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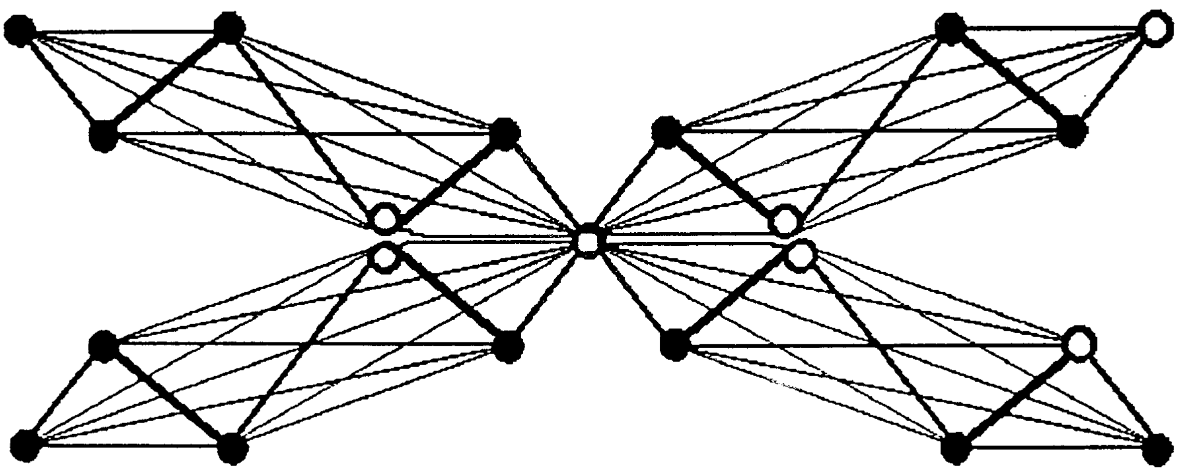

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THE MÉTRO PROBLEM: FARE CLASSES AND CROWDING EQUILIBRIA

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1. Introduction

A typical Paris Métro train consists of five cars. One is first class, and costs 7.80 francs: four are second class and cost 5.20 francs. Apart from the price (and the resulting crowding levels), first and second class are identical. For this reason, the Métro is a paradigm for a fare class system where the only quality differentiation is anticipated crowding. The aim of this paper is to develop a model within which the optimization of such a fare class system, with respect to various objectives, can be analyzed. The applications of our model are not restricted to public transport, a single supplier, or to situations where the word 'classes' is typically used. Many other examples where quality is affected by crowding are equally covered, for example pricing of lift passes at nearby competing ski resorts.

We model the problem of a supplier who offers several classes of service of a quasi-public good. He decides on prices and capacities (e.g. number of cars in a train) for these classes. Heterogeneous consumers then self select themselves into these classes on the basis of price and anticipated crowding levels. Under certain assumptions, a unique equilibrium results in which no consumer wishes to unilaterally move to another class. Since the supplier can predict the equilibrium resulting from his price–capacity choices, he can choose them optimally with respect to profit maximization, passengers travelling, or consumer welfare. We compare one and two class systems with respect to their performance on several of these criteria.

The starting point of our theory is a two stage model of consumer choice. We begin with a primitive notion of 'comfort' as a physical measure of crowding, which is increasing in capacity and decreasing in population. The evaluation of comfort is assumed to be the same for all consumers. What distinguishes consumers is a single parameter called 'income'. Consumer preferences over classes with ex post crowding levels are modeled by a utility function which depends on comfort and remaining income (income minus price). Our single nontrivial hypothesis is that the marginal utility of comfort is increasing in income. From this and other technical hypotheses we are able to derive a number of consequences such as existence and partial uniqueness of equilibrium, and the fact that at equilibrium richer people will be in higher priced classes. Of course this last prediction is violated in practice when some poor people travel in first class, but we feel it is the right first order approximation for a simple model where consumers are distinguished by a single parameter.

When we consider the supplier's optimal response to consumer's predictable choices, we obtain a number of qualitative consequences, at least for specific comfort and utility functions. For example we find that, when facing a profit maximizing supplier, the rich may prefer a one-class system while the middle class prefers a two-class system. Another result is that for low revenue constraints, consumer welfare can be as high in a one-class system as in a two-class system, but not for sufficiently high revenue constraints. The point is not that observations such these have general validity for a fare class system, but rather that these questions can be considered within our simple model, for arbitrary comfort and utility functions.

An important practical question related to our model is whether consumers will actually self select into the equilibrium predicted by our theory. If consumers have perfect information about the preferences (i.e., income) of others, and rational expectation on the behaviour of others, then they can predict equilibrium congestion levels. However neither of these conditions may hold in practice. To deal with this problem we also consider a simulation model of a two class train, where consumers in each station select the class of service myopically based on crowding levels on the incoming train. Here we assume no knowledge of other consumers' income or preferences. We find that the system (train) quickly reaches a dynamic equilibrium whose average properties closely approximate the equilibrium predicted by our theoretical model.

In studying the consumption of quasi-public goods, club theory has proved useful. The literature on entrepreneurial clubs, where an entrepreneur rather than club members decide on membership, is clearly related to what we do in this paper (e.g. Scotchmer 1985). Each fare class in our model can be thought of as an entrepreneurial club. The distinguishing feature of our analysis is that we model the interaction between the classes. In this sense our model integrates club theory ideas with quality differentiation ideas. The main departures from the quality differentiation literature (e.g. Mussa and Rosen) are that in our model, consumer choice is interdependent and quality is a priori identical and only differentiated ex post after consumption. In our model high price is not only an *indicator* of quality, but actually a cause of quality.

The ideas we consider in this paper are also similar to models of queueing in which classes of customers are offered different priorities. The priorities determine the queue discipline and therefore determine how much time a consumer spends in the system. Also, the electricity pricing model of Viswanathan and Tse (1989) is closely related to our work. They consider the problem of designing a menu of prices and corresponding priorities.

The priorities are degrees of reliability of supply given that capacity is uncertain at any point in time. Consumers with a high priority are cut off later than consumers with a low priority. Our model is completely deterministic however and consumers are solely interested in the comfort of travelling, not in securing a seat. In Viswanathan and Tse's model, the uncertainty is the rationale for operating a priority system. Also, in their model, the only decision variables are the prices whereas we allow the supplier to optimize prices and capacities.

The paper is organized as follows. In the next section we assume a given choice of prices and capacities and we develop the demand side in the form of a model of consumer behavior where a consumer's utility depends on the price and comfort of the chosen class. We define 'crowding equilibria' and show existence and uniqueness. This section is fairly formal and rigorous, and the casual reader is advised to go directly to section 3 where a specific example of a crowding equilibrium is given. In section 4 we model the supply side. We derive prices and capacities for both profit and welfare maximization and we analyze the regulatory policy implications of our findings. We also consider a duopoly model where distinct suppliers control and receive the profits of first and second class, and we calculate the unique Nash equilibrium. In section 5 we present a dynamic model of consumer selection into classes of a train. At each stop, consumers of various incomes enter a class based on its price and the comfort levels of classes on the incoming train. A final section 6 has brief concluding remarks.

2. Consumer Choices

In this section we consider a game in which the set X of consumers distribute themselves over a finite set Γ of *compartments* γ. The consumers know the price $p(\gamma)$ and capacity $s(\gamma)$ of each compartment γ (p and s will be regarded as fixed and given in this section). The capacity of a compartment is measured in terms of the fraction of the consumers it can accommodate, formalized via a probability measure μ on X. Thus a partition $\{Y_\gamma\}_{\gamma\in\Gamma}$ of consumers among compartments is called *capacity-feasible* if $\mu(Y_\gamma) \leq s(\gamma)$ for all $\gamma \in \Gamma$. To ensure the existence of at least one such partition, we assume a special compartment γ^* with $p(\gamma^*) = 0$ and $s(\gamma^*) = 1$. The main result of this section (Theorem 1) is the existence and essential uniqueness of a consumer partition $\{Y_\gamma\}_{\gamma\in\Gamma}$ called a *crowding equilibrium* (C.E.) in which no consumer wishes to change compartments. This result is proved under specific assumptions of how consumer preferences are based on crowding and prices.

Our analysis is based on a grouping of compartments into *classes* according to price. Specifically, if $p_1 > p_2 > \cdots > p_m = 0$ are the distinct set of prices, then we say that the compartments $\gamma \in \Gamma_i$ (those with $p(\gamma) = p_i$) are *i'th class*. (We follow the convention that first class has the highest price.) The lowest (m'th) class contains the compartment γ^*. For every consumer partition $\{Y_\gamma\}_{\gamma \in \Gamma}$ into compartments there is an associated partition $\{Y_i\}_1^m$ into classes given by $Y_i = \cup_{\gamma \in \Gamma_i} Y_\gamma$. The distinction between compartments and classes can be overlooked at first reading by assuming all compartments have distinct prices and hence that each class consists of a single compartment.

2.1. Crowding and comfort.

The interaction between customers is incorporated into our model through the use of comfort functions which depend on crowding. The comfort level in each compartment γ is given by a function $c(\gamma) = f_\gamma(n)$, where $n = \mu(Y_\gamma)$ is the fraction of the population in compartment γ. We assume that

$$f_\gamma(n) \text{ is continuous and decreasing in } n \text{ and } f_\gamma(s(\gamma)) = 0 \qquad (1)$$

except for $\gamma = \gamma^*$ for which $c(\gamma^*) = 0$. In examples we will often make the simplifying assumption that all the comfort functions f_γ are obtained from a common formula in terms of population fraction and capacity, $f_\gamma(n) = f(n, s(\gamma))$, where

$$f \text{ is continuous, decreasing in } n \text{, increasing in } s \text{, and } f(s, s) = 0.. \qquad (2)$$

Note that in our definitions of comfort functions we are implicitly allowing situations where $n > s$ and compartments are overfilled. In these situations, which are allowed for theoretical purposes which will be clear later, we require that comfort is negative. It is this requirement, and the existence of a price zero, comfort zero alternative γ^*, that ensures that compartments will not be overfilled at a crowding equilibrium.

The main objective of this section is to show that there is a consumer partition among compartments in which no one wishes to change compartments. This will depend on a subtle interaction between price and comfort *between* classes. However we can now deal with a simpler equilibrium among compartments *within* a given class, where price is constant and comfort is the sole determining factor.

Lemma 1. (Intraclass Equilibrium) *Fix any class i and let $\Gamma_i = \{\gamma_1, \ldots, \gamma_K\}$ be the compartments with price p_i. For any population fraction n, $0 \leq n \leq 1$, there is a unique* *numerical distribution* $z \in \Delta(n)$, *where* $\Delta(n) = \{z : 0 \le z_k \le n, \sum_{k=1,\dots,K} z_k = n\}$, *which equalizes comfort levels across the nonempty compartments of* Γ_i. *This comfort level, denoted* $F_i(n)$, *is called the compound comfort function of class* i *and has the same properties (1) as the simple compartmental comfort functions* f_γ, *with* $s(\gamma)$ *replaced by the class capacity* $s_i = \sum_{k=1,\dots,K} s(\gamma_k)$. *That is,* F_i *is decreasing and continuous, and equals 0 at* s_i.

Proof. For $z \in \Delta(n)$, let

$$h(z) = \min_{k:z_k>0} f_{\gamma_k}(z_k)$$

denote the minimum comfort level among the occupied compartments of class i when the consumer distribution is z. Since each compartment comfort function is continuous, it follows that $h(z)$ is upper semicontinuous. Thus the compactness of $\Delta(n)$ ensures the existence of the maximin,

$$F_i(n) = \max_{z\in\Delta(n)} h(z) = h(\bar{z}(n)),$$

for some $\bar{z}(n) \in \Delta(n)$. It is easy to see that any maximin distribution $\bar{z}(n)$ equalizes comfort levels among occupied compartments, i.e. that $F_i(n) = f_{\gamma_k}(\bar{z}_k)$ for all k with $\bar{z}_k > 0$. (Otherwise small numbers of consumers could be transferred from all compartments to one with higher comfort, raising the minimum comfort level.) It then follows from (1) that $\bar{z}_k(n)$ can be uniquely determined as the population in γ_k which gives a comfort level of $F_i(n)$. If $n = s_i$, the class capacity, then the distribution $\hat{z}_k = s(\gamma_k)$ which fills each compartment with its compartment capacity gives 0 comfort in each compartment. Clearly any other distribution loads some compartment to overcapacity, resulting in a negative minimum comfort. Hence $F_i(s_i) = 0$, as claimed. The remaining assertion, that $F_i(n)$ is continuous and decreasing in n, follows from the hypotheses (1) of these properties for f_γ.

□

The above lemma shows that at a crowding equilibrium all occupied compartments within a class must have the same comfort level. For this reason we will henceforth assume that there is a comfort vector $C = (c_1, \dots, c_m)$ defined over classes, which of course will depend on the number of consumers in each class according to the formula

$$c_i = F_i(\mu(Y_i)) \tag{3}$$

<u>Example 1</u>

As an example, assume comfort in each compartment is the following simple function of the density in the compartment:

$$c(\gamma) = f_\gamma(n) = \frac{s(\gamma) - n}{s(\gamma)} \text{ where } n = \mu(Y_\gamma) \tag{4}$$

This means that comfort is measured as the fraction of the compartment capacity which is not occupied. If all compartments in a class have this comfort function, it is easy to see that the compound comfort function is $F(n) = (s - n)/s$ where s is the total capacity of the class and n is the number of passengers in the class.

Example 2

We now consider the more interesting example of a class with two compartments and quadratic compartment comfort functions:

$$f_\gamma(n) = (s(\gamma))^2 - n^2, \gamma = \gamma_1, \gamma_2. \quad (5)$$

Assume $s(\gamma_1) > s(\gamma_2)$. As long as $s(\gamma_1)^2 - n^2 > s(\gamma_2)^2$, only compartment 1 will be used and therefore the class comfort function is

$$F(n) = s(\gamma_1)^2 - n^2, \quad \text{if} \quad n \leq \sqrt{s(\gamma_1)^2 - s(\gamma_2)^2} \quad (6)$$

When both compartments are used, their comfort levels are equalized i.e. $s(\gamma_1)^2 - z_1^2 = s(\gamma_2)^2 - z_2^2$. Since the total number in the class is $n = z_1 + z_2$, we can rewrite this equation as follows

$$z_1 = \frac{s(\gamma_1)^2 - s(\gamma_2)^2 + n^2}{2n} \quad (7)$$

and the compound comfort function is

$$F(n) = s(\gamma_1)^2 - \left(\frac{s(\gamma_1)^2 - s(\gamma_2)^2 + n^2}{2n}\right)^2 \quad \text{if} \quad n > \sqrt{s(\gamma_1)^2 - s(\gamma_2)^2} \quad (8)$$

In figure 1 the compound comfort function is drawn for capacities $s(\gamma_1) = 0.7$, $s(\gamma_2) = 0.3$, and figure 2 represents the distribution of passengers over compartments for these same capacities.

2.2. Consumer income and preferences.

Up to this point we have only modeled consumers in the physical sense, i.e. that they have bulk which results in crowding. This concept is fully modelled through the measure μ on the consumer set X. We now differentiate between consumers by assigning each an *income* $I(x)$ and then model the utility such an individual obtains in compartment

γ in terms of its price $p(\gamma)$ and comfort $c(\gamma)$. We assume that the income function $I : X \rightarrow [0, I_{max}]$ has a distribution G, where

$$G(I) = \mu\{x : I(x) \leq I\} \quad \text{is continuous in } I. \tag{9}$$

In order to define our notion of a crowding equilibrium, it will be necessary to describe the preferences of individuals over the compartments. We assume that these preferences are determined by a function U that gives the utility of an individual in a compartment in terms of the comfort c of the compartment, and the "remaining income" w after paying the price of the compartment. The income remaining to an individual of income I after paying the compartment price p is $w = I - p$. Thus the utility function is given by

$$U(c, w) = U(c, I - p), \quad \text{where} \tag{10}$$

$$U(c, w) = 0 \quad \text{for } w \leq 0; \quad \frac{\partial U}{\partial c} > 0, \; \frac{\partial U}{\partial w} > 0, \text{for } c, w \geq 0, \text{and} \tag{11}$$

$$\frac{\partial^2 U}{\partial c \, \partial w} > 0, \text{and continuous.} \tag{12}$$

The interpretation of the assumptions on U is as follows: Assumption (11) will ensure that at equilibrium no one will be in a compartment with either negative comfort or a price higher than their income, since the utility they would then obtain is less than they would get from compartment γ^*. (The utility of an individual with income I in compartment γ^* is $U(0, I) \geq 0$, since $c(\gamma^*) = 0$ and $p(\gamma^*) = 0$.) Assumption (12) says that the marginal utility of comfort is increasing in income, and will result in a perfect correlation of comfort, price and income at equilibrium. In other words, this is the assumption that will put the richest individuals in first class. This assumption is similar to the single crossing property in self-selection models (see for example Cooper, 1984). It is certainly true that this assumption is sometimes violated in practice (and that sometimes poor people place a sufficiently high value on comfort so that they travel first class). We make this assumption not only as an approximation to reality but (more importantly) because without it we can guarantee neither the existence nor the uniqueness of a crowding equilibrium. The main consequence of assumption (12) is the following.

Lemma 2. (rarity of indifference between classes) *For any distinct classes $i < j$, with $p_i > p_j$ and positive comfort levels c_i, c_j, there is at most one income level $\bar{I}$ such that individuals with that income are indifferent between class i and class j. If such an $\bar{I}$ exists then individuals with income $I > \bar{I}$ will prefer the (higher priced) class i and those with income $I < \bar{I}$ will prefer class j. In this case $\bar{I}$ is a continuous function of c_i and c_j. If no such income $\bar{I}$ exists, then all individuals prefer the lower priced class j.*

Proof. Let c_i, c_j be the comfort levels in classes i, j. For $k = i, j$ let $u_k(I) = U(c_k, I - p_k)$ denote the utility to an individual with income I of (any compartment of) class k. First observe that if $c_i \leq c_j$ then $u_i(I) < u_j(I)$ for all I, since class j has lower price and higher comfort. So we may assume that $c_j < c_i$. Suppose $u_i(\bar{I}) = u_j(\bar{I})$ for some income $\bar{I}$. It follows from (12) that $\frac{\partial^2 U}{\partial I \partial c} > 0$ which means that classes with higher comfort have higher marginal utility of income. In particular, since $c_j < c_i$ it follows that $u_j'(I) < u_i'(I)$ for all I (where $'$ denotes derivative with respect to income I). Hence $u_j(I) < u_i(I)$ for $I > \bar{I}$ and $u_j(I) > u_i(I)$ for $I < \bar{I}$, as required. To see the upper semicontinuity of $\bar{I}$ as a function of c_i, c_j, fix any $I_0 > \bar{I}$, so that $u_i(I_0) > u_j(I_0)$. Since $u_j(I)$ and $u_i(I)$ are continuous in c_i, c_j, it follows that for sufficiently small changes from c_i, c_j to c_i^*, c_j^* we still have that $u_j^*(I_0) > u_i^*(I_0)$ (where $u_k^*(I) = U(c_k^*, I - p_k)$). Hence the same value I_0 satisfies $I_0 > \bar{I}^* = \bar{I}(c_i^*, c_j^*)$. This argument proves that $\bar{I}$ is upper semicontinuous: a similar argument proves lower semicontinuity. Thus $\bar{I}$ is continuous, as required. □

Let $L_i = L_i(P, C) \equiv \{I : u_i(I) = \max_k u_k(I)\}$ be the set of incomes for which class i has the highest utility for the given price and comfort vectors over classes. It follows immediately from Lemma 2 that, for any price and comfort vectors $P = (p_1, \ldots, p_m)$, $C = (c_1, \ldots, c_m)$ for the classes, the sets L_i are (possibly empty) closed intervals which pairwise intersect at most at endpoints and whose union equals the full income range $[0, I_{max}]$. Furthermore, for $i < j$ $(p_i > p_j)$, the interval L_i lies to the right of L_j ($x_i \in L_i$, $x_j \in L_j$ implies $I(x_i) \geq I(x_j)$).

2.3. Definition and properties of crowding equilibria.

Thus far in this chapter we have been developing a theory of choices among compartments motivated by an intuitive notion of a crowding equilibrium. In this section we formalize that notion and derive some of its elementary properties.

Definition: *Given price and capacity functions $p, s : \Gamma \to R_+$, a partition $\{Y_\gamma\}_{\gamma \in \Gamma}$ of the consumers X into compartments Γ is called a Crowding Equilibrium (C.E.) if it satisfies, for all $\alpha \in \Gamma$, (and with $c(\gamma) = f_\gamma(\mu(Y_\gamma))$)*

$$x \in Y_\alpha \iff U(c(\alpha), I(x) - p(\alpha)) = \max_{\gamma \in \Gamma} U(c(\gamma), I(x) - p(\gamma)). \tag{13}$$

In other words, $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a C.E. if each individual is in a compartment maximizing his utility, given where the other individuals are. Unfortunately we cannot directly prove the existence of a C.E. because certain choices (i.e., best compartment) may be nonunique in our (later) fixed point argument, due to indifference between compartments within a class. However we have shown (Lemma 2) that indifference between classes is rare, so our argument will work at the level of classes. For this reason we will now define a classwise version of a C.E. (called a Classwise Crowding Equilibrium, C.C.E.), show it exists (Theorem 1), and then use it to obtain a full C.E. via Lemma 1. The following definition makes explicit use of the class comfort function F_i derived in Lemma 1.

Definition: *Given price and capacity functions $p, s : \Gamma \to R_+$, a partition $\{Y_i\}_1^m$ of the consumers X into classes $i = 1, \ldots, m$ is called a **Classwise Crowding Equilibrium** (C.C.E.) if it satisfies, for all $i = 1, \ldots, m$, (and with $c_i = F_i(\mu(Y_i))$*

$$x \in Y_i \iff U(c_i, I(x) - p_i) = \max_{1\leq j\leq m} U(c_j, I(x) - p_j). \qquad (14)$$

There is an obvious relation between the two types of crowding equilibria, which follows immediately from their definitions. For the sake of emphasis, we state this relation as follows.

Lemma 3. *Let $\{Y_\gamma\}_{\gamma\in\Gamma}$ be a partition of X into compartments, and let $\{Y_i\}_1^m$ be its associated class partition ($Y_i = \cup_{p(\gamma)=p_i} Y_\gamma$). Then (a) if $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a C.E. then $\{Y_i\}_1^m$ is a C.C.E.: and (b) if $\{Y_i\}_1^m$ is a C.C.E. and comfort is constant within each class. then $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a C.E. .*

We now derive several straightforward but important properties of crowding equilibria.

Lemma 4. (Properties of a Crowding Equilibrium) *Suppose $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a C.E. and $\{Y_i\}_1^m$ is its associated class partition (C.C.E.). Then*

a) $\{Y_\gamma\}_{\gamma\in\Gamma}$ is capacity feasible and price feasible.

($\mu(Y_\gamma) \leq s(\gamma)$ and $I(Y_\gamma) \subseteq [0, p(\gamma)]$, $\gamma \in \Gamma$.)

b) Comfort is constant in (nonempty compartments of) a class; greater in higher classes.

($c(\gamma) = c_i$ if $p(\gamma) = p_i$; $Y_i \neq \emptyset$ and $p_i > p_j$ imply $c_i > c_j$.)

c) Richer individuals are in higher (priced) classes.

($i \neq j$, $x \in Y_i$, $x_j \in Y_j$, $p_i > p_j$ imply $I(x_i) > I(x_j)$.)

Proof. (a) An individual in an overfilled car has negative comfort c, by (1). Hence that individual would prefer the zero comfort and zero price in the alternative compartment γ^*. Similarly, an individual paying a price higher than his income has negative "remaining income" w, and hence zero utility, by the first part of (11). But any individual has positive utility in γ^*, by the final inequality of (11). (b) Within a class, utility depends only on comfort, so an individual in a less comfortable compartment would move to a more comfortable one. (Of course an empty compartment might have lower comfort.) For the second assertion observe that if $c_i \leq c_j$, then for any individual in Y_i, a move to Y_j increases utility by (11). (In other words, a dominated class is empty.) c) An immediate consequence of Lemma 2.

□

2.4. Existence and uniqueness of crowding equilbria.

We are now in a position to establish the existence and uniqueness of a classwise crowding equilibrium. The idea of the existence proof is as follows. We suppose the consumers are distributed arbitrarily among the classes. We interview each individual and ask him where he would like to move to, assuming the others stay put. We call this transformation from one partition to the next (according to preferences expressed at the interview), T. In fact, we work with numerical distributions rather than the partitions themselves. The iterates of T do not converge (cycling is likely) but we can successfully use Brouwer's Theorem to obtain a fixed point.

Theorem 1. *For any given price and capacity functions p, s over the compartments Γ, there exists a unique Classwise Crowding Equilibrium $\{Y_i\}_1^m$.*

Proof. (existence) For numerical distributions z (z_i consumers in class i) in the simplex $\Delta = \{z = (z_1, \ldots, z_m) : \ 0 \leq z_i \leq 1. \sum_{i=1}^m z_i = 1\}$, let $C(z)$ be the comfort vector given by $C_i(z) = F_i(z_i)$. For any comfort vector C, let $D = D(C)$ be the *demand* vector given by $d_i(C) = \mu\{x : I(x) \in L_i(P, C)\}$. Recall that $L_i(P, C)$ is the set of incomes for which class i maximizes utility when the price vector is P and the comfort vector is C. Hence $d_i(C)$ is the fraction of the population whose utility is maximized in class i. Given the possibility of ties (indifference between classes) we must consider whether D(C) belongs to Δ (the entries might sum to more than one). However Lemma 2 guarantees that only finitely many income levels can be indifferent between classes, and then our assumption that the income distribution G is continuous ensures that the fraction of consumers with those incomes is zero. Hence $D(C)$ always belongs to Δ. Hence we may define a map

$$T : \Delta \to \Delta, \quad \text{by} \quad T(z) = D(C(z)). \tag{15}$$

The map $C : \Delta \to R^m$ is continuous by Lemma 1, since $C_i(z) = F_i(z_i)$. The continuity of the map $D : R^m \to \Delta$ follows from the continuity of U in c, Lemma 2 (continuity of I in comfort), and the assumed continuous distribution of income over X. It follows that the composite map T is continuous, and since the domain (and range) Δ is a simplex, we may apply Brouwer's Theorem to obtain a fixed point $\bar{z} = T(\bar{z})$. Once we have the invariant numerical distribution $\bar{z} \in \Delta$ it is an easy matter to find the C.C.E. $\{Y_i\}_1^m$. Begin by defining

$$Y_1 = \{x : 1 - G(I(x)) \leq \bar{z}_1\} = \{x : I(x) \in L_1(P, C(\bar{z}))\}. \tag{16}$$

This simply says we put the $\bar{z}_1$ richest consumers into first class. Similarly, for $i = 2, \ldots, m$, define

$$Y_i = \{x : \sum_{j=1}^{i-1} \bar{z}_j < 1 - G(I(x)) \leq \sum_{j=1}^{i} \bar{z}_j\} = \{x : I(x) \in L_i(P, C(\bar{z}))\} \tag{17}$$

It follows from this construction that $\mu(Y_i) = \bar{z}_i$, $i = 1, \ldots, m$. Hence the comfort vector for the partition $\{Y_i\}_1^m$ is exactly $C(\bar{z})$. Consequently the definition of Y_i (17) is equivalent to the definition (14) of a C.C.E., completing the proof of existence.

(uniqueness) Suppose there are two C.C.E.'s $\{Y_i^A\}_1^m$, and $\{Y_i^B\}_1^m$, with corresponding numerical distributions z^A, z^B given by

$$z_i^A = \mu(Y_i^A), \quad i = 1, \ldots, m \tag{18}$$

$$z_i^B = \mu(Y_i^B), \quad i = 1, \ldots, m. \tag{19}$$

It follows from Lemma 4c that distinct C.C.E.'s have distinct numerical distributions (since the latter determine the former). Hence there is a class k satisfying

$$k = \max\{i : z_i^A \neq z_i^B\}. \tag{20}$$

Without loss of generality we may assume that $z_k^A > z_k^B$. Since the elements of both z^A and z^B sum to one, it follows that there is a class j satisfying

$$j = \max\{i : 1 \leq i \leq k-1, z_i^A < z_i^B\}. \tag{21}$$

The above definitions imply that $z_i^A \geq z_i^B$ for $i = j+1, \ldots, m$, and $z_k^A > z_k^B$, implying that

$$\sum_{i=j+1}^{m} z_i^A > \sum_{i=j+1}^{m} z_i^B. \tag{22}$$

Let x be the poorest individual in Y_j^B. This means that $G(I(x)) = \sum_{i=j+1}^{m} z_i^B$, i.e., that x is richer than exactly those consumers in $Y_i^B, i \geq j+1$. But according to equation (22), there are consumers in Y_{j+1}^A who are richer than x. Hence $x \in Y_{j+r}^A$, for some $r \geq 1$. It now follows from the fact that $x \in Y_j^B \cap Y_{j+r}^A$ and the requirement (14) for a C.C.E., that

$$U(c_j^B,\ I(x) - p_j) \geq U(c_{j+r}^B,\ I(x) - p_{j+r}) \quad \text{and} \tag{23}$$
$$U(c_{j+r}^A,\ I(x) - p_{j+r}) \geq U(c_j^A,\ I(x) - p_j). \tag{24}$$

But it follows from the inequalities on z_i^A and z_i^B, and the monotonicity of comfort in crowding (1), that $c_j^A > c_j^B$ and $c_{j+r}^B \geq c_{j+r}^A$. Hence the monotonicity of utility in comfort (11) gives

$$U(c_j^B,\ I(x) - p_j) < U(c_j^A,\ I(x) - p_j) \quad \text{and} \tag{25}$$
$$U(c_{j+r}^B,\ I(x) - p_{j+r}) \geq U(c_{j+r}^A,\ I(x) - p_{j+r}). \tag{26}$$

But the last two sets of inequalities are inconsistent, so our assumption of distinct C.C.E.'s was wrong.

□

3. An example of a crowding equilibrium.

In general it is not possible to give a closed form expression for the crowding equilibrium resulting from price and capacity decisions of the supplier. However we will do so in this section for specific functional forms of the utility and comfort functions, and for a specific income distribution. We will present the example in a simple and informal manner that is largely independent of the rigorous treatment of the previous section, so that the reader may start reading the paper here.

Consider a train consisting of two cars; one first class, and one second class. Suppose the respective prices are given as p_1, p_2 and the respective capacities are given as s_1, s_2. Individuals who chose not to travel (or are priced out of travelling) are said to belong to third class, which has price 0. Individuals' incomes I are uniformly distributed on the unit interval $[0, 1]$. An individual's preferences among the three classes is based on their respective prices and comfort levels, as follows.

We assume that the comfort levels c_1, c_2 in the two travelling classes is determined by the "unoccupied fraction" comfort function of example 1. This means that if the population of class i is n_i, then the comfort is

$$c_i = \frac{s_i - n_i}{s_i} \quad i = 1, 2. \tag{27}$$

Comfort in third class (not travelling) is assumed to be zero. The utility to any individual of travelling in a class with comfort c and which results (after paying the appropriate fare price) in remaining positive income $w = I - p$, is assumed to be $U(c, w) = c\,w^+$, $(w^+ = \max(w, 0))$. Observe that this utility function satisfies the hypothesis of the previous section, that the marginal utility of comfort is increasing in income. Also observe that an individual travelling in a class he cannot afford ($p > I$) obtains utility zero, which will prevent him from doing so at equilibrium. The class utility functions $u_i(I)$ describing the utility to an individual of income I travelling in class i are given by:

$$u_i(I) = U(c_i, I - p_i) = \begin{cases} c_i\,(I - p_i), & \text{if } I \geq p_i \\ 0, & \text{if } I < p_i \end{cases} \tag{28}$$

The aim of this section is to determine the unique crowding equilibrium resulting from the above functional forms and parameters. That is, we wish to determine passenger numbers n_1, n_2 and n_3, as a function of p_1, p_2, s_1, s_2, so that when

1. the n_1 individuals with incomes in the range $1 - n_1 < I <= 1$ travel in first class,
2. those next richest $n_2 = 1 - n_1 - p_2$ in the range $p_2 < I \leq_1 - n_1$ travel second class, and
3. the poorest $n_3 = p_2$ don't travel at all

then no individual would prefer to be in a different class.

Of course since p_2 is a known parameter, it is clear from the above equations that it is sufficient to find n_1, since it determines n_2 and n_3. (The fact that at equilibrium richer individuals will be in higher classes was demonstrated in the previous section, Lemma 4c.) The reason that, at equilibrium, anyone with income above p_2 will travel is that positive utility always results from affordable travel.

The problem of finding the equilibrium is simply the problem of finding the indifference income $\bar{I}$ between first and second class which is guaranteed by Lemma 2. (Indifference between second and third class is clearly at p_2.) If the price differential between first and second class is sufficiently high, clearly no one will travel in first class at equilibrium. However ignoring this degenerate (and easily analyzed) case, we see that $\bar{I}$ is determined by the equation $c_1\,(\bar{I} - p_1) = c_2\,(\bar{I} - p_2)$, so that at equilibrium we have the following three equations:

$$\bar{I} = \frac{c_1\,p_1 - c_2\,p_2}{c_1 - c_2}$$

$$n_1 = 1 - \bar{I} \tag{29}$$
$$n_2 = \bar{I} - p_2$$

If we replace the comfort levels c_1 and c_2 in the equation for $\bar{I}$ by their values in terms of n_i, s_i we can solve (with some difficulty) the three equations for the equilibrium values $\bar{I}, \tilde{n}_1$ and $\tilde{n}_2$ in terms of the price and capacity parameters. The equilibrium numbers $\tilde{n}_1$, $\tilde{n}_2$ of first and second class travellers are given by the formulae:

$$\tilde{n}_1(p_1, p_2) = \frac{1}{2}\left[2 - 2p_2 + \frac{s_2}{s_1 + s_2} - \frac{p_1\, s_2}{s_1 + s_2} - \sqrt{d}\right], \quad \tilde{n}_2(p_1, p_2) = 1 - p_2 - \tilde{n}_1(p_1, p_2). \tag{30}$$

where

$$d = \frac{-4(s_1 - 2p_2\, s_1 + p_2^2\, s_1 - p_1\, s_1\, s_2 + p_2\, s_1\, s_2)}{s_1 + s_2} + \left(-2 + 2\,p_2 + \frac{s_2}{s_1 + s_2} + \frac{p_1\, s_2}{s_1 + s_2} - \frac{2\,p_2\, s_2}{s_1 + s_2}\right)^2.$$

4. Supplier choices

The theory developed in section two, as illustrated by the example in section three, describes how consumers may react to given prices and capacities for the different fare classes. We say "may" because one cannot of course guarantee that the unique crowding equilibrium will in fact obtain. We will deal with this practical problem in section five, where we use simulation to show that even myopic consumers who are unaware of the preferences or incomes of others will dynamically approximate the crowding equilibrium. For the moment we will deal with the problem confronting the supplier, who chooses the prices and capacities of the fare classes, assuming the unique crowding equilibrium will result. That is, the supplier acts as the Stackelberg leader in a two stage game. We are thus led to the question of what the supplier is trying to achieve.

In this section we will outline how various possible objectives of the supplier can be optimized within the framework of our model. We will do this for the particular model of section 3, since we have explicitly calculated the crowding equilibrium resulting from arbitrary prices and capacities in a two class (plus third nontravelling class) situation. This means that we are assuming that comfort c in a class is given by $(s - n)/s$ where

n and s are the number of passengers and capacity of the class, and utility is given by $c(I-p)$. To make the problem amenable to algebraic solution we shall further assume fixed capacities of 1 for the two classes ($s_1 = s_2 = 1$), so that the supplier's only choice variables are the prices. (In actuality, capacities are indeed often fixed in the short run, so this restriction of convenience is not too damaging.) With this restriction, the crowding equilibrium calculated in equation (30) reduces to

$$\tilde{n}_1 = \frac{p_1 + 2p_2 - 3 + \sqrt{d}}{-4}, \; \tilde{n}_2 = 1 - \tilde{n}_1 - p_2 \quad \text{where } d = (1+p_1)^2 + 4p_2(p_1 - 1 - p_2). \tag{31}$$

We will consider a number of possible objectives for the supplier, who may be a private monopolist or a public utility. In section 4.1 we consider the profit maximization problem. We then (section 4.2) briefly analyze the welfare implications for individiuals of varying income, of profit maximization with one compared with two classes. In section 4.3 we consider the objective of maximizing the total number of travelling passengers ($\tilde{n}_1 + \tilde{n}_2 = 1 - \tilde{n}_3$) subject to an arbitrary revenue constraint. This has been suggested as a possible objective for a public transport system. Section 4.4 analyzes the problem of optimizing (revenue constrained) "total welfare" if we allow interpersonal comparisons of utility. Section 4.5 considers a duopoly model where two suppliers of a similar good (e.g., nearby ski resorts) set prices to maximize their own profits. If we call one of these the first class supplier, then our model applies and we can find the Nash equilibrium prices. We summarize the crowding equilibria corresponding to these various supplier objectives in section 4.6.

4.1. Profit maximization

Assuming that costs are fixed and independent of the numbers travelling, we may set costs equal to zero and maximize the following profit function

$$\pi(p_1, p_2) = p_1 \tilde{n}_1 + p_2 \tilde{n}_2 = p_1 \tilde{n}_1 + p_2 (1 - \tilde{n}_1 - p_2). \tag{32}$$

where $\tilde{n}_1$ and $\tilde{n}_2$ are the numbers of passengers in first and second class in the crowding equilibrium corresponding to the prices p_1 and p_2 as given in equation (31). Fixing p_2, it is possible to solve for the value $p_1 = o(p_2)$ which maximizes the profit function π, so that

$$\zeta(p_2) \equiv \pi\left(\phi(p_2), p_2\right) = \max_{p_1 \geq p_2} \pi(p_1, p_2). \tag{33}$$

The function ϕ may be expressed as a closed form function of p_2, but the expression is too long to print. However we have graphed it, along with the restricted maximum profit function ζ, in Figure 3. The first order condition on ζ then gives the optimal prices $\bar{p}_2 = .447$, and $\bar{p}_1 = \phi(\bar{p}_2) = .593$ and the maximum profit as $\bar{\pi} = .27$. The equilibrium numbers in the fare classes are $\bar{n}_1 = .127$, $\bar{n}_2 = .426$, $n_3 = .447$ and the equilibrium comfort levels are $\bar{c}_1 = .87$ and $\bar{c}_2 = .57$. We will need these numbers for comparative purposes later.

4.2. Welfare consequences of profit maximization

In the previous section we considered how a monopolist would set prices if two classes were allowed. Actually the number of fare classes that may be used is often exogenous. determined by custom, physical constraints, or law. Planes typically have three (economy. business, first) and trains may have one. two, or three. Suppose that a public governing body may restrict the number of fare classes available to a profit maximizing monopolist. What should it do? This is a complicated problem which we cannot fully answer here, but we can certainly analyze it within our framework and get relevant data for our particular model.

We now ask how each income level does under a single class profit maximizing equilibrium compared with the two class maximum derived in the previous section. If the monopolist operated a single class with the same total capacity ($s = 2$) as above. his sole choice variable would be the single price p. Since the utility function $c(I-p)$ that we are using gives positive utility to affordable travel, the equilibrium demand for travel is simply $n(p) = 1 - p$. Hence the single class profit function is $p(1-p)$, which has a maximum of .25 at $\bar{p} = .5$, with $\bar{n} = 1 - .5 = .5$ people travelling. Thus the comfort level in the single class is given by $\bar{c} = (s - \bar{n})/s = 1.5/2 = .75$, and the utility of an individual with income I travelling in this class ($I > .5$) is thus $u(I) = .75(I - .5)$.

Recalling the equilibrium prices and comfort levels derived in the previous section for a two class profit maximization, we see that the class comfort functions u_i in the two classes are given by $u_1(I) = \bar{c}_1\,(I - \bar{p}_1)^+ = .87\,(I - .593)^+$ and $u_2(I) = .57\,(I - .447)^+$. In Figure 4 we have graphed the single class utility function $u(I)$ and the two class utility function $\max(u_1(I), u_2(I))$, which give the utility an individual of income I receives at equilibrium from a profit maximizer operating one or two classes. These two functions intersect at $I = .66$, with the single class utility higher for lower I, and the two class utility greater for higher incomes I. (Those indivuals with income below $\bar{p}_2$, who cannot afford to travel in either scenario, are of course indifferent to the two scenarios.) Thus for the particular comfort and utility functions of this section, we have the counter-intuitive result that the rich prefer one class and the "middle class" prefer two classes. The explanation for this is simply that the extra class enables the monopolist to extract more money from the rich by crowding the second class. We also observe that the extra class allows more people to travel at equilibrium, which may be regarded as socially desirable.

4.3. Maximizing passengers, with revenue constraint

Much of our analysis up to this point, while presented in a "train" metaphor, could apply equally well to the pricing of any good where crowding affects quality. However, for the particular application of trains, or other public transport, it may be socially advantageous to increase the number of passengers. The reasons for this may be ecological or economic (reducing overall travel times). For example in the latest Blackett Memorial Lecture, D. Quarmby (1991) suggests that the objective of London Transport is, approximately, to maximize passenger miles subject to financial limits. Since our model doesn't distinguish between trips of different lengths, we restrict our analysis here to maximizing the number of passengers travelling, subject to an arbitrary revenue constraint.

In our two class model, any individual whose income is above the second class price p_2 will travel. Hence the total number of travellers is given by $1 - p_2 = \tilde{n}_1 + \tilde{n}_2$. So we ask the related question, what is the maximum revenue (profit) achievable when the price of second class is p_2? Luckily we have already calculated this function, which we called $\zeta(p_2)$, and graphed in Figure 3. Therefore the function $N(r) = 1 - \zeta^{-1}(r)$ (where ζ^{-1} is the inverse of ζ on the monotone domain $[0, \bar{p}]$) answers the original question of giving the maximum number of travelling passengers in an equilibrium yielding revenue r. The function $N(r)$ is graphed in Figure 5.

The properties of the "total passengers" function $N(r)$ are easily explained. Its height is 1 if the required revenue can be obtained with $p_2 = 0$. We consider this case. If the price in first class is p_1 the indifference income $\bar{I}$ is determined by the equation $(\bar{I})(\bar{I} - p_1) = (1 - \bar{I})(\bar{I} - 0)$, giving comfort times remaining income in first and second class. Thus we have $\bar{I} = (1 + p_1)/2$ and the profit is $\tilde{n}_1 p_1 = (1 - \bar{I}) p_1 = ((1 - p_1)/2) p_1$, which has a maximum at $p_1 = .5$, in which case $\tilde{n}_1 = .25$ people travel first class giving a profit of .125. Thus $r = .125$ is the highest revenue which can be achieved with everyone travelling. The last point on the graph is $(\bar{\pi}, 1 - \bar{p}_2) = (.27, .553)$, as calculated for the unrestricted profit maximum in section 4.1. Revenues above .27 cannot be achieved at all.

4.4. Maximizing welfare, with revenue constraint

We now consider the problem faced by a supplier who wishes to maximize the welfare of the individuals using the fare class system, subject to a revenue constraint. For this purpose we use the simplest measure of group welfare, the total utitity of the population. This is the only place in the paper where we assume that interpersonal comparisons of utility are possible. Specifically, we define the total welfare of class i to be

$$W_i = \int_{a_i}^{b_i} c_i (I - p_i)\, dI \qquad i = 1, 2, \tag{34}$$

if individuals of income $a_i < I \leq b_i$ are in class i, and the total welfare W to be

$$W = W_1 + W_2. \tag{35}$$

At the crowding equilibrium resulting from the prices p_1, p_2, the total welfare is given by the formula

$$W(p_1, p_2) = \int_{p_2}^{\bar{I}} c_2 (I - p_2)\, dI + \int_{\bar{I}}^{1} c_1 (I - p_1)\, dI, \tag{36}$$

where $c_i = 1 - \tilde{n}_i(p_1, p_2)$ with $\tilde{n}_i$ from (31), and $\bar{I} = 1 - \tilde{n}_1(p_1, p_2)$.

Our aim in this section is to analyze the two class constrained maximum welfare function Ψ given by

$$\Psi(r) = \max_{\pi(p_1, p_2) \geq r} W(p_1, p_2) \tag{37}$$

and compare it with the corresponding maximum for a single class.

$$\Psi^1(r) = \max_{\pi(p,p) \geq r} W(p, p). \tag{38}$$

A theoretical upper bound for all these total welfare functions can be obtained by considering the unconstrained welfare optimization problem which is clearly solved by zero prices in both the single or two-class case. In either case the comfort is $c = (2-1)/2 = (1-(1/2))/1 = .5$, the average remaining income is $.5$, and hence $W = W(0,0) = (.5)(.5) = .25$. Thus the unconstrained welfare maximization problem can be solved with a single class. We shall see below that this property persists for low revenue constraints.

The single class constrained optimum Ψ^1 can be determined analytically, by considering the profit and welfare functions of the single price p,

$$\pi(p) = p(1-p)$$
$$W(p) = \int_p^1 (2-(1-p))(I-p)\,dI = (1+p)(1-p)^2/2. \tag{39}$$

If we eliminate p from the system (39) and solve for $W = \Psi^1$ in terms of revenue (profit) $r = \pi$, we obtain

$$\Psi^1(r) = \frac{1-r+(1+r)\sqrt{1-4r}}{8}. \qquad 0 \leq r \leq .25. \tag{40}$$

(Recall that $r = .25$ is the maximum single class profit.) We graph the single class welfare optimum Ψ^1 in Fig. 6, together with scatter points which will be explained later.

The two class welfare function $W(p_1, p_2)$ may be written explicitly using the equilibrium populations $\tilde{n}_i$ given by equation (31). However, it is not possible to explicitly solve for W in terms of π in the manner we did above for a single class. However we show in Fig. 6 a scatter plot of $(\pi(p_1,p_2), W(p_1,p_2))$ for $0 \leq p_2 \leq p_1 \leq 1$ with $p_i \in \{0, .05, .1, \ldots, .95, 1\}$. Observe that when $r = \pi$ is less than about $.1$, these points are on or below the single class welfare curve Ψ, so that for low revenue constraints the extra class does not create welfare improvements. However, for higher revenue constraints the extra class does allow small welfare improvements. The welfare corresponding to the two-class profit maximizing prices is $W(\bar{p}_1, \bar{p}_2) = W(.59, .45) = .09$, so the rightmost plotted point approximates $(.27, .090)$, where the $.27$ is the two-class maximum profit. However the welfare corresponding to the single-class profit maximizing price of $.5$ (profit $.25$) is the larger number $W(.5) = \Psi^1(.25) = 3/32 = .094$. This means (for the comfort and utility functions of this section) that a public body which can restrict the profit maximizing supplier to a single class should do so, if its aim is maximizing total welfare.

4.5. Nash equilibrium in a duopoly with crowding effects

Up to now we have only considered the case where a single supplier controls first and second class. However our model of consumer choice (section two) makes no assumptions about who is deciding on the price and quantity vectors, and therefore applies equally well to the case of several competing suppliers. An illustrative example might be the case of two nearby ski resorts with fixed lift capacity, where 'comfort' is affected by crowding. To conform with our previous terminology, we call the services provided by the two supplier 'classes'.

In this section we consider the situation where two separate profit maximizers control the prices of first and second class, and we find the unique pure strategy Nash equilibrium price pair $p_2^* < p_1^*$. Each supplier $i \in \{1,2\}$ sets the price p_i and obtains the profit $\pi_i(p_1,p_2) = p_i\, n_i(p_1,p_2)$, the revenue collected in class i with the (crowding) equilibrium number $\tilde{n}_i$ (of equation (31)) of passengers.

In Figure 7 we plot supplier 1's profit π_1 as a function of his choice of price p_1, for various second class prices $p_2 = .1, .2, \ldots, .9$. It can be seen that for $p_2 \geq .4$, supplier 1 would like to respond to a second class price p_2 with an equal or (if allowed) lower first class price. However for $p_2 \leq .3$, there is a unique optimal response $p_1 = R_1(p_2)$ which is higher than p_2. This optimal response function R_1 is given by the formula

$$R_1(p_2) = A(p_2) - \frac{B(p_2)}{C(p_2)^{1/3}} + C(p_2)^{1/3}, \quad 0 \leq p_2 <\approx .34, \quad \text{where}$$

$$A(x) = \frac{-(3+10x)}{6}$$
$$B(x) = \frac{-(3+10x)^2}{36} + \frac{x\,(-10-6x-4x^2)}{6}$$
$$D(x) = \frac{-(3+10x)^3}{216} + \frac{x\,(3+10x)\,(-10-6x-4x^2)}{24} - \frac{-1+5x-x^2+4x^4}{4}$$
$$C(x) = D(x) + \sqrt{B(x)^3 + D(x)^2}.$$

The range of the response function R_1 is approximately the p_1 interval $[.32, .44]$, and we can similarly compute supplier 2's optimal response to any price p_1 in that range, (with C the same in terms of B and D) as

$$R_2(p_1) = A(p_1) - \frac{B(p_1)}{C(p_1)^{1/3}} + C(p_1)^{1/3}, \quad .32 \le p_1 <\approx .44, \quad \text{where}$$

$$A(y) = \frac{-(16-24y)}{48}$$

$$B(y) = \frac{-(16-24y)^2}{2^8\,3^2} + \frac{-3-16y+5y^2}{48}$$

$$D(y) = \frac{-(16-24y)^3}{2^{12}\,3^3} - \frac{y\,(2+4y+2y^2)}{32} + \frac{(16-24y)\,(-3-16y+5y^2)}{2^9\,3}.$$

In Figure 8 we graph the two response equations $p_1 = R_1(p_2)$ and $p_2 = R_2(p_1)$ for the relevant domains. The unique intersection point ($p_1^* = .356$, $p_2^* = .307$) cannot be computed algebraically but can be approximated by iteration of the composed function $R_1 \circ R_2$. At this Nash equilibrium, the crowding equilibrium passenger numbers are $\tilde{n}_1 = .303$ in first class and $\tilde{n}_2 = .389$ in second class. The total welfare function at this duopoly Nash equilibrium (see previous section) is $W(p_1^*, p_2^*) = .150$, which compares very favorably with that computed earlier for the profit maximizing monopolist of $W(\bar{p}_1, \bar{p}_2) =$ .09 for 2 classes or .094 for one class. Thus where feasible, competition is favorable for the consumer.

5. Discrete equilibration dynamics

In the previous sections we have discussed the properties of a crowding equilibrium, but we have not considered the question of how it may be achieved. Since the equilibrium is unique (Theorem 1), it may of course be argued that consumers could do the necessary (fixed point) calculations to correctly predict it. This seems an unlikely scenario, and in any case it requires that consumers know the income distribution. To counter these problems, we now introduce a very simple model of consumer behaviour which quickly leads to a discrete, dynamic approximation to the crowding equilibrium. In brief, we consider a two class train which proceeds from station to station, dropping off and picking up customers. Our basic behavioural assumption is that each customer decides which class to enter based only on the price-comfort pair for the two classes of the train *as it enters the station.*

We now decribe our model more formally. We assume the train has fixed prices p_1, p_2 and capacities s_1, s_2 in the two classes. Since individuals with incomes less than p_2 will not travel, we divide the remaining income interval $[p_2, 1]$ into m equal subintervals, and consider only income levels $j = 1, \ldots, m$ corresponding to incomes $I(j)$ at the center of these intervals. This is a discrete model for the uniform distribution. We assume that at each station there are two consumers of each income level who will enter the train (in one of the two classes) and ride for k stations. Thus the *total* population of the train is always $2mk$. For purposes of comfort calculations, we therefore assume that each individual has mass $\alpha = (1 - p_2)/2mk$. Each individual on the train is characterized by an income level j and an integer $i \in \{1, \ldots, k\}$ describing his remaining number of stations. Since there are only two classes, the composition of the train is fully determined by the matrix A, where $a_{ij} \in \{0, 1, 2\}$ is the number of individuals in first class of income level j who have i stations remaining. Let $\mathcal{A}$ denote the set of all possible train compositions, i.e. the set of all $k \times m$ matrices with entries in $\{0, 1, 2\}$.

The dynamics of the model will be expressed in terms of a transformation $T : \mathcal{A} \to \mathcal{A}$, where a train entering a station with composition A will leave with composition $T(A)$. The action of T with respect to individuals staying on the train is obvious: their number of remaining stops decreases by 1, and their income level is unchanged. If we denote $A' = T(A)$, this can be expressed as

$$a'_{ij} = a_{i+1,j} \quad \text{for } i \leq k - 1.$$

Since the numbers of consumers in both classes of the incoming train, $N1$, $N2$, can be computed from its composition matrix A, the onlookers at the station can compute the two current comfort levels $c_1(A)$ and $c_2(A)$. Each individual of income level j waiting for the train, can calculate the two utilities $u_1(j) = U(c_1(A), I(j) - p_1)$ and $u_2(j) = U(c_2(A), I(j) - p_2)$. He then enters the class yielding the greater utility calculation. Of course, he may regret this decision immediately, if comfort levels change. In summary, our behavioural rule for individuals entering the train (i.e., with $i = k$) is

$$a'_{kj} = \begin{cases} 2, & \text{if } u_1(j) \geq u_2(j) \\ 0, & \text{otherwise.} \end{cases}$$

It should be observed that the two individuals of each income level j entering at each station always enter the same class. The only reason for the doubling of types in our model is for the initialization used below, where the initial composition of the train has these two individuals in different classes.

A full analysis of the discrete dynamical system $T : \mathcal{A} \to \mathcal{A}$. for arbitrary values of the parameters (comfort functions, utility functions, capacities, prices, and the discrete model parameters m and k) is beyond the scope of this paper. We content ourselves with modelling the example given in section 3.1. Recall that example has a train with first and second class cars, each with capacity one. Comfort in each class is given by $1 - n = 1 - N\alpha$, where n denotes the population fraction in that class, N the number of individuals. Utility is given by $U(c, w) = c w^+$. We assume each individual travels $k = 10$ stations and that there are $m = 10$ income levels between p_2 and 1. We assume the prices are the profit maximizers determined in section 3.1, $p_1 = \bar{p}_1 = .593$ and $p_2 = \bar{p}_2 = .447$.

In Table 1 we have run the dynamical system T for the above parameters, starting with a balanced train $A(0)$, which has the two individuals of each type split one each between first and second class. The initial matrix $A(0)$ has its bottom row in the row labeled $t = 0$, and similarly the matrix $A(t) = T^t(A(0))$ has its bottom row in row t. (All matrices are 10×10. The time evolution of the train may be described as follows. Since $A(0)$ is the matrix of ones, it has population $N1 = 100 = N2$. Thus comfort on this train is the same in first and second class, and so all income levels will prefer to enter second class. This is indicated by the row $t = 1$ consisting of all zeroes. The resulting configuration for first class, $A(1)$ has all ones except for a bottom row of zeroes. As this pattern continues, the population in first class decreases. By time 5, the incoming train has $N1 = 50$ people in first class, and those on the platform with income levels 9 and 10 choose to enter the train (the 2's in row $t = 6$, columns $j = 9, 10$). Thereafter, the population in first class oscillates widely, but then settles down to a cycle of length 11 beginning with the matrix $A(32)$ (outlined). We call this cycle $B(0), B(1), \ldots, B(10)$, where $B(q) = A(q + 32)$, $q = 0, \ldots, 10$. In this cycle, all income levels below 8 are never present in first class, and all above 8 are always in first class. The interesting behaviour is restricted to income level eight.

The cyclic behaviour can be explained as follows. When there are $N1 = 44$ people in first class, the indifference income level between first and second class (as in Lemma 2) can be calculated as $\bar{I}_{44} = .861$. Since $I(7) < \bar{I}_{44} < I(8) = .862$, it follows that income levels 8, 9, and 10 will choose to enter the first class of an incoming train with 44 people in first class. Similarly, since $I(8) < \bar{I}_{46} = .87 < I(9) = .92$, only income levels 9 and 10 will enter an incoming train with 46 in first class. Now consider the matrix $B(0) = A(32)$.

It has six income 8's, who will get off two each at the next three stations. Its first class population is 46. Thus no new 8's will enter, and since two leave, the population of $B(1)$ is 44. Thus for the next two stations two 8's leave and two new 8's enter. Then two 8's enter and none leave, bringing the total back to 46, at which point no new 8's enter, resulting eventually in $B(0)$ again (at position $t = 43$). It is perhaps easier to see this pattern in Figure 9, where first class population and lowest income entering first class are graphed together from the data of Table 1. The composition of first class at time t is completely described by the ten data points on the lower curve, ending at t.

It is interesting to see how the limit cycle $B(0), \ldots, B(10)$ approximates the crowding equilibrium calculated for the continuous model of section 3.1, with $n_1 = .127$. In our discrete model, using the same parameters, we obtain an 11-cycle with an average of $45.45 = (3/11)44 + (8/11)46$ individuals in first class. If we multiply this by the individual's mass, $\alpha = (1 - p_2)/200 = .00276$, we find the average fraction of the population in first class to be $45.45 \times .00276 = .126$. This is in fact not only close to the continuous model, it is the closest possible in our model. To see this oberve that if we increased to population 46 for 9/11 of the time, the average fraction of the population in first class would go up to .131, which is a bigger error. (The length of a cycle must divide $k + 1 = 11$.)

We have also checked how long it takes, starting from a random (equiprobable and independent entries 0 or 2) initial train composition A, to reach the limit cycle matrix $B(0)$. The distribution of the stopping time q for which $T^q(A) = B(0)$ is given in the following table, for 50 randomizations of the initial train composition $A(0)$. (The initialization has a_{ij} equiprobably 0 or 2.)

Table 2: Stopping time q for $B(0)$ from random start .

q	29	30	31	32	42	43	44
frequency	1	3	4	13	1	15	13

6. Conclusion

In this paper we analyzed the effects of crowding on demand for a quasi–public good which is offered to consumers in 'classes' which are identical except for price (and perhaps capacity). We modeled consumer preferences among the classes in terms of a common physical notion of 'comfort' and a compound utility function which depends on comfort, price, and a consumer's 'remaining income'. Consumers are identical except for their initial 'income'.

For this model we showed that however the supplier sets prices and capacities for the classes, consumers choices amongst them are uniquely characterized by what we call a "crowding equilibrium". We also showed, by a dynamic model, that this equilibrium can be approximated by consumers with incomplete information about each other's preferences, who behave in a myopic fashion.

Given that this unique consumer equilibrium can be anticipated by the supplier, for arbitrary prices and capacities, we considered the following two stage game (for specific comfort and utility functions). In the first stage, the supplier sets prices and capacities of the classes, acting as a Stackelberg leader. Then consumers form the corresponding crowding equilibrium. We analyzed this game and found explicit solutions for several possible objectives of the supplier: profit maximization, maximizing consumption with revenue constraint, maximizing total consumer welfare. In the case of separate suppliers controlling the classes, we found the unique Nash equilibrium. Some qualitative conclusions of our analysis are the following:

(i) Richer consumers (facing profit maximizer) may prefer a one class system: poorer ones two classes.

(ii) For low revenue constraints, a single class can achieve equal total welfare to two classes, but not for higher constraints.

(iii) Total welfare is higher if the two classes are controlled by competing suppliers.

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