Chromatic Polynomials and the Spectrum of the Kneser Graph

Philipp Reinfeld*
Department of Mathematics
London School of Economics
Houghton St.
London WC2A 2AE England

Abstract

Using the transition matrix method, an explicit expression of the chromatic polynomial for the composition $C_n[K_b]$ of the cycle C_n and the complete graph K_b will be obtained. In order to be able to do so the spectrum of the adjacency matrix of the Kneser graph $K(k, b)$ will be determined.

1 Introduction

We construct a graph $C_n[K_b]$, where C_n is the cycle with n vertices and K_b is the complete graph with b vertices, as follows. The vertex set is given by $V(C_n[K_b]) = V(C_n) \times V(K_b)$ where '$\times$' denotes the cartesian product. Two vertices (u, v), $(u', v') \in V(C_n[K_b])$ are adjacent if either u is adjacent to u' in C_n, or $u = u'$ and $v \neq v'$. This is sometimes called the lexicographic product of C_n and K_b. In other words: Take n copies of the complete graph K_b, and join every vertex in the ith copy to every vertex in the $(i + 1)$th copy. We

*Supported by the Engineering and Physical Sciences Research Council

take $n+1=1$, so that we have a ring of complete graphs K_b, each successive pair defining a complete graph K_{2b}.
Recently, Shrock and Chang [RSC] found an explicit formula for the chromatic polynomial of the graph $C_n[K_b]$. The purpose of this paper is to derive that formula by an alternative method, based on the 'compatibility matrix' [[NB1], [NB2], [NB3]]. It turns out that the calculations required for the compatibility matrix method are equivalent to finding the spectrum of the graph known as the *Kneser graph* $K(k,b)$. The vertex-set of this graph is the set of b-subsets of a k-set, two subsets being adjacent if they are disjoint.
The Kneser graph $K(k,b)$ is closely related to another famous graph, the *Johnson graph* $J(k,b)$. This graph has the same vertex-set as $K(k,b)$, but the adjacency relation is different: two b-subsets are adjacent in $J(k,b)$ if they intersect in a set of size $b-1$.
The Johnson graph is distance-regular, and its eigenvalues and multiplicities can be determined by standard methods [BCN] pp. 255-260. The adjacency matrix of $K(k,b)$ is a known polynomial function of the adjacency matrix of $J(k,b)$, and so (in principle) the spectrum of $K(k,b)$ is 'known'.
The method of calculation given here is has the advantage of being direct, and it explains the multiplicities of the eigenvalues by finding explicit bases for the eigenspaces.

2 The compatibility matrix method

Let $V=\{1,2,\dots,b\}$ be the vertex set of K_b. For any integer $k>2b$, a (proper) k-colouring of K_b is an injection $\alpha: V \to \{1,2,\dots,k\}$. We shall think of b as being fixed, and denote the set of k-colourings of K_b by P_k. Similarly, the vector space of all real-valued functions on P_k will be denoted by $\mathcal{V}_k$. The standard basis for $\mathcal{V}_k$ consists of the functions $[\alpha]$ defined by

$$[\alpha](\beta)=\begin{cases} 1 & \text{if} \quad \alpha=\beta\ ; \\ 0 & \text{otherwise,} \end{cases}$$

for every $\alpha,\ \ \beta\in P_k$ For each $k\geq b$ we define a *compatibility matrix* T_k (and an associated *compatibility operator* on $\mathcal{V}_k$) as follows. The rows and columns of T_k correspond to the elements of P_k, and the entries are given by:

$$(T_k)_{\alpha\beta}=\begin{cases} 1 & \text{if} \quad \mathbf{Im}(\alpha)\cap\mathbf{Im}(\beta)=\emptyset\ ; \\ 0 & \text{otherwise,} \end{cases}$$

Lemma 1 *The number of k-colourings of the graph $C_n[K_b]$ is*

$$\text{trace}(T_k^n)=\sum_{r=1}^{s} m_r(k)\lambda_r(k)^n,$$

where $\lambda_1(k), \lambda_2(k), \ldots \lambda_s(k)$ are the eigenvalues of T_k, with multiplicities $m_1(k), m_2(k), \ldots, m_s(k)$ respectively.

Denote this number, as a function of k, by $P(C_n[K_b]; k)$. Then the corresponding function of a complex variable z, $P(C_n[K_b]; z)$, is the *chromatic polynomial* of $C_n[K_b]$.

3 The spectrum of the Kneser graph

Fix b and $k > 2b$. Let $K = \{1, 2, \ldots, k\}$. We denote by Q_i the set of all i-subsets of K and by $\mathcal{Z}_i$ the vectorspace of real-valued functions on Q_i. For $A \in Q_i$ with $i \leq b$ the natural basis of $\mathcal{Z}_i$ is given by the functions f_A defined as:

$$f_A(B) = \begin{cases} 1 & \text{if} \quad B = A\,; \\ 0 & \text{otherwise,} \end{cases}$$

for any $B \in Q_i$. The dimension of $\mathcal{Z}_i$ is $dim\,(\mathcal{Z}_i) = \binom{k}{i}$.

Define a linear map

$$\theta_i : \mathcal{Z}_i \to \mathcal{Z}_{i+1}$$

by

$$f_A \mapsto \sum_{\substack{D \supset A \\ |D| = i+1}} f_D = \sum_{x \notin A} f_{A \cup x}.$$

Lemma 2 *The linear maps θ_i, defined above, for $i = 1, 2, \ldots, b$ and $k > 2i$ are monomorphisms.*

Proof: Fix i such that $k > 2i$. A basis for $\mathcal{Z}_i$ is given by $\{f_A \big| A \subset K,\ |A| = i\}$ and a basis for $\mathcal{Z}_{i+1}$ is given by $\{f_D \big| D \subset K,\ |D| = i+1\}$. Denote the $\binom{k}{i+1} \times \binom{k}{i}$-matrix, associated with the linear map $\theta_i : \mathcal{Z}_i \to \mathcal{Z}_{i+1}$, by $\Theta^{(i)}$. A typical entry of $\Theta^{(i)}$ is given by:

$$\Theta^{(i)}_{DA} = \begin{cases} 1 & \text{if} \quad A \subset D\,; \\ 0 & \text{otherwise,} \end{cases}$$

Now, θ_i is injective if an only if it has a left inverse. That is if there is a linear map $\psi_i : \mathcal{Z}_{i+1} \to \mathcal{Z}_i$ such that

$$\psi_i\, \theta_i = \mathrm{id}_{\mathcal{Z}_i}.$$

Denote the 'falling factorial' by

$$(z)_0 = 1, \qquad (z)_p = z(z-1)\ldots(z-p+1) \qquad (p \geq 1).$$

Further let $r_B(C) := |B \setminus (B \cap C)|$ for all $B, C \subset K$. Define $\binom{k}{i} \times \binom{k}{i+1}$-matrix $\Psi^{(i)}$ by:

$$\Psi^{(i)}_{AD} = (-1)^{r_A(D)} \frac{r_A(D)!}{(k-i)_{(r_A(D)+1)}}$$

for all $D \in Q_{i+1}$ and $A \in Q_i$ with $i < b$.
Observe that $\Psi^{(i)}_{AD}$ is only defined if $k > 2i$. A typical entry of $\Psi^{(i)}\Theta^{(i)}$ on the diagonal is:

$$\begin{aligned}
\sum_{\substack{D \subset K \\ |D|=i+1}} \Psi^{(i)}_{AD}\Theta^{(i)}_{DA} &= \sum_{\substack{D \supset A \\ |D|=i+1}} \Psi^{(i)}_{AD} \\
&= \sum_{\substack{D \supset A \\ |D|=i+1}} (-1)^{r_A(D)} \frac{r_A(D)!}{(k-i)_{(r_A(D)+1)}} = 1
\end{aligned}$$

for any $A \in Q_i$. A typical entry of $\Psi^{(i)}\Theta^{(i)}$ not on the diagonal is:

$$\begin{aligned}
\sum_{\substack{D \subset K \\ |D|=i+1}} \Psi^{(i)}_{AD}\Theta^{(i)}_{DB} &= \sum_{\substack{D \supset B \\ |D|=i+1}} \Psi^{(i)}_{AD} \\
&= \sum_{\substack{D \supset B \\ |D|=i+1}} (-1)^{r_A(D)} \frac{r_A(D)!}{(k-i)_{(r_A(D)+1)}} \\
&= \sum_{x \in A \setminus B} (-1)^{(r_A(B)-1)} \frac{(r_A(B)-1)!}{(k-i)_{(r_A(B))}} \\
&\qquad + \sum_{x \in K \setminus (A \cup B)} (-1)^{r_A(B)} \frac{r_A(B)!}{(k-i)_{(r_A(B)+1)}} \\
&= (-1)^{(r_A(B)-1)} \left[\frac{(r_A(B)-1)!}{(k-i)_{(r_A(B))}} (r_A(B)) \right. \\
&\qquad \left. - \frac{r_A(B)!}{(k-i)_{(r_A(B)+1)}} (k-i-r_A(B)) \right] = 0
\end{aligned}$$

for any two $A, B \in Q_i$ with $A \neq B$. Hence, we can conclude that θ_i is injective for every i such that $2i < k$. $\square$

Definition 1 *For every $i = 0, 1, \ldots, b-1$ and every $A \in Q_i$ define*

$$\psi_i := (\theta_{b-1} \ldots \theta_i) : \mathcal{Z}_i \to \mathcal{Z}_b$$

and
$$g_A := [(b-|A|)!]^{-1}\psi_i f_A.$$
For $X \subset K$ with $|X| = b$ let $g_X = f_X$.

Since ψ_i is composed of monomorphisms it is itself a monomorphism. For every $i \in \{0, 1, \ldots, b-1\}$ let $\mathcal{Y}_i = \mathbf{Im}(\psi_i)$, so we get the sequence:
$$\mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \ldots \subset \mathcal{Y}_b = \mathcal{Z}_b. \tag{1}$$
Lemma 3 *For every $A \subset K$ with $|A| = i < b$ it holds that:*
$$g_A = \sum_{\substack{Y \supset A \\ |Y|=b}} f_Y.$$

Proof: Let $A \subset K$ with $|A| = i$ then
$$\begin{aligned}
(b-i)!g_A &= (\theta_{b-1}\ldots\theta_i)f_A \\
&= (\theta_{b-1}\ldots\theta_{i+1}) \sum_{\substack{B \supset A \\ |B|=i+1}} f_B \\
&= (\theta_{b-1}\ldots\theta_{i+2}) \sum_{\substack{B \supset A \\ |B|=i+1}} \sum_{\substack{C \supset B \\ |C|=i+2}} f_C \\
&= (\theta_{b-1}\ldots\theta_{i+2}) \sum_{\substack{C \supset A \\ |C|=i+2}} f_C \sum_{A \subset B \subset C} 1 \\
&= 2(\theta_{b-1}\ldots\theta_{i+3}) \sum_{\substack{C \supset A \\ |C|=i+2}} \sum_{\substack{D \supset C \\ |D|=i+3}} f_D \\
&= 2(\theta_{b-1}\ldots\theta_{i+3}) \sum_{\substack{D \supset A \\ |D|=i+3}} f_D \sum_{A \subset C \subset D} 1 \\
&= 3!(\theta_{b-1}\ldots\theta_{i+3}) \sum_{\substack{D \supset A \\ |D|=i+3}} f_D \\
&= \ldots = (b-i)! \sum_{\substack{Y \supset A \\ |Y|=b}} f_Y.
\end{aligned}$$

□

For each $k > 2b$ we define the matrix S (and the associated operator on $\mathcal{Z}_b$) as follows. The row and columns of S correspond to the elements of Q_b, and the entries are given by:
$$(S)_{XY} = \begin{cases} 1 & \text{if} \quad X \cap Y = \emptyset\ ; \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is the adjacency matrix for the Kneser graph $K(k,b)$. The action of the associated operator can be expressed as

$$S(f_X) = \sum_{\substack{Y \subseteq K \setminus X \\ |Y|=b}} f_Y.$$

Lemma 4 *For $A \subset K$ with $|A| \leq b$ we have*

$$S(g_A) = \binom{k-b-|A|}{b-|A|} \sum_{\substack{Y \subseteq K \setminus A \\ |Y|=b}} f_Y.$$

Proof:

$$S(g_A) = \sum_{\substack{X \supseteq A \\ |X|=b}} S(f_X) = \sum_{\substack{X \supseteq A \\ |X|=b}} \sum_{\substack{Y \subseteq K \setminus X \\ |Y|=b}} f_Y.$$

Interchanging the order of summation we get

$$S(g_A) = \sum_{\substack{Y \subseteq K \setminus A \\ |Y|=b}} f_Y \sum_{A \subseteq X \subseteq K \setminus Y} 1.$$

The second sum is just the number of ways of choosing the $b-|A|$ members of $X \setminus A$ from the set $(K \setminus Y) \setminus A$, which has size $k-b-|A|$. □

Lemma 5 *For $A \subset K$ with $|A| \leq b$ we have*

$$\sum_{\substack{Y \subseteq K \setminus A \\ |Y|=b}} f_Y = \sum_{B \subseteq A} (-1)^{|B|} g_B.$$

Proof: The sum over all Y that do not intersect A can be evaluated by the sieve principle:

$$\sum_{Y \subseteq K \setminus A} f_Y = \sum_{B \subseteq A} (-1)^{|B|} \sum_{\substack{Y \supseteq B \\ |Y|=b}} f_Y = \sum_{B \subseteq A} (-1)^{|B|} g_B.$$

□

Putting above together, with $A \subset K$ and $|A| = i$ gives:

$$S(g_A) = \binom{k-b-i}{b-i} \sum_{B \subseteq A} (-1)^{|B|} g_B,$$

and with $\lambda_i := (-1)^i \binom{k-b-i}{b-i}$ for $i = 0, 1, \ldots, b$ it can be rewritten as:

$$S(g_A) = \lambda_i \sum_{B \subseteq A} (-1)^{|A \setminus B|} g_B = \lambda_i \left(g_A + \sum_{B \subset A} (-1)^{|A \setminus B|} g_B \right) \tag{2}$$

Lemma 6 *For every fixed $i \in \{1, 2, \ldots, b\}$, the operator*

$$(S - \lambda_i) : \mathcal{Y}_j \to \mathcal{Y}_j$$

is bijective for all $j = 0, 1, \ldots, i-1$.

Proof: It is sufficient to prove that it is a surjection. Fix $i = 1, 2, \ldots, b$. We will prove the result by induction on j. For $j = 0$ $\ dim\, \mathcal{Y}_0 = 1$ and

$$(S - \lambda_i)(g_\emptyset) = (\lambda_0 - \lambda_i)\, g_\emptyset.$$

Since $\lambda_0 \neq \lambda_i$ we have that $(S - \lambda_i)$ is surjective on $\mathcal{Y}_0$.
Suppose that

$$(S - \lambda_i) : \mathcal{Y}_j \to \mathcal{Y}_j$$

is a surjection for some $j < i - 1$.
Now let $A \subset K$ be such that $|A| = j + 1$. then

$$(S - \lambda_i)(g_A) = (\lambda_{j+1} - \lambda_i)\, g_A + \lambda_{j+1} \sum_{B \subset A} (-1)^{|A \setminus B|}\, g_B.$$

Since all $B \subset A$ are of size less or equal to j we have that

$$\sum_{B \subset A} (-1)^{|A \setminus B|}\, g_B \in \mathcal{Y}_j$$

and by above assumption follows that there are linear combinations $p_B \in \mathcal{Y}_j$, such that

$$(S - \lambda_i) \left(\sum_{B \subset A} (-1)^{|A \setminus B|}\, p_B \right) = \sum_{B \subset A} (-1)^{|A \setminus B|}\, g_B.$$

Observe that $\lambda_i \neq \lambda_{j+1}$ for $j < i - 1$.
By induction on j follows that $g_A \in \mathbf{Im}\,(S - \lambda_i)$. Hence

$$(S - \lambda_i)\, \mathcal{Y}_{j+1} = \mathcal{Y}_{j+1}.$$

$\square$

From the above argument we can further deduce the following lemma.

Lemma 7 *For every $i = 1, 2, \ldots, b$ it holds that:*

$$\mathbf{Im} \left[(S - \lambda_i)\big|_{\mathcal{Y}_i} \right] = \mathcal{Y}_{i-1}.$$

Proof: From lemma 6 and from (1) follows that

$$(S-\lambda_i)(\mathcal{Y}_j) \supseteq \mathcal{Y}_{i-1}.$$

From (2) follows

$$(S-\lambda_i)(\mathcal{Y}_j) \subseteq \mathcal{Y}_{i-1}.$$

$\square$

Theorem 1 *For* $i = 0, 1, \ldots, b \quad \lambda_i = (-1)^i \binom{k-b-i}{b-i}$ *are eigenvalues of* S. *Furthermore, the eigenspaces* $\mathcal{E}_i$ *corresponding to the eigenvalue* λ_i *are of dimension*

$$dim\, \mathcal{E}_i = \begin{cases} \binom{k}{i} - \binom{k}{i-1} & \text{if } \; i \in \{1,2,\ldots,b\}; \\ 1 & \text{if } \; i = 0. \end{cases}$$

Proof: From previous lemma follows that

$$dim\, \mathbf{Ker}(S-\lambda_i)\big|_{\mathcal{Z}_i} = \begin{cases} \binom{k}{i} - \binom{k}{i-1} & \text{if } \; i \in \{1,2,\ldots,b\}; \\ 1 & \text{if } \; i = 0. \end{cases}$$

hence

$$dim\, \mathbf{Ker}(S-\lambda_i) \geq \begin{cases} \binom{k}{i} - \binom{k}{i-1} & \text{if } \; i \in \{1,2,\ldots,b\}; \\ 1 & \text{if } \; i = 0. \end{cases}$$

This means that λ_i are eigenvalues and that

$$dim\, \mathcal{E}_i \geq \begin{cases} \binom{k}{i} - \binom{k}{i-1} & \text{if } \; i \in \{1,2,\ldots,b\}; \\ 1 & \text{if } \; i = 0. \end{cases}$$

for $i = 0, 1, \ldots, b$. Now, since

$$dim\, \mathcal{Z}_b \geq \sum_{i=0}^{b} dim\, \mathcal{E}_i \geq 1 + \sum_{i=1}^{b} \binom{k}{i} - \binom{k}{i-1} = \binom{k}{b} = dim\, \mathcal{Z}_b$$

it follows that the inequalities are in fact equalities. $\square$

Example: Let $k = 5$ and $b = 2$. This gives us the Petersen's graph. Using above we can calculate the eigenvalues:

$$\lambda_0 = 3, \qquad \lambda_1 = -2, \qquad \lambda_2 = 1.$$

Using equation (2) it is easy to find corresponding eigenfunctions e_B where $B \subset K = \{1,2,3,4,5\}$ and $|B| < 2$. For $B = \emptyset$ we get:

$$e_\emptyset = g_\emptyset,$$

for $B = \{x\}$:

$$e_x = 5g_x - 2g_\emptyset$$

and for $B = \{x, y\}$:

$$e_{xy} = 6g_{xy} - 2(g_x + g_y) + g_\emptyset.$$

Hence there are $\binom{5}{|B|}$ eigenfunctions respectively. To get the right multiplicities obtained above there have to be $\binom{5}{|B|-1}$ linear relations between these eigenfunctions for $|B| \geq 1$. Using lemma 3, it follows that

$$\begin{aligned}
\sum_{x \in K} e_x =& 5 \sum_{x \in K} \sum_{\substack{Y \supset (\emptyset \cup \{x\}) \\ |Y|=2}} f_Y - 10 g_\emptyset \\
=& 5 \sum_{\substack{Y \supset \emptyset \\ |Y|=2}} f_Y \sum_{x \in Y} 1 - 10 g_\emptyset = 0
\end{aligned}$$

and for any fixed $y \in K$

$$\begin{aligned}
\sum_{x \in K \setminus \{y\}} e_{xy} =& 6 \sum_{x \in K \setminus \{y\}} \sum_{Y \supseteq \{x,y\}} f_Y - 2 \sum_{x \in K \setminus \{y\}} \sum_{Y \supset \{x\}} f_Y - 8 g_{\{y\}} + 4 g_\emptyset \\
=& 6 \sum_{Y \supset \{y\}} f_Y \sum_{x \in Y \setminus \{y\}} 1 - 4 \sum_{Y \supset \emptyset} f_Y + 2 g_{\{y\}} - 8 g_{\{y\}} + 4 g_\emptyset \\
=& 0,
\end{aligned}$$

which gives exactly the desired linear relations.

4 The relation between S and T_k

Classifying the k-colourings of K_b according to set of colours used gives us $\binom{k}{b}$ classes each of size $b!$. After rearranging the rows and columns of T_k according to these classes we can think of T_k as partitioned into blocks of size $b! \times b!$. Each of these blocks is $O_{b!}$ (the all 0's matrix of size $b!$) or $J_{b!}$ (the all 1's matrix of size $b!$). S is the corresponding $0-1$ matrix. Note that $J_{b!}$ has eigenvalues $b!$ with multiplicity 1 and 0 with multiplicity $b! - 1$.
To get a relation between the spectrum of S and T_k we use a general result given in [Cv] pp. 30-32. Using the there defined *Kronecker product*, denoted by $\otimes$, we get that

$$T_k = S \otimes J_{b!}.$$

It is a known fact that the eigenvalues of $J_{b!}$ are

$$\begin{array}{lll} b! & \text{with multiplicity} & 1 \\ 0 & \text{with multiplicity} & b!-1. \end{array}$$

From [Cv, Theorem 2.3.4] we can deduce that the spectrum of T_k is given by eigenvalues

$$\mu_i(k) = (-1)^i \binom{k-b-i}{b-i} b! = (-1)^i (k-b-i)_{(b-i)} (b)_{(i)},$$

with respective multiplicities

$$m_i(k) = \begin{cases} \binom{k}{i} - \binom{k}{i-1} & \text{if} \quad i \neq 0 \,; \\ 1 & \text{if} \quad i = 0 \end{cases}$$

for $i = 0, 1, \ldots, b$ and eigenvalue 0 with multiplicity $(b! - 1)\binom{k}{b}$. Hence by turning the $\mu_i(k)'$s and $m_i(k)$,s into functions of the complex variable z we get the chromatic polynomial

$$P(C_n[K_b]; z) = \sum_{i=1}^{b} m_i(z) \mu_i(z)^n.$$

References

[NB1] Biggs N.L. (1999).
A matrix method for chromatic polynomials,
CDAM Research Report Series,
LSE-CDAM-99-03

[NB2] Biggs N.L. (1999).
The chromatic polynomial of the $3 \times n$ *toroidal square lattice*,
CDAM Research Report Series,
LSE-CDAM-99-05

[NB3] Biggs N.L. (1999).
Chromatic polynomials for twisted bracelets,
CDAM Research Report Series,
LSE-CDAM-99-08

[BCN] Brouwer A.E., Cohen A.M., Neumaier A. (1989).
Distance-Regular Graphs,
Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 18,
A Series of Modern Surveys in Mathematics,
Springer-Verlag Berlin Heidelberg

[Cv] Cvetović D., Rowlinson P., Simić S. (1997).
Eigenspaces of Graphs,
Encyclopedia of Mathematics and its Applications, Vol.66,
Cambridge University Press

[RSC] Shock R., Shu-Chiuan Chang.
Chromatic Polynomials for Tubes of Complete Graphs,
Preprint, December 1999.