ldots, n$ (as described in the previous section) must all meet at the same node using identical (mixed) strategies. Since $m > 2$, players will sometimes meet with another before the end of the game. We call the part of the game before their first meeting Stage I and the part after this meeting Stage II.

The strategy s^*, which is asymptotically optimal for the games $\Gamma^s_{n,n}$ as n goes to infinity, is defined in [30, Section 2] as follows (the word *start* refers to a player's initial position):

- Rule 1(overriding everything else): If someone you meet says 'follow me', then follow him (adopt same path).
- Stage I: Pick a random direction, independently and equiprobably each time. Go a distance $1/2$ in that direction and then return to start. Then go a distance $1/2$ in the opposite direction and return to start. This takes total time 2, and you are back at start at integer times. Repeat until you are back at start and have met another player This return to start can occur at time 1 or at time 2). Then proceed to Stage II.
- Stage II: Go at speed 1 a maximum distance 1 in the direction away from that in which you met another player earlier (in Stage I). If you meet another player in this time (after time $1/2$ or time 1) then return immediately to your start and wait. If after time 1 you have not met another player, then proceed to Stage III.
- Stage III: Proceed at unit speed towards and then past your start, continuing until the game ends, instructing everyone you meet to follow you.

If all the players use the above strategy s^*, they will have a common $n-$person meeting in expected time no more than $n/2 + 5$, giving the upper bound in the following result [30, Theorem 6].

Theorem 36 *The least expected time $R^s_{n,n}$ required for $n \geq 2$ players to meet together at a single point, using identical strategies and starting from a random placement on $1, 2, \ldots, n$, is asymptotic to $n/2$. In particular, it satisfies the inequality*

$$\frac{n}{2} \leq R^s_{n,n} \leq \frac{n}{2} + 5.$$

The asymmetric rendezvous value $R^a_{n,n}$ satisfies the same bounds.

Variations on the strategy s^* that give improved upper bounds for $R^s_{n,n}$ when n is large are also given in [30].

7.3 Minimax multi-rendezvous on the line

The minimax problem for multi-agent rendezvous on the line seeks the minimum time M_n required to be sure that n players randomly placed on the integers $1, \ldots, n$ can meet together at a single point. This problem, introduced in [29], has been solved for n equal to 2 or 3, and asymptotically for large n. The minimax problem only makes sense in the asymmetric context, so we always will assume that the players may use distinct strategies.

7.3.1 Determination of M_2, M_3

When $n = 2$, the basic problem is the same as in Section 3.1.1, except that maximum rather than expected time is to be minimized. The initial distance between the two players (placed at 1 and 2) is $d = D = 1$ in the notation of that section. The 'modified wait for mummy' strategy $\left(\tilde{g}, \tilde{h}\right)$ given there for $D = 1$ has a maximum meeting time of $3D$, or simply 3. This was shown in [29, Corollary 1] to be a strategy which also minimizes the maximum meeting time: no other strategy pair can ensure a smaller maximum time. The proof of this fact was simplified in [5], using a double linear search problem formulation. In any case, we have the following.

Theorem 37 *The minimum time required for a meeting between two players initially placed a unit distance apart on the line, and faced in random directions, is given by $M_2 = 3$. The 'modified wait for mummy' strategy $\left(\tilde{g}, \tilde{h}\right)$ is a minimax strategy.*

The determination of M_3 is much more difficult and indeed was incorrectly claimed to be 4 in [29, Corollary 1] as the following strategy triple $\tilde{s}$ proposed by Baston [20] demonstrates. We use the bracket [] notation defined in section 7.1. Recall that the Stage I strategy of a player specifies his motion up until the time when he first meets another player. Then the Stage II strategy takes over.

- Stage I strategies: $\tilde{s}_1 = [1, 1, -1, -1, -1, -1, -1]$, $\tilde{s}_2 = [1, -1, -1, 1, 1, -1, -1]$, $\tilde{s}_3 = [1, -1, -1, 1, -1, 1, 1]$.
- Stage II strategies:
  1. If players 2 and 3 meet at time $1/2$, then in the next time unit they first go back to their starting points and then return to their meeting point. They wait there until player 1 meets up with them.
  2. If players 2 and 3 are the first to meet and this meeting occurs after time $1/2$, they remain stationary at this meeting point until player 1 meets up with them.
  3. If players 1 and 2 meet at time $1/2$, then they each go backwards for time $3/2$ (by which time one has met player 3 and keeps him) and then forward for time $3/2$.
  4. If players 1 and 2 are the first to meet and this meeting occurs at time 1, then player 2 moves forwards for time $1/2$ and then backwards for time 2, while player 1 moves backwards for time 2 and then forwards for time $1/2$.
  5. If players 1 and 2 are the first to meet and they meet at time $5/2$, then they both move backwards together (using player 1's orientation) for time 1.
  6. The cases in which players 1 and 3 are the first to meet parallel those of (3.), (4.) and (5.) except that players 2 and 3 must have the opposite orientation to player 1 in the case corresponding to (5.)

By examining the possibilities corresponding to the various possible initial configurations, the following result was established in [20, Section 3].

Lemma 38 *The strategy triple $\tilde{s}$,defined above for three person asymmetric rendezvous on the line, ensures a three-way meeting by time 3.5. Consequently* $M_3 \leq 3.5$.

An earlier result [29, Lemma 5] established the reverse inequality, with some additional partial uniqueness results.

Lemma 39 $M_3 \geq 3.5$. *Furthermore, any rendezvous strategy profile for $\Gamma_{3,3}$ which has a maximum rendezvous time of 3.5 must have all of its Stage I paths (up to time 2) drawn (not necessarily distinctly) from the following three:* $[+1,+1,-1,-1]$, $[+1,-1,-1,+1]$, $[+1,-1,+1,-1]$.

These two lemmas establish $M_3 = 3.5$, as observed in [20]. However the following subsequent result [**?**, Theorem 6] establishes moreover that Baston's strategy $\tilde{s}$ is *uniquely* optimal.

Theorem 40 *The Baston strategy $\tilde{s}$ is the unique (up to labeling the players) minimax strategy for $\Gamma_{3,3}$, and hence $M_3 = 3.5$.*

7.3.2 Asymptotic results for $M_{n,n}$

Clearly a lower bound for $M_{n,n}$ is $(n-1)/2$, since this is the earliest meeting time for the two players initially placed at 1 and n. Using a family of *adjacency search* paths which are useful in meeting all players starting at adjacent integers, an upper bound with maximum meeting time asymptotic to $n/2$ was determined in [29, Theorem 4], giving the following.

Theorem 41 *The minimax rendezvous time M_n, required for n players randomly placed on adjacent integers to all meet at a common point, is asymptotic to $n/2$. That is,* $\lim_{n\to\infty} M_n/n = 1/2$.

We will not give the strategy family used by the author and Lim to establish the above result, as an alternative strategy profile subsequently suggested by Gal [24, Theorem 2] yields a smaller maximum meeting time.

Theorem 42 $M_n \leq n/2 + 2\log_2 n + 3.5$.

Since Gal's strategy is fairly simple yet powerful, we give it here. In the following description k denotes $\lceil \log_2 n \rceil$, α denotes going 1/2 forward and then 1/2 backwards, and β denotes going 1/2 backwards and then 1/2 forwards. (Either α or β takes total time 1). Each player begins (for $t \leq 2k+2$) with a sequence of the following form

$$\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_k, \bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_k,$$

where each $\gamma_i \in \{\alpha, \beta\}$ and $\bar{\gamma}_i = \beta$ if $\gamma_i = \alpha$ and $\bar{\gamma}_i = \alpha$ if $\gamma_i = \beta$. Since we have chosen k so that $2^k \geq n$, there are sure to be enough sequences $\gamma_1, \gamma_2, \ldots, \gamma_k$ so that each of the n players can adopt a different one. If two adjacent players are initially faced in opposite directions, they will have met by time 2 (after both adopt α, β). Otherwise, they are initially facing in the same direction and will meet when one of them uses α while the other uses β. It follows that at time $2k+2$ the players who started at the end positions 1 and n will be back at their initial positions and will each be aware of the direction to all the remaining players. So at this time they should move in that direction and tell anyone they meet to come with them. Consequently there will be an n-way meeting at time $2k + 2 + (n-1)/2 < n/2 + 2\log_2 n + 3.5$.

7.3.3 'Sticky' rendezvous

In multi-agent rendezvous, a common practice is for players who meet to stick together in ever larger groups, until they all meet. (This process occurs in some chemical reactions as well.) If one makes this an assumption of the game, we obtain the sticky form of multi-agent rendezvous. This version of the problem was introduced in [29, Section 4], where the following result was obtained for $n = 3$ as Theorem 2.

Theorem 43 *The sticky minimax rendezvous time $\tilde{M}_3$ required for three adjacent players on the line to meet together without ever separating is* 5.

The easy part of this result is the upper bound $\tilde{M}_3 \leq 5$ which is obtained by nominating one of the players to search while the others wait. The searcher moves forwards until he reaches an unoccupied integer location, at which point he reverses direction. The worst case is when the searcher starts in the middle, reaches an unoccupied integer at time 2, and finds the remaining player at time 5. The lower bound is more difficult to establish. It is somewhat surprising, given the complexity of the minimax strategy $\tilde{s}$ for the non-sticky problem with $n = 3$, that the simple search strategy given here for the sticky problem cannot be improved upon.

7.4 Multi-rendezvous on other regions

The multiple agent rendezvous problem does not have to be restricted to the line. In this section we discuss the problem when the search region is a circle and when it is a complete graph.

7.4.1 Multi-rendezvous on the circle

A variation on the above problem has been proposed by the author (unpublished) in which n players are placed at unit distances apart on a circle of circumference n, faced in random directions, and wish to minimize the expected time taken for all to meet at a single point using the same mixed strategy. The players have no common labeling or common sense of direction on the circle (the given symmetry group is what was called G_3 is the discussion of the circle C). The following simple strategy is quite effective and uses a symmetry-breaking method not previously employed in symmetric rendezvous theory. At time zero, each player i independently performs an experiment which generates a random number q_i in $[0, 1]$. Until another player is met, the strategy is simply a random walk with step length $1/2$. Every player i records the first time t_i that he encounters another player – until such a meeting, he sets $t_i = \infty$. When player i meets player j they do the following

- If $t_i = t_j = \infty$ (first meeting for both), they both reverse directions and q_i and q_j are reset to $\min[q_i, q_j]$.
- If $t_i < t_j$, then i continues in the same direction while j sticks with i forever and vanishes as a player (no future meetings with j are counted)
- If $t_i = t_j < \infty$, then the player with larger value of q follows the other forever and vanishes as a player.

To see how this strategy playes out, suppose $t_i + q_i$ is minimized for player k. (Of all players who are earliest to meet, k has the smallest value of q.) Suppose k's first meeting is with $k+1$ at time $T_0^s = t_k$. Then the strategy implies that they will meet again at the antipodal point to their initial meeting at time $T_0^s + n/2$ together with all the other players, assuming (which happens with probability 1) that $t_k + q_k$ is the unique minimum. The expected value of T_0^s is $\frac{1}{2} + \frac{2^{1-n}}{1-2^{1-n}}$, so the expected meeting time is $T_0^s + n/2 = \frac{n+1}{2} + \frac{2^{1-n}}{1-2^{1-n}}$. If we allow distinct strategies, we can modify the symmetric strategy so that all but one player begin with a random move of step length 1, while the remaining player waits for time 1. Under this strategy profile, the only ways in which there is no meeting at time $1/2$ is that for some location between two non-waiting players, all players move away from that location and towards the waiting player. There are n such locations and for each the probability of such an outcome is 2^{1-n}. Hence the expected time T_0^a for this first meeting is $(1)\, n2^{1-n} + (1/2)\left(1 - n2^{1-n}\right) = \frac{1}{2} + \frac{1}{2}n2^{1-n}$, and the expected meeting time is $T_0^a + n/2 = \frac{1}{2}n2^{1-n} + \frac{1}{2} + \frac{n}{2} = \frac{n+1}{2} + n/2^n$. Hence we have the following estimates on the symmetric and asymmetric rendezvous values $R_{n,n}^s(C)$,$R_{n,n}^a(C)$ which may be exact.

Theorem 44 *$R_{n,n}^s(C, G_3) \leq \frac{n+1}{2} + \frac{1}{2^{n-1}-1}$ and $R_{n,n}^a(C) \leq \frac{n+1}{2} + \frac{n}{2^n}$.*

7.4.2 Multi-rendezvous on a complete graph

This subsection briefly covers the work of Pikounis and Thomas [33] and the 'searchlight' work of Kramarz [45] (see also [41] and [48]) on multirendezvous on the complete graph K_n.

For Pikounis and Thomas, the basic setup is the same as for two players on the complete graph, except that when players meet at a node they can exchange information; for example they can agree to meet again at an a common time and place. They show that in the simpler Markovian scenario, where players "cannot recall anything apart from their present position", any optimal strategy is 'sticky': players who meet remain together. Under these restrictions, they calculate optimal expected times for groups of m players to meet together on K_n. They use simulation techniques to compare the efficacy of sticky strategies and 'split and meet' strategies. Many interesting questions are raised.

Kramarz [45] also considers a Markov scenario for groups of m players to meet together on K_n, but assumes that players know their current position *and the number of players at every other position.* This work extends that of Crawford and Haller on two-person searchlight rendezvous (in our language) discussed above in section 6.2.2. For example there could be a rule that says if there is a unique node with a maximum number of players, then at the next move everyone goes there. Kramarz presents simple rules that are optimal for any m when $n \leq 5$. However a full algorithm that covers all cases, and all n, is not yet determined. The theoretical underpinning of Kramarz's work differs somewhat from our formulation of the rendezvous problem, so it is not possible to give a brief version of his remarkable results from out point of view. For these, the reader is directed to the original paper [45] and to the elegant treatment in [48]. One interesting finding that we can relate, however, is that unique focal points (and hence coordinatio) may arise more rapidly when memory is more limited.

8 Rendezvous with uncertain initial conditions

In all the variations of rendezvous search that have been considered thus far, the method by which the players are initially placed in the search region is assumed to be known. In particular, for rendezvous on the line (which is all we shall deal with in this section) the cumulative distribution function F for the initial distance between the players is common knowledge. In this section we deal with the question of how the rendezvousers should proceed when there is uncertainty about F. This question was vaguely suggested as one of the seven *questions for further work* in the author's initial paper [1, p. 682] on rendezvous, "Adversary-rendezvous games: Suppose the two searchers' initial position in X is chosen by another player who wishes to maximize their meeting time". The question was first attacked by Baston and Gal [21], who presented a number of strategies which work well against arbitrary initial distributions F, in the sense that the expected meeting time against F is small with respect to the mean of F. A subsequent article by the author and Beck [7] determines an optimal strategy for one of the problems introduced by Baston and Gal.

In order to make the various minimax problems considered

8.1 Asymmetric rendezvous

The first result in [21] concerns the asymmetric rendezvous problem on the line, their Theorem 3.1.

Theorem 45 *There exist an asymmetric pair of pure strategies such that the expected meeting time against any distribution F is at most 5.73μ, where μ is the mean of F. That is,*

$$\min_{s_1,s_2\in S}\max_{F\in\mathcal{F}}\frac{\hat{T}(s_1,s_2;F)}{\mu_F}\leq 5.73.$$

The strategies achieving this bound are given explicitly in [21]; they are too complicated for presentation here. Baston and Gal observe that in the case of a known pure point distribution ($d=D=\mu$) the expected meeting time is $\frac{1}{2}\mu+\frac{1}{2}(3\mu)=2\mu$ for the 'wait for mummy' strategy pair, while the optimal pair $\left(\tilde{g},\tilde{h}\right)$ (the 'modified wait for mummy', or 'greedy strategy') gives about a 19% reduction to $(13/8)\,\mu$. Extrapolating this size of reduction from the best 'wait for mummy' (Linear Search) strategy against an unknown distribution of 6.83μ (their Theorem 3.1) gives about 5.54μ for an optimal strategy. This off-the-cuff estimate proved uncannily accurate. The author and Beck [7] considered the following *alternating* strategy (the players alternate reversing directions): In each period n both players move in a fixed direction at maximum speed one. At the end of each period, exactly one of the rendezvousers reverses direction, and they reverse in alternating periods. Each period has length exactly $\hat{a}$ times the length of the previous period, where

$$\hat{a}=\frac{1}{3}+\frac{2^{2/3}}{3\left(8+\sqrt{62}\right)^{1/3}}+\frac{2^{1/3}\left(8+\sqrt{62}\right)^{1/3}}{3}\approx 1.5994.$$

This strategy is used to obtain the following result [7, Corollary 3], which improves the previous theorem and justifies the Baston-Gal extrapolation mentioned above.

Theorem 46 *If the strategy pair described above is used against any distribution F, the expected meeting time is at most $V\mu$, where*

$$V = \frac{1}{2} + (1/8)\left(\frac{8}{\hat{a}-1} + 7 + 5\hat{a} + 3\hat{a}^2 + \hat{a}^3\right) \approx 5.514,$$

and μ is the mean of F. This strategy pair is in fact the minimax alternating strategy. That is

$$\min_{(s_1,s_2)\in S_A} \max_{F\in\mathcal{F}} \frac{\hat{T}\,(s_1, s_2; F)}{\mu_F} = V$$

where S_A denotes the set of all alternating strategy pairs, as defined in [7].

Note that against a known distribution F, there is no need for the players in an asymmetric problem to randomize — pure strategies are sufficient. However against an unknown F the players are essentially playing a zero-sum game against Nature (who chooses F). In this case it is well known that randomization may help. In the asymmetric rendezvous context, they may pick their *pair* of strategies randomly, which means their individual strategies will be correlated.

Theorem 47 *There is a probability distribution over pairs of distinct rendezvous strategies, so that if this 'correlated' strategy is used against any distribution F, the expected meeting time is at most 4.42μ, where μ is the mean of F. That is,*

$$\min_{\omega\in(\mathcal{S}\times\mathcal{S})^*} \max_{F\in\mathcal{F}} \frac{\int_{\mathcal{S}\times\mathcal{S}} \hat{T}\,((s_1, s_2)\,; F)\ d\omega\,(s_1, s_2)}{\mu_F} \leq 4.42.$$

8.2 Symmetric rendezvous

We now consider the scenario where the two players are constrained to use the same mixed strategy. The first result of [21] for this case is their Theorem 5.2 which we state as follows.

Theorem 48 *Their exists a mixed search strategy which, if adopted with independent randomization by the two players against any distribution F, gives an expected meeting time at most $\left(7 + 2\sqrt{10}\right)\mu \approx 13.33\mu$. That is,*

$$\min_{\nu\in\mathcal{S}^*} \max_{F\in\mathcal{F}} \frac{\int_{\mathcal{S}} \hat{T}\,(s, s; F)\ d\nu\,(s)}{\mu_F} \leq 7 + 2\sqrt{10}.$$

Again, the rendezvousers can improve this bound by using joint randomization, as in [21, Theorem 5.2].

Theorem 49 *There exists a probability distribution over mixed search strategies which has the following property. If a mixed strategy is chosen according to this distribution and then adopted by both rendezvousers with independent randomization, then the expected meeting time against any initial distance distribution $F \in \mathcal{F}$ is at most 11.4μ, where μ is the mean of F. That is,*

$$\min_{\gamma \in (\mathcal{S}^*)^*} \max_{F \in \mathcal{F}} \frac{\int_{\mathcal{S}^*} \hat{T}\left(s^*, s^*; F\right) \, d\gamma\left(s^*\right)}{\mu_F} \leq 11.4.$$

9 Asymmetric Rendezvous with bounded resources

An important aspect of the formulation of the rendezvous problem given in Sections 1 and 2 is the assumption that players who have not found each other can always continue the search. That is, rendezvous search is an infinite horizon problem.

However in practice players may have restrictions on say the total distance they can travel. In this section we model this restriction in the context of rendezvous search on the line, following the model introduced by the author and Beck in [4]. The basic formulation is the same as that given in Section 3, except that players I and II cannot travel total distances greater than given constraints a and b, respectively, where $a \geq b$. For example, the rendezvousers may be driving cars on the same road, and a and b might represent the initial fuel levels. The notion of *total distance* can be made precise using the notion of the *total variation* of a function, defined by

$$\operatorname{Var}(s) = \sup \sum \left| s\left(x_i\right) - s\left(x_{i-1}\right) \right|,$$

where the supremum is taken over all finite sequences of points $0 = x_0 < x_1 < \cdots < x_n$. Thus the strategy set for player I is given by

$$\mathcal{S}_a = \left\{ s \in \mathcal{S} : \operatorname{Var}(s) \leq a \right\},$$

where $\mathcal{S} = \mathcal{S}_\infty$ denotes the unrestricted strategy set for rendezvous on the line given in (5). Thus for each pair of resource bounds (a, b) there is a corresponding rendezvous problem with pure strategy profiles $\mathcal{S}_a \times \mathcal{S}_b$ which we denote by $\Gamma_F^{a,b}$, where F is the given cumulative probability distribution of the initial distance between the players. In this section we will always assume that F is a bounded distribution with some maximum distance D, which we will normalize to 1.

We begin this analysis with a few simple observations. If a and b are sufficiently large, then the players can ensure a meeting. The optimization problem considered in this case is usual one of minimizing the expected meeting time, and will be presented in Section 9.1 based on [4]. However if a and b are not large enough, then the players may have no way of ensuring a meeting. In this case they may wish to maximize the probability of an eventual meeting, a problem considered in Section 9.2, based on [6].

The cutoff values of a and b that determine which case is to be considered were determined in [4, Theorem 9] although the proof was simplified in [6].

Theorem 50 *Players in the bounded resources rendezvous problem $\Gamma_F^{a,b}$ can ensure an eventual meeting if and only if $5a + 3b \geq D$, where D is maximum initial distance between the players for the distribution F.*

An important special case of this problem, where one of the players is stationary ($b = 0$) was introduced by Foley, Hill and Spruill [?] as 'linear search with bounded resources'.

The bounded resources problem has thus far only been considered in an asymmetric version of rendezvous as the players could use this information (except in the case $a = b$) to distinguish between themselves.

9.1 Expected time minimization

We now assume that the initial distance between the two players is $D = 1$ and that the they have enough resources ($5a + 3b \geq 1$) so that they can ensure a meeting. Under these assumptions we determine the restricted resources asymmetric rendezvous value $R(a, b)$ which gives the least expected time for a meeting. As in [4], we define

$$R(a,b) = \min_{(s_1,s_2)\in\mathcal{S}_a\times\mathcal{S}_b} \hat{T}(s_1, s_2).$$

It is worth recalling that the 'modified wait for mummy' strategy $\left(\tilde{g}, \tilde{h}\right)$ given in (6) and (7) required each player to move a total distance of 3. Consequently if $a \geq b \geq 3$ then $\left(\tilde{g}, \tilde{h}\right) \in \mathcal{S}_a \times \mathcal{S}_b$ and is available in the restricted problem, so that $R(a, b) = 13/8$ in this case by Theorem 5(ii). However for smaller resources the rendezvous value will be affected. Another easy case is the linear search version arising when $b = 0$. For $a \geq 3$ the optimal solution is clearly for the moving player to search the possible positions of the stationary player in turns, with expected time $\frac{1}{2}(1) + \frac{1}{2}(3) = 2 = \frac{16-b}{8}$. These are the easy cases of the following result [4, Theorem 9].

Theorem 51 *Suppose two players are placed a unit distance apart on the line and faced in random directions so that neither knows the direction of the other. Suppose the players can move at maximum speed 1 and have resources a, b which bound the total distance each can travel, where $a \geq b$. Then they can guarantee a meeting if and only if $5a + 3b \geq 15$. For such resource pairs, the least expected meeting time $R(a, b)$ satisfies*

$$R(a,b) = \begin{cases} (31 - 5a - 3b)/8 & \text{if } 2a + b \leq 6, \\ (19 - a - b)/8 & \text{if } 2a + b \geq 6 \text{ and } a \leq 3, \\ (16 - b)/8 & \text{if } b \leq 3 \leq a, \\ 13/8 & \text{if } 3 \leq b. \end{cases}$$

9.2 Meeting-probability maximization

In the case ($5a + 3b < 15$) where the players cannot ensure a rendezvous, they try to at least maximize the probability that they meet. This problem was introduced in by the author and Beck in [6], and we follow the treatment given there. The Rendezvous Probability Value $P(F; a, b)$ for distribution F and resource constraints a, b is defined by

$$P(F; a, b) = \max_{(f,g)\in\mathcal{S}_a\times\mathcal{S}_b} P_F(f, g), \tag{8}$$

where $P_F(f,g)$ is the probability meeting. We will say that a strategy pair is *optimal* if the above maximum is attained on it. As observed in [6], the existence of such a maximum on the compact set determined by the total variation constraints follows from the fact that P_F is upper semi continuous in f, g. The rendezvous probability value $P(F; a, b)$ represents the highest probability of meeting that the players can ensure, given their resources and the distribution of their initial distance.

The treatment given in begins by determining geometrical properties of an optimal path $p(t) = (f(t), g(t))$, that is, for (f, g) optimal. Then the relatively easy problem of the atomic distribution A with $d = 1$ is analyzed [6, Theorem 3], as in the previous theorem but with meeting–probability maximization.

Theorem 52 *Suppose two players are placed a unit distance apart (the atomic distribution A) on the line and faced in random directions so that neither knows the direction of the other. Suppose the players can move at maximum speed 1 and have resources a, b which bound the total distance each can travel, where $a \geq b$. Then the maximum probability with which they can meet is given by*

$$P(A; a, b) = \begin{cases} 0, & \text{if} \quad a + b < 1, \\ 1/4, & \text{if} \quad a + b/3 < 1 \leq a + b, \\ 1/2, & \text{if} \quad a/3 + b/3 < 1 \leq a + b/3, \\ 3/4, & \text{if} \quad a/3 + b/5 < 1 \leq a + b, \\ 1 & \text{if} \quad 1 \leq a/3 + b/5. \end{cases}$$

Stronger optimality conditions were found in [6] for the case where F is concave (such distributions are called *central* there). As a consequence of these conditions, which we will not give here, it can be shown [6, Lemma 4] that for a given total amount of fuel it is best if one player has as much as possible.

Theorem 53 *If F is concave, and the total resources $a + b = r$ as fixed, then the rendezvous probability value $P(F; a, r - a)$ is nondecreasing in a.*

For concave F, the evaluation of the rendezvous probability value $P(F; a, b)$ is reduced in [6, Theorem 7] to the following explicit optimization problem, where the parameters x, y, z correspond to geometric parameters of the optimal strategy.

Theorem 54 *For concave distributions F, and resources $a \geq b$, the rendezvous probability value is given by*

$$P(F; a, b) = \phi(a, b)/4, \text{ where}$$

$$\phi(a, b) = \max\left[F(x + z) + F(y + z) + F(y + a - 2z) + F(b - 2x - 2y + a - 2z)\right],$$

and the maximum is taken over the set

$$0 \leq z \leq a/2, \ \ 0 \leq x \leq b/2, \quad 0 \leq y \leq b - 2x.$$

We now give the explicit results obtained in [8] for the rendezvous probability value $P(F; a, b)$ as a function of the resource pair (a, b) when F is uniform, triangular, or exponential. These were announced earlier in [6].

9.2.1 The uniform distribution

When F is the uniform distribution on $[0,1]$, the determination of the rendezvous probability value $P(F;a,b)$ reduces to a linear programming problem which can be solved in its dual form. The general solution is given in the following table.

Table I: Optimal Strategy (x,y,z) and $P(F;a,b)$, for F uniform.				
domain	(x,y,z)			$P(F;a,b)$
$a+b \le 1$	$(0,0,0)$			$\frac{2a+b}{4}$
$3a+b \le 3 \le 3a+3b$	$\left(0, \frac{a+b-1}{2}, 0\right)$			$\frac{2a+b}{4}$
$a+b \le 3 \le 3a+b$	$\left(0, \frac{b}{3}, \frac{3a+b-3}{6}\right)$			$\frac{3a+2b+3}{12}$
$\frac{3a+5b}{5} \le 3 \le a+b$	$\left(\frac{a+b-3}{2}, \frac{3-a}{3}, \frac{a}{3}\right)$			$\frac{5a+3b+9}{24}$

9.2.2 The triangular distribution

Consider the triangular distribution given by $F(t) = t(2-t)$, $0 \le t \le 1$, with density $F'(t) = 2-2t$. The optimal strategies belong to different strategy classes, depending on the position of the resource pair $(a,b), a \ge b$ in ab space. The regions are determined by polygons with vertices at the points $A = (0,0)$, $B = (0,5)$, $C = (3,0)$, $D = (1/3, 5/6)$, $E = (8/15, 2/15)$, $F = (0, 3/2)$, and $G = (1/2, 0)$.

Table II: Optimal Strategies, Triangular Distribution			
region	x	y	z
BDC	$(5a+16b-15)/65$	$(2b-a+3)/13$	$(25a+2b-10)/65$
DEC	0	$(14b-2a+6)/51$	$(7a+2b-4)/17$
GEC	0	b	$(4a-b-2)/10$
FDEA	0	$(a+2b)/6$	0
AEG	0	b	0

9.2.3 Negative exponential distribution

We now consider the situation where the initial distance between the players is drawn from the negative exponential distribution, whose cumulative distribution function is $F(t) = 1 - e^{-t}$, for all $t > 0$. The regions where the various classes dominate are given in the table below, where the functions are defined by $\rho(a) = \ln(54) - 3a - \ln(e^a + 1)$, $\tau(b) = \ln(2) + \ln\left(e^{2b}+1\right) - \ln\left(e^b + 1\right)$, $\theta(b) = 2b + \ln\left(2 - e^{-2b}\right)$.

Table III: Strategy Class Regions for Exponential Distribution	
I	$\ln(9/4) \le b \le a$
II	$\max[b^*, \rho(a)] \le b \le \ln(9/4),\ b \le a$
III	$\tau(b) \le a,\ 0 \le b \le b^*$
IV	$b \le \max[a, \rho(a)],\ a \le \theta(b)$
V	$\theta(b) \le a \le \tau(b),\ 0 \le b \le \rho(a)$

The optimal strategies within each of these classes, as derived above, are given in the following table.

Table IV: Optimal Strategies for the Exponential Distribution	
I	$x = (b - \ln(9/4))/5, y = (b - \ln(8/27)/5), z = (a - \ln(2))/3$
II	$x = 0,\ b = \ln(2) - \ln\left(3e^{-4y} + e^{-3y}\right),\ z = (y + a - \ln(3))/3$
III	$x = 0,\ y = b, z = \left(a + \ln\left(\left(1 + e^{-b}\right)/\left(2 + 2e^{-2b}\right)\right)\right)/3$
IV	$x = 0,\ z = 0,\ y = (a + b - \ln(2) + \ln(1 + e^{-a}))/3$
V	$x = 0,\ y = b,\ z = 0$

10 Rendezvous-evasion games

This section deals with a problem posed by the author, in which the common aim of two rendezvousers R_1 and R_2 is to meet strictly before either of them encounters an enemy searcher S (who has the opposite aim). This game models the situation for example where a rescuer R_2 is sent to find a downed mobile pilot R_1 before the enemy on the ground S finds him. The game could in principle be played on any search region, including the line or circle. However it has only been successfully attacked in the setting of a complete graph K_n with an imposed time limit for the rendezvousers to meet.

In this setting the game begins with an initial placement of the three agents R_1, R_2 and S on distinct nodes at time 1 (that is, the starting method of Anderson and Weber [19]) and continues until time m. This is a two *player* zero-sum game of kind between the minimizing team of rendezvousers R_1, R_2 and the enemy searcher S. The maximizing searcher wins the game, and the payoff is 1, unless the two rendezvousers occupy the same node by time m (which may be infinity) and before either occupies the same node as S; otherwise the rendezvousers win and the payoff is zero. There are three *agents,* R_1, R_2, S. The value $v_{n,m}$ of the game is therefore the probability of the searcher winning, with best play on both sides. There are two cases, depending on whether the rendezvousers can play asymmetrically or must play symmetrically, and corresponding values v_n^a, (no limit m is required for this case) and $v_{n,m}^s$. In the following two subsections we discuss the work of Lim [27] on the asymmetric version and the work of the author and Lim [13] on the symmetric version.

10.1 Asymmetric rendezvous-evasion on K_n.

The main result of Lim [27] is the complete determination of optimal play for both sides when $n = 3$. Each agent P labels his initial position as $P(1)$, his next *new* positions as $P(2)$, $P(3)$, and so on. When going to a new position, he can only move randomly (equiprobably) among the locations not already visited.

Using this notation, the optimal mixed strategy for the rendezvous team is as follows:

Step 1 R_1 moves to location $R_1(2)$, while R_2 stays at location $R_2(1)$.

Step 2 Choose the following with probabilities $(7/19, 6/19, 6/19)$:

1. R_1 visits location $R_1(3)$ (uniquely determined), R_2 stays at $R_2(1)$.
2. R_1 returns to $R_1(1)$; R_2 visits $R_2(2)$.

3. R_1 stays at $R_1(2)$; R_2 visits $R_2(2)$.

Step 3 At this time it can be shown [27, Lemma 2.1] that R_1 and R_2 have a common labeling of the three locations. Hence they arrange to meet equiprobably at each of the three locations.

The optimal strategy for the searcher is given as follows.

Step 1 S moves to a new location $S(2)$.

Step 2 Choose the following with probabilities $(5/19, 5/19, 9/19)$:

1. Return to location $S(1)$
2. Stay at location $S(2)$
3. Go to location $S(3)$

Step k, $k \geq 3$ Visit each location with probability $1/3$.

The main result is [27, Thereom 2.3]:

Theorem 55 *The value v_3^a of the unlimited time, asymmetric rendezvous evasion game on the complete graph K_3 is given by $v_3^a = 47/76$. The strategies given above are optimal.*

Lim also gives a strategy for the searcher which guarantees him a win on K_4 with probability $31/54$. She further shows [27, Theorem 3.3] that when all the agents have a common circular ordering of the locations (but may move between any two) this knowledge helps the rendezvousers and the value is $\left((1-2/n)^{n-1}+1\right)/2$. The version of this game in which the actions of the players are revealed after each move is considered in [28].

10.2 Symmetric rendezvous evasion on K_3

We now consider the symmetric version of the above rendezvous-evasion game, where the two rendezvouser must be given the same mixed strategy to follow. It is easiest to think of a controller who broadcasts a strategy for the two agents to follow. This broadcast cannot be received by the searcher. Unlike the asymmetric version in the previous subsection, the rendezvousers cannot be certain to have met by a given time, and so to make the problem tractable we must impose a limit $m = 3$ of two additional time units after the initial placement.

We assume that all agents have a common notion of clockwise direction around the three nodes, which we view as vertices of a triangle. (The given group of symmetries as described in Section 2 is the group of the three rotations of the vertices of the triangle.) This assumption allows the agents to label their initial location as 0, and the remaining two as 1 and 2 in a clockwise direction from 0. In this notation there are nine pure strategies

$$\{(0,0),(1,2),(2,1),(0,1),(1,0),(2,2),(0,2),(1,1),(2,0)\}.$$

We will be concerned with four distribution over these strategies, which we call behavioural strategies.

$$\begin{aligned}
b^1 &= \frac{1}{3}(1,1,1,0,0,0,0,0,0), \\
b^2 &= \frac{1}{3}(0,0,0,1,1,1,0,0,0), \\
b^3 &= \frac{1}{3}(0,0,0,0,0,0,1,1,1), \\
b^4 &= \frac{1}{9}(1,1,1,1,1,1,1,1,1).
\end{aligned}$$

These strategies must be mixed for optimality, as shown in the following result [13, Corollary 2.6]. The version with no common direction is [13, Theorem 4.1]

Theorem 56 *The value of the common-direction two-step symmetric rendezvous-evasion game on K_3 is given by $v^s_{3,3} = 57/81$. The optimal strategy for the rendezvouser is to each use (the same) behavioural strategy b^i, $i = 1, 2, 3$, equiprobably. The optimal strategy for the searcher is to use the random strategy b^4. In the version with no common notion of direction the value lies in the interval* $[58/81, 59/81]$

The interpretation of the rendezvous strategy is as follows. The controller broadcasts a number 1, 2,3, equiprobably. When number i is broadcast (and received), both rendezvousers adopt the behavioural strategy b^i. The need for such randomizations over behavioural strategies is explained in [13]. In fact, if the rendezvouser must adopt simply a common distribution over the nine pure strategies they cannot ensure that the searcher wins with probability no more than 57/81 (this is [13, Theorem 3.1]).

It can be shown [13, Theorem 3.3] that the common-direction symmetric rendezvous evasion value is in fact equivalent to the following pure rendezvous problem (with no enemy searcher): Two indistinguishable agents rendezvous on n locations for m steps. They (each) get a fixed prize when they meet, whose value they discount by a common factor.

Rendezvous Articles

References

[1] S. Alpern. The rendezvous search problem. *LSE CDAM Research Report*, **53** (1993). *SIAM J. Control & Optimization*, **33** (1995), 673-683.

[2] S. Alpern. Asymmetric rendezvous search on the circle. *Dynamics and Control*, to appear.

[3] S. Alpern, V. Baston, and S. Essegaier. Rendezvous search on a graph. *Journal of Applied Probability*, **36** (1999), No. 1, 223-231.

[4] S. Alpern and A. Beck. Rendezvous search on the line with bounded resources: expected time minimization. *European Journal of Operational Research*, **101** (1997), 588-597.

[5] S. Alpern and Anatole Beck. Asymmetric rendezvous on the line is a double linear search problem. *Mathematics of Operations Research*, **24** (1999), No. 3, 604-618.

[6] S. Alpern and Anatole Beck. Rendezvous search on the line with bounded resources: Maximizing the probability of meeting. *Operations Research*, **47** (1999), no. 6, scheduled.

[7] S. Alpern and Anatole Beck. Pure strategy asymmetric rendezvous on the line with an unknown initial distance . *Operations Research*, to appear.

[8] S. Alpern and Anatole Beck. Rendezvous search on the line with limited resources: meeting probability maximization for concave distributions. Preprint, Center for Discrete and Applied Mathematics, London School of Economics.

[9] S. Alpern and S. Gal. Rendezvous search on the line with distinguishable players. *SIAM J. Control & Optimization*, **33** (1995), 1270-1276.

[10] S. Alpern and S. Gal. Search for an agent who may or may not want to be found. Preprint, London School of Economics (1999).

[11] S. Alpern and S. Gal. *Search Games and Rendezvous Theory*. Kluwer. Expected publication date 2001.

[12] S. Alpern, and J.V. Howard. Alternating search at two locations. *London School of Economics Operational Research Working Paper Series*, LSEOR 99.30 (1999).

[13] S. Alpern and W. S. Lim. The symmetric rendezvous-evasion game. *SIAM J. Control & Opt.* **36** (1998), No. 3, 948-959.

[14] S. Alpern and W. S. Lim. Minimax triple rendezvous on the line: corrections and extensions. LSE-CDAM-97-7, London School of Economics (1997).

[15] S. Alpern and M. Pikounis. The telephone coordination game. *Game Theory and Applications*, **5**, (1999).

[16] S. Alpern and D. J. Reyniers. The rendezvous and coordinated search problems. *Proceedings of IEEE, 33rd CDC* (1994), No. 1, 513-517

[17] E. J. Anderson and S. Essegaier (1995). Rendezvous search on the line with indistinguishable players, *SIAM Journal of Control and Optimization*, **33**, 1637-1642.

[18] E. J. Anderson and S. Fekete (1998). Two-dimensional rendezvous search. *Proceedings, 14'th ACM Symposium on Computational Geometry* (1998), 365-373 (extended abstract). *Operations Research*, to appear (1999).

[19] E. J. Anderson and R. R. Weber. The rendezvous problem on discrete locations. *Journal of Applied Probability* **28**, (1990) 839-851.

[20] V. J. Baston. Two rendezvous search problems on the line, *Naval Research Logistics* **46** (1999) 335-340.

[21] V. J. Baston, S. Gal . Rendezvous on the line when the players' initial distance is given by an unknown probability distribution. *SIAM Journal of Control and Optimization* **36** (1998), no. 6, 1880-1889.

[22] C. Di Cairano. Asymmetric rendezvous search on the line. Ph.D. Dissertation, Faculty of Mathematical Studies, University of Southampton (1999).

[23] S. Essegaier. The rendezvous problem: a survey on a pure coordination game of imperfect information, *LSE CDAM Research Report*, **61** (1993).

[24] S. Gal. Rendezvous search on the line. *Operations Research,* to appear (1999).

[25] S. Gal. Complexity of linear search and rendezvous on the line ??.

[26] J. V. Howard. Rendezvous search on the interval and circle. *Operations Research,* **47**, No. 4 (1999), 550-558.

[27] W. S. Lim. A rendezvous-evasion game on discrete locations with joint randomization. *Advances in Applied Probability* **29** (1997), 1004-1017.

[28] W. S. Lim. Rendezvous-evasion as a multi-stage game with observed actions. LSE-CDAM-96-05 (1996), The London School of Economics.

[29] W. S. Lim and S. Alpern. Minimax rendezvous search on the line. *SIAM J. Control & Optimization* **34** (1996), 1650-1665.

[30] W. S. Lim, S. Alpern, and A. Beck (1997). Rendezvous search on the line with more than two players. *Operations Research* 45, No. 3, 357-364.

[31] R. Matthews. How to find your children when they are lost. *Sunday Telegraph*, Science page, 26 March 1995.

[32] M. Pikounis. Rendezvous search simulation. Project Report, Department of Computer Science, University of Edinburgh, 1997.

[33] M. Pikounis and L.C. Thomas. Many player rendezvous search: stick together or split and meet? Preprint (1998), Management School, University of Edinburgh.

[34] L. C. Thomas and P. B. Hulme. Searching for targets who want to be found. *Journal of the O. R. Society* **48** (1997), Issue 1, Pages 44 - 50.

Related Articles

[35] R. Aleniunas, R. M. Karp, R. J. Lipton, L. Lovasz and C. Rackoff, Random walks, universal traversal sequences, and the complexity of maze problems, 20'th Annual symposium on Foundations of Computer Science, San Juan, Puerto Rico (1979), 218-223.

[36] Alspach, The search for long paths and cycles in vertex-transitive graphs and digraphs, in: K.L. McAveney, ed., Combinatorial Mathematics VIII, Lecture Notes #884, Springer, Berlin,1981, 14-22.

[37] V. J. D. Baston and F. A. Bostock. A basic searchlight game. *Journal of Optimization Theory and Applications*, **71** (1991), 47-66.

[38] Anatole Beck. On the linear search problem. *Israel Journal of Mathematics* **2** (1964) , 221-228.

[39] R. Bellman. An optimal search problem. *SIAM Review* **5** (1963), p. 274.

[40] S. Benkoski, M. Monticino, and J. Weisinger. A survey of the search theory literature. *Naval Research Logistics* **38** (1991), 469-494.

[41] V. P. Crawford and H. Haller. Learning how to cooperate: optimal play in repeated coordination games. *Econometrica* **58** (1990), No. 3, 571-596.

[42] Foley, R. D., Hill, T. P. and Spruill, M.C. 1991. Linear search with bounded resources. Naval Research Logistics, **38**, 555-565.

[43] Gal, S. (1980). *Search Games*. Academic Press, New York.

[44] R. Isaacs. *Differential Games*. John Wiley, New York, 1965.

[45] F. Kramarz. Dynamic focal points in N-person coordination games. *Theory and Decision* **40** (1996), 277-313.

[46] L. Lovasz, Unsolved problem II in: Combinatorial Structures and their Applications (Proceedings of the Calgary International Conference on Combinatorial Structures and their Applications, 1969), ed. R. Guy et al, Gordon and Breach, New York, 1970.

[47] A. McNab. Bravo Two Zero. Corgi Books, London (1994).

[48] J-P Ponssard. Formalisation des connaissances, apprentisage organisationnel et rationalité interactive (Chapitre 7). In: Analyse Economiques des Conventions, A Orléan ed., P.U.F., Paris (1994).

[49] Ruckle, W. H., (1983). *Geometric Games and Their Applications*. Pitman, Boston.

[50] W. H. Ruckle, Pursuit on a cyclic graph, *International Journal of Game Theory*, **10** (1983), 91-99.

[51] Schelling, T. 1960. *The Strategy of Conflict*. Harvard University Press, Cambridge.

[52] Stone, L. D. (1989). *Theory of Optimal Search*, 2nd edition. Operations Research Society of America, Arlington, VA.

[53] R. Sugden, A Theory of focal points, *The Economic Journal*, **105** (1995), 533-550.

[54] P. Tetali and P. Winkler, Simultaneous Reversible Markov Chains, *Bolyai Society Mathematical Studies, Combinatorics, Paul Erdos is Eighty*, Vol. 1, 1993, 433-451.