Reserving Resilient Capacity with Upper Bound Constraints

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Abstract

Continuing research begun in [2], we investigate problems of reserving capacity in the arcs of a network, subject to the constraint that, on the failure of any one arc, there is enough reserved capacity on the remaining arcs to support a flow of value T from a source s to a destination t. We also impose upper bounds on the amount of capacity we may reserve on the arcs: this alters the nature of the problem.

In the case where each arc has the same upper bound, we investigate the strategy of finding the minimum-cost reservation that is itself an acyclic (s, t) flow: we show that such a reservation is easy to find, always has a simple form, and has a cost at most twice that of the optimal solution.

We investigate other related problems.

1 Introduction and Definitions

We investigate various problems of a type we call *resilience*. The theme is that we are to reserve capacity in the arcs of a network so that, if any one arc of the network fails, there is sufficient capacity reserved in the remaining arcs to allow a flow of value T from a single source s to a single destination t. Each arc has a cost, which is to be interpreted as a per-unit cost for reserving capacity in that arc.

We observed in a previous paper [2] that, while the problem as stated can be solved in polynomial time using linear programming techniques, there seems to be no simple combinatorial algorithm. Also, the integer version of the problem is NP-complete. In [2], we discussed the utility of finding the optimal reservation of a certain particularly simple type, namely one derived by reserving capacity along a set of *diverse* (arc-disjoint) paths from s to t. We showed that finding such a *diverse paths reservation* was particularly simple in practice, and was at worst twice the cost of the optimum. Furthermore, the simplicity of the solution obtained may be an asset in many practical settings.

Here we discuss the more general scenario where the arcs of the network are also equipped with upper bounds on the capacity we may reserve, possibly uniform over all arcs and possibly not. In this case, the strategy of looking for a diverse paths reservation does not have such pleasant properties: we consider a slightly more general type of reservation, namely a reservation that is itself an acyclic (s,t) flow. We show that a minimum cost acyclic flow reservation can easily be found, and always takes a simple form, when there is a uniform upper bound on the maximum capacity of an arc; furthermore its cost is at most twice that of an optimum solution.

In the case where each arc has its own upper bound, we explain why no such results can hold, but we do give an efficient algorithm to find the optimum solution in the case where we are asked for a diverse paths reservation with the set of paths pre-specified.

One of the eventual aims is to use the strategies and algorithms we are developing as part of an algorithm to solve multi-commodity flow problems where there is a resilience requirement.

We start with some basic definitions and facts concerning flows in networks.

Throughout, we suppose (sometimes implicitly) that we are given a directed graph (network) $D = (V, A)$ with *node set* V and *arc set* A. We shall always assume that D comes with two nodes permanently fixed as the *source* s and the *destination* t. We denote by $\kappa(D)$ the (s,t)-edge connectivity of the network. We also suppose that we are given a rational number T (usually an integer) representing the required traffic flow from s to t through the network D in the case of failure. Finally, we are also given a vector (c_a) of non-negative rational (again, usually integer) costs on the arcs a of D. Even though there may be parallel arcs, we often use the notation (u,v) for an arc from u to v, when no confusion arises.

For any $S \subseteq V$, let $\delta^+_D(S)$ (or simply $\delta^+(S)$ if the context is clear) denote the set of arcs with tail in S and head in $V - S$. Let $\delta^-(S)$ denote the set of arcs $\delta^+(V-S)$. For a node $v \in V$, we write $\delta^+(v)$ for $\delta^+(\{v\})$, etc. We call $S \subseteq V$ an (s,t)-*set* if $s \in S$ and $t \in V - S$.

A vector $x \in \mathcal{Q}_+^A$ of capacities is *T-resilient* if:

$$x(\delta^+(S) - e) \geq T \quad \text{for each } (s,t)\text{-set } S \text{ and } e \in \delta^+(S). \tag{1}$$

We often refer to constraint (1) as the *partial T-cut constraint* associated with the pair (S, e). We also refer to capacity vectors as *reservations.*

A collection of paths from s to t is called *diverse* if no two have a common arc. Examples of T-resilient reservations are obtained by taking any collection of $m \geq 2$ diverse paths, and reserving capacity $T/(m-1)$ on each arc in any of the paths. Any reservation obtained by choosing a collection $\{P_1, \ldots, P_m\}$ of diverse paths and reserving capacity z_i on each arc of path P_i is called a *diverse paths reservation.*

A vector $x \in \mathcal{Q}_+^A$ is an (s,t) *flow-vector* (or simply a *flow*) if $x(\delta^+(v)) = x(\delta^-(v))$ for all $v \neq s,t$ and (therefore) $x(\delta^+(s)) - x(\delta^-(s)) = x(\delta^-(t)) - x(\delta^+(t))$. In this case, the *value* of the flow is $x(\delta^+(s)) - x(\delta^-(s))$. A flow-vector of value M is also called an *M-flow.*

For any vector $x \in \mathcal{Q}_+^A$ and $A' \subseteq A$, we denote by $x(A')$ the sum $\sum_{a \in A'} x_a$. We let $I(x)$ denote the *support* of x, that is $I(x) = \{a \in A : x_a > 0\}$. For a set S of arcs, we denote by $e(S)$ the incidence vector of S, i.e., $e(S)_i = 1$ if $i \in S$ and 0 otherwise. For a single arc a, we write e_a for $e(\{a\})$. (A path P is considered as a set of arcs, and so also has an associated incidence vector $e(P)$.)

We say a vector $x \in \mathcal{Q}_+^A$ is *acyclic* if there are no cycles in the support $I(x)$ of x.

The paper is organized as follows.

We start Section 2 by reviewing some results from our earlier paper [2] on diverse paths reservations in unbounded networks – that is, networks where we may reserve any (rational) amount of capacity on each arc. These results suggest that using a diverse paths reservation is a good strategy for building resilient networks.

We then move to the scenario where there are upper bounds λ_a on the capacity we may reserve on each arc a of the network. A resilient reservation satisfying such upper bounds will be called a *bounded* resilient reservation.

We show that then finding a minimum cost bounded diverse paths reservation is NP-hard, and furthermore is not k-approximable for any $k \geq 1$ even in the fractional case, and even if all the λ_a are equal (unless $\text{P} = \text{NP}$).

In Section 3 we discuss acyclic flow-reservations. Diverse paths reservations are particular examples of acyclic flow-reservations; we show that there always exists a minimum cost acyclic flow-reservation which is a convex combination of *two* diverse paths reservations. This extends to acyclic flow reservations many of the nice practical features of diverse paths. Moreover, we show that finding a minimum cost, fractional or integral, acyclic flow-reservation is easy both in the unbounded case, and when all the bounds λ_a are equal. In the latter case, we give a very simple algorithm which outputs a bounded acyclic flow-reservation whose cost is at most twice the cost of any bounded T-resilient vector.

Then, in Section 4, we move to the scenario where the bounds λ_a are different. We start by giving an example which shows that in this case the ratio between the cost of any bounded resilient flow reservation (even non-acyclic) and the cost of an optimal bounded T-resilient reservation can be arbitrarily large. Moreover, the problem of finding a minimum cost bounded diverse paths reservation is still NP-hard and not k-approximable if the set of paths is not fixed. We therefore deal with the case where the set of paths is fixed and we give an $O(m \log m)$ combinatorial algorithm for both fractional and integral cases.

We close by considering, in Section 5, the case where, for each arc a of the network, capacity has to be reserved in bundles of μ_a units, even if there is no upper bound on the *total* capacity we may reserve on each arc.

2 Diverse Paths Reservations

In [2] we have studied in detail the case where it is required that our T-resilient reservation vector be a diverse paths reservation, i.e., formed from a collection of diverse (s, t) paths by reserving a capacity along each path. We briefly review some results; for details see [2]. The scenario is that of unbounded reservations, i.e., we are allowed to reserve as much capacity as we want on each path. We consider two cases: in the former we assume that the set of paths is pre-specified, in the latter they are not. In the former case, we may as well treat the network as consisting of a set of arcs from s to t, and so the problem may be formulated as follows.

Problem (Fixed Unbounded Problem). Given a demand T and a non-decreasing sequence of costs $c_1, c_2, \ldots, c_m$, find non-negative real numbers $x_1, x_2, \ldots, x_m$ to minimize $c_1x_1 + c_2x_2 + \ldots + c_mx_m$ subject to the constraints

$$\sum_{i \neq j} x_i \geq T \quad \text{for any } j. \tag{2}$$

In [2], we give extremely simple algorithms to solve this problem, even in the integral case. In fact, we show that an optimal (fractional) T-resilient reservation is obtained at one of the solutions z^i:

$$z^i: \quad z^i_j = \begin{cases} T/(i-1) & j \leq i \\ 0 & j > i, \end{cases} \tag{3}$$

for some i between 2 and m. Moreover, since the cost-sequence $cz^2, \ldots, cz^m$ is unimodal, we may obtain the integral optimum through a simple rounding of the fractional one.

```
PATHS
{
    j = 0; D_0 = D; c^0 = c; P_0 = ∅;
    While (D_j contains a directed (s,t) path)
        Let Q_j be a minimum c^j-cost directed (s,t) path in D_j
        Set P_{j+1} = (P_j − R) ∪ F
            where R = {a ∈ A : a^- ∈ E(Q_j)}
            and F = A ∩ E(Q_j)
        Let z^{j+1} be the vector obtained by assigning T/j to each arc
            in each path of P_{j+1}
        If j ≥ 2 and cost(z^{j+1}) ≥ cost(z^j)
            then Output(z^j) and Quit
        D_{j+1}, c^{j+1} are the same as D_j, c^j except
            if a ∈ R
                remove a^-, and include a with cost c_a^{j+1} = c_a
            if a ∈ F
                remove a, and include a^- with cost c_{a^-}^{j+1} = −c_a
        Set j = j + 1;
    EndWhile
    Output(z^j)
}
```

If the set of diverse paths is not given from the outset, there is a difference between the fractional and the integral case, since the latter turns out to be NP-hard. We show that a minimum cost (fractional) T-resilient reservation is obtained at one of the solutions w^i:

$$w^i = \sum_{j=1}^{i} \frac{T}{i-1} e(P_j^i),$$

for some i between 2 and $\kappa(D)$, where $\mathcal{P}^i = \{P_1^i, \ldots, P_i^i\}$ is a minimum cost collection of i diverse (s,t) paths.

Since unimodality holds again, the optimal reservation may be found by the preceding algorithm PATHS which is based on the *successive shortest path method* - see [1]. We recall it since it will be the basis for algorithms we introduce later[1].

For the integral case, we give in [2] a polytime $\frac{1}{14}$-approximate algorithm, and we show that this bound is best possible (if P $\neq$ NP).

Also in [2], we show that the naive algorithm that allocates capacity T to the arcs of a cheapest pair of diverse (s,t) paths gives at most twice the cost of any T-resilient reservation.

2.1 Reservations with a Uniform Bound

In the rest of this section we consider the scenario where the capacity we may reserve on any arc is uniformly bounded by an upper bound $\lambda \in \mathcal{Q}^+$, that is (resilient) vectors must satisfy the additional constraint:

$$x_a \leq \lambda \quad \text{for each } a \in A. \tag{4}$$

Of course, if we are interested in a minimum cost T-resilient reservation, this bound is effective only if $\lambda \leq T$. We call a vector satisfying constraints (4) *uniformly bounded (by λ)*. Conversely, we refer to vectors which do not necessarily satisfy constraints (4) as *unbounded*. We discuss here the extension of diverse path reservations to this scenario.

We start by examining the case where we are given a fixed collection of diverse paths to use. Finding a minimum cost bounded diverse paths reservation then amounts to solving the following problem, where again each path is regarded as a single arc from s to t.

[1]For an arc $a = (u,v) \in A$, we let a^- denote an "artificial" arc (v,u), not present in D. In the course of the algorithm, we construct a series of auxiliary digraphs D_j, each of which will contain exactly one of each pair a, a^-. Moreover, in iteration $j = 0$, we take the value T/j as some suitably large number.

Problem (Fixed Uniformly Bounded Problem). Given a demand T and a non-decreasing sequence of costs $c_1, c_2, \ldots, c_m$, find non-negative real numbers $x_1, x_2, \ldots, x_m \leq \lambda$ to minimize $c_1x_1 + c_2x_2 + \ldots + c_mx_m$ subject to the constraints (2).

Clearly if $\lambda(m-1) < T$ then there is no feasible solution; in other words we need to use at least $\lceil \frac{T+\lambda}{\lambda} \rceil$ paths. For the remainder of this section, we set $n = \lceil \frac{T+\lambda}{\lambda} \rceil$, and assume that $n \leq m$. For $i = 2, \ldots, m$ let z^i be defined by (3), and let $z^{(T;\lambda)}$ be defined by:

$$z_i^{(T;\lambda)} = \begin{cases} \lambda & i < n \\ T - (n-2)\lambda & i = n \\ 0 & i > n. \end{cases}$$

Notice that, if T is an integer multiple of λ, then n is an integer and $z^n = z^{(T;\lambda)}$. The proof of the next lemma is routine, and in any case it comes later as a corollary of Lemma 7.

Lemma 1 *Let $n = \lceil \frac{T+\lambda}{\lambda} \rceil$ and suppose that $n \leq m$. Let h be such that $cz^h = \min_{i \in \{2,\ldots,m\}} cz^i$ (that is, z_h is an optimal solution for the fixed unbounded problem). Then an optimal solution for the fixed uniformly bounded problem is given by $z^{(T;\lambda)}$ if $h < n$ and by z^h if $h \geq n$.*

Lemma 1 and algorithm PATHS provide then a very simple algorithm to solve the *fixed* uniformly bounded problem. In contrast, we next show that the version of the problem where we are free to choose the set of paths is NP-hard, and indeed is not k-approximable for any $k \geq 1$, even in the fractional case, unless P = NP.

Denote by BDP the problem each instance of which consists of a network with two specified nodes s and t, together with non-negative integer costs on the arcs, an integer T, and a rational λ. An (optimal) solution for an instance of BDP will be a minimum cost uniformly bounded T-resilient diverse paths reservation, with upper bound λ. Note that problem BDP does not require the reservation vector to be integer.

Lemma 1 shows that the support of an optimal such solution will consist of a collection of diverse paths $P_1, P_2, \ldots, P_j$, where the arcs of the first $j-1$ paths will reserve a common amount r of capacity, and the last path's arcs will reserve capacity $T-(j-2)r$, equal to r unless $r = \lambda$. We now show that the subproblem of BDP with the input T fixed to 1, denoted by 1-BDP, is NP-hard if $\frac{1}{2} < \lambda < 1$; the proof is similar to one in [2].

Let 2DIV-PATHS denote the problem of determining whether a given digraph D, with four distinct nodes s_1, t_1, s_2, t_2, contains a pair of arc-disjoint paths P_1, P_2, where P_i joins s_i and t_i $(i = 1, 2)$. Fortune, Hopcroft and Wyllie [3] show that this problem is NP-complete.

Suppose that we are given an instance of 2DIV-PATHS, that is a digraph D, with four distinct nodes s_1, t_1, s_2, t_2. Construct a digraph from D by adding new nodes s, t as well as arcs (s,t), (s,s_1), (s,s_2), (t_1,t) and (t_2,t) with cost 0, 0, 1, 0 and 1 respectively. All remaining arcs will have cost zero. Moreover, let $T = 1$ and assume that $\frac{1}{2} < \lambda < 1$: this is our instance of 1-BDP.

We know that an optimal 1-resilient reservation will have support consisting of three arc-disjoint paths; two of the paths will have reserved capacity λ and the third capacity $1 - \lambda$.

Note that if there exists a solution to 2DIV-PATHS, with P_i a path between s_i, t_i $(i = 1, 2)$, then by assigning capacity λ to the arcs (s,t), (s,s_1), (t_1,t) and the arcs of P_1, and capacity $1 - \lambda$ to (s,s_2), (t_2,t) and the arcs of P_2 we obtain a solution to 1-BDP of cost $2(1 - \lambda)$. Conversely, if 2DIV-PATHS has no solution, then any solution to 1-BDP will use one path of cost 0 and two paths of cost 1, from which we deduce that the reservation will cost at least 1. Thus the optimal solution to the instance of 1-IDP is $2(1 - \lambda)$ if and only if the instance of 2DIV-PATHS has a solution.

Moreover, the above result implies that, unless P = NP, 1-BDP is not k-approximable for any $k \geq 1$. Indeed, suppose there exists a polynomial time k-approximation algorithm $\mathcal{A}$ for 1-BDP. Now consider any instance of 2DIV-PATHS and, as above, associate with it an instance of 1-BDP. Set $\lambda = 1 - \frac{1}{2(k+2)}$. Now, it is easy to check that the solution returned by algorithm $\mathcal{A}$ is less than 1 if and only if the instance of 2DIV-PATHS has a solution.

The hardness of this problem is extremely sensitive to the values of T and λ given. Indeed, suppose that we are given an instance of BDP where $T = (n-2)\lambda + \mu$, with $0 < \mu \leq \lambda$. Then the algorithm PATHS can be used to find the optimal solutions of the form $w^{(n)}, w^{(n+1)}, \ldots$. Furthermore, it is easy to see that the cost of any solution of the form $w^{(T;\lambda)}$ (analagous to $z^{(T;\lambda)}$, so having reserved capacity λ on $n-1$ paths, and μ on one) is at least $\mu\frac{n-1}{T}$ times the cost of $w^{(n)}$, (the extreme case being when all the cost of an optimal collection of n paths falls on one path, which is given capacity μ under $w^{(T;\lambda)}$ and $T/(n-1)$ under $w^{(n)}$). Hence a trivial adaptation of algorithm PATHS achieves an approximation ratio of $\frac{(n-2)\lambda+\mu}{(n-1)\mu} = 1 + \frac{n-2}{n-1}\frac{\lambda-\mu}{\mu}$. It is fairly easy to adapt the construction given above to show that this is best possible for the given values of T and λ, unless P = NP.

Roughly speaking, the reason for the hardness of BDP is to be found in Lemma 1. If $h < n$, we get an "asymmetric" optimal solution, and then it could even be the case that any optimal diverse paths solution is attained at a collection of j paths which is not a minimum cost collection.

The conclusion is that upper bounds on the capacity make diverse paths reservations hard to handle in general (this, of course, extends to the case when the bounds λ_a are different). In the next section we deal with acyclic flow-reservations, which have many of the nice properties of diverse paths reservations, yet do allow easy optimization in the bounded case.

3 Acyclic Flow-Reservations

In this section we show that seeking an acyclic flow-reservation is a good strategy for building resilient networks when we are given a uniform upper bound on the capacity we may reserve on each arc. We start by discussing acyclic flow-reservations in unbounded networks; later we move to the uniformly bounded scenario.

We recall that a reservation vector x is an *acyclic flow-reservation* if x is an (s,t) flow-vector and there are no cycles in the support $I(x)$ of x. This is a severe restriction on the types of reservation we consider: in [2], we gave several examples of optimal T-resilient reservations that were not flows, or that contained cycles in the support. Consider for instance the digraph in Figure 1, which is an optimal T-resilient reservation for suitable cost vectors. If arcs of cost 0 are added from t to the intermediate nodes, and capacity $T/3$ reserved on them, we get an optimal T-resilient flow reservation with cycles.

Lemma 2 *Let x be an acyclic flow of value M in a network D. Then x is T-resilient if and only if $x_a \leq M - T$ for every arc a.*

Proof. Suppose that x is an acyclic flow of value M, and consider an ordering $s =$

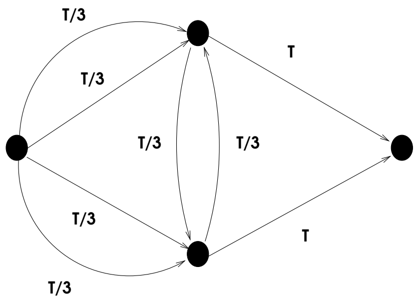

Figure 1: *A basic resilient reservation with a cycle*

$v_1, v_2, \ldots, v_n = t$ of V such that all arcs in the support of x go from a lower numbered node to a higher.

Suppose all arcs a have $x_a \leq M - T$, and consider an (s,t)-set S and an arc $a \in \delta^+(S)$. We have $x(\delta^+(S) - a) = x(\delta^+(S)) - x_a \geq M - x_a \geq T$, so x satisfies the partial-T cut constraint for (S, a).

Conversely, if there is some arc $a = (v_i, v_j)$ with $x_a > M - T$, then let $S = \{v_1, \ldots, v_i\}$. The flow out of S in $D - a$ is $M - x_a < T$, so x violates the partial T-cut constraint for (S, a). □

We now consider the problem of finding a minimum cost T-resilient acyclic flow, and we start with the problem of finding an optimal one of value M, that is an M-flow.

Lemma 2 implies that, to find a minimum cost T-resilient acyclic M-flow, we just have to find a minimum cost M-flow subject to capacity constraints $x_a \leq M - T$ for each arc a. Even better, the structure of an optimal solution to this minimum cost flow problem can be described very simply in terms of the output of algorithm Paths.

Recall that a minimum cost T-resilient diverse path reservation is always attained at one of the vectors w^i:

$$w^i = \sum_{j=1}^{i} \frac{T}{i-1} \chi(P_j^i), \tag{5}$$

for some i between 2 and $\kappa(D)$, where $\mathcal{P}^i = \{P_1^i, \ldots, P_i^i\}$ is a minimum cost collection of i diverse (s,t) paths. Notice that the vectors w^i are T-resilient acyclic flows.

From Lemma 2, it follows that there is *some* T-resilient acyclic M-flow, provided that the

(s,t)-connectivity $\kappa(D)$ is at least $\lceil \frac{M}{M-T} \rceil$. From now on in this section, we set $n = \lceil \frac{M}{M-T} \rceil$, which again should be thought of as the minimum number of paths in a T-resilient reservation, and we assume that $n \leq \kappa(D)$.

From the definition of n, it follows that $\frac{n}{n-1}T \leq M < \frac{n-1}{n-2}T$. Let α, $0 \leq \alpha < 1$, be such that $M = \alpha\frac{n-1}{n-2}T + (1-\alpha)\frac{n}{n-1}T$; finally let $w^{(M)}$ be the following vector:

$$w^{(M)} = \begin{cases} w^2 + (M-2T)\chi(P_1) & \text{if } n = 2 \\ \alpha w^{n-1} + (1-\alpha)w^n & \text{if } n \geq 3. \end{cases} \tag{6}$$

It is easy to see that $w^{(M)}$ is an M-flow such that $x_a \leq M - T$ for every arc a. If $\mathcal{P}^{n-1} \subset \mathcal{P}^n$, then $w^{(M)}$ is itself a diverse paths reservation vector, but in general it is not.

Our claim is that the acyclic vector $w^{(M)}$ is a minimum cost M-flow, subject to the capacity constraints. It then follows that $w^{(M)}$ is a minimum cost T-resilient acyclic M-flow. Of course, a consequence is that the algorithm PATHS can then be used to generate minimum cost T-resilient acyclic M-flows efficiently.

Theorem 3 *Let D be a network equipped with non-negative costs on its arcs, and let M and T be such that $n = \lceil \frac{M}{M-T} \rceil \leq \kappa(D)$. Then $w^{(M)}$ is a minimum cost T-resilient acyclic M-flow.*

Proof. One may check that $w^{(M)}$ is alternatively defined as follows (we omit the details). Start with the vector obtained by assigning $M - T$ to each arc in the collection $\mathcal{P}_{n-1}$ found by PATHS. Then augment by the amount $M - (n-1)(M-T)$ along the path Q_{n-1}. One checks that this is precisely a solution found by the successive shortest path method for the minimum cost M-flow problem with capacities $M - T$, and hence $w^{(M)}$ is optimal as required. □

As a consequence of the previous theorem we have the following corollary showing us that minimum cost T-resilient acyclic M-flows have a quite simple form.

Corollary 4 *There always exists a minimum cost T-resilient acyclic M-flow which is a convex combination either of a pair of diverse paths T-resilient reservations, or of one such reservation and a reservation using a single shortest path.*

Theorem 3 gives the structure of a minimum cost T-resilient acyclic M-flow. But what should we do if we just want the minimum cost T-resilient acyclic flow, when we are free to choose the value of the flow?

Since each $w^{(M)}$ is a convex combination of some pair of diverse paths vectors w^i, for $2T \geq M \geq \frac{\kappa(D)}{\kappa(D)-1}T$, and $w^{(M)}$ dominates w^2 for $M > 2T$, the minimum of the cost over all the $w^{(M)}$ is attained at some w^i. This gives us the following result.

Corollary 5 *Let D be a network equipped with non-negative costs on its arcs and let T be an integer. There is some minimum cost T-resilient acyclic flow-reservation that is a diverse paths reservation.*

INTEGRAL RESERVATIONS

As we have mentioned, the problem of finding a minimum cost *integral* T-resilient diverse paths reservation is NP-hard. We will now show that finding a minimum cost integral T-resilient acyclic flow-reservation is quite easy. We need a simple lemma.

Let $c(M) = cw^{(M)}$, that is $c(M)$ is equal to the cost of a minimum cost T-resilient acyclic M-flow. We show that the function $c(M)$ is unimodal along the interval $[\frac{\kappa(D)}{\kappa(D)-1}T, \infty)$.

Lemma 6 *For three points $M_1 < M_2 < M_3$ in the interval $[\frac{\kappa(D)}{\kappa(D)-1}T, \infty)$, if $c(M_1) \leq c(M_2)$ then $c(M_2) \leq c(M_3)$.*

Proof. Suppose the contrary: there exist $M_1 \leq M_2 \leq M_3$ in the interval $[\frac{\kappa(D)}{\kappa(D)-1}T, \infty)$ such that $c(M_1) \leq c(M_2)$ and $c(M_2) > c(M_3)$.

Let $\alpha \in (0, 1)$ be such that $M_2 = \alpha M_1 + (1-\alpha)M_3$. Then $w = \alpha w^{(M_1)} + (1-\alpha)w^{(M_3)}$ is a T-resilient acyclic M_2-flow and hence $cw \geq c(M_2)$. But of course $cw = \alpha c(M_1) + (1-\alpha)c(M_3) < c(M_2)$, a contradiction. □

Now, observe that, when T is an integer and the allocations must be integers, M is necessarily an integer too and so are the optimal vectors $w^{(M)}$. So one of these will be optimal for the integral problem. Then, from Lemma 6, we know that if w^h is the optimal fractional solution, then the value of an optimal integer solution will be either $\lfloor \frac{hT}{h-1} \rfloor$ or $\lceil \frac{hT}{h-1} \rceil$. Then, from Theorem 3, either $w^{(\lfloor \frac{hT}{h-1} \rfloor)}$ or $w^{(\lceil \frac{hT}{h-1} \rceil)}$ is an optimal solution.

3.1 Reservations with Bounds

We now go back to the scenario considered in Section 2.1 where the capacity we may reserve on each arc is uniformly bounded by $\lambda \in \mathcal{Q}^+$. There we proved that the problem of finding a minimum cost uniformly bounded T-resilient *diverse path* reservation is not k-approximable for any $k \geq 1$, even in the fractional case; we now show that, if we go instead for acyclic flow-reservations, the problem is easy, even in the integral case. This is, at first sight, remarkable, since we know from Corollary 4 that there always exists a minimum cost T-resilient acyclic flow-reservation which is a convex combination of *two* T-resilient diverse paths reservations.

Indeed, Lemma 2 essentially reduces the problem of finding a minimum cost T-resilient acyclic flow, uniformly bounded by λ, to the problem of finding a minimum cost *unbounded* T-resilient acyclic flow of some value at most $T + \lambda$.

This latter problem is easy. Let h be such that $cw^h = \min_{i \in \{2,\ldots,\kappa(D)\}} cw^i$, where the w^i are defined by (5), that is, w^h is an optimal unbounded diverse paths reservation. Note that w^h is feasible for the bounded problem if $T \leq \lambda(h-1)$. The next lemma follows easily from Theorem 3 and Lemma 6.

Lemma 7 *If c is a vector of non-negative costs, a minimum cost unbounded T-resilient acyclic flow of value at most $T + \lambda$ is given by*

$$w^{(T,\lambda,-)} = \begin{cases} w^{(T+\lambda)} & \text{if } T \geq \lambda(h-1) \\ w^h & \text{if } T \leq \lambda(h-1). \end{cases}$$

We point out that the integral version of the previous lemma follows as before. Indeed, if $T \geq \lambda(h-1)$ then $w^{(T,\lambda,-)} = w^{(T+\lambda)}$, which is integral, and otherwise the flow value of the optimal integral solution is either $\lfloor \frac{hT}{h-1} \rfloor$ or $\lceil \frac{hT}{h-1} \rceil$; in both cases by Theorem 3 we are done. The solution $w^{(T,\lambda,-)}$ may easily be found with a slight modification of the algorithm Paths.

Finally, as the next proposition shows, a simple 1-approximate algorithm for uniformly bounded T-resilience in a network D – in the fractional or integral case – is to find and output the minimum cost T-resilient acyclic $(T + \lambda)$-flow.

Proposition 8 *If x is a T-resilient vector bounded by λ, then $cx \geq \frac{1}{2}cw^{(T+\lambda)}$.*

Proof. As usual, we may assume that $T \leq \lambda$. Let x be a minimum cost T-resilient vector bounded by λ. Define x' such that, for each arc a, $x'_a = \min(2x_a, \lambda)$; it follows that $x' \geq x$.

Claim. For each (s,t)-set S, $x'(\delta^+(S)) \geq T + \lambda$.

Let S be any (s,t)-set. Suppose that there exists $e \in \delta^+(S)$ such that $x_e \geq \frac{\lambda}{2}$, it follows that $x'_e = \lambda$. Then $x'(\delta^+(S)) = x'_e + \sum_{a \in \delta^+(S)-e} x'_a \geq \lambda + \sum_{a \in \delta^+(S)-e} x_a \geq \lambda + T$.

If, on the other hand, $x_e < \frac{\lambda}{2}$ for each $e \in \delta^+(S)$, then $x'(\delta^+(S)) = 2x(\delta^+(S)) > 2T > T + \lambda$. *(End of Claim.)*

Since $x'(\delta^+(S)) \geq T + \lambda$ for each (s,t)-set S, then there exists a flow-vector x'' of value $T + \lambda$ such that $x'' \leq x'$; moreover, since $x''_a \leq x'_a \leq \lambda$, x'' is a T-resilient vector bounded by λ. It follows that $cw^{(T+\lambda)} \leq cx'' \leq cx' \leq 2cx$. □

We close this section with some remarks. First, it is easy to check that, if λ is an integer, then $w^{(T+\lambda)}$ is integral too. Moreover, if $\lambda = T$, than $w^{(T+\lambda)} = w^{(2T)}$ and we have the analogous result of [2] for the unbounded case. There we prove that the cost of the solution $w^{(2T)}$ is at most twice the cost of any (unbounded) T-resilient solution.

In fact, we may show that the constant $\frac{1}{2}$ in Proposition 8 is best possible using the following example from [2]. The network D has three nodes s, u and t, and $\lambda = T$. There are two arcs of cost 1 from s to u, and r arcs of large cost c from u to t. The minimum cost 1-resilient reservation involves giving capacity 1 to the arcs between s and u, and capacity $\frac{1}{r-1}$ to the arcs between u and t, at a cost of $2 + cr/(r-1)$. However, the minimum cost diverse paths reservation uses only two of the arcs from u to t, at a total cost of $2 + 2c$. For large c and r, the ratio between these two costs can be made arbitrarily close to 2: the optimal diverse paths reservation is almost twice as expensive as the optimal 1-resilient reservation.

On the other hand, one would expect that the normal situation is even better than suggested by Proposition 8; in fact, we may output the vector $w^{(T,\lambda,-)}$ (see Lemma 7), that is a minimum cost T-resilient acyclic flow with value at most $T + \lambda$, whose cost is typically lower than that of $w^{(T+\lambda)}$.

It follows that in many practical settings $w^{(T,\lambda,-)}$ will not be very far from the overall optimal solution in cost, and we recall that this vector is either a diverse paths reservation or a convex combination of two diverse paths reservations. So, given its simplicity, it is likely that this is a good solution to adopt.

4 Resilient Reservation with Differing Upper Bounds

We now consider the scenario where there are bounds on the capacity we may reserve on each arc of the network which are, in general, not the same for each arc. Hence any (resilient) vectors must satisfy the additional constraint:

$$x_a \leq \lambda_a \quad \text{for each } a \in A. \tag{7}$$

We call a vector satisfying constraints (7) *bounded* while we again refer to vectors which do not necessarily satisfy constraints (7) as *unbounded*.

One might wonder if the approximation algorithm we gave at the end of the previous section, for the case where the upper bound is the same for all the arcs of the network, extends to this case. The answer is negative: consider the following instance. The network D has three nodes s, u and t. There are k arcs with upper bound $u_a = T/(k-1)$ and cost 1 from s to u, $k+1$ arcs with upper bound $u_a = T/k$ and cost 1 from u to t and one more arc with upper bound $u_a = T/k$ and large cost c from u to t. Then the minimum cost bounded 1-resilient reservation involves giving capacity $T/(k-1)$ to the arcs between s and u, and capacity T/k to the $k+1$ arcs between u and t with cost 1, at a total cost of $\frac{2k^2-1}{k(k-1)}$. However, the cost of an optimal flow-reservation, even non-acyclic, involves giving also capacity $\frac{T}{k(k-1)}$ to the arc with cost c, at a total cost of $\frac{2k^2-1}{k(k-1)} + c\frac{T}{k(k-1)}$. By increasing c, the ratio between these two costs can be made arbitrarily large.

Hence, when the bounds are different, it is hopeless to look for an approximation algorithm which involves flow-reservations (acyclic or not). Moreover, as we have shown in Section 2.1, the problem of finding a minimum cost bounded T-resilient diverse paths reservation is hard if the set of paths is not given from the outset.

Finally, the reduction we used to deal with uniformly bounded resilience (from Lemma 2) does not even extend to the case where we ask for diverse paths reservation on a fixed set of paths.

The problem of finding a strategy for general bounded resilience with some or all of the good properties that we have been dealing with in other scenarios (simplicity of the solution type sought, good approximation properties, good optimization properties) remains open. However, we are able to treat the case where we require a diverse paths reservation with the paths pre-specified.

Hence, in the rest of this section we suppose that we are given a set of paths $P_1, \ldots, P_m$, considered as single arcs from s to t, and for each arc P_i an upper bound λ_i on the capacity we may reserve (in general, λ_i is equal to the smallest upper bound among those of the arcs of a path P_i). The problem of finding a minimum cost resilient reservation is as follows.

Problem (Fixed Bounded Problem). Given a demand T, a vector $\lambda \in \mathcal{Q}_+^m$ and a non-decreasing sequence of costs $c_1, c_2, \ldots, c_m$, find non-negative real numbers $x_1, x_2, \ldots, x_m$ to minimize $c_1x_1 + c_2x_2 + \ldots + c_mx_m$ subject to the partial cut constraints (2) and such that $x_i \leq \lambda_i$ for each i.

First, notice that the set of bounded T-resilient vectors is non-empty if and only if $\sum_j \lambda_j \geq T + \max_j\{\lambda_j\}$. We assume from now on that this holds.

As before, to find a minimum cost bounded T-resilient vector, we start with the problem of finding a minimum cost T-resilient M-flow (that is, a T-resilient vector x with *value* $\sum_i x_i = M$). Observe that, if $\sum_{i=1}^m \min\{M-T, \lambda_i\} < M$ then no T-resilient vector has value M. Otherwise, let n be the smallest index such that $\sum_{i=1}^n \min\{M - T, \lambda_i\} \geq M$.

Lemma 9 *The vector $x^{(M)}$ is a minimum cost T-resilient M-vector, where:*

$$x_i^{(M)} = \begin{cases} \min\{M - T, \lambda_i\} & i < n \\ M - \sum_{i=1}^{n-1} x_i^{(M)} & i = n. \end{cases}$$

Proof. Suppose the contrary. Then for any minimum cost solution y there exists a minimum index $i \leq n$ such that $y_i < x_i$ where x_i is as above. Without loss of generality assume that we are given an optimal y maximizing the value of i.

It follows that there exists $J \subseteq \{i+1, \ldots, m\}$ such that $\sum_{j\in J} y_j \geq x_i - y_i$; hence it is possible to define a new minimum cost solution y' such that $y'_j = x_j$ for $j \leq i$: a contradiction. $\square$

This gives the structure of a bounded minimum cost T-resilient M-vector $x^{(M)}$; from now on let $c(M) = cx^{(M)}$. The following lemma states that $c(M)$ is a unimodal function along the interval $[M^*, 2T]$, where M^* is the minimum rational such that $\sum_{i=1}^{m} \min\{M^* - T, \lambda_i\} \geq M^*$. Its proof is almost the same as that of Lemma 6, and we skip it.

Lemma 10 *For any rationals $M_1 < M_2 < M_3$ in the interval $[M^*, \infty)$, if $c(M_1) \leq c(M_2)$, then $c(M_2) \leq c(M_3)$.*

We now know that to find a minimum cost T-resilient vector amounts to minimizing the unimodal function $c(M)$, and it is easy to check that $c(M)$ is piecewise linear with at most $2m$ "turning points", so the problem can be solved by generating the successive turning points and evaluating $c(M)$ until the function starts to rise. However, generating the successive turning points of $c(M)$ is not entirely straightforward, and we present instead an alternative approach resulting in an $O(m \log m)$ algorithm for the problem.

Our approach involves solving at most m *unbounded* problems with a lower bound on the value of the solution. We say that a (bounded) T-resilient vector of value greater than or equal to M is (bounded) (T, M)*-resilient*. We are interested in the problem of finding a minimum cost bounded (T, M)-resilient reservation; note that if $M = T$ then this is certainly a minimum cost bounded T-resilient reservation.

Observe that we already have a solution to the unbounded problem, which is as follows.

Problem. Given a demand T and a non-decreasing sequence of costs $c_1, c_2, \ldots, c_m$, find non-negative real numbers $x_1, x_2, \ldots, x_m$ to minimize $c_1x_1 + c_2x_2 + \ldots + c_mx_m$ subject to the partial cut constraints (2) and $\sum_i x_i \geq M$.

The next corollary follows from Theorem 3 and Lemma 6.

Corollary 11 *Suppose the optimal T-resilient reservation is z^j, the solution having value $T/(j-1)$ on the cheapest j arcs. Let $n = \lceil M/(M-T) \rceil$. An optimal (T, M)-resilient* *reservation is then given by:*

$$z^{(M,+)}: \quad z_i^{(M,+)} = \begin{cases} M-T & i \le n-1 \\ T-(n-2)(M-T) & i = n \\ 0 & i \ge n+1 \end{cases} \qquad \text{if } j \ge n$$

$$z^{(M,+)} = z^j \qquad \text{if } j < n.$$

Observe that if $z^{(M,+)}$ satisfies the constraints (7), then it is also a minimum cost bounded (T,M)-resilient vector. Otherwise, let $h \in \{1,\dots,m\}$ be such that:

$$\lambda_h = \min_{i \in \{1,\dots,m\}: z_i^{(M,+)} > \lambda_i} \lambda_i. \tag{8}$$

In this case we have the following theorem.

Theorem 12 *There exists a minimum cost bounded (T,M)-resilient vector x such that $x_h = \lambda_h$.*

Proof. Suppose the theorem does not hold. Let x be a minimum cost bounded (T,M)-resilient vector such that any other minimum cost solution x' has $x'_h \le x_h < \lambda_h$.

Claim. (i) $\sum_{i \ne h} x_i > T$; (ii) $\sum_i x_i > M$.

(i) Let $z = \mu x + (1-\mu) z^{(M,+)}$, $0 \le \mu \le 1$. Since x and $z^{(M,+)}$ are both (T,M)-resilient, it follows that z is (T,M)-resilient too. Moreover we have that $cz^{(M,+)} \le cz \le cx$; hence, because of the definition of x, z is not feasible for any $\mu < 1$. It follows that there exists j such that $x_j = \lambda_j$ and $z_j^{(M,+)} > \lambda_j$; from the definition of h it follows that $\lambda_j \ge \lambda_h$. Then $x_j = \lambda_j \ge \lambda_h > x_h$, that is $x_j > x_h$, and hence $\sum_{i \ne h} x_i > T$.

(ii) If there exists $l > h$ such that $x_l > 0$, then from (i) it follows that we can move a positive amount of reserved capacity from arc l to arc h and obtain a new reservation $\bar{x}$ that is (T,M)-resilient, and such that $c\bar{x} \le cx$ and $\bar{x}_h > x_h$: this is a contradiction.

Now since $\sum_{i \ne 1} z_i^{(M,+)} = T$ (from Corollary 11), $I(x) \subseteq \{1,\dots,h\}$ and $x_h < z_h^{(M,+)}$, it follows that there exists $j \in \{2,\dots,h-1\}$ such that $x_j > z_j^{(M,+)}$. Since $j \in \{2,\dots,h-1\}$, we have that $z_j^{(M,+)} \ge M-T$ (again from Corollary 11); hence $x_j > M-T$. But then $\sum_i x_i = \sum_{i \ne j} x_i + x_j > M$. *(End of Claim.)*

Let $S = I(x) \setminus h$. Let $\bar{x} = x + \varepsilon(e_h - \frac{1}{|S|-1}e(S))$. Then for ε small enough it is easy to check that $\bar{x}$ is bounded (T, M)-resilient. Moreover, $\bar{x}_h > x_h$ and

$$c\bar{x} - cx = \varepsilon(c_h - \frac{\sum_{i \in S} c_i}{|S| - 1}) > 0$$

because of the definition of x. But since $z^{(M,+)}$ is optimal for the unbounded problem, we have:

$$\frac{\sum_{i \in S} c_i}{|S| - 1} \geq \frac{\sum_{i=1..h-1} c_i}{h - 2} \geq c_h.$$

□

We use the previous theorem to design an algorithm to find a minimum cost bounded T-resilient vector. The algorithm at each step drops out a variable according to the value of a minimum cost solution for the unbounded problem as suggested by Theorem 12.

Before going into the details we need a simple lemma. Consider the following sets (polyhedra) $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$\mathcal{R}_1 = \left\{ x \in \mathcal{Q}_+^m : \begin{array}{rcll} \sum_{i \neq j} x_i & \geq & T & \text{for any } j \\ \sum_i x_i & \geq & M & \\ x_i & \leq & \lambda_i & \text{for any } i \\ x_h & = & \alpha & \end{array} \right\};$$

$$\mathcal{R}_2 = \left\{ x \in \mathcal{Q}_+^{m-1} : \begin{array}{rcll} \sum_{l \neq j} x_l & \geq & T - \alpha & \text{for any } j \\ \sum_l x_l & \geq & \max\{T, M - \alpha\} & \\ x_l & \leq & \lambda_l & \text{for any } l \end{array} \right\};$$

where $\alpha \leq \lambda_h$ and the index l runs over $\{1, \ldots, h-1, h+1, \ldots, m\}$ (while i runs as usual from 1 to m). Moreover, for any $x \in \mathcal{R}_1$ let $p(x) \in \mathcal{Q}_+^{m-1}$ be the vector such that $p(x)_i = x_i$ for each $i \neq h$; conversely for any $y \in R_2$ let $l^\alpha(y) \in \mathcal{Q}_+^m$ be the vector such that $l_i^\alpha = y_i$ for $i \neq h$ and $l_h^\alpha = \alpha$.

The following lemma is straightforward.

Lemma 13 *If $x \in \mathcal{R}_1$, then $p(x) \in \mathcal{R}_2$. Conversely, if $y \in \mathcal{R}_2$, then $l^\alpha(y) \in \mathcal{R}_1$.*

As a consequence of the previous lemma and of Theorem 12 we have the following corollary.

Corollary 14 *Let $z^{(M,+)}$ be a minimum cost (T, M)-resilient vector over the set $\{1, \dots, m\}$ and let h be such that (8) holds. Let y be a minimum cost bounded $(T - \lambda_h, \max\{T, M - \lambda_h\})$-resilient vector over the set $I \setminus h$. Then the vector $l^{\lambda_h}(y)$ is a minimum cost bounded (T, M)-resilient vector.*

Proof. From Lemma 13 we know that $l^{\lambda_h}(y)$ is bounded (T, M)-resilient. On the other hand, from Theorem 12 we know that there exists a minimum cost bounded (T, M)-resilient vector x' such that $x'_h = \lambda_h$. Now suppose that $cx' < cl^{\lambda_h}(y)$. It follows that $\sum_{i \in I \setminus h} c_i p(x')_i < \sum_{c \in I \setminus h} c_i y_i$, but since $p(x')$ is $(T - \lambda_h, \max\{T, M - \lambda_h\})$-resilient, we have a contradiction. □

BASIC FIXED SET OF PATHS ALGORITHM

{

1. $M = T$; $I = \{1, \dots, m\}$
2. If there is no (T, M)-resilient vector then
   Output(''Infeasible'') and Quit
3. Find a minimum cost (T, M)-resilient vector $z^{(M,+)}$ over the set I
4. If $z_i^{(M,+)} \leq \lambda_i$ for each $i \in I$ then
   let $x_i^* = z_i^{(M,+)}$ for each $i \in I$
   Output(x^*) and Quit
5. Let $h \in I$ be such that $\lambda_h = \min_{i \in I : z_i^{(M,+)} > \lambda_i} \lambda_i$
6. Let $I = I \setminus h$; $x_h^* = \lambda_h$; $T = T - \lambda_h$ and $M = \max\{T, M - \lambda_h\}$
7. Goto 2

}

Finally, the algorithm above, BASIC FIXED SET OF PATHS ALGORITHM, finds a minimum cost bounded T-resilient vector x^*.

We point out that also the previous algorithm can be used in the integral case. In fact, in that case it follows from Lemma 10 that there exists a minimum cost reservation x^I such that $\sum_j x_j^I \in \{\lfloor \sum_j x_j^* \rfloor, \lceil \sum_j x_j^* \rceil\}$. In turn, Lemma 9 gives us the form of a minimum cost T-resilient M-vector. Hence to get a minimum cost integral T-resilient vector, we just have to set M equal to $\lfloor \sum_j x_j^* \rfloor$ and $\lceil \sum_j x_j^* \rceil$, define the corresponding minimum cost T-resilient M-vector, as from Lemma 9, and finally choose a minimum cost one.

4.1 An Efficient Algorithm

It is easy to see that the algorithm Basic Fixed Set of Paths Algorithm can be implemented so that its running time is $O(m^2)$. In fact, the number of iterations is bounded by m and in each iteration we do the following.

1. Find an optimal unbounded (T, M)-resilient vector $z^{(M,+)}$ over a set $I \subseteq \{1, \dots, m\}$. From Corollary 11 it follows that we have to find an optimal T-resilient reservation: this can be done in $O(m)$-time.
2. Check if $z_i^{(M,+)} \leq \lambda_i$ for each $i \in I$ and else find h such that (8) – restricted to the set I – holds: this can be done in $O(m)$-time.
3. If not, update I, M, T and start a new iteration: this can be done in constant time.

We shall give a more efficient implementation, resulting in a running time of $O(m \log m)$. The trick will be that of lowering the *average* cost of an iteration. This implementation is based on the following simple idea.

At each iteration, in order to get an optimal (T, M)-resilient vector over the current set I, we have to find an optimal T-resilient reservation z^j given by 3. If we assume, without loss of generality, that $I = \{1, \dots, m\}$ and $c_1 \leq c_2 \leq \dots \leq c_m$, we have to scan the cost vector $(c_1 + c_2)T$; $(c_1 + c_2 + c_3)\frac{T}{2}$ ... up to when either $c_i \geq (c_1 + c_2 + c_{i-1})\frac{T}{i-2}$ or $i = n = \lceil \frac{M}{M-T} \rceil$ (in fact, unimodality holds as usual).

As we show in the following, when in the next iteration, (after having dropped out the variable x_h) we scan this vector again, we may scan it starting from c_i. Moreover, even if this does not affect the cost the cost of a single iteration (in the wo rst case), it does affect its average cost.

We need two simple propositions. As usual we assume that $c_1 \leq c_2 \leq \dots \leq c_m$. Let $z^{(M,+)}$ be a minimum cost unbounded (T, M)-resilient reservation, and denote by q the size of its support (that is, $q = |I(z^{(M,+)})|$).

Proposition 15 *If* $\frac{1}{i-2} \sum_{j<i} c_j > c_i$ *and* $h \in \{1, \dots, i-1\}$, *then* $\frac{1}{i-3} \sum_{j<i, j \neq h} c_j > c_i$.

Proof. Since $\frac{1}{i-2} \sum_{j<i} c_j > c_i$, it follows that $\frac{1}{i-3} \sum_{j<i-1} c_j > \frac{1}{i-2} \sum_{j<i} c_j$. Then

$\frac{1}{i-3}\sum_{j<i,j\neq h} c_j \geq \frac{1}{i-3}\sum_{j<i-1} c_j > \frac{1}{i-2}\sum_{j<i} c_j > c_i$. $\square$

Proposition 16 *Suppose that the set* $I = \{i \in I(z_i^{(M,+)}) : z_i^{(M,+)} > \lambda_i\}$ *is non-empty and let* $\lambda = \min_{i\in I} \lambda_i$. *Then* $\lceil \frac{max(M-\lambda,T)}{max(M-\lambda,T)-(T-\lambda)} \rceil \geq q-1$.

Proof. Using corollary 11, we deduce that $q \leq n$ (recall that the value of $z^{(M,+)}$ is at least M and that $n = \lceil \frac{M}{M-T} \rceil$). Moreover, notice that if $\lambda \leq M-T$ then $\frac{\max\{M-\lambda,T\}}{\max\{M-\lambda,T\}-(T-\lambda)} = \frac{M-\lambda}{M-T}$, while if $\lambda > M-T$ then $\frac{\max\{M-\lambda,T\}}{\max\{M-\lambda,T\}-(T-\lambda)} = \frac{T}{\lambda}$.

In the former case, we have $\lceil \frac{M-\lambda}{M-T} \rceil \geq \lceil \frac{T}{M-T} \rceil = \lceil \frac{M}{M-T} - 1 \rceil \geq q-1$. In the latter case, since $\lambda > M-T$, it follows that the value of any non-zero component of $z^{(M,+)}$ is $\frac{T}{q-1}$ and then $\lambda \leq \frac{T}{q-1}$. But then $\frac{T}{\lambda} \geq q-1$. $\square$

¿From the previous propositions it is possible to assume inductively that, at the beginning of each iteration, we are given a set Q and an integer i such that the following properties hold:

(i) $I = Q \cup \{i, \ldots, m\}$ and $Q \subset \{1, \ldots, i-1\}$;

(ii) for each $j \in Q$, $c_j < \frac{\sum_{q\in Q:q<j} c_q}{|\{q\in Q:q<j\}|-1}$;

(iii) $|Q| \leq n$ where $n = \lceil \frac{M}{M-T} \rceil$.

Notice that at the first iteration we have to set $i = 3$ and $Q = \{1, 2\}$. We now show how this allows us to get a more efficient implementation.

1. From (ii), it easily follows that $Q \subseteq I(z^{(M,+)})$. Hence, at each iteration, to find $z^{(M,+)}$, we may scan the vector c starting from c_i.

Then, we check if $c_i < \frac{1}{|Q|-1}\sum_{q\in Q} c_q$ and $|Q| \leq n-1$; in this case add i to Q, increase i by one, consider the next c_i, and so on up to when either $c_i \geq \frac{1}{|Q|-1}\sum_{q\in Q} c_j$ or $|Q| = n$.

Incidentally, we observe that the cost of the single iteration could then increase up to $O(m \log m)$. In fact, we record the set Q as ordered according to the bounds on its arc. Hence, using an appropriate data structure, e.g., a heap, the operation of insertion of a new element i or deletion of an element j from Q (denoted as Insert(Q, i) and Delete(Q, j) in the algorithm) can be performed in $O(\log |Q|)$-time, while a minimum element in Q (returned by the function $Pop(Q)$ in the algorithm) may be picked up in constant time.

2. From Corollary 11 it follows that the non-zero elements of $z^{(M,+)}$ may take at most two different values and, in particular, one of these values may be taken only by the last non-zero element.

In particular, we will have that either $I(z^{(M,+)}) = Q$ and all the non-zero elements have the same value equal to $\frac{T}{|Q|-1}$; or $I(z^{(M,+)}) = Q \cup i$, $z_q^{(M,+)} = M - T$ for each $q \in Q$ and $z_i^{(M,+)} = T - (n-2)(M-T)$.

$O(m \log m)$ Fixed Set of Paths Algorithm

```
{
    For i ∈ {1,...,m} let x*_i = 0
    Let M = T; Q = {1,2}; i = 3; n = m+1; c_{m+1} = ∞
    While |Q ∪ {i,...,m}| ≥ 2 do
        While (c_i < 1/(|Q|-1) Σ_{q∈Q} c_q) and (|Q| ≤ n-1) do
            Insert(Q,i)
            i = i+1
        EndWhile
        If (|Q| = n) and (c_i < 1/(|Q|-1) Σ_{q∈Q} c_q) then
            let j = Pop(Q)
            let z_i = T - (n-2)(M-T)
            if (λ_i ≥ z_i) and (λ_j ≥ M-T) then
                for each q ∈ Q let x*_q = M-T
                let x*_i = z_i
                Output(x*) and Quit
            else if (λ_i < z_i) and (λ_i ≤ λ_j) then
                x*_i = λ_i; T = T - λ_i; M = max(M - λ_i, T)
                i = i+1
            else Delete(Q,j)
                x*_j = λ_j; T = T - λ_j; M = max(M - λ_j, T)
        else if c_i ≥ 1/(|Q|-1) Σ_{q∈Q} c_q then
            let j = Pop(Q)
            if λ_j ≥ T/(|Q|-1) then
                for each q ∈ Q let x*_q = T/(|Q|-1)
                Output(x*) and Quit
            else Delete(Q,j)
                x*_j = λ_j; T = T - λ_j; M = max(M - λ_j, T)
        Let n = ⌈(M/T)/((M/T)-1)⌉
    EndWhile
    Output(``Infeasible'') and Quit
}
```

In the former case, we just have to check if $\min_{q \in Q} \lambda_q \geq \frac{T}{|Q|-1}$. In the latter case, we have to check both that $\min_{q \in Q} \lambda_q \geq M - T$ and that $\lambda_i \geq T - (n-2)(M-T)$. In any case, this can be done in constant time.

3. We set $x^*_h = \lambda_h$, update T and M and remove h from I: if $h \in Q$ this may be done in $O(\log |Q|)$-time; while if $h = i$ we simply increase i by one. Then a new iteration starts.

The correctness of the algorithm relies on properties (i)–(iii), which hold by construction, and because of Propositions 15 and 16.

To evaluate the running time of the algorithm, we first observe that the number of iterations is equal to the number of times the external *While* loop is executed, which is $O(m)$. Then we observe that in each iteration an inner *While* loop is executed: we evaluate separately the time-cost contribution of this internal loop, which is $O(m)$ since in any iteration we increase i by one (and i goes from 3 to $m+1$ and never decreases). Discarding the inner *While* loop, at each iteration the most time-consuming operations are the insertions and deletions of elements in/from the set Q, which can be done in $O(\log m)$. Hence the overall running time of the algorithm is $O(m \log m)$.

5 Bundled resilience

In this section we adopt the viewpoint that on each arc a of the network we are free to reserve as much capacity as we want but only in bundles of μ_a units, and each bundle comes with cost c_a. As usual, our objective is to minimize the total cost of the capacity reservations we make, subject to supporting a given target amount T of traffic from s to t, even if one of the arcs in the network fails.

Here there is a dramatic difference between the case (*homogeneous bundles*) where the size of the bundles is the same for all the arcs in the network, that is $\mu_a = \mu$ for each arc a, and the case (*non-homogeneous bundles*) where there are different sizes. In fact, as we show, in the former case any resilient reservation problem reduces to an integral unbounded resilient reservation problem (and hence it is easy for diverse paths reservation and flow reservation) while in the latter even the problem of finding a minimum cost resilient reservation on a fixed set of paths is NP-complete.

HOMOGENEOUS BUNDLES

In this case a resilient reservation is a vector $x \in \mathcal{Z}_+^m$ such that for any $s-t$ cut S and any $e \in \delta^+(S)$ then $\sum_{f\in\delta^+(S)\setminus e} \mu x_f \geq T$. It follows that, without loss of generality, we may assume that $T = h\mu$, where h is a positive integer. Then, trivially, any T-resilient reservation problem with homogeneous bundles reduces to the corresponding integer T'-resilient reservation problem (i.e., without bundles) with $T' = \frac{T}{\mu}$. If the original problem has bounds, so does the transformed problem.

NON-HOMOGENEOUS BUNDLES

Generally speaking, resilience problems with bundles of different sizes are hard to solve. In fact, finding a minimum cost reservation is hard even if we are looking for a diverse paths reservation on a *given* set of paths. Denote this problem by BFP.

Any instance of BFP consists of integers m and T, and integer m-vectors c and μ. The problem may be formulated as follows.

$$
\begin{array}{ll}
\min\ c_1x_1 + c_2x_2 + \ldots + c_mx_m & \\
\sum_{i\neq j}\ \mu_i x_i \geq T & \text{for any } j \\
x_i \geq 0 & \text{an integer for any } i.
\end{array}
$$

The previous program is very similar to the maximization version of the INTEGER KNAPSACK problem. An instance of this problem (which has been proved to be NP-complete both in the minimization and maximization versions [4]) consists of integers p and B, and positive size-p integral vectors w and a. The problem may be formulated as follows:

$$
\begin{array}{ll}
\min\ w_1x_1 + w_2x_2 + \ldots + w_px_p & \\
\sum_i a_i x_i \geq B & \\
x_i \geq 0 & \text{an integer for any } i,
\end{array}
$$

where w and a are vectors of integers and B is an integer too.

It is routine to show that any instance of INTEGER KNAPSACK, may be reduced to BFP in polynomial time: namely set $T = B$; $m = p+1$; $c_i = a_i$ for $i \leq p$ and $c_{p+1} = 0$; $\mu_i = w_i$ for $i \leq p$ and $\mu_{p+1} = B$. Then if x is a minimum cost B-resilient vector, and y is such that $y_i = x_i$ for $i \leq p$, then y is an optimal solution to INTEGER KNAPSACK.

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